IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received March 2, 2022, accepted March 18, 2022, date of publication March 30, 2022, date of current version April 6, 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3163399

Kalman Filtering With Linear State Equality Constraints: A Method Based on Separating Variables

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This work was a part of ASELSAN's self-funded research project entitled ''Positioning in Global Navigation Satellite Systems (GNSS) Denied Environments''.

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ABSTRACT This article considers the minimum variance state estimation of linear dynamic systems with linear state equality constraints. The proposed method uses singular value decomposition to divide the constrained system states into deterministic and stochastic parts. The deterministic part can be independently determined through the constraint equations. The stochastic part consists of random variables which have to be determined through a filtering process. The measurement and dynamic equations of the system are also divided into stochastic and deterministic parts. In order to update the mean and covariance of the state vector's stochastic part, the measurement updating phase of the unconstrained Kalman Filter is used. Then, the deterministic part of the dynamic equation is used as a noisy measurement for the proposed method. Finally, the stochastic part of the dynamic equation is used to predict the mean and covariance of state vector's stochastic part. Simulations show that the proposed method provides superior performance compared to other methods in the literature especially for estimating the constrained unobservable states.

INDEX TERMS Constrained estimation, Kalman filters, optimal estimation, system decomposition.

I. INTRODUCTION

The well-known Kalman Filter (KF) has been used for decades as an optimal state estimator from noisy measurements [1]. When the state evolution and measurement equations are linear and all probability density functions of noises and errors are Gaussian, the KF is the optimal minimum variance estimator [1]. However, there are some cases in which the states of the system are subjected to some linear or nonlinear constraints due to conservation of a physical quantity or imposition of a mathematical property in the modeling of the system [2]–[4]. Sometimes developing dynamic equations with constraints has some benefits. For example, in attitude determination problems when the attitude dynamic equations are parametrized with Euler angles, the problem is nonlinear. However, when it is parameterized with quaternion, the dynamic equation will be linear with a nonlinear constraint [5].

The associate editor coordinating the review of this manuscript and approving it for publication was Larbi Boubchir.

Although constrained state estimation is a widespread problem, there are a few studies about this problem [4]. The idea behind constrained filters comes from using the constraint information to reduce the errors of estimation. In other words, there is information in constraint equations which could lead to reduce the covariance of the estimation error [4].

One of the simplest methods to overcome with the constrained estimation is considering the constraint equations as pseudo-measurement equations [6]–[8]. This method is referred to as perfect measurements method, and will lead to a singular covariance matrix in the KF. A singular covariance matrix, however, may cause numerical problems in the implementation of the KF [9], [10].

Another well-known constrained estimation method is based on the reduction of system states and then using the unconstrained KF (uncKF) to solve the problem [11], [12]. One of the advantages of this method is that it can be easily implemented. However, the physical meaning of the states are lost due to the structural change in the states of the system, and then this will cause some difficulties in the interpretation of the original problem [13].

Another approach is to project the estimations of the uncKF onto the constraint space [12], [14], [15]. This approach is called the estimate projection KF (epKF) [12], [16]. Providing the state estimation using the uncKF, a weighted least squares problem is solved to project the estimation onto the constraint space. When the weights are the inverse of the covariance matrix of the estimation error, the maximum probability solution subjected to the constraints is obtained [12]. When the weighting matrix is equal to the unit matrix, the least squares solution subjected to the constraints is conducted [12].

The generalization of epKF is presented in [17] which is called the equality constrained KF (ecKF) [12], [16]. This method adds error of constraints to the uncKF cost function using of Lagrange multipliers [17]. The last method that we consider is the system projection KF (spKF) which projects the system onto the null space of the constraints and discussed in [10], [12], [16]. The method projects not only the prediction estimation but also the updated estimation onto the constraints' null space.

The methods mentioned above use a linear model for the process and do not take into account any uncertainty in the model. In real applications, the performance of the filter will be reduced due to uncertainties in the system's model and drift in noise parameters. Newly an alternative method based on trajectory function of time (T-FoT) is introduced in [18], [19]. The method includes state equality and inequality constraints and models target motion by curves to avoid the difficulty of process noise modeling [18].

The main idea of the proposed method comes from using the system's dynamic equation more efficiently. The proposed method is based on the separation of system state variables into deterministic and stochastic parts. The null space of the constraint matrix is utilized to separate the system variables. In addition, the system's dynamic equation is used more efficiently to reduce the effect of the system's noise. For the implementation of the proposed method, the original states are projected onto both the null space and its complementary space. Then, the system's new states are divided into two types, one with deterministic behavior and the other with stochastic behavior. Deterministic part of the new state is solved by linear constraint equations, and stochastic part is first estimated by the measurement equation. In the prediction phase the system equations are divided into two parts again. The first part predicts the stochastic part of the state and the second part acts as a new noisy measurement equation.

A two-dimensional navigation problem with linear state equality constraints is considered in order to compare the estimation performance of the proposed method with of the others in the literature developed for constrained systems. When the system is fully observable all compared methods have almost the same performances. However, when the system is partially observable (some states are unobservable) the performances of the compared methods are different. The simulations show that the proposed method outperforms uncKF, epKF, spKF and ecKF methods especially for estimating the constrained unobservable states.

II. UNCONSTRAINED KALMAN FILTER

Consider a linear discrete-time time-varying system as follows

$$\begin{aligned} x_{k+1} &= F_k x_k + B_k u_k + G_k w_k \\ z_k &= H_k x_k + D_k u_k + v_k \end{aligned} \tag{1}$$

Here, k is the time index, $F_k \in \Re^{n\times n}$, $B_k \in \Re^{n\times r}$, $G_k \in \Re^{n\times q}$, $H_k \in \Re^{m\times n}$ and $D_k \in \Re^{m\times r}$ are system, deterministic input, stochastic input, measurement, and input-output matrices while $x_k \in \Re^n$, $u_k \in \Re^r$, $w_k \in \Re^q$, $z_k \in \Re^m$ and $v_k \in \Re^m$ are state, deterministic input, system noise, measurement, and measurement noise vectors, respectively. We suppose that the system noise and measurement noise vectors are Gaussian random vectors with zero mean and with covariances Q_k and R_k, respectively. The initial condition of the state vector at time zero is a random vector with known mean and variance given by

$$\boldsymbol{E}[x_0] = \hat{x}_{0|0}, \quad P_{0|0} = \boldsymbol{E}\left[\left(x_0 - \hat{x}_{0|0}\right)\left(x_0 - \hat{x}_{0|0}\right)^T\right] \tag{2}$$

Here $\hat{x}_{0|0}$ and $P_{0|0}$ are estimated initial values of the state vector and its covariance, respectively. The operator $\boldsymbol{E}$ stands for the expectation operator. Suppose $\hat{x}_{k|k-1}$ (prior estimation) and $\hat{x}_{k|k}$ (posterior estimation) are estimated state for time t_k by mesurements up to time t_{k-1} and t_k, and their covariances are $P_{k|k-1}$ and $P_{k|k}$, respectively. To find the optimal estimation of the state vector at time t_k, Problem I which is an unconstrained optimization problem is defined as

Problem I:

$$\text{Minimize } J_k = \frac{1}{2}\left[\begin{array}{c} \left(x_k - \hat{x}_{k|k-1}\right)^T P_{k|k-1}^{-1}\left(x_k - \hat{x}_{k|k-1}\right) \\ + \left(z_k - D_k u_k - H_k x_k\right)^T R_k^{-1}\left(z_k - D_k u_k - H_k x_k\right) \end{array}\right] \tag{3}$$

Here prior estimation $\hat{x}_{k|k-1}$ and measurement vector z_k are known vectors and the weighting matrices $P_{k|k-1}^{-1}$ and R_k^{-1} are the inverse of covariance matrices of prior estimation error and measurement noise, respectively.

Solution: To determine the optimal estimation of the state vector, shown by $\hat{x}_{k|k}$, we need to calculate the derivative of the performance index J_k with respect to x_k, and setting it to zero [1]

$$\frac{dJ_k}{dx_k^T} = \left[P_{k|k-1}^{-1}\left(x_k - \hat{x}_{k|k-1}\right) - H_k^T R_k^{-1}\left(z_k - D_k u_k - H_k x_k\right)\right] = 0 \tag{4}$$

Then, the solution of (4) is

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k\left(z_k - D_k u_k - H_k \hat{x}_{k|k-1}\right) \\ K_k &= P_{k|k-1} H_k^T \left(H_k P_{k|k-1} H_k^T + R_k\right)^{-1} \end{aligned} \tag{5}$$

Here K_k is the Kalman gain. The error of posterior estimation is computed as

$$\hat{e}_{k|k} = \hat{x}_{k|k} - x_k = \hat{e}_{k|k-1} + K_k \left(-H_k \hat{e}_{k|k-1} + v_k\right) \quad (6)$$

Here $\hat{e}_{k|k-1}$ is the error of prior estimation of x_k and its covariance is

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} \quad (7)$$

The covariance of estimation error after measurement update is lower than the prior covariance [1]. ■

When probability density functions of all random vectors are Gaussian, and measurement and system noises are white and uncorrelated, the uncKF is the optimal unbiased minimum variance estimator [1]. The uncKF has predictor-corrector structure. It means that in the first phase, the filter updates the mean and covariance of the state at time t_k using the measurement equation. In the second phase, it predicts mean and covariance of state vector at the time t_{k+1} by using the system equation. Each time the covariance decreases in update phase and increases in prediction phase [1]. The uncKF algorithm is summarized in Algorithm 1 [1], [20].

Algorithm 1 Unconstrained KF Algorithm

Initialization:

$$\boldsymbol{E}[x_0] = \hat{x}_0, cov\left(\hat{x}_0\right) = P_0$$

Measurement Update:

$$\begin{aligned} K_k &= P_{k|k-1} H_k^T \left(H_k P_{k|k-1} H_k^T + R_k\right)^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k \left(z_k - D_k u_k - H_k \hat{x}_{k|k-1}\right) \\ P_{k|k} &= (I - K_k H_k) P_{k|k-1} \end{aligned}$$

Prediction:

$$\begin{aligned} \hat{x}_{k+1|k} &= F_{k-1} \hat{x}_{k|k} + B_{k-1} u_{k-1} \\ P_{k+1|k} &= F_k P_{k|k} F_k^T + G_k Q_k G_k^T \end{aligned}$$

III. PROPOSED CONSTRAINED KALMAN FILTER

In this section, a new method for implementing a KF algorithm for systems with linear time-varying state equality constraints is introduced. The proposed method has two steps like the uncKF – update and prediction. In the following two subsections, the update and prediction phases of the uncKF will be expanded to cover the state equality constraints by the separating state variables approach.

A. CONSTRAINED UPDATE PHASE

Now suppose that the system states satisfy some equality constraints as

$$C_k x_k = d_k, \quad k = 0, 1, 2, \ldots \quad (8)$$

where $d_k \in \Re^p$ is a known vector and $C_k \in \Re^{p \times n}$ is a known and full row rank matrix. We assume that the number of constraints is lower than the number of states, i.e., $p < n$ and all constraints are independent. When $p = n$, and constraint matrix has full rank, then the system does not need any estimator and one can calculate the state vector using (8) without any error.

We are considering a factorization for the constraint matrix C_k as

$$C_k = \begin{bmatrix} S_k & \mathbf{0}_{p \times (n-p)} \end{bmatrix} V_k \quad (9)$$

where S_k is a full rank matrix with the dimension of $p \times p$, $\mathbf{0}$ is zero matrix, and V_k is an orthonormal matrix with dimensions of $n \times n$. The above matrix factorization can be done by well-known QR or Singular Value Decomposition (SVD) methods [20]. Here we are defining a new state vector y_k as

$$y_k = \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix} = V_k x_k = \begin{bmatrix} \mathcal{U}_k \\ \mathcal{V}_k \end{bmatrix} x_k = \begin{bmatrix} \mathcal{U}_k x_k \\ \mathcal{V}_k x_k \end{bmatrix} \quad (10)$$

where ξ_k and η_k are two parts of the new state vector y_k with dimensions p and $n - p$, $\mathcal{U}_k$ and $\mathcal{V}_k$ are two parts of transformation matrix V_k with dimensions $p \times n$ and $(n - p) \times n$, respectively. We assume that the mean and covariance of prior estimation of the state vector are available.

The relation between mean and covariance of the original and new state vector of prior estimation is

$$\begin{aligned} \hat{y}_{k|k-1} &= V_k \hat{x}_{k|k-1} \\ \mathcal{P}_{k|k-1} &= V_k P_{k|k-1} V_k^T \end{aligned} \quad (11)$$

Here $\hat{y}_{k|k-1}$ and $\mathcal{P}_{k|k-1}$ are mean and covariance of prior estimation of the new state vector. Substituting (10) and (9) into (8) gives

$$S_k \xi_k = d_k \quad (12)$$

Then using (12), the first part of new state ξ_k is calculated as

$$\xi_k = S_k^{-1} d_k \quad (13)$$

Here ξ_k is a deterministic vector due to the errorless solution in (13) via the known parameters S_k and d_k. As a result, its covariance is zero (i.e., $\hat{\xi}_{k|k-1} = \hat{\xi}_{k|k} = \xi_k$). Then, the first part of the new state vector y_k has a deterministic behavior, but its second part has a random behavior. After updating the constraint equation and using (10) and (11)

$$\begin{aligned} \hat{\eta}_{k|k-1} &= \mathcal{V}_k \hat{x}_{k|k-1} \\ \mathbb{P}_{k|k-1} &= \mathcal{V}_k P_{k|k-1} \mathcal{V}_k^T \end{aligned} \quad (14)$$

are obtained. Here, $\hat{\eta}_{k|k-1}$ and $\mathbb{P}_{k|k-1}$ are prior estimation of the second part of the new state vector and its covariance. Lets $\mathcal{L}_k$ and $\mathfrak{L}_k$ be the two parts of transformed measurement matrix with appropriate dimensions as

$$H_k V_k^T = \begin{bmatrix} \mathfrak{L}_k & \mathcal{L}_k \end{bmatrix} \quad (15)$$

Here the matrices $\mathcal{L}_k$ and $\mathfrak{L}_k$ are known matrices with suitable dimensions. Then using (15), the measurement in (1), can be written as

$$\mathcal{Z}_k = z_k - D_k u_k - \mathfrak{L}_k \xi_k = \mathcal{L}_k \eta_k + v_k \quad (16)$$

Here $\mathcal{Z}_k$ is a new known vector with dimension of m. The equivalent problem for the estimation of η_k is defined by Problem II as follows

Problem II:

$$\text{Minimize } J_k = \frac{1}{2}\left[\begin{array}{c}(\eta_k - \hat{\eta}_{k|k-1})^T \mathbb{P}_{k|k-1}^{-1}(\eta_k - \hat{\eta}_{k|k-1}) \\ + (\mathcal{Z}_k - \mathcal{L}_k\eta_k)^T R_k^{-1}(\mathcal{Z}_k - \mathcal{L}_k\eta_k)\end{array}\right] \tag{17}$$

Similar to Problem I, the solution of Problem II can be obtained as

$$\begin{aligned}\hat{\eta}_{k|k} &= \hat{\eta}_{k|k-1} + \mathcal{K}_k(\mathcal{Z}_k - \hat{\eta}_{k|k-1}) \\ \mathcal{K}_k &= \mathbb{P}_{k|k-1}\mathcal{L}_k^T\left(\mathcal{L}_k\mathbb{P}_{k|k-1}\mathcal{L}_k^T + R_k\right)^{-1} \\ \mathbb{P}_{k|k} &= (I - \mathcal{K}_k\mathcal{L}_k\mathbb{P}_{k|k-1})\end{aligned} \tag{18}$$

Here $\mathcal{K}_k$ is the Kalman gain, $\hat{\eta}_{k|k}$ is posterior estimation of η_k and $\mathbb{P}_{k|k}$ is its covariance.

Then the posterior estimation of y_k and its covariance are

$$\hat{y}_{k|k} = \begin{bmatrix} S_k^{-1}d_k \\ \hat{\eta}_{k|k}\end{bmatrix}, \quad \mathcal{P}_{k|k} = \begin{bmatrix}\mathbf{0} & \mathbf{0}^T \\ \mathbf{0} & \mathbb{P}_{k|k}\end{bmatrix} \tag{19}$$

It should be mentioned that, as shown is (19), the cross-covariance of two parts of the new state vector and the first part of the state vector are set to be zero. Now, using the transformation described by (14), the mean and covariance of the original state can be calculated as

$$\hat{x}_{k|k} = V_k^T\hat{y}_{k|k}, \quad P_{k|k} = V_k^T\mathcal{P}_{k|k}V_k \tag{20}$$

Then substituting (10) and (19) into (20) results in

$$\hat{x}_{k|k} = \mathcal{U}_k^T\xi_k + \mathcal{V}_k^T\hat{\eta}_{k|k}, \quad P_{k|k} = \mathcal{V}_k^T\mathbb{P}_{k|k}\mathcal{V}_k \tag{21}$$

Thus, the estimated state is the summation of two vectors. The first one is a deterministic vector with zero covariance and the second one is a random vector with known covariance. ■

B. CONSTRAINED PREDICTION PHASE

In the prediction stage, the moments of x_{k+1} will be predicted using the first two statistical moments of x_k via the first equation of (1). Here again, it is supposed that the constraints in (8) are valid for the time t_{k+1}, hence

$$C_{k+1}x_{k+1} = d_{k+1} \tag{22}$$

The constraint matrix C_{k+1} is factorized as

$$C_{k+1} = \begin{bmatrix} S_{k+1} & \mathbf{0}_{p\times(n-p)}\end{bmatrix}V_{k+1} \tag{23}$$

We are considering a transformation as

$$y_{k+1} = \begin{bmatrix}\xi_{k+1} \\ \eta_{k+1}\end{bmatrix} = V_{k+1}x_{k+1} = \begin{bmatrix}\mathcal{U}_{k+1} \\ \mathcal{V}_{k+1}\end{bmatrix}x_{k+1} \tag{24}$$

Then, using the transformation matrix V_{k+1}, the first equation of (1) can be written as

$$y_{k+1} = \bar{F}_k y_k + \bar{B}_k u_k + \bar{G}_k w_k \tag{25}$$

where $\bar{F}_k$, $\bar{B}_k$ and $\bar{G}_k$ are system, input and noise input of the transformed system matrices and they are defined as

$$\bar{F}_k = V_{k+1}F_kV_{k+1}^T, \quad \bar{B}_k = V_{k+1}B_k, \quad \bar{G}_k = V_{k+1}G_k \tag{26}$$

Then, (25) can be separated in two equations as

$$\begin{aligned}\xi_{k+1} &= \bar{F}_{11,k}\xi_k + \bar{F}_{12,k}\eta_k + \bar{B}_{1,k}u_k + \bar{G}_{1,k}w_k \\ \eta_{k+1} &= \bar{F}_{21,k}\xi_k + \bar{F}_{22,k}\eta_k + \bar{B}_{2,k}u_k + \bar{G}_{2,k}w_k\end{aligned} \tag{27}$$

Here $\bar{F}_{11,k}$, $\bar{F}_{12,k}$, $\bar{F}_{21,k}$ and $\bar{F}_{22,k}$ are partitions of $\bar{F}_k$, $\bar{B}_{1,k}$ and $\bar{B}_{2,k}$ are partitions of $\bar{B}_k$, $\bar{G}_{1,k}$ and $\bar{G}_{2,k}$ are partitions of $\bar{G}_k$ with appropriate dimensions. Using (22), (23) and (24), the deterministic part of the transformed state can be obtained as

$$\xi_{k+1} = S_{k+1}^{-1}d_{k+1} \tag{28}$$

Due to the deterministic behavior of ξ_{k+1}, its covariance is zero. Besides, the correlation between ξ_{k+1} and η_{k+1} is zero. Using the first equation of (27), we can update the estimation of η_k and using the second equation of (27), we can predict the random vector η_{k+1}. The first equation of (27) can be rewritten as

$$\mathbb{Z}_k = \xi_{k+1} - \bar{F}_{11,k}\xi_k - \bar{B}_{1,k}u_k = \bar{F}_{12,k}\eta_k + \bar{G}_{1,k}w_k \tag{29}$$

Here $\mathbb{Z}_k$ is the new known measurement vector. Then, using information $\hat{\eta}_{k|k}$ and $\mathbb{P}_{k|k}$, we can re-update the estimation of η_k. A new optimization problem can be defined as

Problem III:

$$\text{Minimize } J_k = \frac{1}{2}\left[\begin{array}{c}(\eta_k - \hat{\eta}_{k|k})^T \mathbb{P}_{k|k}^{-1}(\eta_k - \hat{\eta}_{k|k}) \\ + (\mathbb{Z}_k - \bar{F}_{12,k}\eta_k)^T \bar{Q}_k^{-1}(\mathbb{Z}_k - \bar{F}_{12,k}\eta_k)\end{array}\right] \tag{30}$$

Here, we suppose that the covariance of the new defined measurement is $\bar{Q}_k = \bar{G}_{1,k}Q_k\bar{G}_{1,k}^T$ and it is a full rank matrix. When the matrix $\bar{Q}_k$ is a singular matrix then we can have another constraint and it will increase the accuracy of the estimation.

Similar to Problem I, the solution of Problem III can be obtained as

$$\begin{aligned}\check{\eta}_{k|k} &= \hat{\eta}_{k|k} + \mathbb{K}_k(\mathbb{Z}_k - \bar{F}_{12,k}\hat{\eta}_{k|k}) \\ \mathbb{K}_k &= \mathbb{P}_{k|k}\bar{F}_{12,k}^T\left(\bar{F}_{12,k}\mathbb{P}_{k|k-1}\bar{F}_{12,k}^T + \bar{Q}_k\right)^{-1} \\ \check{\mathbb{P}}_{k|k} &= (I - \mathbb{K}_k\bar{F}_{12,k}\mathbb{P}_{k|k})\end{aligned} \tag{31}$$

Here $\check{\eta}_{k|k}$ is the new updated estimation of $\hat{\eta}_k$, $\check{\mathbb{P}}_{k|k}$ is its covariance, and $\mathbb{K}_k$ is the new Kalman gain. The prediction of the mean of the new state's second part is done by using (27) and (31)

$$\hat{\eta}_{k+1|k} = \bar{F}_{22,k}\check{\eta}_{k|k} + \bar{F}_{21,k}\xi_k + \bar{B}_{2,k}u_k \tag{32}$$

The errors of $\check{\eta}_{k|k}$ and $\hat{\eta}_{k|k}$ are represented by $\check{e}_{k|k}$ and $\hat{e}_{k|k}$, respectively. Then, the error of the estimation is calculated using the second equation of (27) and (31) as follows

$$\check{e}_{k|k} = \hat{e}_{k|k} + \mathbb{K}_k(\xi_{k+1} - \bar{F}_{11,k}\xi_k - \bar{B}_{1,k}u_k - \bar{F}_{12,k}\hat{\eta}_{k|k}) \tag{33}$$

Substituting ξ_{k+1} from the first equation of (27) into (32) results in

$$\check{e}_{k|k} = (I - \mathbb{K}_k \bar{F}_{12,k})\hat{e}_{k|k} + \mathbb{K}_k \bar{G}_{1,k} w_k \tag{34}$$

Since the system noise w_k does not affect the estimation at time t_k, cross-covariance between $\hat{e}_{k|k}$ and w_k is zero. However, it will affect the estimation $\check{e}_{k|k}$, then

$$\boldsymbol{E}\left[\check{e}_{k|k} w_k^T\right] = \mathbb{K}_k \bar{G}_{1,k} \boldsymbol{E}\left[w_k w_k^T\right] = \mathbb{K}_k \bar{G}_{1,k} Q_k \tag{35}$$

Using (27) and (31), one can write

$$\hat{e}_{k+1|k} = \bar{F}_{22,k}\check{e}_{k|k} + \bar{G}_{2,k} w_{k_k} \tag{36}$$

Then, the covariance of $\hat{e}_{k+1|k}$ is

$$\begin{aligned}
&\mathbb{P}_{k+1|k} \\
&= \bar{F}_{22,k}\check{\mathbb{P}}_{k|k}\bar{F}_{22,k}^T + \bar{G}_{2,k} Q_k \bar{G}_{2,k}^T + \bar{F}_{22,k}\mathbb{K}_k \bar{G}_{1,k} Q_k \bar{G}_{2,k}^T \\
&\quad + \bar{G}_{2,k} Q_k \bar{G}_{2,k}^T \mathbb{K}_k^T \bar{F}_{22,k}^T
\end{aligned} \tag{37}$$

Now, using (27) and (31), the mean of the predicted transformed state and its covariance can be obtained as

$$\hat{y}_{k+1|k} = \begin{bmatrix} S_{k+1}^{-1} d_{k+1} \\ \hat{\eta}_{k+1|k} \end{bmatrix}, \quad \mathcal{P}_{k+1|k} = \begin{bmatrix} \mathbf{0} & \mathbf{0}^T \\ \mathbf{0} & \mathbb{P}_{k+1|k} \end{bmatrix} \tag{38}$$

Moreover, the mean and covariance of the original state can be predicted as

$$\begin{aligned}
\hat{x}_{k+1|k} &= V_{k+1}^T \hat{y}_{k+1|k} \\
P_{k+1|k} &= V_{k+1}^T \mathcal{P}_{k+1|k} V_{k+1}
\end{aligned} \tag{39}$$

Substituting (24) and (37) into (38) gives

$$\begin{aligned}
\hat{x}_{k+1|k} &= \mathcal{U}_{k+1}^T \xi_{k+1} + \mathcal{V}_{k+1}^T \hat{\eta}_{k+1|k} \\
P_{k+1|k} &= \mathcal{V}_{k+1}^T \mathbb{P}_{k+1|k} \mathcal{V}_{k+1}
\end{aligned} \tag{40}$$

For the implementation of the proposed method, steps in Algorithm 2 can be followed.

IV. SIMULATIONS

A simple two-dimensional navigation problem as shown in Fig. 1 will be used to evaluate the performance of the proposed method. A rover restricted to moving on an elliptical path will be considered. The position of the rover with respect to the inertial frame i is represented by (x, y). Suppose that a gyroscopic stable platform keeps the body frame parallel to the inertial frame. Two accelerometers measure the accelerations in the body frame. The major and minor axes of the ellipse are a and b, respectively. We can parameterize the equations of the elliptic motion as

$$r = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} \mathrm{a}\sin(t) \\ b\,cos(t) \end{bmatrix} \tag{41}$$

Then, the velocity of the rover is

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \mathrm{a}\cos(t) \\ -b\,sin(t) \end{bmatrix} \tag{42}$$

and the acceleration of the system is

$$f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} -\mathrm{a}\sin(t) \\ -b\,cos(t) \end{bmatrix} \tag{43}$$

The system given above can be represented in discrete-time state-space form as follows

$$\begin{aligned}
x_k &= F x_{k-1} + B u_{k-1} + G w_{k-1} \\
z_k &= H x_k + \upsilon_k
\end{aligned} \tag{44}$$

Here k is the time index, $x_k = \begin{bmatrix} r_k^T & v_k^T \end{bmatrix}^T$ is the state vector of the system, $u_{k-1} = f_{k-1}$ is the input vector, w_{k-1} is the accelerometers' noise, z_k and υ_k are the measurement vector and its white noise, respectively. We suppose that w_{k-1} and υ_k are independent Gaussian random vectors with zero mean and have covariances Q and R, respectively. In addition, the matrices F, B, G, and H are,

$$F = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{45}$$

$$B = G = \begin{bmatrix} \Delta t^2/2 & 0 \\ 0 & \Delta t^2/2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \tag{46}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{47}$$

The simulation parameters are given in Table 1. We will consider two types of measurement noise with standard deviation values of 5m as high noise and 0.1m as low noise measurements.

The simulated nominal position, velocity, and acceleration of the rover are shown in Fig. 2. It is known that the tangential velocity v_t and normal velocity v_n are

$$\begin{aligned}
v_t &= \sqrt{a^2\cos^2(t) + b^2\sin^2(t)} \\
v_n &= 0
\end{aligned} \tag{48}$$

In the defined problem, the velocity at time t_k is

$$v_k = \begin{bmatrix} \mathrm{a}\cos(t_k) \\ -b\sin(t_k) \end{bmatrix} \tag{49}$$

Then, one can define the following time-varying constraint

$$\begin{aligned}
C_k x_k &= d_k \\
C_k &= \begin{bmatrix} 0 & 0 & b\sin(t_k) & a\cos(t_k) \end{bmatrix} \\
d_k &= 0
\end{aligned} \tag{50}$$

We can see from (50) that the velocity related elements of the constraint matrix C_k oscillate and are sometimes zero. Thus, at the time of a velocity related element be zero, the constraint cannot improve the estimation of the corresponding state.

In the simulations, we are examining five filters which are the unconstrained KF (uncKF), proposed constrained KF (proKF), estimation projection KF (epKF), system projection KF (spKF), and equality constrained KF (ecKF). In simulations, the performances of compared methods will be evaluated for two different cases: Estimating observable and unobservable states. In all simulations, the errors in the x-direction are not considered since velocity and position

Algorithm 2 Proposed Constrained KF Algorithm

Initialization:

$$\boldsymbol{E}[x_0] = \hat{x}_0, cov(\hat{x}_0) = P_0$$

Measurement Update:

- Factorize the matrix C_k and extract the matrices S_k and V_k using (9).
- Partition matrix V_k as $\mathcal{U}_k$ and $\mathcal{V}_k$ using (10).
- Estimate the deterministic part of the new state (ξ_k) using (13).
- Use (14) to calculate the mean and covariance of the stochastic part of the new state.
- Use (16) to calculate the new measurement vector ($\mathcal{Z}_k$).
- Use (18) to update the mean and covariance of the deterministic part of the new state ($\hat{\eta}_{k|k}, \mathbb{P}_{k|k}$).
- Use (19) to estimate the new state's mean and covariance ($\hat{y}_{k|k}, \mathcal{P}_{k|k}$).
- Use (21) to estimate the original state ($\hat{x}_{k|k}, P_{k|k}$).

Prediction:

- Factorize the matrix C_{k+1} and extract the matrices S_{k+1} and V_{k+1} using (22).
- Partition matrix V_{k+1} as $\mathcal{U}_{k+1}$ and $\mathcal{V}_{k+1}$ using (24).
- Estimate the deterministic part of the new state (ξ_{k+1}) using (28).
- Using (26) and (27) extract the following matrices $\bar{F}_{11,k}, \bar{F}_{12,k}, \bar{F}_{21,k}, \bar{F}_{22,k}, \bar{B}_{1,k}, \bar{B}_{2,k}, \bar{G}_{1,k}, \bar{G}_{2,k}$
- Calculate the new measurement vector ($\mathbb{Z}_k$) in (29).
- Update the mean and covariance of the stochastic part of the new state vector ($\check{\eta}_{k|k}, \check{\mathbb{P}}_{k|k}$) using (31).
- Predict the mean and covariance of the stochastic part of the new state ($\hat{\eta}_{k+1|k}, \mathbb{P}_{k+1|k}$) by (32) and (37).
- Calculate the prediction of the new state ($\hat{y}_{k+1|k}, \mathcal{P}_{k+1|k}$) using (38).
- Calculate the mean and covariance of the original state vector ($\hat{x}_{k+1|k}, P_{k+1|k}$) using (40).
- Go to the measurement update phase.

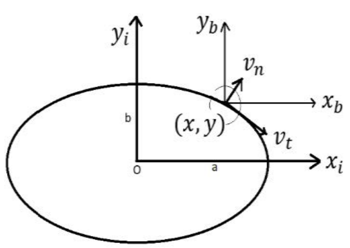

FIGURE 1. Inertial navigation with a reference path.

errors in x-direction are stable and remain small. In all figures, there are five subplots which are the errors of the uncKF, proKF, epKF, spKF, and ecKF, respectively. The error initial condition of the navigation system will have zero mean and a covariance P_0 as

$$P_0 = diag([4 \quad 4 \quad 0.01 \quad 0.01]) \tag{51}$$

TABLE 1. Simulation parameters.

Major axes	$a = 150m$
Minor axes	$b = 100m$
System noise covariance	$Q = I_{2\times2}$
Measurement noise covariances	$R = 5^2 I_{2\times2}$ or $R = 0.1^2 I_{2\times2}$
Sampling time	$\Delta t = 0.01sec$

A. OBSERVABLE CONSTRAINED STATE ESTIMATION

Physically, we know that all navigation parameters are observable when position measurements in x and y- directions are available. Observability matrix is a full rank matrix and then all states are observable. After 1000 times Monte Carlo simulations the root mean square of velocity and position errors in y-direction as a function of time for compared methods are shown in Fig. 3 and Fig. 4.

Fig.3 shows that uncKF has steady error close to 0.03m/sec, while other filters have oscillatory errors at steady condition. Maximum errors for constrained filters at the steady condition are 0.0089, 0.0107, 0.1092 and 0.0107m/sec for proKF, epKF, spKF and ecKF, respectively.

Fig. 4 shows that the position error in y-direction is stable but oscillating for all filters. The error peaks after 14 seconds for the different filters are 0.1751, 0.0267, 0.0467, 0.1536, and 0.0467m for uncKF, proKF, epKF, spKF and ecKF, respectively. The computational loads for the 20-second simulations are 0.0240, 0.1430, 0.0386, 0.065, and 0.0381sec for uncKF, proKF, epKF, spKF and ecKF, respectively.

Fig. 3 and Fig. 4 show that when the system is observable, the performances of all filters are almost the same, but the computational burden of proKF is greater than of the others and uncKF has the minimum computational load.

In the second simulation for the observable case, we are reducing the measurement noise to be $R = 0.1^2 I_{2\times2}$. Thus, the input noise is very high but measurement noise is very low for this condition, and the results of the simulation are depicted in Fig. 5 and Fig. 6.

Fig. 5 shows that again uncKF has steady root mean square error about 0.005m/sec and the errors of other filters are oscillatory with maximum magnitudes of 0.004, 2.0, 2.9, and 2m/sec for proKF, epKF, spKF and ecKF, respectively. The computational loads are almost the same with the ones obtained in the previous simulation.

Fig. 6 shows that the position error of uncKF in the steady condition has stationary with 0.0005m error while maximum errors of constrained filters are 0.0004, 2.1, 2.3, and 2.15m for proKF, epKF, spKF, and ecKF, respectively. Although the measurement noise is reduced from $5^2 I_{2\times2}$ to $0.1^2 I_{2\times2}$, the errors increase in constrained filters except proKF, and the performance of proKF is better than of uncKF.

B. UNOBSERVABLE CONSTRAINED STATE ESTIMATION

Now suppose that only the measurement in x-direction is available. In this case, only the position and velocity in x-direction are observable. The velocity in y-direction

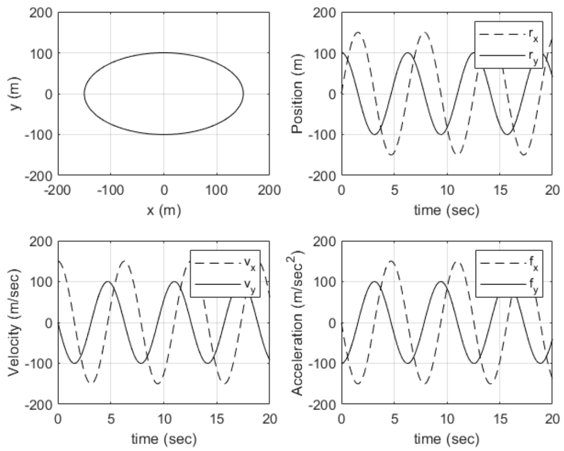

FIGURE 2. Nominal navigation parameters and reference path.

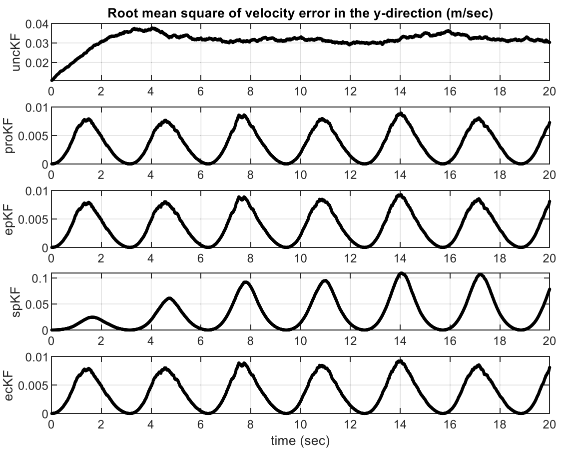

FIGURE 3. Root mean square of v_y error when measurements of position in x and y-directions are available and the covariance of measurement noise is $R = 5^2 I_{2\times 2}$.

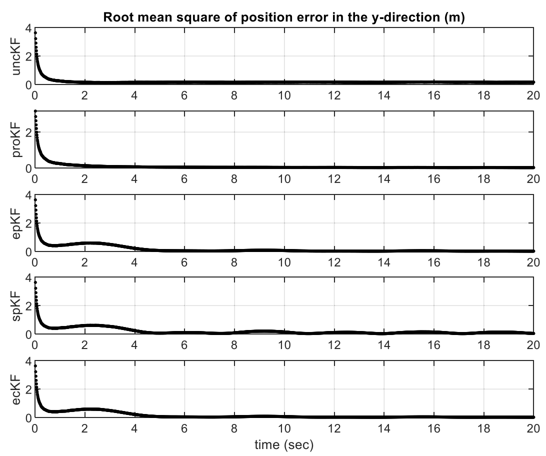

FIGURE 4. Root mean square of r_y error when measurements of position in x and y-directions are available and the covariance of measurement noise is $R = 5^2 I_{2\times 2}$.

remains unobservable for the uncKF. However, the constrained KF methods take the advantage of using the constraint equation to stabilize the velocity error and then to reduce the position error in y-direction.

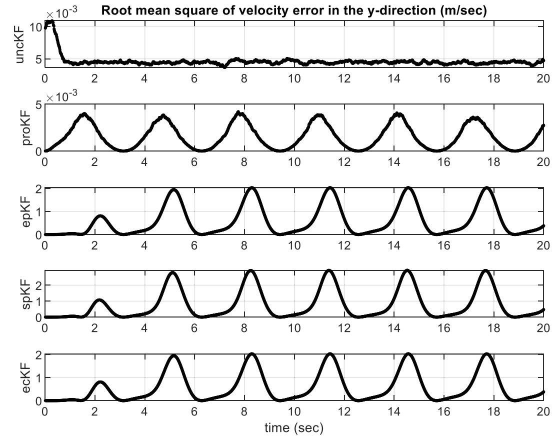

FIGURE 5. Root mean square of v_y error when measurements of position in x and y-directions are available and the covariance of measurement noise is $R = 0.1^2 I_{2\times 2}$.

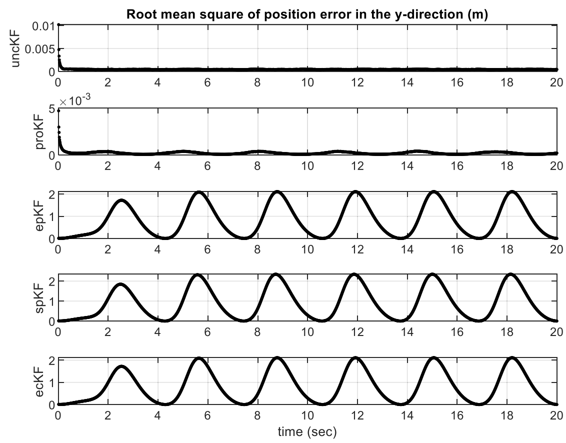

FIGURE 6. Root mean square of r_y error when measurements of position in x and y-directions are available and the covariance of measurement noise is $R = 0.1^2 I_{2\times 2}$.

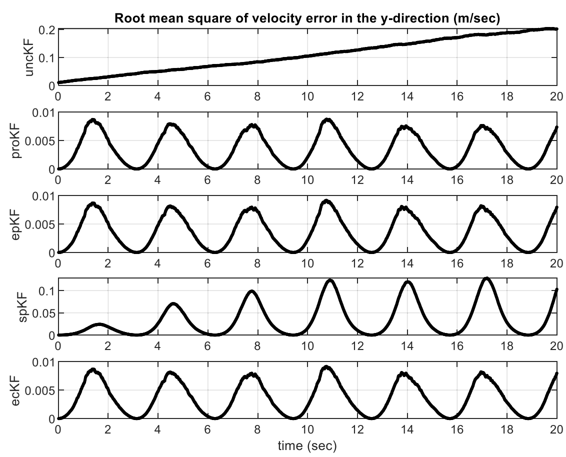

FIGURE 7. Root mean square of v_y error when measurements of position in x and y-directions are available and the covariance of measurement noise is $R = 5^2$.

In the first simulation of the unobservable case, the measurement matrix and the covariance of the measurement noise are chosen as $H = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ and $R = 5^2$. The obtained

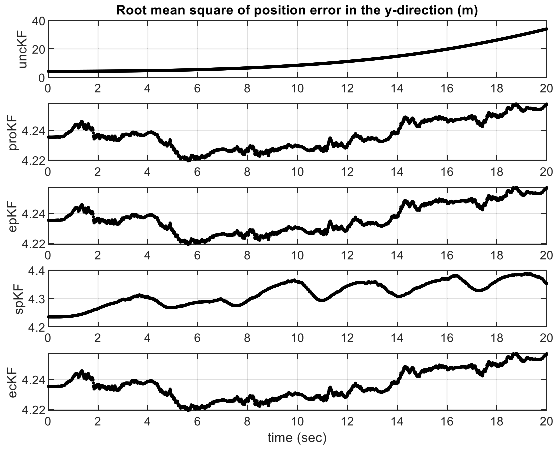

FIGURE 8. Root mean square of r_y error when measurements of position in x and y-directions are available and the covariance of measurement noise is $R = 5^2$.

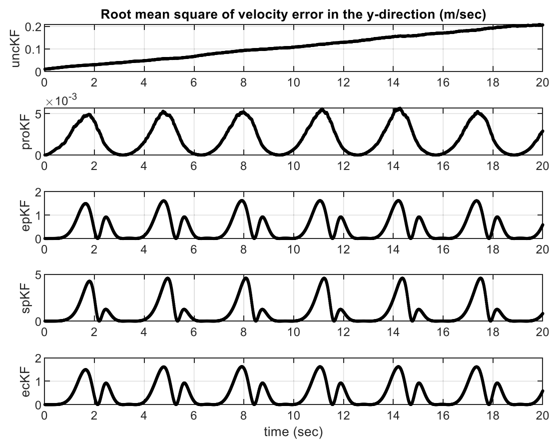

FIGURE 9. Root mean square of v_y error when measurements of position in x-direction are available and the covariance of measurement noise is $R = 0.1^2$.

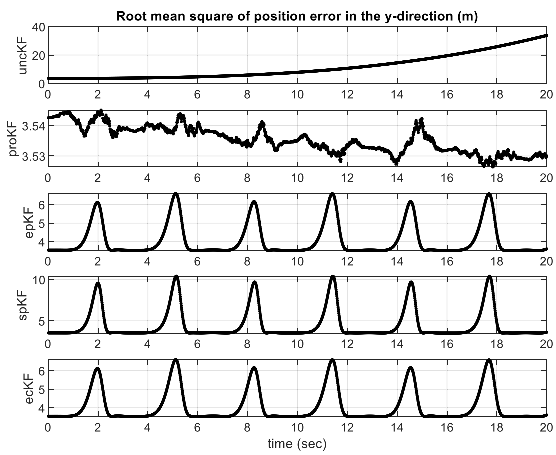

FIGURE 10. Root mean square of r_y error when measurements of position in the x-direction are available and the covariance of measurement noise is $R = 0.1^2$.

velocity and position errors in y-direction for the compared filters are shown in Fig. 7 and Fig. 8, respectively. Fig. 7 shows that the velocity error is unstable for uncKF due to the unobservability in y-direction. However, it is oscillatory and stable for all constrained filters. Maximum velocity errors in y-direction for constrained filters are 0.007, 0.008, 0.125, and 0.008m/sec for proKF, epKF, spKP and ecKF, respectively. Computational loads for all compared filters are 0.0116, 0.0868, 0.0186, 0.0409 and 0.0186sec for uncKF, proKF, epKF, spKP and ecKF, respectively. We can see that the lowest computational burden belongs to uncKF and the highest one belongs to proKF.

Fig. 8 shows that the position error in y-direction is again unstable for uncKF. For the constrained filters, maximum position errors in y-direction are 4.25, 4.257, 4.38, and 4.257m for proKF, epKF, spKP and ecKF, respectively. It can be seen that proKF has the lowest error.

Again, for better evaluating the performance of filters in different condition we are considering another simulation. In that simulation, the measurement matrix and the covariance of the measurement noise are chosen as $H = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ and $R = 0.1^2$. The obtained velocity and position errors in y-direction for the compared filters are shown in Fig. 9 and Fig. 10, respectively.

As we can see from Fig. 9 and Fig. 10, the velocity and position errors in y-direction have oscillatory behaviors for all constrained filters while uncKF has an unstable errors due to the unobservability of velocity and position in y-direction. Fig. 9 shows that the maximum velocity errors of constrained filters in steady state condition are 0.0056, 1.6, 4.5 and 1.6 m/sec for proKF, epKF, spKP and ecKF, respectively.

Fig. 10 shows that the maximum position errors in y-direction are 3.545m, 6.58m, 10.45m, 6.58m for proKF, epKF, spKF and ecKF, respectively.

Simulations show that the proposed filter performs better than other filters for estimating unobservable constrained states since the method uses the deterministic part of the dynamic equation as an extra measurement to reduce the prediction position error in y-direction.

V. CONCLUSION

In this paper, a new method based on the separation of the state vector into two parts is proposed. The method uses the deterministic part of the dynamic equation as an extra measurement to reduce the effects of system noise in the constraint direction. The proposed method is successfully implemented and compared with four other filters from the literature. Independent Monte Carlo simulations are done 1000 times and the results of these simulations are presented by figures.

The simulations show that the uncKF error is stable for all cases where the system is observable, but when the system loses its observability in y-direction, the position and velocity errors in that direction have unlimited growth. Unlike uncKF, all constrained Kalman filters contain limited errors for velocity in y-direction even if they lose their observability. Limited velocity error in y-direction for constrained filters comes from the constrained equation which has information related to components of velocity.

If the standard deviation of the measurement noise in the Kalman filter is reduced, the estimation error will normally be reduced. In an unconstrained Kalman filer we can see this behavior by comparing the Fig. 3 and Fig. 5. Regarding the constrained filters, when the measurement noise is high all constrained filters have almost the same performances. Due to the oscillatory behavior of constraint matrix elements, velocity error in y-direction has oscillatory behavior for all filters. In all simulations, the magnitude of velocity error in y-direction for the proposed filter is lower than the other filters. When we are reducing the measurement's standard deviation, the velocity errors of other constrained filters are increasing due to numerical degeneracy. It is amplified for unobservable cases, but in all simulations the proposed method has less error than the others.

The position error in y-direction is due to the velocity error in y-direction, and the position error is large for those with large velocity error. Therefore, the position error of the proposed filter was lower than the other filters due to having a lower velocity error.

It can be concluded that the proposed filter works very well and has good accuracy, while the other filters have low performances, especially in estimating unobservable states of the system. Whenever there is a deterministic constraint, we can decompose the constraint matrix and then the state separation is always present. Thus, the proposed method is applicable for all linear systems when the constrained equation is deterministic and linear. However, the computational load becomes higher.

REFERENCES

[1] J. L. Crassidis and J. L. Junkins, *Optimal Estimation of Dynamic Systems*, vol. 2. Boca Raton, FL, USA: Chapman & Hall, 2004.
[2] H. Hemami and B. Wyman, "Modeling and control of constrained dynamic systems with application to biped locomotion in the frontal plane," *IEEE Trans. Autom. Control*, vol. AC-24, no. 4, pp. 526–535, Aug. 1979.
[3] R. Tomovich, "Role of variable kinematic structure in control of skeletal activities," *Avtomatika Telemekhanika*, vol. 6, pp. 117–124, Mar. 1975.
[4] R. J. Hewett, M. T. Heath, M. D. Butala, and F. Kamalabadi, "A robust null space method for linear equality constrained state estimation," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 3961–3971, Aug. 2010.
[5] F. L. Markley and J. L. Crassidis, *Fundamentals of Spacecraft Attitude Determination and Control*. New York, NY, USA: Springer, 2014.
[6] S. D. Hayward, J. McWhirter, and I. Proudler, "Constrained Kalman filter for least-squares estimation of time-varying beamforming weights," *Institute of Mathematics and its Applications Conference Series*, vol. 67. Oxford, U.K.: Oxford Univ. Press, 1998.
[7] J. Porrill, "Optimal combination and constraints for geometrical sensor data," *Int. J. Robot. Res.*, vol. 7, no. 6, pp. 66–77, Dec. 1988.
[8] J. De Geeter, H. Van Brussel, J. De Schutter, and M. Decreton, "A smoothly constrained Kalman filter," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 19, no. 10, pp. 1171–1177, Oct. 1997.
[9] R. F. Stengel, *Optimal Control and Estimation*, 2nd ed. New York, NY, USA: Dover, 1994.
[10] S. Ko and R. R. Bitmead, "State estimation for linear systems with state equality constraints," *Automatica*, vol. 43, no. 8, pp. 1363–1368, Aug. 2007.
[11] W. Wen and H. F. Durrant-Whyte, "Model-based multi-sensor data fusion," in *Proc. IEEE Int. Conf. Robot. Automat.*, vol. 2, May 1992, pp. 1720–1726.
[12] D. Simon, "Kalman filtering with state constraints: A survey of linear and nonlinear algorithms," *IET Control Theory Appl.*, vol. 4, no. 8, pp. 1303–1318, Aug. 2010.
[13] P. Stoica and B. C. Ng, "On the Cramér–Rao bound under parametric constraints," *IEEE Signal Process. Lett.*, vol. 5, no. 7, pp. 177–179, Jul. 1998.
[14] D. Simon and T. L. Chia, "Kalman filtering with state equality constraints," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 38, no. 1, pp. 128–136, Jan. 2002.
[15] T. L. Chia, "Parameter identification and state estimation of constrained systems," Ph.D. dissertation, Dept. Syst. Control Eng., Case Western Reserve Univ., Cleveland, OH, USA, 1985.
[16] L. E. Andersson, L. Imsland, E. F. Brekke, and F. Scibilia, "On Kalman filtering with linear state equality constraints," *Automatica*, vol. 101, pp. 467–470, Mar. 2019.
[17] B. O. S. Teixeira, J. Chandrasekar, L. A. B. Tôrres, L. A. Aguirre, and D. S. Bernstein, "State estimation for linear and non-linear equality-constrained systems," *Int. J. Control*, vol. 82, no. 5, pp. 918–936, Apr. 2009.
[18] J. Zhou, T. Li, X. Wang, and L. Zheng, "Target tracking with equality/inequality constraints based on trajectory function of time," *IEEE Signal Process. Lett.*, vol. 28, pp. 1330–1334, 2021.
[19] T. Li, H. Chen, S. Sun, and J. M. Corchado, "Joint smoothing and tracking based on continuous-time target trajectory function fitting," *IEEE Trans. Autom. Sci. Eng.*, vol. 16, no. 3, pp. 1476–1483, Jul. 2019.
[20] P. S. Maybeck, *Stochastic Models, Estimation and Control*. New York, NY, USA: Academic, Aug. 1982.

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