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Stabilization of a Class of Nonlinear ODE/Wave PDE Cascaded Systems

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ABSTRACT We investigate stabilization of a class of cascaded systems of nonlinear ordinary differential equation (ODE)/wave partial differential equation (PDE) with time-varying propagation speed based on a two-step PDE backstepping transformation. A time-varying propagation velocity of wave PDE leads to two difficulties. One is how to prove the well-posedness and uniqueness of the time-varying kernel PDEs in the first-step backstepping transformation, the other is how to construct a backstepping transform to map the original system into a suitable target system during the second-step transformation. We prove that there exists a unique continuous 2×2 matrix-valued solution to the time-varying kernel PDEs, and design a predictor control for the original cascaded system. An example is provided to illustrate the feasibility of the proposed design.

INDEX TERMS Nonlinear system, predictor control, time-varying propagation speed, two-step backstepping transformation.

I. INTRODUCTION

Since the pioneering works [1], [2] revealed PDE backstepping method as a new way to stabilize input delay systems, various interesting results [3]–[11] have been achieved. PDE backstepping design has also been utilized to control various PDE-ODE cascaded systems [12]–[15]. Boundary stabilization of one-dimensional linear hyperbolic PDEs with time and space varying parameters is presented, the well-posedness of time and space varying kernel PDEs has been solved in [16]. Global stabilization of a class of switched nonlinear systems is investigated in [17], in which a state feedback sampled-data controller is constructed by backstepping design.

In oil drilling, torsional vibrations of a drill string caused by the friction between drill bit and rock will seriously damage the drilling facilities [18]. The torsional dynamics of a drill string is modeled as a wave PDE, which is coupled with a nonlinear ODE that describes dynamics of angular velocity of the drill bit at the bottom of the drill string [19]. This engineering application inspires researchers to study how to control cascaded systems of nonlinear ODE/wave PDE.

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A predictor control is presented for nonlinear ODE/wave PDE cascaded system [20]. Stabilization of wave PDE dynamics with a moving controlled/uncontrolled boundary has been solved in [21] and [22], respectively. Boundary control of a nonlinear ODE actuated through a wave PDE with spatially-varying propagation speed is explored in [23].

Note that a variable propagation speed increases the intricacies arising in this class of problems and also causes difficulties in the analysis of control design. Therefore, the study of a time-varying propagation speed is challenging and practical. This motivates us to investigate nonlinear ODE/wave PDE cascaded system with time-varying propagation speed.

In this paper, we develop a stabilization design for cascaded system of nonlinear ODE/wave PDE with time-varying propagation speed. Based on a two-step PDE backstepping transformation and Lyapunov arguments, we prove globally asymptotical stability of the closed-loop system. In addition, the time-varying propagation speed leads to two difficulties. One is how to prove the well-posedness and uniqueness of time-varying kernel PDEs during the first-step backstepping transformation. The other is how to construct predictors during the second-step backstepping transformation. We prove that there exists a unique continuous 2×2

matrix-valued solution to the time-varying kernel PDEs, and design predictors for the second-step backstepping transformation.

This paper is organized as follows: System description and main results are in Section II. Coordinate transformations and backstepping transformations are in Section III. Stability analysis of the closed-loop system is in Section IV, and an example is in V. Concluding remarks are shown in Section VI.

Notation: The definitions of $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions are from [2]. For a scalar function $u(\cdot,t)$, $\|u(t)\|_\infty = \sup_{x\in[0,L]} |u(x,t)|$ denotes the supremum norm, and $|\cdot|$ denotes the Euclidean norm.

II. SYSTEM DESCRIPTION AND MAIN RESULTS

Consider the nonlinear ODE/wave PDE cascaded system

$$\dot{X}(t) = f(X(t), u(0,t)) \tag{1}$$
$$\partial_{tt}u(x,t) = v(t)\partial_{xx}u(x,t) \tag{2}$$
$$\partial_x u(0,t) = 0 \tag{3}$$
$$\partial_x u(L,t) = U(t), \tag{4}$$

where $X \in \mathbb{R}^n$, $u \in \mathbb{R}$, $U \in \mathbb{R}$ are ODE state, PDE state, and control input, respectively, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0) = 0$, and v is a propagation speed satisfying the following Assumption.

Assumption 1: Propagation speed $v : R \to R$ is continuously differentiable, and there are $\underline{v} > 0, \overline{v} > 0$, $M_1 > 0$ such that

$$\underline{v} = \inf v(t), \quad \overline{v} = \sup v(t), \tag{5}$$

and

$$|\dot{v}(t)| \leq M_1, \tag{6}$$

for all $t \in \mathbb{R}$.

Remark 1: Propagation speed $v > 0$ is bounded, and its rate of change is also bounded.

Denote

$$\phi(t) = \int_0^t \sqrt{v(s)}ds, \tag{7}$$

for all $t \geq 0$, and $\phi^{-1}(t)$ is the inverse function of $\phi(t)$.

A. CONTROL DESIGN

If a nominal controller $\kappa : \mathbb{R}^n \to \mathbb{R}$ is such that $\dot{X}(t) = f(X(t), \kappa(X(t)))$ is globally asymptotically stable, then a predictor control for system (1)–(4) is designed as

$$\begin{aligned} U(t) &= -\frac{\partial_t u(L,t)}{2\sqrt{v(t)}} + \frac{1}{2}\partial_x u(L,t) + \frac{1}{2\sqrt{v(t)}} \\ &\quad \times \int_0^L k_{11}(L,s,t)\left(\partial_t u(s,t) + \sqrt{v(t)}\,\partial_s u(s,t)\right)ds \\ &\quad + \frac{1}{2\sqrt{v(t)}}\int_0^L k_{12}(L,s,t)(\partial_t u(s,t) - \sqrt{v(t)}\,\partial_s u(s,t))ds \\ &\quad - \frac{e^{-\int_t^{\phi^{-1}(\phi(t)+L)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}}{2\sqrt{v(t)}} c_1(p_2(L,t) - \kappa(p_1(L,t))) \\ &\quad + \frac{e^{-\int_t^{\phi^{-1}(\phi(t)+L)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}}{2\sqrt{v(t)}} \frac{\partial\kappa(p_1(L,t))}{\partial p_1} f(p_1(L,t), p_2(L,t)), \end{aligned} \tag{8}$$

where $p_1 \in \mathbb{R}^n$, $p_2 \in \mathbb{R}$ are given by

$$\begin{aligned} &p_1(x,t) \\ &= X(t) + \int_0^x \frac{f(p_1(y,t), p_2(y,t))}{\sqrt{v(\phi^{-1}(\phi(t)+y))}}dy, \end{aligned} \tag{9}$$

$$\begin{aligned} &p_2(x,t) \\ &= u(0,t) + \int_0^x \frac{e^{\int_t^{\phi^{-1}(\phi(t)+y)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau} s_1(y,t)}{\sqrt{v(\phi^{-1}(\phi(t)+y))}}dy \\ &\quad - \int_0^x \int_y^x \frac{e^{\int_t^{\phi^{-1}(\phi(t)+y)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau} s_1(y,t)k_{11}(\alpha,y,t)}{\sqrt{v(\phi^{-1}(\phi(t)+\alpha))}}d\alpha dy \\ &\quad - \int_0^x \int_y^x \frac{e^{\int_t^{\phi^{-1}(\phi(t)+y)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau} s_2(y,t)k_{12}(\alpha,y,t)}{\sqrt{v(\phi^{-1}(\phi(t)+\alpha))}}d\alpha dy, \end{aligned} \tag{10}$$

with

$$s_1(y,t) = \partial_t u(y,t) + \sqrt{v(t)}\,\partial_y u(y,t), \tag{11}$$
$$s_2(y,t) = \partial_t u(y,t) - \sqrt{v(t)}\,\partial_y u(y,t), \tag{12}$$

for all $x \in [0,L]$. The initial conditions of (9) and (10) are given as

$$\begin{aligned} &p_1(x,0) \\ &= X(0) + \int_0^x \frac{f(p_1(y,0), p_2(y,0))}{\sqrt{v(\phi^{-1}(y))}}dy, \end{aligned} \tag{13}$$

$$\begin{aligned} &p_2(x,0) \\ &= u(0,0) + \int_0^x \frac{e^{\int_0^{\phi^{-1}(y)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau} s_1(y,0)}{\sqrt{v(\phi^{-1}(y))}}dy \\ &\quad - \int_0^x \int_y^x \frac{e^{\int_0^{\phi^{-1}(y)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau} s_1(y,0)k_{11}(\alpha,y,0)}{\sqrt{v(\phi^{-1}(\alpha))}}d\alpha dy \\ &\quad - \int_0^x \int_y^x \frac{e^{\int_0^{\phi^{-1}(y)} \frac{\dot{v}(\tau)}{4v(\tau)}d\tau} s_2(y,0)k_{12}(\alpha,y,0)}{\sqrt{v(\phi^{-1}(\alpha))}}d\alpha dy, \end{aligned} \tag{14}$$

for all $x \in [0,L]$. The gain $c_1 > 0$ in (8), and the kernel gains k_{11} and k_{12} are solutions to the following kernel PDEs:

$$\begin{aligned} \partial_t K(x,s,t) &= \mathcal{A}(t)\partial_x K(x,s,t) + \partial_s K(x,s,t)\mathcal{A}(t) \\ &\quad - K(x,s,t)\mathcal{B}(t) \end{aligned} \tag{15}$$
$$K(x,x,t)\mathcal{A}(t) = \mathcal{A}(t)K(x,x,t) + \mathcal{B}(t) \tag{16}$$
$$k_{11}(x,0,t) = k_{12}(x,0,t) \tag{17}$$
$$k_{21}(x,0,t) = k_{22}(x,0,t), \tag{18}$$

where (15)–(18) is defined on $\{(x, s, t) : 0 \le s \le x \le L, t \ge 0\}$, $K(x, s, t) = [k_{ij}(x, s, t)] \in \mathbb{R}^{2\times 2}$ and

$$\mathcal{A}(t) = \begin{pmatrix} \sqrt{v(t)} & 0 \\ 0 & -\sqrt{v(t)} \end{pmatrix}, \tag{19}$$

$$\mathcal{B}(t) = \begin{pmatrix} 0 & -\frac{\dot{v}(t)}{4v(t)} \\ -\frac{\dot{v}(t)}{4v(t)} & 0 \end{pmatrix}. \tag{20}$$

Remark 2: For an implementation of control law (8), we have to numerically integrate a finite interval in (9) and (10) by one of the numerical quadratures. In the simulations, we use the composite left-endpoint rectangle rule.

B. MAIN RESULTS

We will prove that system (1)–(4), with (8)–(10) is globally asymptotically stable under Assumptions 1–4.

Assumption 2: System $\dot{X} = f(X, \kappa(X) + \upsilon)$ is input-to-state stable with respect to υ and the function $\kappa : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with locally Lipschitz derivative $\frac{\partial\kappa(X)}{\partial X}$ and satisfies $\kappa(0) = 0$.

Let

$$Z(t) = \begin{pmatrix} X(t) \\ u(0, t) \end{pmatrix}, \quad \varphi(Z(t), \upsilon) = \begin{pmatrix} f(X, u(0, t)) \\ \upsilon \end{pmatrix}, \tag{21}$$

and μ is given as

$$\mu(Z) = -c_1(Z_2 - \kappa(Z_1)) + \frac{\partial\kappa(Z_1)}{\partial Z_1} f(Z_1, Z_2), \tag{22}$$

where $c_1 > 0$, and $Z = [Z_1, Z_2] \in \mathbb{R}^n \times \mathbb{R}$.

Remark 3: Assumption 2 guarantees that there exists a locally Lipschitz feedback law $\mu : R^{n+1} \to R$, with $\mu(0) = 0$ such that system $\dot{Z} = \varphi(Z, \mu(Z)+\nu)$ is input-to-state stable with respect to ν.

Assumption 3: For system $\dot{Z} = \varphi(Z, \upsilon)$ there exist smooth positive definite functions R_1, R_2 and class $\mathcal{K}_\infty$ functions $\alpha_1, \cdots, \alpha_6$ such that

$$\alpha_1(|Z|) \le R_1(Z) \le \alpha_2(|Z|), \tag{23}$$

$$\frac{\partial R_1(Z)}{\partial Z}\varphi(Z, \upsilon) \le R_1(Z) + \alpha_3(|\upsilon|), \tag{24}$$

$$\alpha_4(|Z|) \le R_2(Z) \le \alpha_5(|Z|), \tag{25}$$

$$-\frac{\partial R_2(Z)}{\partial Z}\varphi(Z, \upsilon) \le R_2(Z) + \alpha_6(|\upsilon|), \tag{26}$$

for all $Z \in \mathbb{R}^{n+1}$ and $\upsilon \in \mathbb{R}$.

Remark 4: Condition (23), (24), (or (25), (26)) guarantees that for every initial condition and every measurable locally essentially bounded input signal υ, the corresponding solution of $\dot{Z} = \varphi(Z, \upsilon)$ is defined for all $t \ge 0$ (or $t \le 0$).

Assumption 4: For system $\dot{Z} = \varphi(Z, \mu(Z)+\upsilon)$, there exist a smooth positive definite function R_3 and class $\mathcal{K}_\infty$ functions $\alpha_7, \alpha_8, \alpha_9$ such that

$$\alpha_7(|Z|) \le R_3(Z) \le \alpha_8(|Z|), \tag{27}$$

$$-\frac{\partial R_3(Z)}{\partial Z}\varphi(Z, \mu(Z) + \upsilon) \le R_3(Z) + \alpha_9(|\upsilon|), \tag{28}$$

for all $Z \in \mathbb{R}^{n+1}$ and $\upsilon \in \mathbb{R}$.

Remark 5: Condition (27), (28) guarantees that for every initial condition and every measurable locally essentially bounded input signal υ, the corresponding solution of $\dot{Z} = \varphi(Z, \mu(Z) + \upsilon)$ is defined for all $t \le 0$.

Theorem 1: Under Assumptions 1–4, consider system (1)–(4) together with the control law (8)–(10), for any initial condition $u(\cdot, 0) \in C_1[0, L]$, $u_t(\cdot, 0) \in C_1[0, L]$, which is compatible with the feedback law (8)–(10) and satisfies $\partial_x u(0, 0) = 0$, the closed-loop system has a unique solution $X(t) \in C_1([0, \infty), \mathbb{R}^n)$, $u_t(x, t), u_x(x, t) \in C_1([0, L] \times [0, \infty))$. Moreover, there is a $\mathcal{KL}$ function $\overline{\beta}$ such that

$$\Omega(t) \le \overline{\beta}(\Omega(0), t) \tag{29}$$

$$\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty, \tag{30}$$

for all $t \ge 0$.

III. COORDINATE TRANSFORMATIONS AND BACKSTEPPING TRANSFORMATIONS

First, introducing the following change of coordinate

$$\overline{\zeta}(x, t) = \partial_t u(x, t) + \sqrt{v(t)}\,\partial_x u(x, t), \tag{31}$$

$$\overline{\eta}(x, t) = \partial_t u(x, t) - \sqrt{v(t)}\,\partial_x u(x, t), \tag{32}$$

the reverse is

$$\partial_t u(x, t) = \frac{\overline{\zeta}(x, t) + \overline{\eta}(x, t)}{2}, \tag{33}$$

$$\partial_x u(x, t) = \frac{\overline{\zeta}(x, t) - \overline{\eta}(x, t)}{2\sqrt{v(t)}}. \tag{34}$$

Using change of coordinate (31), (32), system (1)–(4) is expressed as

$$\dot{X} = f(X, u(0, t)) \tag{35}$$

$$\partial_t\overline{\xi}(x, t) = \mathcal{A}(t)\partial_x\overline{\xi}(x, t) + \mathcal{B}_0(t)\overline{\xi}(x, t) \tag{36}$$

$$\partial_t u(0, t) = \overline{\zeta}(0, t) \tag{37}$$

$$\overline{\eta}(0, t) = \overline{\zeta}(0, t) \tag{38}$$

$$\overline{\zeta}(L, t) = \overline{\eta}(L, t) + 2\sqrt{v(t)}\,U(t), \tag{39}$$

where

$$\overline{\xi}(x, t) = \begin{pmatrix} \overline{\zeta}(x, t) \\ \overline{\eta}(x, t) \end{pmatrix}, \quad \mathcal{B}_0(t) = \begin{pmatrix} \frac{\dot{v}(t)}{4v(t)} & -\frac{\dot{v}(t)}{4v(t)} \\ -\frac{\dot{v}(t)}{4v(t)} & \frac{\dot{v}(t)}{4v(t)} \end{pmatrix}, \tag{40}$$

and $\mathcal{A}(t)$ is given by (19).

Secondly, applying the state transformations

$$\zeta(x, t) = e^{-\int_0^t \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\,\overline{\zeta}(x, t), \tag{41}$$

$$\eta(x, t) = e^{-\int_0^t \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\,\overline{\eta}(x, t), \tag{42}$$

system (35)–(39) is rewritten in the following form

$$\dot{Z}(t) = \varphi\left(Z(t), e^{\int_0^t \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\, \zeta(0,t)\right) \tag{43}$$

$$\partial_t \xi(x,t) = \mathcal{A}(t)\partial_x \xi(x,t) + \mathcal{B}(t)\xi(x,t) \tag{44}$$

$$\xi(0,t) = Q_0 \xi(0,t) \tag{45}$$

$$\zeta(L,t) = \eta(L,t) + 2\sqrt{v(t)}e^{-\int_0^t \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\, U(t), \tag{46}$$

where $\xi(x,t) = \left(\zeta(x,t),\ \eta(x,t)\right)^T$, $Q_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are defined in (19).

A. FIRST-STEP BACKSTEPPING TRANSFORMATION

We make the first-step backstepping transformation

$$\omega(x,t) = \xi(x,t) - \int_0^x K(x,s,t)\xi(s,t)ds, \tag{47}$$

for all $0 \le x \le L$, $t \ge 0$ and $K(x,s,t) = [k_{ij}(x,s,t)] \in \mathbb{R}^{2\times 2}$, to map system (43)–(46) into the following target system

$$\dot{Z}(t) = \varphi\left(Z(t), e^{\int_0^t \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\, \omega_1(0,t)\right) \tag{48}$$

$$\partial_t \omega_1(x,t) = \sqrt{v(t)}\partial_x \omega_1(x,t) \tag{49}$$

$$\partial_t \omega_2(x,t) = -\sqrt{v(t)}\partial_x \omega_2(x,t) \tag{50}$$

$$\omega_2(0,t) = \omega_1(0,t) \tag{51}$$

$$\omega_1(L,t) = \eta(L,t) + 2\sqrt{v(t)}e^{-\int_0^t \frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\, U(t) - \int_0^L (k_{11}(L,s,t)\zeta(s,t) + k_{12}(L,s,t)\eta(s,t))ds, \tag{52}$$

for $0 \le x \le L$, $t \ge 0$, and $\omega(x,t) = [\omega_1(x,t), \omega_2(x,t)]^T$.

Taking the time and spatial derivatives of (47), and with the help of (44), one can straightforwardly derive the kernel PDEs (15)–(18) to satisfy the mapping of (44) into (49), (50). In addition, it is easy to obtain (48) and (51), (52) according to (43) and (45), (46). The kernel gains k_{11} and k_{12} are solutions to the kernel PDEs (15)–(18).

The inverse transform of (47) is designed as

$$\xi(x,t) = \omega(x,t) + \int_0^x \Gamma(x,s,t)\omega(s,t)ds, \tag{53}$$

for all $0 \le x \le L$, $t \ge 0$ and $\Gamma(x,s,t) = [\Gamma_{ij}(x,s,t)] \in \mathbb{R}^{2\times 2}$, to map the target system (48)–(52) to system (43)–(46), with the inverse kernel PDEs as follows:

$$\partial_t \Gamma(x,s,t) = \mathcal{A}(t)\partial_x \Gamma(x,s,t) + \partial_s \Gamma(x,s,t)\mathcal{A}(t) + \mathcal{B}(t)\Gamma(x,s,t) \tag{54}$$

$$\Gamma(x,x,t)\mathcal{A}(t) = \mathcal{A}(t)\Gamma(x,x,t) + \mathcal{B}(t) \tag{55}$$

$$\Gamma_{11}(x,0,t) = \Gamma_{12}(x,0,t) \tag{56}$$

$$\Gamma_{21}(x,0,t) = \Gamma_{22}(x,0,t), \tag{57}$$

defined on $\{(x,s) : 0 \le s \le x \le L,\ t \ge 0\}$, with $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are given by (19).

Based on the successive approximation method, Coron, *et al.* solved the existence and uniqueness of time-varying kernel PDEs in Theorem 2.6 in [16], but the boundary conditions of the kernel PDE (15)–(18) are different from those of the kernel PDEs in Theorem 2.6 in [16]. So the result of [16] cannot be directly applied to the kernel PDEs (15)–(18). Following the method in [16], we prove the existence and uniqueness of the kernel PDEs (15)–(18).

First, writing (15)–(18) component-wise, it is equivalent to the following four sub-systems:

$$\partial_t k_{11}(x,s,t) - \sqrt{v(t)}\partial_x k_{11}(x,s,t) - \sqrt{v(t)}\partial_s k_{11}(x,s,t) = \frac{\dot{v}(t)}{4v(t)}k_{12}(x,s,t), \tag{58}$$

$$k_{11}(x,0,t) = k_{12}(x,0,t), \tag{59}$$

$$\partial_t k_{12}(x,s,t) - \sqrt{v(t)}\partial_x k_{12}(x,s,t) + \sqrt{v(t)}\partial_s k_{12}(x,s,t) = \frac{\dot{v}(t)}{4v(t)}k_{11}(x,s,t), \tag{60}$$

$$k_{12}(x,x,t) = \frac{\dot{v}(t)}{8(v(t))^{3/2}}, \tag{61}$$

$$\partial_t k_{21}(x,s,t) + \sqrt{v(t)}\partial_x k_{21}(x,s,t) - \sqrt{v(t)}\partial_s k_{21}(x,s,t) = \frac{\dot{v}(t)}{4v(t)}k_{22}(x,s,t), \tag{62}$$

$$k_{21}(x,x,t) = -\frac{\dot{v}(t)}{8(v(t))^{3/2}}, \tag{63}$$

$$\partial_t k_{22}(x,s,t) + \sqrt{v(t)}\partial_x k_{22}(x,s,t) + \sqrt{v(t)}\partial_s k_{22}(x,s,t) = \frac{\dot{v}(t)}{4v(t)}k_{21}(x,s,t), \tag{64}$$

$$k_{22}(x,0,t) = k_{21}(x,0,t). \tag{65}$$

Remark 6: Noting that $\mathcal{A}(t)$, $\mathcal{B}(t)$ are given by (19), (20), respectively, and $K(x,s,t) = [k_{ij}(x,s,t)] \in \mathbb{R}^{2\times 2}$, using matrix multiplication, (15) is equivalent to (58), (60), (62), (64), and (16) is equivalent to (61) and (63). So (15)–(18) is equivalent to the four sub-systems (58), (59); (60), (61); (62), (63); (64), (65).

Remark 7: The subsystem satisfied by (k_{21}, k_{22}) is similar to that satisfied by (k_{11}, k_{12}). Thus, from now on we only focus on the subsystem satisfied by (k_{11}, k_{12}).

Remark 8: The vales of k_{11}, k_{12} at a point $(x,s,t) \in \{(x,s,t) \in (0,L) \times (0,L) \times (0,+\infty)\}$ for sufficiently small t can not be obtained from its values on the planes $s = x$ or $x = L$. We study (58)–(59), (60)–(61) on the domain extended in time

$$P = \{(x,s,t) \in (0,L) \times (0,L) \times \mathbb{R},\ s < x\}, \tag{66}$$

in order to avoid such a condition.

For each $(x,s,t) \in \mathbb{R}^3$ fixed, $\chi_{1j}(\cdot; x,s,t), j = 1,2$, are denoted the characteristic curves associated with systems (58)–(59), (60)–(61) passing through the point (x,s,t), respectively, i.e.

$$\chi_{1j}(\tau; x,s,t) = (\chi_1(\tau; x,t), \chi_j(\tau; s,t), \tau), \quad \forall \tau \in \mathbb{R}, \tag{67}$$

where $\chi_1(\cdot; x,t)$ is the solution to the ODE

$$\partial_\tau \chi_1(\tau; x,t) = -\sqrt{v(\tau)} \tag{68}$$

$$\chi_1(t; x,t) = x, \tag{69}$$

and $\chi_2(\cdot; s, t)$ is the solution to the ODE

$$\partial_\tau \chi_2(\tau; s, t) = \sqrt{v(\tau)} \quad (70)$$
$$\chi_2(t; s, t) = s. \quad (71)$$

The existence and uniqueness of the solutions to the ODEs (68), (69); (70), (71) follow from the (local) Cauchy-Lipschitz theorem and solutions are global since $\sqrt{v(\tau)}$ is bounded. The uniqueness of the solutions to the ODE (68), (69); (70), (71) yield

$$\chi_i(\sigma; \chi_i(\tau; x, t), \tau) = \chi_i(\sigma; x, t), \quad (72)$$

for $\sigma \in \mathbb{R}$, $i = 1, 2$. It is easy to know that χ_i, $i = 1, 2$ have the regularity $\chi_i \in C^1(\mathbb{R}^3)$, $i = 1, 2$ and

$$\partial_t \chi_1(\tau; x, t) = \sqrt{v(t)}, \quad \partial_x \chi_1(\tau; x, t) = 1, \quad (73)$$

and

$$\partial_t \chi_2(\tau; s, t) = -\sqrt{v(t)}, \quad \partial_s \chi_2(\tau; s, t) = 1. \quad (74)$$

Let $\tau_1^{out}(x, t)$, $\tau_2^{out}(s, t)$ be the exit times of the flows $\chi_1(\cdot; x, t)$, $\chi_2(\cdot; s, t)$ from the domain $[0, L]$, i.e. the respective unique solutions to

$$\chi_1(\tau_1^{out}(x, t); x, t) = 0, \quad \chi_2(\tau_2^{out}(s, t); s, t) = L. \quad (75)$$

Differentiating (75) and using (73), (74), (5), it holds

$$\partial_x \tau_1^{out}(x, t) > 0, \quad \partial_s \tau_2^{out}(s, t) < 0, \quad (76)$$

and

$$\partial_t \tau_1^{out}(x, t) > 0, \quad \partial_t \tau_2^{out}(s, t) > 0, \quad (77)$$

for every $t \in \mathbb{R}$ and $x, s \in [0, L]$.

Proposition 1: There exists a unique $\tau_{11}^{out} \in C^0(\overline{P})$ with $(x, s, t) \to \tau_{11}^{out}(x, s, t) - t \in L^\infty(P)$ such that for every $t \in \mathbb{R}$, and $0 < s \le x \le L$, one has $\tau_{11}^{out}(x, s, t) > t$ (and $\tau_{11}^{out}(x, s, t) = t$ otherwise) and if $s < x$, then it holds

$$\chi_{11}(\tau; x, s, t) \in P, \quad \forall \tau \in (t, \tau_{11}^{out}(x, s, t)), \quad (78)$$

and

$$\chi_1(\tau_{11}^{out}(x, s, t); s, t) = 0. \quad (79)$$

Proof: The uniqueness follows from the properties that have to be satisfied, we will prove the existence. Integrating the ODE (68) from two sides, we get

$$\chi_1(\tau_1^{out}(x, t); x, t) - \chi_1(t; x, t) = -\int_t^{\tau_1^{out}(x,t)} \sqrt{v(\tau)} d\tau, \quad (80)$$

with the help of (75), (69), (5), from (80), one has

$$\sqrt{\overline{v}}(\tau_1^{out}(x, t) - t) \ge x \ge \sqrt{\underline{v}}(\tau_1^{out}(x, t) - t),$$

so

$$0 \le (\tau_1^{out}(x, t) - t) \le \frac{L}{\sqrt{\underline{v}}}, \quad (81)$$

clearly, $\tau_1^{out}(x, t) - t \in L^\infty(P)$. In addition, from (76), we see that for $s < x$

$$\tau_1^{out}(s, t) < \tau_1^{out}(x, t). \quad (82)$$

Using (73), we know

$$\chi_1(\tau; s, t) < \chi_1(\tau; x, t), \quad (83)$$

for any $t \le \tau \le \tau_1^{out}(x, s, t)$, $s < x$, and by (75), we have

$$\chi_1(\tau_1^{out}(x, s, t); s, t) = 0. \quad (84)$$

Clearly, $\tau_1^{out}(x, s, t)$ belongs to $C^0(\overline{P})$, so there exists $\tau_{11}^{out}(x, s, t)$ defined as follows:

$$\tau_{11}^{out}(x, s, t) = \tau_1^{out}(s, t), \quad (85)$$

which satisfies all the properties stated in the Proposition 1. □

Proposition 2: There exists a unique $\tau_{12}^{out} \in C^0(\overline{P})$ with $(x, s, t) \to \tau_{12}^{out}(x, s, t) - t \in L^\infty(P)$ such that for every $t \in \mathbb{R}$, and $0 \le s < x \le L$, one has $\tau_{12}^{out}(x, s, t) > t$ (and $\tau_{12}^{out}(x, s, t) = t$ otherwise) with

$$\chi_{12}(\tau; x, s, t) \in P, \quad \forall \tau \in (t, \tau_{12}^{out}(x, s, t)), \quad (86)$$

and

$$\chi_2(\tau_{12}^{out}(x, s, t); s, t) = \chi_1(\tau_{12}^{out}(x, s, t); x, t). \quad (87)$$

Proof: Since the uniqueness readily follows from the properties that have to be satisfied, we only prove the existence. Integrating the ODE (70) from two sides, we get

$$\chi_2(\tau_2^{out}(s, t); s, t) - \chi_2(t; s, t) = \int_t^{\tau_2^{out}(s,t)} \sqrt{v(\tau)} d\tau, \quad (88)$$

with the help of (75), (71), (5), from (88), one has

$$\sqrt{\overline{v}}(\tau_2^{out}(s, t) - t) \ge L - s \ge \sqrt{\underline{v}}(\tau_2^{out}(s, t) - t),$$

so

$$0 \le (\tau_2^{out}(s, t) - t) \le \frac{L - s}{\sqrt{\underline{v}}}, \quad (89)$$

for $t \in \mathbb{R}$, $0 \le s \le L$. We see that $\tau_2^{out}(s, t) - t \in L^\infty(P)$.

For each $(x, s, t) \in \overline{P}$, such that $0 < s \le x \le L$, we denote the C^1 function

$$g(\tau) = \chi_1(\tau; x, t) - \chi_2(\tau; s, t), \quad (90)$$

for $\tau \in [t, \min\{\tau_1^{out}(x, t), \tau_2^{out}(s, t)\}]$. Note that the interval has a non empty interior since $0 < s \le x \le L$. Using (68), (70), we have

$$g'(\tau) = -2\sqrt{v(\tau)} < 0, \quad (91)$$

for any $\tau \in \mathbb{R}$.

We will prove that the existence of $\tau_{12}^{out}(x, s, t)$ with

$$t \le \tau_{12}^{out}(x, s, t) \le \min\{\tau_1^{out}(x, t), \tau_2^{out}(s, t)\}, \quad (92)$$

and such that

$$\chi_2(\tau; s, t) < \chi_1(\tau; x, t), \tag{93}$$

for any $\tau \in (t, \tau_{12}^{out}(x, s, t))$.

It is clear that $\tau_{12}^{out}(x, s, t) - t \in L^\infty(P)$ since (81), (89). Using (90), (91), from the implicit function theorem, we get

$$\begin{cases} \tau_{12}^{out}(x, s, t) = t, & s = x, \\ g(\tau_{12}^{out}(x, s, t)) = 0, & s < x. \end{cases} \tag{94}$$

Clearly, $\tau_{12}^{out}(x, s, t)$ defined by (94) belongs to $C^0(\overline{P})$, and τ_{12}^{out} satisfies all the properties stated in the Proposition 2. □

Using Proposition 1, integrating (58), (59), along the characteristic curve $\chi_{11}(\tau; x, s, t)$ for $\tau \in (t, \tau_{11}^{out}(x, s, t))$ yields the following integral equation

$$k_{11}(x, s, t) = k_{11}(\chi_1(\tau_{11}^{out}(x, s, t); x, t), 0, \tau_{11}^{out}(x, s, t)) - \int_t^{\tau_{11}^{out}(x,s,t)} \frac{\dot{v}(\tau)}{4v(\tau)} k_{12}(\chi_{11}(\tau; x, s, t))d\tau. \tag{95}$$

From Proposition 2, integrating (60), (61) along the characteristic curve $\chi_{12}(\tau; x, s, t)$ for $\tau \in (t, \tau_{12}^{out}(x, s, t))$ yields the following integral equation

$$k_{12}(x, s, t) = \frac{\dot{v}(\tau_{12}^{out}(x, s, t))}{8(v(\tau_{12}^{out}(x, s, t)))^{3/2}} - \int_t^{\tau_{12}^{out}(x,s,t)} \frac{\dot{v}(\tau)}{4v(\tau)} k_{11}(\chi_{12}(\tau; x, s, t))d\tau. \tag{96}$$

Define the vector space B as

$$B = \{K = (k_{11}, k_{12})^T, k_{11}, k_{12} \in C^0(\overline{P}) \cap L^\infty(P)\}.$$

It can be checked that B is a Banach space when equipped with the L^∞ norm. We introduce the mapping

$$\Phi : B \to B, \tag{97}$$

for every $K \in B$, defined by

$$\Phi(K) = K^0 + \Phi_1(K), \tag{98}$$

for every $(x, s, t) \in \overline{P}$, where $K^0 = (k_{11}^0, k_{12}^0)^T$ is defined by

$$k_{11}^0(x, s, t) = k_{11}(\chi_1(\tau_{11}^{out}(x, s, t); x, t), 0, \tau_{11}^{out}(x, s, t)), \tag{99}$$

$$k_{12}^0(x, s, t) = \frac{\dot{v}(\tau_{12}^{out}(x, s, t))}{8(v(\tau_{12}^{out}(x, s, t)))^{3/2}}, \tag{100}$$

and $\Phi_1(K) = (\Phi_1(K)_{11}, \Phi_1(K)_{12})^T$ is defined by

$$\Phi_1(K)_{11}(x, s, t) = -\int_t^{\tau_{11}^{out}(x,s,t)} \frac{\dot{v}(\tau)}{4v(\tau)} k_{12}(\chi_{11}(\tau; x, s, t))d\tau, \tag{101}$$

$$\Phi_1(K)_{12}(x, s, t) = -\int_t^{\tau_{12}^{out}(x,s,t)} \frac{\dot{v}(\tau)}{4v(\tau)} k_{11}(\chi_{12}(\tau; x, s, t))d\tau. \tag{102}$$

Thanks to (99), (100), (59), and (5), (6), we see that $k_{1j}^0(x, s, t) \in C^0(\overline{P}) \cap L^\infty(P)$, for $j = 1, 2$.

Lemma 1: There exists a function $\Upsilon_1 \in C^1(\overline{P}) \cap L^\infty(P)$ and $\varepsilon_0 > 0$ such that for every $(x, s, t) \in \overline{P}$, it holds $\Upsilon_1 \geq 0$, with

$$\partial_t\Upsilon_1(x, s, t) - \sqrt{v(t)}\partial_x\Upsilon_1(x, s, t) - \sqrt{v(t)}\partial_s\Upsilon_1(x, s, t) \leq -\varepsilon_0, \tag{103}$$

$$\partial_t\Upsilon_1(x, s, t) - \sqrt{v(t)}\partial_x\Upsilon_1(x, s, t) + \sqrt{v(t)}\partial_s\Upsilon_1(x, s, t) \leq -\varepsilon_0. \tag{104}$$

Proof: Let

$$\Upsilon_1(x, s, t) = \gamma_1^L(x, t) - \gamma_1^\nu(s, t), \tag{105}$$

where for any $\nu \in (0, L]$, γ_1^ν is the solution to the following linear hyperbolic equation:

$$\partial_t\gamma_1^\nu(x, t) - \frac{L\sqrt{v(t)}}{\nu}\partial_x\gamma_1^\nu(x, t) = 0, \tag{106}$$

$$\gamma_1^\nu(0, t) = t, \tag{107}$$

for $x \in [0, L]$, $t \in \mathbb{R}$. The solution of (106)–(107) is

$$\gamma_1^\nu(x, t) = \gamma_1^\nu(0, \tau_1^{out,\nu}(x, t)) = \tau_1^{out,\nu}(x, t), \tag{108}$$

where $\tau_1^{out,\nu}(x, t) \geq t$ (with $\tau_1^{out,\nu}(x, t) = t \Leftrightarrow x = 0$) is the unique number such that

$$\chi_1^\nu(\tau_1^{out,\nu}(x, t); x, t) = 0, \tag{109}$$

where $\tau \to \chi_1^\nu(\tau; x, t)$ is the solution to the ODE

$$\partial_\tau\chi_1^\nu(\tau; x, t) = \frac{-L\sqrt{v(\tau)}}{\nu}, \tag{110}$$

$$\chi_1^\nu(t; x, t) = x, \tag{111}$$

for any $\tau \in R$. It can be checked that the map $(x, \nu, t) \to \Upsilon_1^\nu(x, t)$ belongs to $C^1([0, L] \times (0, L] \times \mathbb{R})$.

Next, we will prove that there exists $\delta > 0$ such that for every $t \in \mathbb{R}$, $x \in [0, L]$ and $\nu \in (0, L]$, it holds

$$\partial_t\gamma_1^\nu(x, t) \geq \sqrt{\underline{\upsilon}}\delta, \quad \partial_x\gamma_1^\nu(x, t) \geq \nu\delta, \quad \partial_\nu\gamma_1^\nu(x, t) \geq 0. \tag{112}$$

Taking the derivative of (109) with respect to x, it holds

$$\frac{-L\sqrt{v(\tau_1^{out,\nu}(x, t))}}{\nu}\partial_x\tau_1^{out,\nu}(x, t) + \partial_x\chi_1^\nu(\tau_1^{out,\nu}(x, t); x, t) = 0. \tag{113}$$

In view of (110), (111), it is easy to know

$$\partial_x\chi_1^\nu(\tau_1^{out,\nu}(x, t); x, t) = 1. \tag{114}$$

With the help of Assumption 1, from (113), (114), it holds

$$\partial_x\tau_1^{out,\nu}(x, t) \geq \frac{\nu}{L\sqrt{\overline{\upsilon}}}. \tag{115}$$

Using (108), (106), from (115), it is easy to know

$$\partial_x\gamma_1^\nu(x, t) \geq \frac{\nu}{L\sqrt{\overline{\upsilon}}}, \quad \partial_t\gamma_1^\nu(x, t) \geq \frac{\sqrt{\underline{\upsilon}}}{\sqrt{\overline{\upsilon}}}. \tag{116}$$

Denote $\Gamma^{\nu}(x,t)=\partial_{\nu}\gamma_1^{\nu}(x,t)$, from (106), it can be deduced that

$$\partial_t\Gamma^{\nu}(x,t)-\frac{L\sqrt{v(t)}}{\nu}\partial_x\Gamma^{\nu}(x,t)=-\frac{L\sqrt{v(t)}}{\nu^2}\partial_x\gamma_1^{\nu}(x,t), \tag{117}$$

$$\Gamma^{\nu}(0,t)=0. \tag{118}$$

With the help of Assumption 1, by (116), from (117), (118), we get

$$\Gamma^{\nu}(x,t)\geq 0, \tag{119}$$

that is, $\partial_{\nu}\gamma_1^{\nu}(x,t)\geq 0$. Choose $0<\delta\leq\min\{\frac{1}{L\sqrt{\underline{\upsilon}}},\frac{1}{\sqrt{\underline{\upsilon}}}\}$, we get (112). From (112), for any $0<\nu\leq L$, it is clear that

$$\begin{aligned}\Upsilon_1(x,s,t)&=\gamma_1^{L}(x,t)-\gamma_1^{\nu}(s,t)\\&\geq\gamma_1^{\nu}(x,t)-\gamma_1^{\nu}(s,t)\\&\geq 0,\end{aligned} \tag{120}$$

with $0\leq s\leq x\leq L$, $t\in\mathbb{R}$.

Finally, we will prove that (103), (104) hold. Using (106), it is easy to know that

$$\partial_t\gamma_1^{L}(x,t)=\sqrt{v(t)}\gamma_1^{L}(x,t). \tag{121}$$

From (105), (121), it can be deduced that

$$\begin{aligned}&\partial_t\Upsilon_1(x,s,t)-\sqrt{v(t)}\partial_x\Upsilon_1(x,s,t)-\sqrt{v(t)}\partial_s\Upsilon_1(x,s,t)\\&=\partial_t\gamma_1^{L}(x,t)-\partial_t\gamma_1^{\nu}(s,t)-\sqrt{v(t)}\partial_x\gamma_1^{L}(x,t)\\&\quad+\sqrt{v(t)}\partial_s\gamma_1^{\nu}(s,t)\\&=-\partial_t\gamma_1^{\nu}(s,t)+\sqrt{v(t)}\partial_s\gamma_1^{\nu}(s,t).\end{aligned} \tag{122}$$

With (106), (112), from (122), it holds

$$\begin{aligned}&\partial_t\Upsilon_1(x,s,t)-\sqrt{v(t)}\partial_x\Upsilon_1(x,s,t)-\sqrt{v(t)}\partial_s\Upsilon_1(x,s,t)\\&=-(\frac{L}{\nu}-1)\sqrt{v(t)}\partial_s\gamma_1^{\nu}(s,t)\\&\leq-(L-\nu)\sqrt{\underline{\upsilon}}\delta.\end{aligned} \tag{123}$$

Choosing $0<\varepsilon_0\leq(L-\nu)\sqrt{\underline{\upsilon}}\delta$, we have

$$\partial_t\Upsilon_1(x,s,t)-\sqrt{v(t)}\partial_x\Upsilon_1(x,s,t)-\sqrt{v(t)}\partial_s\Upsilon_1(x,s,t)\leq-\varepsilon_0, \tag{124}$$

for $0<\nu\leq L$, that is (103). Using the same argument as proof (103), it holds

$$\begin{aligned}&\partial_t\Upsilon_1(x,s,t)-\sqrt{v(t)}\partial_x\Upsilon_1(x,s,t)+\sqrt{v(t)}\partial_s\Upsilon_1(x,s,t)\\&=-(\frac{L}{\nu}+1)\sqrt{v(t)}\partial_s\gamma_1^{\nu}(s,t)\\&\leq-(L+\nu)\sqrt{\underline{\upsilon}}\delta\\&\leq-\varepsilon_0,\end{aligned} \tag{125}$$

that is (104). □

For $N\in\mathcal{N}$, let us define

$$\Phi_1^{N}(K)_{1j}(x,s,t)=\Phi_1(\Phi_1^{N-1}(K)_{1j}(x,s,t)),\quad j=1,2,$$

for $(x,s,t)\in\overline{P}$, we have

Lemma 2: There exists $C>0$ such that for every $N\in\mathcal{N}$, and $K,H\in B$,

$$\|\Phi_1^{N}(K)_{11}-\Phi_1^{N}(H)_{11}\|\leq\frac{C^N}{N!}\|K-H\|, \tag{126}$$

$$\|\Phi_1^{N}(K)_{12}-\Phi_1^{N}(H)_{12}\|\leq\frac{C^N}{N!}\|K-H\|. \tag{127}$$

Proof: We start with the proof of (127). From the definition (102) of $\Phi_1^N(K)_{12}$, and with the help of (5), (6), we have

$$\begin{aligned}&|\Phi_1(K)_{12}(x,s,t)-\Phi_1(H)_{12}(x,s,t)|\\&\leq\frac{M_1}{4\underline{\upsilon}}\int_t^{\tau_{12}^{out}(x,s,t)}|k_{11}(\chi_{12}(\tau;x,s,t))\\&\quad-h_{11}(\chi_{12}(\tau;x,s,t))|d\tau\end{aligned} \tag{128}$$

$$\leq\frac{M_1}{4\underline{\upsilon}}\int_t^{\tau_{12}^{out}(x,s,t)}d\tau\|K-H\|. \tag{129}$$

Thanks to the estimate (104), we make the change of variable $\theta(\tau)=\Upsilon_1(\chi_{12}(\tau;x,s,t))$ and get

$$\begin{aligned}\varepsilon_0\int_t^{\tau_{12}^{out}(x,s,t)}d\tau&\leq-\int_t^{\tau_{12}^{out}(x,s,t)}\frac{d\theta(\tau)}{d\tau}d\tau\\&=\theta(t)-\theta(\tau_{12}^{out}(x,s,t))\\&\leq\theta(t)=\Upsilon_1(\chi_{12}(t;x,s,t)).\end{aligned} \tag{130}$$

This gives the estimate

$$\begin{aligned}&|\Phi_1(K)_{12}(x,s,t)-\Phi_1(H)_{12}(x,s,t)|\\&\qquad\leq\frac{M_1}{4\varepsilon_0\underline{\upsilon}}\Upsilon_1(\chi_{12}(t;x,s,t))\|K-H\|.\end{aligned} \tag{131}$$

Calculating $\Phi_1^2(K)_{12}(x,s,t)-\Phi_1^2(H)_{12}(x,s,t)$, and using the previous estimate, we obtain

$$\begin{aligned}&|\Phi_1^2(K)_{12}(x,s,t)-\Phi_1^2(H)_{12}(x,s,t)|\\&=|\Phi_1(\Phi_1(K)_{12}(x,s,t))-\Phi_1(\Phi_1(H)_{12}(x,s,t))|\\&\leq\frac{M_1^2}{\varepsilon_0(4\underline{\upsilon})^2}\int_t^{\tau_{12}^{out}(x,s,t)}\Upsilon_1(\chi_{12}(\tau;x,s,t))d\tau\|K-H\|.\end{aligned} \tag{132}$$

Using the change of variable $\theta(\tau)=\Upsilon_1(\chi_{12}(\tau;x,s,t))$ again and (104), it holds

$$\begin{aligned}&\varepsilon_0\int_t^{\tau_{12}^{out}(x,s,t)}\theta(\tau)d\tau\\&\leq-\int_t^{\tau_{12}^{out}(x,s,t)}\theta(\tau)\frac{d\theta(\tau)}{d\tau}d\tau\\&=\frac{\theta(t)^2}{2}-\frac{(\theta(\tau_{12}^{out}(x,s,t)))^2}{2}\\&\leq\frac{\theta(t)^2}{2}=\frac{\Upsilon_1(\chi_{12}(t;x,s,t))^2}{2}.\end{aligned} \tag{133}$$

From (132), (133), we obtain

$$\begin{aligned}&|\Phi_1^2(K)_{12}(x,s,t)-\Phi_1^2(H)_{12}(x,s,t)|\\&\qquad\leq\frac{M_1^2}{\varepsilon_0^2(4\underline{\upsilon})^2}\frac{\Upsilon_1(\chi_{12}(t;x,s,t))^2}{2!}\|K-H\|.\end{aligned} \tag{134}$$

By applying the induction, it can be deduced that

$$|\Phi_1^N(K)_{12}(x,s,t)-\Phi_1^N(H)_{12}(x,s,t)| \leq \frac{M_1^N}{\varepsilon_0^N(4\underline{\upsilon})^N}\frac{\Upsilon_1(\chi_{12}(t;x,s,t))^N}{N!}\|K-H\|. \quad (135)$$

Using the estimate (103) instead of (104), and with the help of (5), (6), we can deduce the following estimate:

$$|\Phi_1^N(K)_{11}(x,s,t)-\Phi_1^N(H)_{11}(x,s,t)| \leq \frac{M_1^N}{\varepsilon_0^N(4\underline{\upsilon})^N}\frac{\Upsilon_1(\chi_{11}(t;x,s,t))^N}{N!}\|K-H\|. \quad (136)$$

From (103), (104), (120), it can be seen that $\tau \to \Upsilon_1(\chi_{1j}(\tau;x,s,t)), j=1,2$ are strictly decreasing and $\Upsilon_1(\chi_{1j}(\tau;x,s,t)) \geq 0, j=1,2$, so $\Upsilon_1(\chi_{1j}(\tau;x,s,t)), j=1,2$ are bounded. It follows that there exists some $C>0$ independent of N and K, H,

$$\|\Phi_1^N(K)_{1j}(x,s,t)-\Phi_1^N(H)_{1j}(x,s,t)\| \leq \frac{C^N}{N!}\|K-H\|, \quad (137)$$

for $j=1,2$. □

Therefore, Φ_1^N is a contraction for $N \in \mathcal{N}$ large enough, the Banach fixed-point theorem can be applied, giving the existence and uniqueness of $K \in B$ such that $K=\phi(K)$. So, we have the following conclusion.

Conclusion 1: Denote $\top=\{(x,s,t)\in(0,L)\times(0,L)\times(0,\infty)\}$, there exists a unique continuous 2×2 matrix-valued solution $K=(k_{ij})_{1\leq i,j\leq 2}\in L^\infty(\top)\cap C^0(\top)$ to the kernel PDEs (15)–(18).

B. SECOND-STEP BACKSTEPPING TRANSFORMATION

In order to compensate the PDE actuator dynamics of the target system (48)–(52), the vector functions $p(x,t)$ and $q(x,t)$ are defined as follows:

$$p(x,t)=Z(t)+\int_0^x \frac{\varphi\left(p(y,t), e^{\int_0^{\phi^{-1}(\phi(t)+y)}\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\omega_1(y,t)\right)}{\sqrt{v(\phi^{-1}(\phi(t)+y))}}dy, \quad (138)$$

where $0\leq x\leq L,\ t\geq 0$, and $p(x,t)=[p_1(x,t),p_2(x,t)]^T$, with the initial condition

$$p(x,0)=Z(0)+\int_0^x \frac{\varphi\left(p(y,0), e^{\int_0^{\phi^{-1}(y)}\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\omega_1(y,0)\right)}{\sqrt{v(\phi^{-1}(y))}}dy, \quad (139)$$

where ϕ is given by (7). Next, let us define

$$q(x,t)=Z(t)-\int_0^x \frac{\varphi\left(q(y,t), e^{\int_0^{\phi^{-1}(|\phi(t)-y|)}\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\omega_2(y,t)\right)}{\sqrt{v(\phi^{-1}(|\phi(t)-y|))}}dy, \quad (140)$$

where $q(x,t)=[q_1(x,t),q_2(x,t)]^T$, with the initial condition

$$q(x,0)=Z(0)-\int_0^x \frac{\varphi\left(q(y,0), e^{\int_0^{\phi^{-1}(y)}\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\omega_2(y,0)\right)dy}{\sqrt{v(\phi^{-1}(y))}}, \quad (141)$$

for $0\leq x\leq L,\ t\geq 0$. We introduce the following variable

$$\mu_1(\theta,\chi)=e^{-\int_0^\theta\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\mu(\chi), \quad (142)$$

where $\theta\in\mathbb{R}$, $\chi\in\mathbb{R}^{n+1}$, and μ is given by (22).

Lemma 3: For all $0\leq x\leq L,\ t\geq 0$, the second-step backstepping transform is designed as follows:

$$\varpi(x,t)=\omega_1(x,t)-\mu_1(\phi^{-1}(\phi(t)+x),\ p(x,t)), \quad (143)$$
$$\lambda(x,t)=\omega_2(x,t)-\mu_1(\phi^{-1}(|\phi(t)-x|),\ q(x,t)), \quad (144)$$

where μ_1 is defined in (142), $p(x,t)$ and $q(x,t)$ are given as (138) and (140), respectively, and $U(t)$ is

$$U(t)=-\frac{e^{\int_0^t\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}}{2\sqrt{v(t)}}(\eta(L,t)-\mu_1(\phi^{-1}(\phi(t)+L),p(L,t)))-\int_0^L k_{11}(L,s,t)\zeta(s,t)+k_{12}(L,s,t)\eta(s,t)ds, \quad (145)$$

map system (48)–(52) into the target system given below

$$\dot{Z}(t)=\varphi\left(Z(t), e^{\int_0^t\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}(\varpi(0,t)+\mu_1(t,Z(t)))\right) \quad (146)$$
$$\partial_t\varpi(x,t)=\sqrt{v(t)}\,\partial_x\varpi(x,t) \quad (147)$$
$$\partial_t\lambda(x,t)=-\sqrt{v(t)}\,\partial_x\lambda(x,t) \quad (148)$$
$$\lambda(0,t)=\varpi(0,t) \quad (149)$$
$$\varpi(L,t)=0, \quad (150)$$

Proof: Note that $p(0,t)=Z(t)$, using (138), (143), (48), we obtain (146). From (143), (144), and knowing that $p(0,t)=q(0,t)=Z(t)$, with the help of (51), it is easy to derive (149). Using (52), (142), (143), (145), relation (150) can be deduced. In what follows, we will prove (148).

Differentiating (140) with respect to t and x, we get

$$\begin{aligned}\partial_t q(x,t) &= \varphi\left(Z(t),\ e^{\int_0^t\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\omega_1(0,t)\right) - \int_0^x \frac{\partial_q\varphi(q(y,t),\delta_2(y,t)\omega_2(y,t))\partial_t q(y,t)}{\sqrt{v(\phi^{-1}(|\phi(t)-y|))}}dy \\ &\quad - \int_0^x \frac{\partial_{\delta_2\omega_2}\varphi\left(q(y,t),\ \delta_2(y,t)\omega_2(y,t)\right)}{\sqrt{v(\phi^{-1}(|\phi(t)-y|))}}\Bigg(\delta_2(y,t)\partial_t\omega_2(y,t) \\ &\quad + e^{\int_0^{\phi^{-1}(|\phi(t)-y|)}\frac{\dot{v}(\tau)}{4v(\tau)}d\tau}\frac{\dot{v}(\phi^{-1}(|\phi(t)-y|))}{4v(\phi^{-1}(|\phi(t)-y|))} \\ &\quad \times\frac{\partial(\phi^{-1}(|\phi(t)-y|))}{\partial(|\phi(t)-y|)}sgn\{\phi(t)-y\}\sqrt{v(t)}\omega_2(y,t)\Bigg)dy\end{aligned}$$

$$+\int_0^x \frac{\varphi\left(q(y,t),\ \delta_2(y,t)\omega_2(y,t)\right)}{2\sqrt{v(\phi^{-1}(|\phi(t)-y|))^3}}\frac{\partial v(\phi^{-1}(|\phi(t)-y|))}{\partial(\phi^{-1}(|\phi(t)-y|))} \times\frac{\partial(\phi^{-1}(|\phi(t)-y|))}{\partial(|\phi(t)-y)|}sgn(\phi(t)-y)\sqrt{v(t)}dy, \quad (151)$$

with $\delta_2(y,t)=e^{\int_0^{\phi^{-1}(|\phi(t)-y|)}\frac{\dot v(\tau)}{4v(\tau)}d\tau}$ and

$$\begin{aligned}
&\partial_x q(x,t)\\
&=-\int_0^x \frac{\partial_q\varphi(q(y,t),\delta_2(y,t)\omega_2(y,t))\partial_y q(y,t)}{\sqrt{v(\phi^{-1}(|\phi(t)-y|))}}dy\\
&\quad-\int_0^x \frac{\partial_{\delta_2\omega_2}\varphi\left(q(y,t),\ \delta_2(y,t)\omega_2(y,t)\right)}{\sqrt{v(\phi^{-1}(|\phi(t)-y|))}}\Big(\delta_2(y,t)\partial_y\omega_2(y,t)\\
&\quad-e^{\int_0^{\phi^{-1}(|\phi(t)-y|)}\frac{\dot v(\tau)}{4v(\tau)}d\tau}\frac{\dot v(\phi^{-1}(|\phi(t)-y|))}{4v(\phi^{-1}(|\phi(t)-y)|))}\\
&\quad\times\frac{\partial(\phi^{-1}(|\phi(t)-y|))}{\partial(|\phi(t)-y|)}sgn\{\phi(t)-y\}\omega_2(y,t)\Big)dy\\
&\quad-\int_0^x \frac{\varphi\left(q(y,t),\ \delta_2(y,t)\omega_2(y,t)\right)}{2\sqrt{v(\phi^{-1}(|\phi(t)-y|))^3}}\frac{\partial v(\phi^{-1}(|\phi(t)-y|))}{\partial(\phi^{-1}(|\phi(t)-y|))}\\
&\quad\times\frac{\partial(\phi^{-1}(|\phi(t)-y)|))}{\partial(|\phi(t)-y|)}sgn\{\phi(t)-y\}dy\\
&\quad-\frac{\varphi\left(Z(t),\ e^{\int_0^t\frac{\dot v(\tau)}{4v(\tau)}d\tau}\omega_2(0,t)\right)}{\sqrt{v(t)}},
\end{aligned}\quad (152)$$

respectively, for $0\le x\le L,\ t\ge 0$. Defining

$$\overline{H}(x,t)=\partial_t q(x,t)+\sqrt{v(t)}\partial_x q(x,t),$$

and combining (151) and (152), noting $\omega_1(0,t)=\omega_2(0,t)$, by virtue of (50), we arrive at

$$\overline{H}(x,t)=-\int_0^x \frac{\partial_q\varphi(q(y,t),\delta_2(y,t)\omega_2(y,t))\overline{H}(y,t)}{\sqrt{v(\phi^{-1}(|\phi(t)-y|))}}dy \quad (153)$$

for $0\le x\le L,\ t\ge 0$. Differentiating (153) with respect to x, we have

$$\partial_x\overline{H}(x,t)=-\frac{\partial_q\varphi(q(x,t),\delta_2(x,t)\omega_2(x,t))\overline{H}(x,t)}{\sqrt{v(\phi^{-1}(|\phi(t)-x|))}}, \quad (154)$$

and $\overline{H}(0,t)=0$, for all $0\le x\le L,\ t\ge 0$. Hence, we get

$$\partial_t q(x,t)=-\sqrt{v(t)}\partial_x q(x,t). \quad (155)$$

Taking the time and the spatial derivatives of (144), and from (50) and (155), we obtain (148). Relation (147) can be deduced similarly. □

Remark 9: Using (41)*, substituting* (31), (32) *and* (22) *into* (145) *gives the equivalent control action $U(t)$ defined in* (8)*. In addition, it can be deduced that $p(x,t)$ defined by* (138) *is equal to $[p_1(x,t),p_2(x,t)]^T$, where $p_1(x,t),p_2(x,t)$ are given by* (9), (10)*, respectively.*

In order to derive the inverse backstepping transformation, we define the vector functions $\pi(x,t)$ and $\iota(x,t)$ as

$$\pi(x,t)=Z(t)+\int_0^x\varphi\Big(\pi(y,t),e^{\int_0^{\phi^{-1}(\phi(t)+y)}\frac{\dot v(\tau)}{4v(\tau)}d\tau}\times\left(\varpi(y,t)+\mu_1(\phi^{-1}(\phi(t)+y),\pi(y,t))\right)\Big)\times\frac{1}{\sqrt{v(\phi^{-1}(\phi(t)+y))}}dy, \quad (156)$$

where $\pi(x,t)=[\pi_1(x,t),\pi_2(x,t)]^T$, with the initial condition

$$\pi(x,0)=Z(0)+\int_0^x\varphi\Big(\pi(y,0),e^{\int_0^{\phi^{-1}(y)}\frac{\dot v(\tau)}{4v(\tau)}d\tau}\times\left(\varpi(y,0)+\mu_1(\phi^{-1}(y),\pi(y,0))\right)\Big)\times\frac{1}{\sqrt{v(\phi^{-1}(y))}}dy, \quad (157)$$

and

$$\iota(x,t)=Z(t)-\int_0^x\varphi\Big(\iota(y,t),e^{\int_0^{\phi^{-1}(|\phi(t)-y|)}\frac{\dot v(\tau)}{4v(\tau)}d\tau}\times\left(\lambda(y,t)+\mu_1(\phi^{-1}(|\phi(t)-y|),\iota(y,t))\right)\Big)\times\frac{1}{\sqrt{v(\phi^{-1}(|\phi(t)-y|))}}dy, \quad (158)$$

where $\iota(x,t)=[\iota_1(x,t),\iota_2(x,t)]^T$ with the initial condition

$$\iota(x,0)=Z(0)-\int_0^x\varphi\Big(\iota(y,0),e^{\int_0^{\phi^{-1}(y)}\frac{\dot v(\tau)}{4v(\tau)}d\tau}\times\left(\lambda(y,0)+\mu_1(\phi^{-1}(y),\iota(y,0))\right)\Big)\times\frac{1}{\sqrt{v(\phi^{-1}(y))}}dy, \quad (159)$$

where $\varpi,\lambda,\ \mu_1$ are defined in (143), (144), (142), respectively.

Inverse Backstepping Transforms: The inverse backstepping transformations of ϖ,λ are defined as

$$\omega_1(x,t)=\varpi(x,t)+\mu_1(\phi^{-1}(\phi(t)+x),\pi(x,t)), \quad (160)$$
$$\omega_2(x,t)=\lambda(x,t)+\mu_1(\phi^{-1}(|\phi(t)-x|),\iota(x,t)), \quad (161)$$

where $\pi(x,t),\ \iota(x,t),\ 0\le x\le L,\ t\ge 0$, are given as (156), (158), respectively.

The inverse backstepping transformation (160), (161), and the control law (145) transform the target system (146)–(150) into system (48)–(52) and the proof can be derived from straightforward computations.

IV. STABILITY ANALYSIS OF THE CLOSED-LOOP SYSTEM

Lemma 4: Under assumptions 1 and 2, consider system (146)–(150), there exists a class $\mathcal{KL}$ function β, such that

$$|Z(t)|+\|\varpi(t)\|_\infty+\|\lambda(t)\|_\infty \le\beta(|Z(0)|+\|\varpi(0)\|_\infty+\|\lambda(0)\|_\infty,t), \quad (162)$$

for all $t\ge 0$.

Proof: We introduce a new variable $z(x,t)$, $x \in [-L, L]$, $t \geq 0$, such that

$$z(x,t) = \begin{cases} \varpi(x,t), & \text{for all } x \in [0, L], \\ \lambda(-x,t), & \text{for all } x \in [-L, 0]. \end{cases} \quad (163)$$

Let $\Gamma_{g,n}(t)$ be the following norm

$$\Gamma_{g,n}(t) = \int_{-L}^{L} e^{2ng(L+x)} z(x,t)^{2n} dx, \quad t \geq 0, \quad (164)$$

for any $g > \frac{M_1}{2\underline{\upsilon}}$ and positive integer n, the derivative of $\Gamma_{g,n}(t)$ satisfies

$$\dot{\Gamma}_{g,n}(t) \leq -2ng\sqrt{\underline{\upsilon}}\,\Gamma_{g,n}(t), \quad t \geq 0. \quad (165)$$

Using Assumption 2, from (165), it is easy to get (162). □

To establish stability of the closed-loop system (1)–(4), (8)–(10), we show the boundedness of predictors.

Lemma 5: Under Assumptions 1–4, there exist class $\mathcal{K}_\infty$ functions ρ_1, ρ_2, ρ_3 and ρ_4 such that

$$\sup_{0\leq x\leq L} |p(x,t)| \leq \rho_1(|Z(t)| + \|\omega_1(t)\|_\infty), \quad (166)$$
$$\sup_{0\leq x\leq L} |q(x,t)| \leq \rho_2(|Z(t)| + \|\omega_2(t)\|_\infty), \quad (167)$$
$$\sup_{0\leq x\leq L} |\pi(x,t)| \leq \rho_3(|Z(t)| + \|\varpi(t)\|_\infty), \quad (168)$$
$$\sup_{0\leq x\leq L} |\iota(x,t)| \leq \rho_4(|Z(t)| + \|\lambda(t)\|_\infty), \quad (169)$$

for all $t \geq 0$.

Proof: Defining $\rho_1(s) = \alpha_1^{-1}(e^{\frac{L}{\sqrt{\underline{\upsilon}}}}\alpha_2(s) + (e^{\frac{L}{\sqrt{\underline{\upsilon}}}} - 1)\alpha_3(\sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}s))$, and $\rho_2(s) = \alpha_4^{-1}(e^{\frac{L}{\sqrt{\underline{\upsilon}_1}}}\alpha_5(s) + (e^{\frac{L}{\sqrt{\underline{\upsilon}_1}}} - 1)\alpha_6(\sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}s))$, we derive (166), (167). The proof of (168) can be established following [23]. Define $\rho_4(s) = \alpha_7^{-1}(e^{\frac{L}{\sqrt{\underline{\upsilon}}}}\alpha_8(s) + (e^{\frac{L}{\sqrt{\underline{\upsilon}}}} - 1)\alpha_9(\sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}s))$, we derive (169). □

Equivalence of norms of original and target PDE states are shown in Lemma 7 and Lemma 8.

Lemma 6: Under Assumptions 1, 2 and 4, consider system (146)–(150), and the output maps (160), (161), there exists a class $\mathcal{K}_\infty$ function γ_2 such that the following holds:

$$|Z(t)| + \|\omega_1(t)\|_\infty + \|\omega_2(t)\|_\infty \leq \gamma_2(|Z(t)| + \|\varpi(t)\|_\infty + \|\lambda(t)\|_\infty), \quad (170)$$

for all $t \geq 0$.

Proof: Using (5), from (142), there exists a class $\mathcal{K}_\infty$ function Λ_1 such that

$$|\mu_1(t,\chi)| \leq \sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}\Lambda_1(|\chi|), \quad (171)$$

for all $t \geq 0$, $\chi \in \mathbb{R}^{n+1}$. Defining $\gamma_2(s) = s + \sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}\Lambda_1(\rho_3(s)) + \sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}\Lambda_1(\rho_4(s))$ we get (170). □

Lemma 7: Under Assumptions 1, 3 and 4, consider system (48)–(52), and the output maps (143), (144), there exists a class $\mathcal{K}_\infty$ function γ_3 such that

$$|Z(t)| + \|\varpi(t)\|_\infty + \|\lambda(t)\|_\infty \leq \gamma_3(|Z(t)| + \|\omega_1(t)\|_\infty + \|\omega_2(t)\|_\infty), \quad (172)$$

for all $t \geq 0$.

Proof: Defining $\gamma_3(s) = s + \sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}\Lambda_1(\rho_1(s)) + \sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}\Lambda_1(\rho_2(s))$, we get (172). □

Proof of Theorem 1: Using Lemma 4, Lemma 6, Lemma 7, with the help of (53), we get

$$|Z(t)| + \|\xi(t)\|_\infty \leq (1+\overline{L})\gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K})(|Z(0)| + \|\xi(0)\|_\infty), t)), \quad (173)$$

for all $t \geq 0$, and

$$\overline{L} = \sup_{(x,y,t)\in[0,L]\times[0,L]\times[0,\infty)} |\Gamma(x,y,t)|,$$
$$\overline{K} = \sup_{(x,y,t)\in[0,L]\times[0,L]\times[0,\infty)} |K(x,y,t)|.$$

Finally, from (5), (31), (32), (33), (34), (41), (42), and (173), we get (29) by defining a class $\mathcal{K}_\infty$ function

$$\overline{\beta}(s,t) = (1+\overline{L})\max\left\{\sqrt{2}, \frac{\sqrt{2}}{2}(1+\frac{1}{\sqrt{\underline{\upsilon}}})\sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}\right\} \times \gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K}) + 2\sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}(1+\sqrt{\overline{\upsilon}}))s), t)). \quad (174)$$

Following [23], it can be proved that under Assumptions 1–4 and $u(\cdot,0) \in C_1[0,L]$, $u_t(\cdot,0) \in C_1[0,L]$, which is compatible with the feedback law (8)–(10), the closed-loop system has a unique solution $X(t) \in C_1([0,\infty),\mathbb{R}^n)$, $u_t(x,t), u_x(x,t) \in C_1([0,L]\times[0,\infty))$. □

V. EXAMPLE

Example 1: For a second-order system

$$\dot{X}_1 = X_2 + X_3^2 \quad (175)$$
$$\dot{X}_2 = X_3 \quad (176)$$
$$\dot{X}_3 = -X_2 - 2X_3 + u(0,t), \quad (177)$$

a nominal control law [24] is

$$u(0,t) = -X_3 - (X_1 + 2X_2 + X_3 + 0.25X_2^2 + 0.25X_3^2)(1 + 0.5X_3). \quad (178)$$

Now system (175)–(177) cascaded with (2)–(4) is controlled by (8)–(10). In simulation, let

$$v(t) = \left(1 + \frac{1}{1+t^2}\right)^2. \quad (179)$$

It is clear that $v(t)$ satisfies Assumption 1. Following [25], a forward finite difference scheme is used for the explicit time

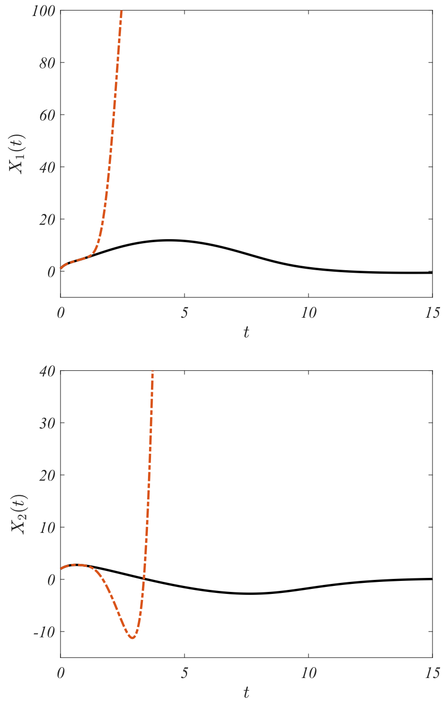

FIGURE 1. Response of $X_1(t)$ and $X_2(t)$ with compensated (solid line) and uncompensated control laws (178) (dotted line).

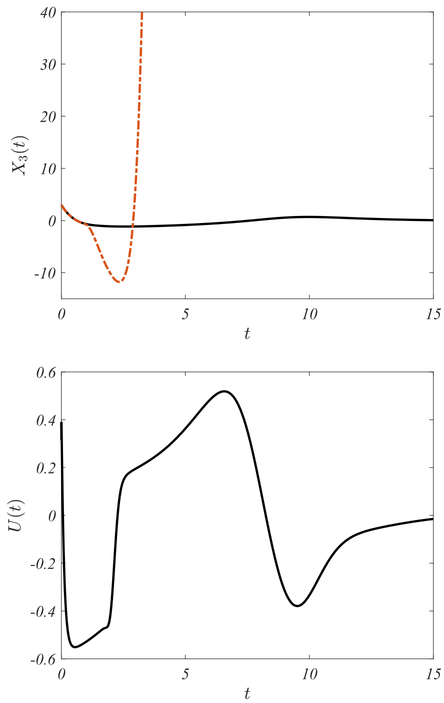

FIGURE 2. Response of $X_3(t)$ with compensated (solid line) and uncompensated control laws (178) (dotted line), and the dynamics of the proposed control.

integral with a negative time step to archive a backward in time computation of kernel PDEs (15)–(18).

Responses of the states X_1, X_2 and X_3 of the closed-loop system under the proposed control law, the uncompensated control law are shown in Fig. 1 and Fig.2. Wave dynamics

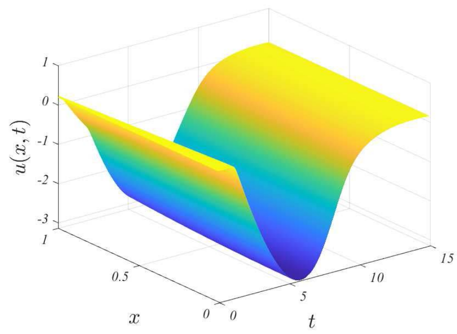

FIGURE 3. Response of the state of compensated wave PDE.

under the compensated control is in Fig.3. One can conclude that the proposed control law ensures asymptotic stability of the closed-loop system while the uncompensated control (178) leads to instability.

VI. CONCLUSION

We consider a class of nonlinear ODE/wave PDE cascaded systems. A predictor control is designed such that the closed-loop system is globally asymptotically stable. One difficulty is how to prove the well-posedness and uniqueness of time-varying kernel PDEs (15)–(18), the other is how to construct predictors $p(x, t)$, $q(x, t)$ in backstepping transforms (143), (144). Stability of the closed-loop system is proved using a two-step backstepping transformation and Lyapunov-like arguments. Generalization of the result to a wider class of propagation speed and robustness analysis with respect to disturbances will be considered in our future work.

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