Opinion Dynamics With Antagonistic Relationship and Multiple Interdependent Topics

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ABSTRACT In this paper, the opinion dynamics model with antagonistic (competitive) relationship, multiple interdependent topics and the stubborn individuals is investigated. Different from the classical Friedkin-Johnsen (FJ) model, in our model, antagonistic and cooperative relationships are considered simultaneously. Furthermore, multiple logically interdependent topics are discussed in our model where the logical interdependence between the different topics is captured by a logic matrix. For structurally balanced and unbalanced network topologies, we rigorously examine the stability and convergence of our proposed model. A few conditions of stability and convergence are obtained. These conditions completely depend on network topology, stubborn coefficient and the logic matrix. They fully show that these factors jointly affect the evolution of opinions. Finally, a few numerical simulations are provided to illustrate our obtained results.

INDEX TERMS Opinion dynamics, antagonistic relationship, multiple interdependent topics, stability.

I. INTRODUCTION

In recent years, opinion dynamics has received much attention for researchers in various fields. As an interdisciplinary subject of sociology, biology, economics, physics and control theory, opinion dynamics plays a great role in the development of human society [1]–[10]. In the society, the reciprocity between individuals constitutes the social network. As a special complex network, social networks have been studied extensively and many great achievements have been obtained [11]–[14]. Opinion dynamics is an important branch of social network research, it is concerned with the fundamental question of how individuals are influenced by other individuals in a social group. Here the term ''opinion'' is used to denote individuals' displayed attitudes to objects (i.e., topics or issues). The key problem of opinion dynamics lies in the constructing of models. In social networks, individuals interact and discuss opinions on a topic or a set of topics according to certain rules. Establishing mathematical models to analyze the evolution of opinions in social networks not only reveals the basic laws of the development of human

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society and animal population, but also plays an important role in the development of scientific knowledge and engineering technology [15], [16].

In order to investigate the evolution of opinions and the impact of interactions between individuals, many opinion dynamics models have been proposed over the past few decades. Among them, the classic DeGroot model [17] was proposed in the 1970s. The DeGroot model examines how a group of individuals reach an agreement on a common topic by exchanging their subjective opinions in a social network. In the DeGroot model, the opinion of individual i is determined by a weighted average of the opinions of individual i and its neighbors at the previous moment. If the network contains a spanning tree and the strong connected subgraph which consists of all root nodes is aperiodic, then the DeGroot model always reaches opinion consensus for the arbitrary initial opinions [18].

Subsequently, investigators made a further analysis of the DeGroot model, and a lot of variable models were proposed to capture the properties of opinion evolution [19]–[21]. Sociologist found that social actors accept readily opinions of like-minded individuals, and accept the more deviant opinions with discretion. This phenomenon is called the

biased assimilation [22]. Combining the updated rules of the DeGroot model with the idea of biased assimilation, the bounded confidence (BC) models were proposed [22], [23]. In BC modes, individuals only accept opinions within their confidence intervals but ignore the opinion which are outside the confidence intervals. Compared with the DeGroot model, BC models capture the basic rules of individual interaction in human society and may model more complex phenomena, such as, animal flocking [24]. In fact, different individuals may have different confidence intervals. Heterogenous BC models were used to represent this phenomenon [25]. In BC models, the network topologies are dependent on the system states, so it is more difficult to perform a strict mathematical analysis on it. So far, the rigorous mathematical analysis of BC models is still an open problem. For example, the statistical characteristics for the distribution of clusters are still unclear [25].

Another classic extension of the DeGroot model is the FJ model [26]. In the FJ model, the presence of the stubborn individuals was considered. The so-called stubborn individuals are willing to maintain their initial opinions as their prejudices. Different from the DeGroot model where each individual updates its opinion based on its own and neighbors' opinions, in the FJ model, the stubborn individuals also factor their initial opinions into every iteration of opinions. In other words, the stubborn agents never forget their prejudices, and are influenced by exogenous conditions. Therefore, it is difficult to reach opinion consensus for the FJ models. In fact, if the stubborn agents exist in the social network, social groups often form multiple clusters [27]–[29]. In [30] and [31], sufficient conditions for the stability of the FJ model was obtained. In [27], the FJ model was referred to a best-response game. By minimizing its own local function, the individuals update their opinions according to opinions of themselves and their neighbors [27]. For further extensions of the FJ model, please refer to references [31]–[33].

In all the literatures mentioned above, only cooperative relationship between individuals was investigated, while the possible antagonistic relationship between individuals was ignored. However, in fact, no matter in human society or in nature, confrontation and competition can be found [34], [35]. In human society, people compete with each other for resources. In nature, animals fight for water and territory. In biological system, cooperative and antagonistic relationships exist in the form of activators/inhibitors [34]. Therefore, the opinion dynamics models with antagonistic relationship have attracted extensive attention recently [36], [37]. Due to the existence of confrontation, it is difficult for social networks to achieve consensus. In this case, the concept of bipartite consensus was proposed in literature [35] to represent a kind of special disagreement of opinions. The signed graphs were used to characterize the cooperative and antagonistic relationship between individuals [35]. In [38], both state-dependent susceptibility to persuasion and antagonistic interactions were investigated. For three specializations of state-dependent susceptibility, some sufficient conditions of bipartite consensus were obtained. In [39], the competition between two stubborn agents was examined, the convergence and stability were discussed for the structurally balanced signed network. It should be pointed out that most existing literatures focus on the structurally balanced social networks. In this paper, for the structurally balanced and unbalanced networks, we will try to examine how opinions evolve.

Many of the previous models focus on a specific topic, but in the social networks, the agents always discuss a few topics at the same time [33]. A corresponding multidimensional FJ model was proposed in [33] where each topic was assumed to be inter independent. However, the topics may be interdependent in our real life, and the dynamics of the topic specific opinions are entangled. The multiple interdependent topics (the attitude to fruit and the attitude to watermelon) make models more complex. In [40], the authors proposed two continuous-time opinion dynamics models where the topics are multi interdependent. Besides, the necessary and sufficient conditions for the network to reach a consensus on each separate topic were obtained. If the dependencies between topics were ignored, then those models in [40] may not reach a consensus. This means that it is very important to consider the dependencies between topics when modelling opinion dynamics. As we all know, so far, the antagonistic relationship, stubbornness and multiple interdependent topics have not been considered simultaneously in any paper. In fact, this situation is possible in real social networks. So, in this paper, we consider the antagonistic relationship, stubbornness and multiple interdependent topics simultaneously to model the opinion evolution. Of course, this undoubtedly increases the difficulty of mathematical analysis model. In particular, when the network topology is structurally unbalanced, the existing results can not provide more meaningful reference. For a structurally unbalanced network, the relationship between individuals may be very chaotic. The evolution of opinions is often unexpected. This invisibly increases the difficulty of discussion. Therefore, the most existing literature pays more attention to the structurally balanced network. However, in this paper, by using the spectrum analysis and graph theory, we give a sufficient condition of the network convergence for the structurally unbalanced network, i.e., the network is connected. Of course, this condition may be relatively conservative.

The main contributions of this paper are as follows. Firstly, we have extended the classical FJ model and proposed a more general model. Competitive relationship and multiple interdependent topics, stubborn individuals are considered simultaneously in the proposed model. Secondly, by using the matrix theory and graph theory, the conditions of convergence and stability have been obtained for the structurally balanced and unbalanced network topologies. Meanwhile, these conditions depending on stubborn coefficient, network topology and logic matrix also fully show how these factors affect the opinion evolution together. In particular, we find that the structural unbalance will not destroy the convergence of the

connected network. Finally, numerical simulation have been used to show that the dependence between topics can not only affect the final opinion value, but also can influence the convergence of opinions.

This paper is organized as follows. In section II, we introduce the related knowledge of graph theory and make a brief introduction to our model. Main results and proofs are given in III. Section IV provides four simulation examples to illuminate our results. Finally, our conclusion is given in V.

II. PRELIMINARIES
In this section, model and mathematical preliminaries are provided to derive the main results of this paper.

A. NOTATIONS
Throughout this paper, $\mathbb{R}^{m \times n}$ and $\mathbb{R}^n$ denote, respectively, the $m \times n$ real matrix space and the n-dimensional real vector space. $|A| = [|a_{ij}|]_{i,j=1}^n$ means that each entry of matrix A takes an absolute value. The notation 1_n denotes the column vector of $[1, 1, \ldots, 1]^T \in \mathbb{R}^n$, and I_n is the identity matrix of size n. Given a square matrix A, $\rho(A)$ denotes its spectral radius. Given a pair of matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, their Kronecker product is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp \times nq}.$$

B. GRAPH THEORY
The graph consists of nodes and arcs (or edges). A node can be represented by a dot in the plane. And an arc is a directed line or curve which connecting a dot to another dot or itself. A graph is denoted by $G = (V, E)$, where V stands for the finite set of nodes and $E \in V \times V$ is the set of arcs. The node i connected to the node j is defined as the arc $e_{ij} = (i, j)$. An arc $e_{ii} = (i, i)$ is called a self-loop, and it connects a node to itself. In our paper, we use nodes, agents and individuals interchangeably to represent the same concept. For any arc $e_{ij} = (i, j) \in V$ the inverse arc $e_{ji} = (j, i)$ exists, we call that the graph is a undirected graph, otherwise, the graph is directed. Sequence of nodes $v_0 \mapsto v_1, v_1 \mapsto v_2, \ldots, v_{r-1} \mapsto v_r \in V$ is called a walk from v_0 to v_r; the node v_r is reachable from the node v_0 if there at least exists one walk from v_0 to v_r. A walk without repeating nodes is referred to a path. In the graph $G(V, E)$, if there is a node i which has a directed path from i to other nodes, then the graph $G(V, E)$ contains a spanning tree and the node i is named the root node. Suppose a matrix $A \in \mathbb{R}^{n \times n}$ satisfies: $a_{ij} \neq 0 \Leftrightarrow (j, i) \in E$, then the matrix is called the weighted adjacency matrix of the graph $G(V, E)$. In this situation, $G(A)$ is used to represent the graph $G(V, E)$. If the adjacency matrix A is allowed to take both positive and negative values, then it is called the signed adjacency matrix and its associated graph is called the signed graph. Suppose $A \in \mathbb{R}^{n \times n}$, if there exist two sets $V_1 \bigcap V_2 = \emptyset$ where $\emptyset$ denotes the empty set and $V_1 \bigcup V_2 = V$ such that

$a_{ij} \geq 0$ for $\forall i, j \in V_l (l \in \{1, 2\})$ and $a_{ij} \leq 0$ for $\forall i \in V_p, j \in V_q, p \neq q, (p, q \in \{1, 2\})$, we claim that the signed graph $G(A)$ is structurally balanced [35]. In other words, the signed graph $G(A)$ is structurally balanced if and only if there exists a diagonal matrix $P = diag[d_1, d_2, \ldots, d_n]$ such that $PAP \geq 0$ where $d_i \in \{1, -1\}$ [35].

C. MODEL DESCRIPTION
Opinion dynamics models on a particular topic have been widely researched and many conclusions have been drawn [41]–[44]. With the progress of communication technology, communication online is becoming more and more common. The amount of information in social networks has increased dramatically [19]. As a result, the spread and diffusion of many topics on social networks happen at the same time. Furthermore, these topics may be always interdependent. So in this paper, we examine the evolution of individuals' opinions under multiple interdependent topics. This implies that in our model the individuals' opinions on one specific topic are influenced by the opinions about the other topics. Now, let's consider the following scenario: a group of people discuss two topics, for example, fruit and watermelon. Watermelon is a very common and popular fruit in summer. If an individual hates fruit, then he dislikes to eat watermelon. If the influence process changes the individuals' attitudes toward fruit, saying that eating more fruit is good for health, then as a kind of fruit, there will be more people like to eat watermelon. On the contrary, the influence process changes the individuals' attitudes against fruit, warning that fruit now has high levels of the hormone detected, then fewer people choose watermelon even on a hot day. So it can be observed that there exists usually a certain logical relationship between multiple topics in social networks.

To ensure the consistency of the belief system, individuals may need to adjust their positions on multiple related topics at the same time. Tension and discomfort caused by inconsistencies can be resolved through an individual's introspective process. This introspective process was researched in cognitive dissonance and cognitive congruence theory [40]. It is thought to be an automatic process in the brain and can enable individuals to develop a system of consistent attitudes and beliefs. The external form of this introspective process can be reflected by the logical matrix mentioned below to a certain extent [40].

An opinion dynamics model describing multiple logically interdependent topics in discrete time was firstly proposed by Parsegov [14]. In [14], the authors characterized the logic interdependent relationships through a matrix of multi-issues dependence structure(MiDS). In [40], the authors defined a logical matrix to encode the logical coupling relationships between issues. Similarly, in this paper, we introduce a logic matrix to represent the dependencies between different topics. It should be pointed out that only cooperative relationship between individuals was considered in [14], [40]. To our knowledge, there is no literature to research the impact of competition on the evolutions of individuals opinions

with multiple interdependent topics. Therefore, in this paper, we simultaneously consider both the competitive relationship and multiple interdependent topics, and propose a novelty opinion dynamics model based on the classic FJ model.

$$x_i(k+1) = \lambda_i C \sum_{j=1}^{n} w_{ij} x_j(k) + (1-\lambda_i) x_i(0),$$
$$i = 1, \ldots, N, \quad (1)$$

where $x_i(k) \in \mathbb{R}^m$ represents the opinions of individual i at time k. $C \in \mathbb{R}^{m \times m}$ is the logical matrix. It is used to describe the dependencies between topics. The constant $\lambda_i \in [0,1]$ expresses the susceptibility of the individual i to interpersonal influence. Naturally, $(1-\lambda_i)$ represents the stubbornness of individual i regarding initial opinions $x_i(0)$. If $\lambda_i = 0$, we call the individual i a totally stubborn agent. It implies that the individual i refuses to communicate with others or ignores the other individuals' opinions. So his/her opinions will never change. If $0 < \lambda_i < 1$, we call the individual i a stubborn agent, which implies he/she is not completely "open-minded" [26]. The stubborn agents communicate with the others, but retain their original opinions to a certain extent. If $\lambda_i = 1$, we call the individual i a non-stubborn agent. The non-stubborn agent ignores the initial opinions and is influenced by the opinions of other individuals. In our article, we consider a special class of non-stubborn agents: they can be influenced by other stubborn agents. This implies that there exists a path from the stubborn agent to the non-stubborn agent. Such individual can be considered as "an implicitly stubborn agent".

In order to facilitate our research, we reclassify all individuals. We use the uniform name "non-opening agents" to represent totally stubborn agents, stubborn agents and implicitly stubborn agents. The remaining individuals are called "opening agents". They are not the stubborn agents. Furthermore, they are also not influenced by the stubborn agents.

$W = [w_{ij}] \in \mathbb{R}^{N \times N}$ describes the network topology. In this paper, both cooperative and competitive relationships between individuals are considered simultaneously. It implicitly indicates that $G(W)$ is a signed graph. If $w_{ij} \neq 0$ shows that the agent j can send information to the agent i at k instant. In other words, the individual j is a neighbor of the individual i. This means that the individual i can be affected by the individual j, i.e., when making a decision the individual i will consider the opinions of individual j. $|w_{ij}|$ represents the degree of impact and $\sum_{j=1}^{N} |w_{ij}| = 1$ are assumed for all $i = 1, 2, \ldots, N$. And $w_{ij} > 0$ represents the cooperative relationship and $w_{ij} < 0$ indicates the competitive or confrontational relationship.

We denote the entry of the matrix C with c_{pq}, which represents the influence of the issue q to the issue p. If $c_{pq} > 0$, it means that the issue p is closed to the issue q. For example, the following two topics: (*a*) eating fish is good for one's health and (*b*) people should eat more salmons. On the contrary, $c_{pq} < 0$ means the issue q is opposite to the issue p. Similarly, we give two issues: (*a*) eating fish is good for one's health and (*b*) fishes are now contaminated

by toxic chemical. A natural assumption in this paper is that $\sum_{q=1}^{m} |c_{pq}| = 1, p = 1, \ldots, m$ and $c_{ii} > 0$, which implies that the matrix $|C|$ is a row-stochastic matrix.

In fact, both competition and logical relationship between topics can impact on the finial opinions of individuals, we will illustrate this with a few examples.

Example 1: The model (1) will regress to the classical FJ model if $C = I_m$. And the compact form of the classical FJ model is:

$$x(k+1) = \Lambda W x(k) + (I_N - \Lambda) x(0), \quad (2)$$

where $\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_N]$. $x(k) = [x_1(k)^T, \ldots, x_N(k)^T]^T$ represents the opinions of all individuals at time k. We assume that there are five individuals in the network and two topics are discussed at the same time. We set the stubborn coefficient as $\Lambda = \text{diag}[0.7, 0.6, 0.8, 0, 0.9]$. Obviously, the agent 4 is a totally stubborn agent and the individuals 1,2,3 and 5 are stubborn agents. The adjacent matrix is as follows

$$W_1 = \begin{bmatrix} 0.4 & 0.15 & 0.2 & 0.15 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.1 & 0.3 \\ 0.4 & 0.1 & 0.3 & 0.1 & 0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0.15 & 0.15 & 0.2 & 0.3 & 0.2 \end{bmatrix}. \quad (3)$$

The opinions of two independent topics (a) and (b) are represented by $x_i^1(k)$ and $x_i^2(k)$ respectively. Here, we choose the initial opinions:

$$x(0) = \begin{bmatrix} 20,-20 & -20,5 & 50,20 & 75,-50 & 85,5 \end{bmatrix}^T. \quad (4)$$

Through simple calculation, the final opinions of all individuals are:

$$x_1' = \begin{bmatrix} 35,-22 & 16,-11 & 43,-13 & 75,-50 & 53,-24 \end{bmatrix}^T. \quad (5)$$

Comparing the initial opinions with the final opinions, one can find that the opinions of all individuals have changed except the individual 4. Especially, the individuals 2,3 and 5 have opposite attitudes towards the topic (b). Besides, for the topic (a), the individual 2 changes from negative attitude to positive attitude. By taking a comprehensive observation, it is obvious that the attitudes of other agents are consistent with individual 4. These changes fully indicate that stubbornness has a significant impact on individuals' opinions. If both cooperative relationship and competitive relationship exist between individuals, we replace the adjacent matrix W_1 with W_2:

$$W_2 = \begin{bmatrix} 0.4 & -0.15 & 0.2 & -0.15 & -0.1 \\ 0.2 & 0.3 & -0.1 & 0.1 & 0.3 \\ 0.4 & 0.1 & 0.3 & 0.1 & -0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0.15 & -0.15 & 0.2 & -0.3 & 0.2 \end{bmatrix}. \quad (6)$$

And the other conditions remain the same, we obtain the individuals final opinions

$$x_2' = \begin{bmatrix} -4,1 & 2,-4 & 16,1.6 & 75,-50 & 38,-15 \end{bmatrix}^T. \quad (7)$$

Compared (5) and (7), it can be found that when there exists competitive relationship between individuals, the attitude of individual 1 has a significant change about topic (a) and topic (b). This implies that competitive relationship can also affect the individuals' attitude towards events.

Example 2: Different from example 1, in this example, we assume that topics are interdependent. For instance, we have the topic (a): the attitudes about fruit and the topic (b): the attitudes about watermelon (as a part of fruit). In our cognition, people who don't like fruit generally don't like watermelon. Conversely, people who don't like watermelon may also don't like fruit. It implies that the opinions of all individuals on these two topics should be consistent in symbols. In other words, there is a positive interaction between the two topics. We introduce the matrix C_1 to describe the relationship of the topics (a) and (b).

$$C_1 = \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix}. \tag{8}$$

Next, we use the same initial opinions (4) and obtain the final opinions of the five individuals

$$x'_{C_1} = \left[\underbrace{-1, -8} \quad \underbrace{0.2, 4} \quad \underbrace{15, 3} \quad \underbrace{75, -50} \quad \underbrace{33, 4} \right]^T. \tag{9}$$

Comparing (5) and (7), we find that when the interdependence between individuals is not considered, the attitudes of individual 1 towards the two events are opposite. When the dependency is considered, the attitudes of individual 1 towards the two events are the same. It implies that considering dependencies makes it easier to achieve consistency in belief systems. Therefore, it is easier for social networks to reach agreement on belief systems by considering dependencies. That is to say, topic interdependence plays an important role in the evolution of opinions.

Remark 1: The Example 1 illustrates competitive relationship between individuals and stubbornness can cause the inconsistency of agent's opinions. The Example 2 shows that the interdependent relationships between different topics can affect the final opinions of agents too. In this paper, competitive relationship, stubbornness and multiple-interdependent are investigated. To some extent, our model is more popular.

Remark 2: Different from the classic FJ model [26], where a special topic was considered, in our model, multiple interdependent topics are researched simultaneously. Furthermore, we allow elements of the adjacency matrix W to have negative values, it implies cooperative relationship and competitive relationship are considered at the same time. However, the classic FJ model ignored the competitive relationship between individuals.

D. MATHEMATICAL PRELIMINARIES

The following definitions and lemmas are needed for the derivation of our main results in this paper.

Definition 1: If for any initial value $x_i(0)(i = 1, 2, \ldots N)$, the sequence $x_i(k)(i = 1, 2, \ldots N)$ has a limit, the model (1)

is convergent. Especially, if $\rho((\Lambda W) \otimes C) < 1$, the model (1) is stable. A stable model (1) is convergent.

Lemma 1 ([45]): Suppose $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$. The spectrum of the matrix $A \otimes B$ consists of all products $\lambda_i \mu_j$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A and $\mu_1, \ldots, \mu_m$ are those of B.

Lemma 2 ([18]): A is a stochastic matrix, $G(A)$ has a spanning tree and its strongly connected component of all its root nodes is aperiodic. Then, 1 is the only maximum-modulus eigenvalue of A, and its algebraic multiplicity is 1.

Lemma 3 ([15]): Let $C \in \mathbb{R}^{m \times m}$ be an irreducible matrix, and there exists at least a diagonal entry $c_{ii} > 0$, then $\rho(C) = \rho(|C|)$ if and only if $G(C)$ is structurally balanced.

Lemma 4 ([19]): Suppose that the matrix W is a row-stochastic matrix. $\lim_{k \to \infty} W^k$ exists if and only if each independent strongly connected component of $G(W)$ contains at least one node whose the lengths of all loops are coprime.

III. MAIN RESULTS

In this section, we discuss the stability and convergence of system (1). For the convenience of analyzing the problem, we will rewrite the system (1).

Let $x(k) = [x_1(k)^T, x_2(k)^T, \ldots, x_N(k)^T]^T$, $W = [w_{ij}]_{N \times N}$, $x(0) = [x_1(0)^T, x_2(0)^T, \ldots, x_N(0)^T]^T$, $\Lambda = diag[\lambda_1, \lambda_2, \ldots, \lambda_N]$. Then the model (1) can be rewritten by following compact from:

$$x(k + 1) = [(\Lambda W) \otimes C]x(k) + [(I_N - \Lambda) \otimes I_m]x(0). \tag{10}$$

We assume that the index set of non-opening individuals is expressed as $\{1, 2, \ldots, n_1\}$, the index set of opening agents is $\{n_1 + 1, n_1 + 2, \ldots, N\}$. Now we can rewrite the matrix ΛW with the following from:

$$\Lambda W = \begin{bmatrix} \Lambda^{11} W^{11} & \Lambda^{11} W^{12} \\ 0 & W^{22} \end{bmatrix}, \tag{11}$$

where $\Lambda^{11} = diag[\lambda_1, \lambda_2, \ldots, \lambda_{n_1}]$, $W^{11} = [w_{ij}]_{n_1 \times n_1}$, $i, j \in \{1, 2, \ldots, n_1\}$, $W^{12} = [w_{ij}]_{n_1 \times (N-n_1)}$, $i \in \{1, 2, \ldots, n_1\}$, $j \in \{n_1 + 1, n_1 + 2, \ldots, N\}$. $W^{22} = [w_{ij}]_{(N-n_1) \times (N-n_1)}$, $i, j \in \{n_1 + 1, n_1 + 2, \ldots, N\}$.

In our paper, the considered graph is a signed graph. Here, we firstly consider the structurally balanced graph.

(A) $G(W)$ is a structural balanced graph.

Because $G(W)$ is structurally balanced, there exists a diagonal matrix $P = diag[d_1, d_2, \ldots, d_N]$ such that $PAP \geq 0$ where $d_i \in \{1, -1\}$.

Let

$$y(k + 1) = (P \otimes I_m)x(k + 1). \tag{12}$$

As $P^{-1} = P$, we also obtain that $x(k+1) = (P \otimes I_m)y(k+1)$. At this time

$$\begin{aligned} &y(k + 1) \\ &= (P \otimes I_m)x(k + 1) \\ &= (P \otimes I_m)\{[(\Lambda W) \otimes C]x(k) + [(I_N - \Lambda) \otimes I_m]x(0)\} \\ &= [(P\Lambda WP) \otimes C]y(k) + [(I_N - \Lambda) \otimes I_m]y(0). \end{aligned} \tag{13}$$

Let $y^1(k) = [y_1(k)^T, y_2(k)^T, \ldots, y_{n_1}(k)^T]^T$, $P^1 = diag[d_1, d_2, \ldots, d_{n_1}]$, $y^2(k) = [y_{n_1+1}(k)^T, y_{n_1+2}(k)^T, \ldots, y_N(k)^T]^T$, $P^2 = diag[d_{n_1+1}, d_{n_1+2}, \ldots, d_N]$. Then we have

$$P = \begin{bmatrix} P^1 & \\ & P^2 \end{bmatrix}$$

Therefore, we can rewrite model (13) with the following form:

$$\begin{bmatrix} y^1(k+1) \\ y^2(k+1) \end{bmatrix}$$
$$= \begin{bmatrix} (P^1\Lambda^{11}W^{11}P^1)\otimes C & (P^1\Lambda^{11}W^{12}P^2)\otimes C \\ 0 & (P^2W^{22}P^2)\otimes C \end{bmatrix} \begin{bmatrix} y^1(k) \\ y^2(k) \end{bmatrix}$$
$$+ \begin{bmatrix} (I_{n_1}-\Lambda^{11})\otimes I_m \\ 0 \end{bmatrix} \begin{bmatrix} y^1(0) \\ y^2(0) \end{bmatrix}$$
$$= \begin{bmatrix} (P^1\Lambda^{11}W^{11}P^1)\otimes C & (P^1\Lambda^{11}W^{22}P^2)\otimes C \\ 0 & (P^2W^{22}P^2)\otimes C \end{bmatrix} \begin{bmatrix} y^1(k) \\ y^2(k) \end{bmatrix}$$
$$+ \begin{bmatrix} [(I_{n_1}-\Lambda^{11})\otimes I_m]y^1(0) \\ 0 \end{bmatrix}.$$

That is to say:

$$y^1(k+1) = [(P^1\Lambda^{11}W^{11}P^1)\otimes C]y^1(k)$$
$$+[(P^1\Lambda^{11}W^{12}P^2)\otimes C]y^2(k)$$
$$+[(I_{n_1}-\Lambda^{11})\otimes I_m]y^1(0), \quad (14)$$
$$y^2(k+1) = [(P^2W^{22}P^2)\otimes C]y^2(k). \quad (15)$$

Theorem 1: If the opening agents are absent, then the model (1) is stable. At this time, the final opinion

$$x'_C := \lim_{k\to\infty} x(k)$$
$$= P^*\{[(I_{mN}-P\Lambda WP)\otimes C]^{-1}(I_N-\Lambda)\}P^*x(0), \quad (16)$$

where $P^* = P\otimes I_m$.

Proof: We construct a matrix B as following:

$$B_{(N+1)\times(N+1)} = \begin{bmatrix} 1 & 0_{1\times N} \\ \beta & P\Lambda WP \end{bmatrix},$$

where $\beta = [\beta_1, \beta_2, \ldots, \beta_N]^T$, $\beta_i = 1 - \lambda_{ii}$. Obviously, the matrix B is row-stochastic, $G(B)$ contains a spanning tree and an unique root node.

In other words, strongly connected component consisting of all root nodes is aperiodic. According to Lemma 2, we know that 1 is the only maximum-modulus eigenvalue of B, and its algebraic multiplicity is 1.

Furthermore, we have

$$|\lambda I_{N+1} - B|$$
$$= \left| \begin{bmatrix} \lambda & \\ & \lambda E \end{bmatrix} - \begin{bmatrix} 1 & 0_{1\times N} \\ \beta & P\Lambda WP \end{bmatrix} \right|$$
$$= \begin{vmatrix} \lambda-1 & 0_{1\times N} \\ -\beta & \lambda I_N - P\Lambda WP \end{vmatrix}$$
$$= (\lambda-1)|\lambda I_N - P\Lambda WP|. \quad (17)$$

According to (17), it can be obtained that $\rho(P\Lambda WP) < 1$. We know that $\rho(C) \le 1$, hence we have $\rho[(P\Lambda WP)\otimes C] < 1$,

if this holds:

$$y(k+1) = [(P\Lambda WP)\otimes C]y(k) + [(I_N-\Lambda)\otimes I_m]y(0)$$
$$= [(P\Lambda WP)\otimes C]\{[(P\Lambda WP)\otimes C]y(k-1)$$
$$+[(I_N-\Lambda)\otimes I_m]y(0)\} + [(I_N-\Lambda)\otimes I_m]y(0)$$
$$= \ldots$$
$$= [(P\Lambda WP)\otimes C]^{k+1}y(0)$$
$$+\{[(P\Lambda WP)\otimes C]^k + \ldots + I_{mN}\}$$
$$\times [(I_N-\Lambda)\otimes I_m]y(0). \quad (18)$$

As $\rho[(P\Lambda WP)\otimes C] < 1$, we have $\lim_{k\to\infty}[(P\Lambda WP)\otimes C]^{k+1} = 0$. Besides, $\{[(P\Lambda WP)\otimes C]^k + \ldots + I_{mN}\}[I_{mN} - (P\Lambda WP)\otimes C] = I_{mN} - [(P\Lambda WP)\otimes C]^{k+1}$, so $\lim_{k\to\infty}\{[(P\Lambda WP)\otimes C]^k + \ldots + I_{mN}\} = [I_{mN}-(P\Lambda WP)\otimes C]^{-1}$. Therefore, $\lim_{k\to\infty} y(k+1) = [I_{mN} - (P\Lambda WP)\otimes C]^{-1}[(I_N-\Lambda)\otimes I_m]y(0)$.

Notice that $x(k+1) = (P\otimes I_m)y(k+1)$ and our proof has been completed.

Corollary 1: If the graph $G(W)$ is strongly connected and $\Lambda \ne I_N$, then the model (1) is stable.

Proof: The strong connectivity of $G(W)$ implies that the agents are either stubborn or reachable by the other stubborn agents. It indicates that the opening agents disappear. According to Theorem 1, the conclusion is obviously true.

When the logic matrix is structurally unbalanced, we have the conclusion as follows.

Theorem 2: If the matrix C is irreducible and $G(C)$ is structurally unbalanced, then the model (1) is stable and the vector of final opinions is (16).

Proof: According to Lemma 3, if the matrix C is irreducible and $G(C)$ structurally unbalanced, then $\rho(C) < \rho(|C|) = 1$. Noticing that $\rho((P\Lambda WP)\otimes C) = \rho(P\Lambda WP)\rho(C)$ and $\rho(P\Lambda WP) = \rho(\Lambda W) \le 1$, then we obtain that $\rho((P\Lambda WP)\otimes C) < 1$. Next, according to the proof of Theorem 1, one can complete the proof of Theorem 2.

Next, we consider the case where there exist opening agents and the matrix C is structurally balanced. And the conclusion is as follows.

Theorem 3: When there exist the opening agents. Suppose the matrix C is structurally balanced and $G(W)$ has a self-loop. The model (1) is convergent if and only if each independent strongly connected component of $G(C)$ and $G(W^{22})$ contains at least one node whose the lengths of all loops are coprime. The final opinions

$$x'_C = P^* \begin{bmatrix} [(I_{n_1}-P^1\Lambda^{11}W^{11}P^1)\otimes C]^{-1} & 0 \\ 0 & I_{(N-n_1)m} \end{bmatrix} QP^*x(0), \quad (19)$$

where

$$Q = \begin{bmatrix} (I_{n_1}-\Lambda^{11})\otimes I_m & (P^1\Lambda^{11}W^{12}P^2W^{22}_*)\otimes CC_* \\ 0 & W^{22}_*\otimes C_* \end{bmatrix},$$

$W^{22}_* = \lim_{k\to\infty}(P^2W^{22}P^2)^k$ and $C_* = \lim_{k\to\infty} C^k$.

Proof: It can be seen from the above that $x(k+1)$ and $y(k+1)$ have the same convergence. If each independent

strongly connected component of $G(C)$ and $G(W^{22})$ contains at least one node which the length of all loops is coprime, then each independent strongly connected component of $G(P^2W^{22}P^2)$ also contains at least one node which the length of all loops is coprime. According to the Lemma 4, we know that $\lim_{k\to\infty}(P^2W^{22}P^2)^k$ exists. Obviously, $\lim_{k\to\infty}C^k$ also exists. Then we have $y^2(k) \to (W_*^{22} \otimes C_*)y^2(0)$ as $k \to \infty$, if this holds, according to (14) and (15), similar to the proof of [14], one immediately obtains (19). The sufficient part has proved.

Proof of necessity: According to (14) and (15), the convergence of the model (1) implies that $\lim_{k\to\infty}[(P^2W^{22}P^2) \otimes C]^k$ exists. Furthermore, the existence of $\lim_{k\to\infty}[(P^2W^{22}P^2) \otimes C]^k$ shows that both $\lim_{k\to\infty}[(P^2W^{22}P^2)]^k$ and $\lim_{k\to\infty}C^k$ exist. Otherwise, we suppose that $\lim_{k\to\infty}[(P^2W^{22}P^2)]^k$ doesn't exist. Noticing that the matrix $P^2W^{22}P^2$ is a stochastic matrix, 1 is an eigenvalue of it. On the other hand, when $\lim_{k\to\infty}[(P^2W^{22}P^2)]^k$ doesn't exist, there must exist a eigenvalue α of the matrix $P^2W^{22}P^2$ such that $\alpha \neq 1$ and its modulus is 1. Since the matrix C is structurally balanced, it means that 1 is the maximum-modulus eigenvalue of it. According to Lemma 1, $1 \cdot \alpha = \alpha$ is the maximum-modulus eigenvalues of $P^2W^{22}P^2 \otimes C$. Noticing $\alpha \neq 1$, therefore we conclude that $\lim_{k\to\infty}[(P^2W^{22}P^2) \otimes C]^k$ doesn't exist. This is obviously contradictory. Similarly, if we suppose that $\lim_{k\to\infty}C^k$ doesn't exist, we can obtain that $\lim_{k\to\infty}[(P^2W^{22}P^2) \otimes C]^k$ also doesn't exist. Noting that both $G(C)$ and $G(W^{22})$ have a self-loop, it implies that each independent strongly connected component of $G(C)$ and $G(W^{22})$ contains at least one node with the length of all loops is coprime. According to Lemma 4, we know that both $\lim_{k\to\infty}C^k$ and $\lim_{k\to\infty}[(P^2W^{22}P^2)]^k$ exist. The proof of Theorem 3 has been completed.

(B) $G(W)$ is a structurally unbalanced graph.

In this section, we reclassify the individuals. We divide all individuals into non-totally stubborn individuals which is represented by index set $\{1, 2, \ldots, n_2\}$ and totally stubborn individuals with $\{n_2+1, n_2+2, \ldots, N\}$. Then the system (1) can be represented as following:

$$x_i(k + 1) = \lambda_i C \sum_{j=1}^{n_2} w_{ij}x_j(k) + (1 - \lambda_i)x_i(0)$$
$$+ \lambda_i C \sum_{j=n_2+1}^{n} w_{ij}x_j(0) (i \in \{1, 2, \ldots, n_2\}), \quad (20)$$
$$x_j(k + 1) = x_j(0), (j \in \{n_2+1, n_2+2, \ldots, N\}). \quad (21)$$

We rewrite the matrix ΛW as follows:

$$\Lambda W = \begin{bmatrix} \Lambda^{22}W^{11} & \Lambda^{22}W^{12} \\ 0 & 0 \end{bmatrix}, \quad (22)$$

where $\Lambda^{22} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_{n_2}]$, $W^{11} = [W_{ij}]_{n_2 \times n_2}$, $W^{12} = [W_{ij}]_{n_2 \times (N-n_2)}$, $(i \in \{1, 2, \ldots, n_2\}, j \in \{n_2+1, n_2+2, \ldots, N\})$. Then the model (20), (21) can be rewritten

as follows:

$$\begin{bmatrix} x^1(k + 1) \\ x^2(k + 1) \end{bmatrix} = \begin{bmatrix} (\Lambda^{22}W^{11}) \otimes C & (\Lambda^{22}W^{12}) \otimes C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^1(k) \\ x^2(k) \end{bmatrix}$$
$$+ [(I_N - \Lambda) \otimes I_m] \begin{bmatrix} x^1(0) \\ x^2(0) \end{bmatrix}. \quad (23)$$

Remark 4: If $\Lambda^{22}W^{11}$ is structurally balanced, similar to Theorem 2, we can obtain the conclusion of the convergence of the model. As the length of our paper, we will not repeat here. Next, the main thing that we focus on is the convergence of system (23) when $\Lambda^{22}W^{11}$ is structurally unbalanced. In this case, our conclusion is as follows.

Theorem 4: The system (23) is convergent if $\Lambda^{22}W^{11}$ is structurally unbalanced and strongly connected. At this time, $x^2(k) \to x^2(0); x^1(k) \to [(I_{n_2} - \Lambda^{22}W^{11}) \otimes C]^{-1}\{[(I_{n_2} - \Lambda^{22}) \otimes I_{n_2}]x^1(0) + [(\Lambda^{22}W^{12}) \otimes C]x^2(0)\}$.

Proof: As $G(\Lambda^{22}W^{11})$ is structurally unbalanced and strong connected, according to Lemma 3, we have $\rho(\Lambda^{22}W^{11}) < \rho(|\Lambda^{22}W^{11}|) \le 1$. As $\rho(C) \le 1$, we can get $\rho[(\Lambda^{22}W^{11}) \otimes C] < 1$. According to the proof of Theorem 2, we can obtain $\lim_{k\to\infty}x^1(k) = [(I_N - \Lambda^{22}W^{11}) \otimes C]^{-1}\{[(I_N - \Lambda^{22}) \otimes I_{n_2}]x^1(0) + [(\Lambda^{22}W^{12}) \otimes C]x^2(0)\}$.

IV. NUMERICAL EXAMPLE

In this section, we will give four examples to verify the validity of our conclusions. In the following simulations, the blue and red lines are used to represent the opinions of stubborn agents and non-stubborn agents, respectively.

Example 3: In this example, we consider a network with 7 agents. The network topology, logic matrix and the initial opinions of 7 agents are given as follows:

$$W_3 = \begin{bmatrix} 0.3 & 0 & 0 & 0.3 & 0.1 & 0 & -0.3 \\ 0.3 & 0.2 & 0.1 & 0 & 0.2 & -0.2 & 0 \\ 0 & 0.3 & 0.2 & 0.4 & 0.1 & 0 & 0 \\ 0.2 & 0 & 0.2 & 0.4 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0.3 & 0 & -0.2 \\ 0 & -0.3 & 0 & 0 & 0.2 & 0 & 0.5 \\ -0.1 & 0 & -0.4 & 0 & 0 & 0.3 & 0.2 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{bmatrix},$$

$$x(0)_a = [0.3, -0.1, 0.6, 0.5, 0.1, -0.6, 0.4],$$
$$x(0)_b = [0.5, 0.3, -0.8, 0.1, 0.7, 0.6, 0.2]. \quad (24)$$

$x(0)_a$ and $x(0)_b$ represent the initial opinions of 7 agents about the topic (a) and the topic (b), respectively. According to W_3, we know that $G(W_3)$ is structurally balanced and has a spanning tree. Let $\Lambda_1 = diag[0, 0.7, 1, 1, 1, 1, 1]$, then the individual 1 is a totally stubborn agent and the individual 2 is a stubborn agent. The non-stubborn agents 3,4,5,6 and 7 are influenced by the agents 1 and 2, which implies that the conditions of Theorem 1 are satisfied. The evolutions of opinions are shown in Fig.1 and Fig.2. According to Fig.1 and Fig.2, we find that the system reaches the equilibrium point quickly.
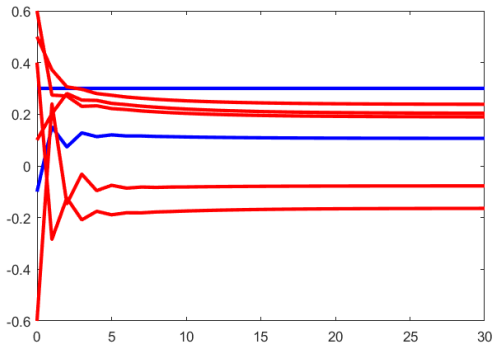
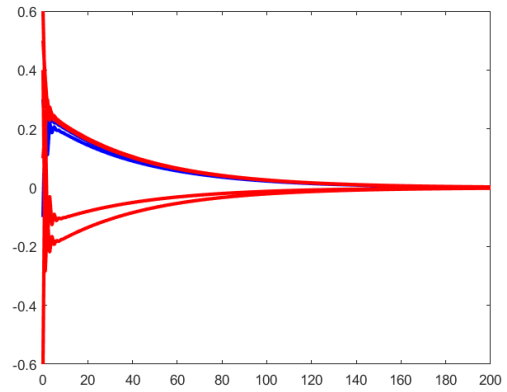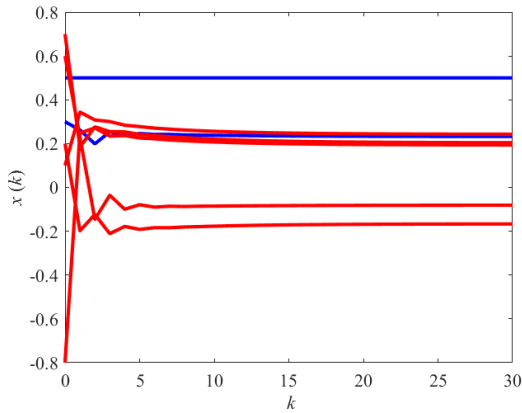**FIGURE 1.** Evolution of opinions for the topic a in Example 3.

FIGURE 2. Evolution of opinions for the topic b in Example 3.

FIGURE 3. Evolution of opinions of the similar DeGroot model for topic a in Example 3.

FIGURE 4. Evolution of opinions of the similar DeGroot model for topic b in Example 3.
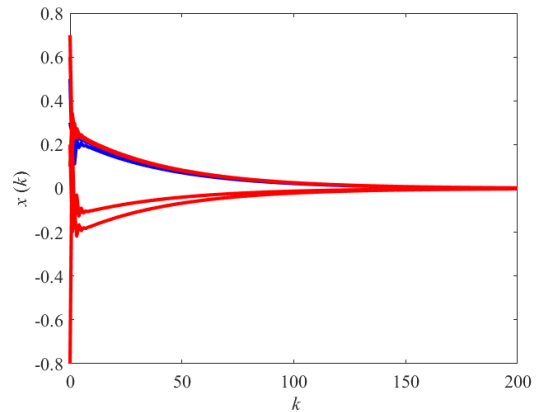

FIGURE 5. When $C = I_m$, evolution of opinions for topic a in Example 3.

Let $\Lambda = I_N$, we can obtain a similar DrGroot model. In the model, we consider the same initial opinions (24) and the evolutions of opinions are shown in Fig.3 and Fig.4. It is obvious that the final opinions of the similar DeGroot model reach an agreement. It implies that the stubborn agents play an important role in social networks. Furthermore, if let $C_2 = I_2$, our model degenerates into the classical FJ model. Under these circumstances, for the initial opinions (24), the evolutions of opinions are shown in Fig.5 and Fig.6. Compared Figs.1.2 with Figs.5.6, one can find that logical matrix C can affect the final opinion value. So, for the same initial opinions, our presented model and the classical FJ model will generally converge to the different opinions.

Example 4: In this example, we replace the network topology and logic matrix in example 3 with W_4 and C_3. Other conditions remain the same.

$$W_4 = \begin{bmatrix} 0.3 & 0 & 0.3 & 0 & 0 & 0 & -0.4 \\ 0.3 & 0.4 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0.1 & 0.1 & 0 & -0.3 & 0 \\ 0.2 & 0 & 0.3 & 0.3 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & -0.8 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0.3 \end{bmatrix}.$$

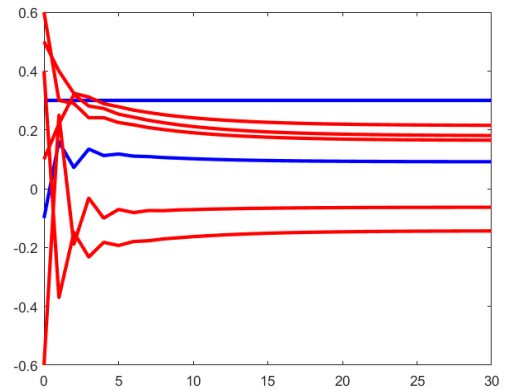$$C_3 = \begin{bmatrix} 0.9 & -0.1 \\ -0.8 & 0.2 \end{bmatrix}.$$

According to W_4, it can be easily found that the agents 6 and 7 are opening agents and the agents 3,4,5 are influenced by the agents 1 and 2. The network topology W^{22} consisting of the agents 6 and 7 is given as follows:

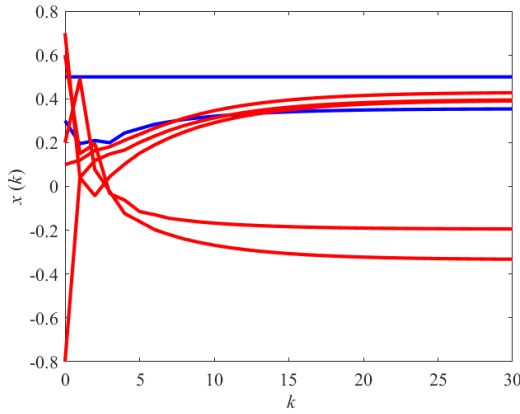$$W^{22} = \begin{bmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{bmatrix}.$$

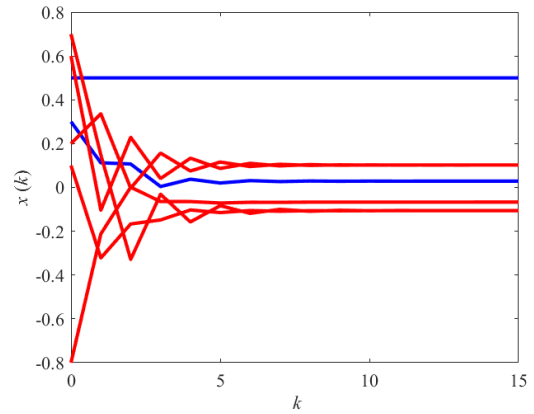**FIGURE 6.** When $C = I_m$, evolution of opinions for topic b in Example 3.

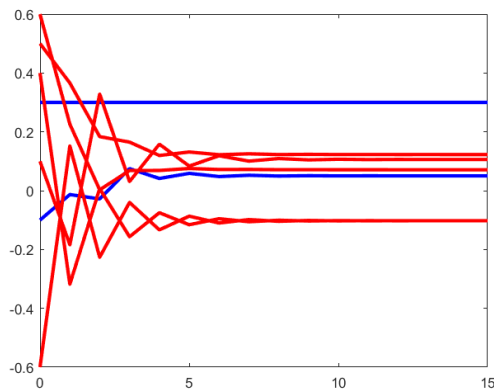**FIGURE 8.** Evolution of opinions for the topic b in Example 4.

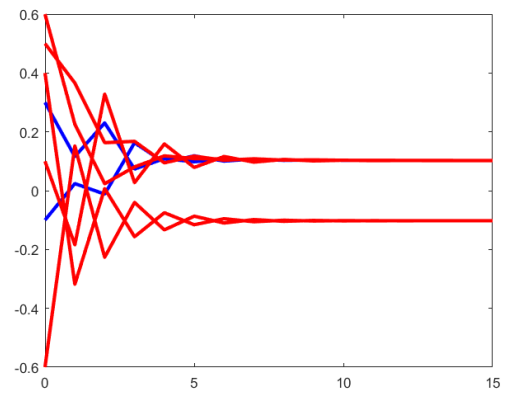**FIGURE 7.** Evolution of opinions for the topic a in Example 4.

FIGURE 9. Evolution of opinions of the similar DeGroot model at topic a in Example 4.

According to C_3 and W^{22}, we know that each independent strongly connected component of $G(C_3)$ and $G(W^{22})$ contains at least one node which the length of all loops is coprime. Besides, $G(C_3)$ is structurally balanced. Therefore, the conditions of Theorem 3 are satisfied. It implies that the system (1) is convergent for any initial opinions. We use the initial opinions (24) in Example 3. The evolutions of opinions are shown in Fig.7 and Fig.8. Similar to Example 3, we assume that $\Lambda = I_N$. According to Fig.9 and Fig.10, the similar DeGroot model can achieve bipartite consensus.

Example 5: In this example, we still consider a network with 7 agents. The network topology and the initial opinions are as follows:

$$W_5 = \begin{bmatrix} 0.3 & 0 & 0 & 0.2 & 0 & -0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.4 & 0 & 0.1 & 0 \\ 0.5 & 0 & 0 & 0 & 0.3 & 0 & -0.2 \\ 0 & 0.4 & 0 & 0.5 & 0 & -0.1 & 0 \\ 0 & 0.5 & 0 & 0 & 0.1 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & -0.2 & 0 & 0.4 & 0 & 0 & 0.4 \end{bmatrix}.$$

$$x(0)_{a'} = [0.6, 0.5, 0.1, -0.6, 0.4, -0.1, 0.3],$$
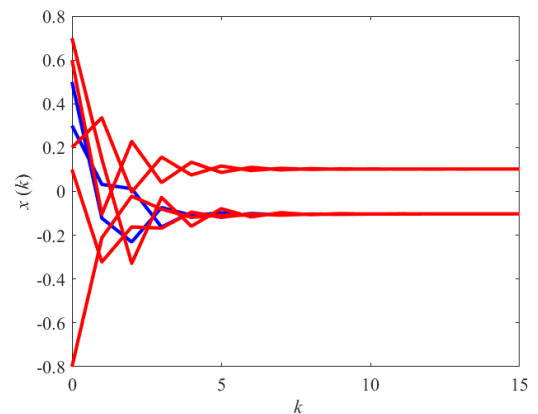$$x(0)_{b'} = [-0.8, 0.1, 0.7, 0.6, 0.2, 0.3, 0.5]. \tag{25}$$

FIGURE 10. Evolution of opinions of the similar DeGroot model at topic b in Example 4.

Let $\Lambda = diag[1, 1, 1, 1, 1, 0.7, 0]$ and $C = C_2$. In this case, the individual 7 is a totally stubborn agent. According to $G(W_5)$, we know that the graph consisting of all agents except the agent 7 is structurally unbalanced and strongly connected. According to Theorem 4, the system (23) is convergent for any initial opinions. We use the initial opinions (25) and the evolutions of opinions are shown in Fig.11 and Fig.12.
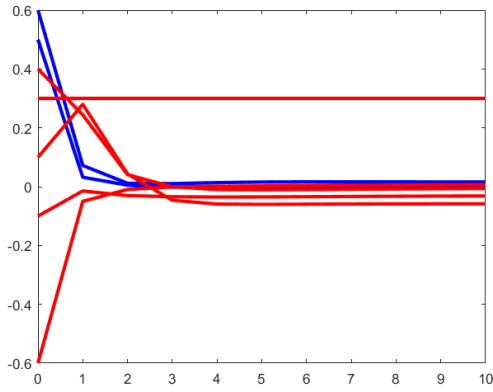
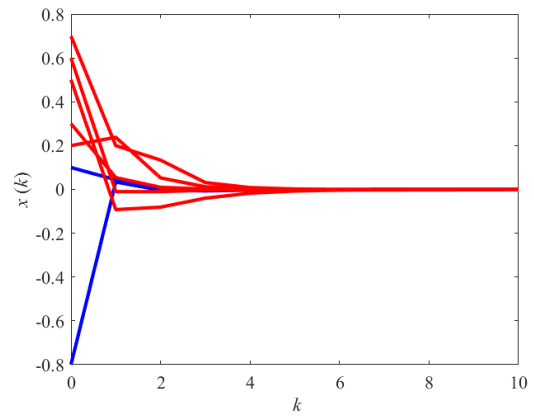**FIGURE 11.** Evolution of opinions for the topic a in Example 5.

FIGURE 12. Evolution of opinions for the topic b in Example 5.

FIGURE 13. Evolution of opinions of the similar DeGroot model at topic a in Example 5.
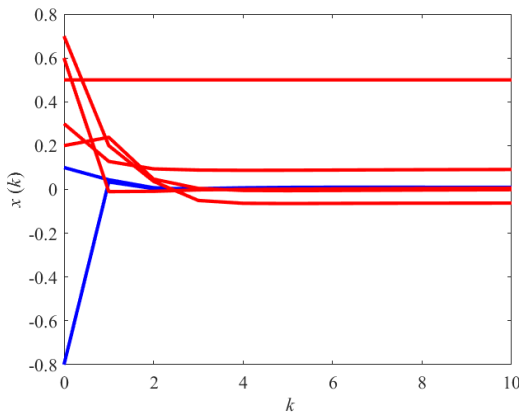
According to Fig.11 and Fig.12, one can easily find that the system (23) converges to the stable point quickly. We also assume that $\Lambda = I_N$, at this time, the similar DeGroot can achieve consensus which is shown in Fig.13 and Fig.14.

Through the above three examples, we find that both stubborn individuals and competitive relationship between individuals can create inconsistent opinions.

Example 6: In this example, we still consider a network with 7 agents. The network topology and logic matrix are

FIGURE 14. Evolution of opinions of the similar DeGroot model at topic b in Example 5.

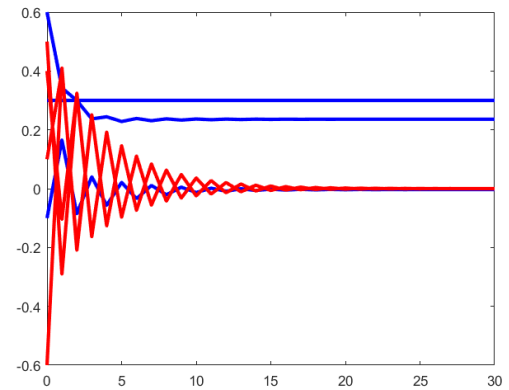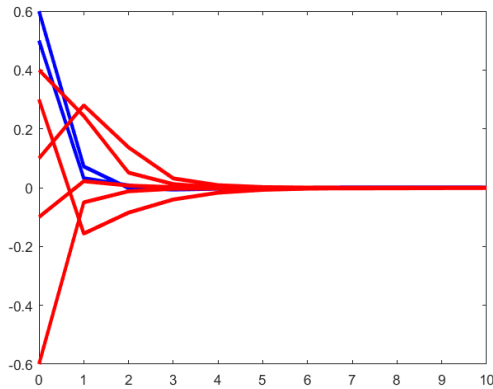**FIGURE 15.** Evolution of opinions for the topic a in Example 6.

as follows:

$$W_6 = \begin{bmatrix} 0.8 & 0 & 0 & 0 & -0.1 & 0 & -0.1 \\ 0.3 & 0.2 & 0 & 0 & 0 & -0.5 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.7 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0.8 & 0.2 & 0 \end{bmatrix}.$$

$$C_4 = \begin{bmatrix} 0.9 & -0.1 \\ 0.8 & 0.2 \end{bmatrix}.$$

Let $\Lambda_1 = diag[0, 0.6, 0.8, 1, 1, 1, 1]$. Obviously, the matrix C_4 is irreducible and $G(C_4)$ is structurally unbalanced. And there exists at least a diagonal entry $c_{ii} > 0$. So the condition of theorem 2 is satisfied, i.e., the system is stable. For randomly initial values, Fig.15 and Fig.16 illustrate the evolutions of opinions. According to Fig.15 and Fig.16, one can easily find that the system (10) converges to the stable point quickly. Let $C_4 = I_2$, i.e., our model becomes the classic FJ model. The evolutions of opinions are represented by Fig.17 and Fig.18. Obviously, through Fig.17 and Fig.18, the system (10) does not converge. It implies that considering the dependent relationship between topics may affect the convergence of opinions. So, according to Example 3 and 6, it can be concluded the dependent relationship between topics
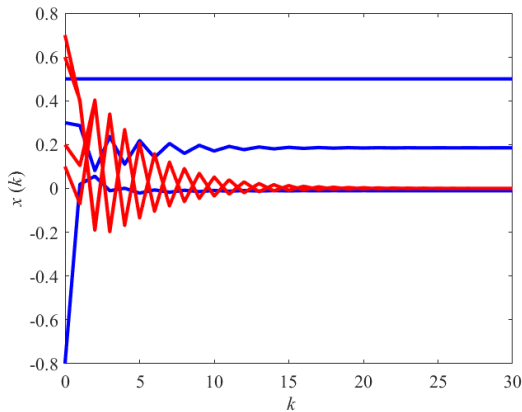
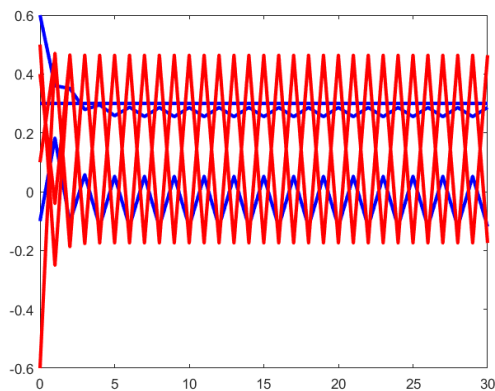**FIGURE 16.** Evolution of opinions for the topic b in Example 6.

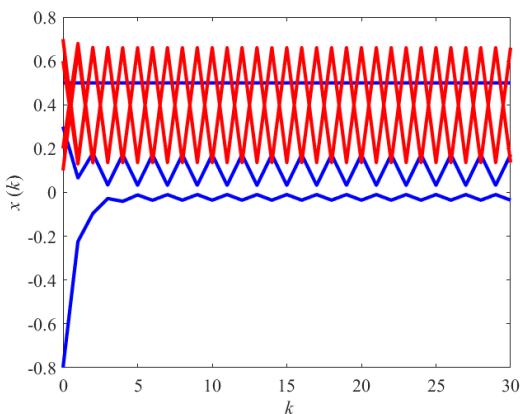**FIGURE 17.** When $C_4 = I_2$, evolution of opinions for the topic a in Example 6.

FIGURE 18. When $C_4 = I_2$, evolution of opinions for the topic b in Example 6.

can not only affect the final opinion value, but also affect the convergence of opinions. Therefore, it is very important to consider the dependencies between topics.

V. CONCLUSION
In this paper, by extending the classic FJ model, a new opinion dynamics model has been presented, in which the

individuals' opinions are affected by the competitive mechanism, stubborn agents and the interdependent relationship of multiple topics. For structurally balanced network topologies, we have obtained the conditions of stability and convergence of the model respectively. For structurally unbalanced network topologies, we have achieved the conditions of convergence. These conditions fully illustrate the influence of topic interdependent relations and individual competition mechanism on the evolutions of opinions. Finally, several examples have been used to verify our conclusions.

REFERENCES
[1] Y. Dong, M. Zhan, G. Kou, Z. Ding, and H. Liang, "A survey on the fusion process in opinion dynamics," *Inf. Fusion*, vol. 43, pp. 57–65, Sep. 2018.
[2] Z. Bu, H.-J. Li, C. Zhang, J. Cao, A. Li, and Y. Shi, "Graph K-means based on leader identification, dynamic game, and opinion dynamics," *IEEE Trans. Knowl. Data Eng.*, vol. 32, no. 7, pp. 1361–1438, Jul. 2020.
[3] Y. Tang, D. Zhang, P. Shi, W. Zhang, and F. Qian, "Event-based formation control for nonlinear multiagent systems under DoS attacks," *IEEE Trans. Autom. Control*, vol. 66, no. 1, pp. 452–459, Jan. 2021.
[4] G. Chen, X. Duan, N. E. Friedkin, and F. Bullo, "Social power dynamics over switching and stochastic influence networks," *IEEE Trans. Autom. Control*, vol. 64, no. 2, pp. 582–597, Feb. 2019.
[5] L. Vassio, F. Fagnani, P. Frasca, and A. Ozdaglar, "Social power dynamics over switching and stochastic influence networks," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 1, pp. 109–120, Apr. 2014.
[6] V. Gazi and K. M. Passino, "Stability analysis of swarms," *IEEE Trans. Autom. Control*, vol. 48, no. 4, pp. 692–697, 2003.
[7] Y. Tang, D. Zhang, D. W. C. Ho, W. Yang, and B. Wang, "Event-based tracking control of mobile robot with denial-of-service attacks," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 50, no. 9, pp. 3300–3310, Sep. 2020.
[8] H. Liang, Y. Yang, and X. Wang, "Opinion dynamics in networks with heterogeneous confidence and influence," *Phys. A, Stat. Mech. Appl.*, vol. 392, no. 9, pp. 2248–2256, 2013.
[9] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo, "Opinion dynamics and the evolution of social power in influence networks," *SIAM Rev.*, vol. 57, no. 3, pp. 367–397, 2015.
[10] P. Jia, N. E. Friedkin, and F. Bullo, "Opinion dynamics and social power evolution: A single–timescale model," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 2, pp. 899–911, Jun. 2020.
[11] W. Zhang, Q.-L. Han, Y. Tang, and Y. Liu, "Sampled-data control for a class of linear time-varying systems," *Automatica*, vol. 103, pp. 126–134, May 2019.
[12] X. Wu, Y. Tang, and J. Cao, "Input-to-state stability of time-varying switched systems with time delays," *IEEE Trans. Autom. Control*, vol. 64, no. 6, pp. 2537–2544, Jun. 2019.
[13] X. Wu, Y. Tang, J. Cao, and X. Mao, "Stability analysis for continuous-time switched systems with stochastic switching signals," *IEEE Trans. Autom. Control*, vol. 63, no. 9, pp. 3083–3090, Sep. 2018.
[14] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, "Novel multidimensional models of opinion dynamics in social networks," *IEEE Trans. Autom. Control*, vol. 62, no. 5, pp. 2270–2285, May 2017.
[15] G. He, H. Ruan, Y. Wu, and J. Liu, "Opinion dynamics with competitive relationship and switching topologies," *IEEE Access*, vol. 9, pp. 3016–3025, 2021.
[16] Y. Tang, F. Qian, H. Gao, and J. Kurths, "Synchronization in complex networks and its application—A survey of recent advances and challenges," *Annu. Rev. Control*, vol. 38, no. 2, pp. 184–198, 2014.
[17] M. H. DeGroot, "Reaching a consensus," *J. Amer. Statist. Assoc.*, vol. 69, no. 345, pp. 118–121, Mar. 1974.
[18] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.
[19] L. Wang, Y. Tian, and J. Du, "Opinion dynamics in social networks," (in Chinese), *Scientia Sinica, Inf.*, vol. 48, no. 1, pp. 3–23, Jan. 2018.
[20] M. Ye, *Opinion Dynamics and the Evolution of Social Power in Social Networks*. Berlin, Germany: Springer, 2019.
[21] Z. Xu, J. Liu, and T. Basar, "On a modified DeGroot-Friedkin model of opinion dynamics," in *Proc. Amer. Control Conf. (ACC)*, Jul. 2015, pp. 1047–1052.

[22] P. Dandekar, A. Goel, and D. T. Lee, "Biased assimilation, homophily, and the dynamics of polarization," *Proc. Nat. Acad. Sci. USA*, vol. 110, no. 15, pp. 5791–5796, 2013.

[23] A. Nedic and B. Touri, "Multi-dimensional Hegselmann-Krause dynamics," in *Proc. IEEE 51st IEEE Conf. Decis. Control (CDC)*, Dec. 2012, pp. 68–73.

[24] E. Girejko, L. Machado, A. B. Malinowska, and N. Martins, "Krause's model of opinion dynamics on isolated time scales," *Math. Methods Appl. Sci.*, vol. 39, no. 18, pp. 5302–5314, Dec. 2016.

[25] S. E. M. Tadmor, "Heterophyllous dynamics enhanced consensus," *SIAM Rev.*, vol. 56, no. 4, pp. 577–621, 2014.

[26] N. Friedkin and E. Johnsen, *Social Influence Networks and Opinion Change*. New York, NY, USA: HarperCollins, 1999.

[27] D. Bindel, J. Kleinberg, and S. Oren, "How bad is forming your own opinion?" *Games Econ. Behav.*, vol. 92, pp. 248–265, Jul. 2015.

[28] J. Ghaderi and R. Srikant, "Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate," *Automatica*, vol. 50, no. 12, pp. 3209–3215, 2014.

[29] S. Stephens-Davidowitz and A. Pabon, *Everybody Lies: Big Data, New Data, and What the Internet Can Tell us About Who we Really are*. New York, NY, USA: Harper-Collins, 2017.

[30] P. Frasca, C. Ravazzi, R. Tempo, and H. Ishii, "Gossips and prejudices: Ergodic randomized dynamics in social networks," *IFAC Proc. Volumes*, vol. 46, no. 27, pp. 212–219, 2013.

[31] C. Ravazzi, P. Frasca, R. Tempo, and H. Ishii, "Ergodic randomized algorithms and dynamics over networks," *IEEE Trans. Control Netw. Syst.*, vol. 2, no. 1, pp. 78–87, Mar. 2015.

[32] N. A. F. P. Jia and A. Mirtabatabaei, "Opinion dynamics and the evolution of social power in influence networks," *SIAM Rev.*, vol. 57, no. 3, pp. 367–391, 2015.

[33] N. E. Friedkin, "The problem of social control and coordination of complex systems in sociology: A look at the community cleavage problem," *IEEE Control Syst. Mag.*, vol. 35, no. 3, pp. 40–51, Jun. 2015.

[34] Y. Zhu, S. Li, J. Ma, and Y. Zheng, "Bipartite consensus in networks of agents with antagonistic interactions and quantization," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 65, no. 12, pp. 2012–2016, Dec. 2018.

[35] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, Apr. 2013.

[36] D. Xue, S. Hirche, and M. Cao, "Evolution of social power over influence networks containing antagonistic interactions," *Inf. Sci.*, vol. 540, pp. 449–468, Nov. 2020.

[37] Z. Meng, G. Shi, K. H. Johansson, M. Cao, and Y. Hong, "Behaviors of networks with antagonistic interactions and switching topologies," *Automatica*, vol. 73, pp. 110–116, Nov. 2016.

[38] S. Zhai and W. X. Zheng, "Dynamic behavior for social networks with state-dependent susceptibility and antagonistic interactions," *Automatica*, vol. 129, Jul. 2021, Art. no. 109652.

[39] X. Lin, Q. Jiao, and L. Wang, "Competitive diffusion in signed social networks: A game-theoretic perspective," *Automatica*, vol. 112, Feb. 2020, Art. no. 108656.

[40] M. Ye, M. H. Trinh, Y.-H. Lim, B. D. O. Anderson, and H.-S. Ahn, "Continuous-time opinion dynamics on multiple interdependent topics," *Automatica*, vol. 115, May 2020, Art. no. 108884.

[41] F. Dietrich, S. Martin, and M. Jungers, "Control via leadership of opinion dynamics with state and time-dependent interactions," *IEEE Trans. Autom. Control*, vol. 63, no. 4, pp. 1200–1207, Apr. 2018.

[42] L. Li, A. Scaglione, A. Swami, and Q. Zhao, "Consensus, polarization and clustering of opinions in social networks," *IEEE J. Sel. Areas Commun.*, vol. 31, no. 6, pp. 1072–1083, Jun. 2013.

[43] X. L. Shi, J. Cao, G. Wen, and M. Perc, "Finite-time consensus of opinion dynamics and its applications to distributed optimization over digraph," *IEEE Trans. Cybern.*, vol. 49, no. 10, pp. 3767–3779, Oct. 2019.

[44] V. Amelkin, F. Bullo, and A. K. Singh, "Polar opinion dynamics in social networks," *IEEE Trans. Autom. Control*, vol. 62, no. 11, pp. 5650–5665, Nov. 2017.

[45] R. S. Varga, *Matrix Iterative Analysis*. New York, NY, USA: Springer, 1999.
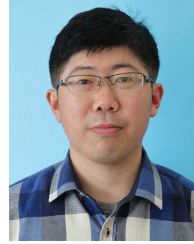
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