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Construction of Standard Solid Sudoku Cubes and 3D Sudoku Puzzles

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ABSTRACT In this paper, standard solid Sudoku cubes (SSSCs), a three-dimensional (3D) extension of Sudoku tables, are introduced, and a method to construct these cubes is presented. This is the first class of standard solid Sudoku cubes. An SSSC of order m is a solid Latin cube of order m with solid subcubes of order $x \times y \times z$ in which each element occurs exactly once in each row, column, depth, and subcube. The structure of these cubes is based on cyclotomic cosets of $\mathbf{Z}_n$, and we make use of a vector Z and a basic table T to construct SSSCs. We obtain m tables by multiplying all entries of T by a number from the vector Z. Then, these tables are converted to an SSSC by stacking them in order. Based on this method of construction, a perfect set of strongly mutually distinct standard solid Sudoku cubes is designed. We also provide a two-dimensional (2D) representation of these SSSCs in a table with numbers placed on the main diagonal of its subtables. Finally, a new class of 3D Sudoku puzzles based on SSSCs is presented as standard solid Sudoku puzzles (SSSPs).

INDEX TERMS Latin cube, solid Sudoku cube, Sudoku puzzle, Sudoku table.

I. INTRODUCTION

Sudoku tables are a special class of Latin squares which are very popular among researchers [3], [8], [9], [13], [14]. A Sudoku table of order m with subtables of order $s \times d$ is a Latin square in which m different numbers occur exactly once in each row, column, and subtable [6]. The most common Sudoku table is a 9×9 table with 3×3 subtables in which 9 different numbers occur in the entries of this table and any number appears exactly once in each row, column, and subtable. As a consequence, a three-dimensional (3D) standard solid Sudoku cube of order m is defined as an $m \times m \times m$ cube in which m different numbers appear in the entries of the cube such that any number appears only once in each row, column, depth, and $x \times y \times z$ subcube, $x \cdot y \cdot z = m$. This is called a standard solid Sudoku cube (SSSC) of order m with subcubes of order $x \times y \times z$, and denote it by SSSC(x, y, z). An SSSC(x, y, z) can be divided into subcubes of order $x \times y \times z$ along the X, Y, and Z axes. Without loss of generality, throughout this paper we assume that $1 \leq x \leq y \leq z \leq m$. In the design of these Sudoku cubes, we fill the m^3 entries of an SSSC of order m with only m different numbers such that the properties are satisfied.

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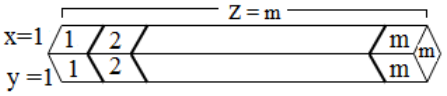

FIGURE 1. A solid subcube of an SSSC(1, 1, m) having entries filled with m different numbers.

In the case $x = 1, y = 1$, and $z = m$, any solid subcube of an SSSC(1, 1, m) is a solid column of length m as shown in Figure 1. For the case $x = 1$ and $1 < y, z < m$, any solid subcube of order $y \times z$ from an SSSC(1, y, z) is a solid table in which any of the m different numbers appear exactly once in this subcube as shown in Figure 2. Therefore, we consider that for parameters in the range $1 < x, y, z < m$, any subcube of order $x \times y \times z$ from an SSSC(x, y, z) is a solid cube in which any of the m different numbers appears exactly once. Figure 3 shows a subcube of order $2 \times 2 \times 2$ from an SSSC(2, 2, 2) of order $m = 8$. It is obvious that any of the 8 different numbers appear exactly once in this subcube.

In this paper, we use cyclotomic cosets of the group $\mathbf{Z}_n$ which have previously been used to design Sudoku tables [6], twin Sudoku tables, and triplet solid Sudoku cubes [7], and also LDPC codes [5]. Sudoku tables/puzzles have applications in other fields of research such as image encryption [21]. They have also been used in cryptographic protocols such as

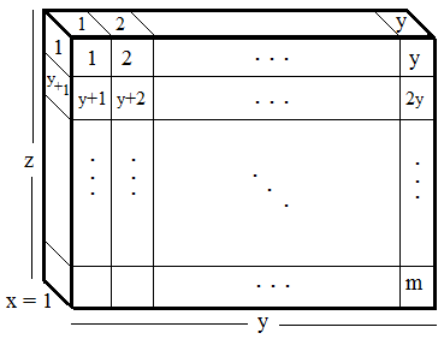

FIGURE 2. A solid subcube of an SSSC(1, y, z), $y \cdot z = m$, having entries filled with m different numbers.

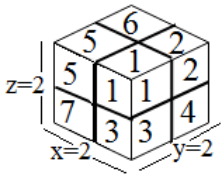

FIGURE 3. A 2 × 2 × 2 solid subcube of an SSSC(2, 2, 2) having entries filled with 8 different numbers.

zero-knowledge proofs to prove knowledge without revealing the secret [10], [15], and they can be used in data encryption [22].

To construct an SSSC, we generate a basic table T of order $m \times m$ using the elements of the cyclotomic cosets of $\mathbf{Z}_n$. A vector Z is designed which contains the m different numbers of the union of all cyclotomic cosets. Then multiplying T by numbers from the vector Z, we obtain other tables. Stacking these solid tables in order gives a standard solid Sudoku cube SSSC(x, y, z). We also show that for the constructed SSSC(x, y, z) (denoted Q) of order m, there exist m strongly mutually distinct (SMD) SSSC(x, y, z) of order m as a perfect set of SMDSSSC(x, y, z) of order m. The perfect set of SMDSSSC(x, y, z) is created by multiplying Q by the numbers in the vector Z.

In [6], methods to construct Sudoku tables and solid Sudoku cubes were presented. Sudoku tables of order m with $s \times d$ subtables were constructed and solid Sudoku cubes of order m with subcubes of order $d \times d \times d$ were given. In their construction, the solid Sudoku cubes had $d \times d \times d$ subcubes such that all d^3 entries are filled with m different numbers where $d \cdot d = m$. In other words, each number occurs exactly d times in each subcube. Thus, the m different numbers fill all d^3 entries while any number occurs exactly once in each row, column, and depth, and d times in each solid subcube.

In [7], twin Sudoku tables and triplet solid Sudoku cubes were introduced, and a method to construct them was presented. Twin Sudoku tables of order m were constructed such that they can be divided into subtables of order $s \times d$ and $d \times s$ simultaneously. Triplet solid Sudoku cubes were designed so that they can be divided into subcubes of order $s \times s \times d$, $s \times d \times s$ and $d \times s \times s$, at the same time, where $s \cdot d = m$. Furthermore, the solid subcubes are filled with m different numbers, and any number appears exactly s times in each subcube. Thus, the m different numbers fill the $s^2 \cdot d$ entries of each solid subcube and any number appears exactly once in each row, column, and depth, and s times in each solid subcube.

In [4], linear magic squares were introduced, and the existence and construction of orthogonal magic Sudoku solutions of order p^2 where p is a prime power were investigated. In [12], a set of mutual orthogonal Sudoku Latin squares (MOSLS) was presented, and for a prime power p, $p^2 - p$ MOSLS of order p^2 were constructed. It was shown that for integer k, there exist MOSLS of order k^2.

Sudoku puzzles as incomplete Sudoku tables are quite interesting for researchers, and they are also very popular games among people. These puzzles should be solved so that a complete Sudoku table is obtained. Few researchers have investigated solving Sudoku puzzles [1], but some have studied solutions and developed algorithms to solve them [11], [17], [19], [20]. Solving Sudoku puzzles is an NP-complete problem [16] so obtaining solutions is very complex. Sudoku puzzles as a cube such that six Sudoku puzzles are arranged on the six faces of a cube were introduced in [18] as a 3D version of Sudoku puzzles.

In this paper, a new class of solid Sudoku cubes called standard solid Sudoku cubes is constructed. This class of solid Sudoku cubes of order $m \times m \times m$ with subcubes of order $x \times y \times z$ are created along the X, Y, and Z axes, and the m^3 entries of an SSSC(x, y, z) are filled with m different numbers such that each number appears exactly once in each row, column, depth, and subcube. We show that these SSSC(x, y, z) can be used to create m SMDSSSC(x, y, z). As it is difficult to visualize the inside of an SSSC(x, y, z) as a solid cube, we present a two-dimensional (2D) representation for this class of SSSC(x, y, z) to facilitate their investigation and use. In this representation, the columns of an SSSC(x, y, z) occur on the diagonals of subtables of a table of order $m^2 \times m^2$ where the subtables have order $m \times m$. Finally, we generate a new class of 3D Sudoku puzzles from these SSSCs and their 2D representations which are called standard solid Sudoku puzzles (SSSPs).

II. PRELIMINARIES

Let $\mathbf{Z}_n$ be the group of integers modulo n. Assume $1 < q < n$, $\gcd(q, n) = 1$, is a number of order z, i.e. $q^z = 1 \bmod n$, and z is the least positive integer that satisfies this congruence. Then the set $C_{a_0} = C_1 = \{1, q, q^2, \ldots, q^{z-1}\}$ is a cyclotomic coset with coset leader $a_0 = 1$. For any positive integer t, the number a_t, where a_t is the least positive integer that has not appeared in the previous cyclotomic cosets, is the tth coset leader and creates a cylotomic coset $C_{a_t} = \{a_t \cdot c \bmod n | c \in C_1\}$ [2]. We choose b different cyclotomic cosets such that they have cardinality $|C_1| = z$, and satisfy the properties

$\gcd(a_t, n) = 1$, and for any $0 \le t_1, t_2 \le b-1$

$$a_{t_1} \cdot a_{t_2} \bmod n \in \hat{C} := \bigcup_{0 \le t \le b-1} C_{a_t}. \quad (1)$$

Lemma 1: For any $c \in \hat{C}$, let $M_c = \{c \cdot e \bmod n | \ e \in \hat{C}\}$. If relation (1) holds, then $M_c = \hat{C}$.

Proof: This follows from [6, Lemma 1]. □

Lemma 1 proves that if we multiply all numbers in $\hat{C}$ by any number $c \in \hat{C}$, then all numbers of $\hat{C}$ are generated, i.e. for any $c \in \hat{C}$ we have $c \cdot \hat{C} \bmod n = \hat{C}$.

Remark 1: While we are working with the m different numbers in $\hat{C}$ they may not be consecutive numbers. However, they can be replaced with m consecutive numbers after the SSSC(x, y, z) is created.

Definition 1: A solid Latin cube of order m is an $m \times m \times m$ cube having entries filled with m different elements such that each element occurs exactly once in each row, column, and depth of the cube.

Definition 2: A standard solid Sudoku cube of order m with subcubes of order $x \times y \times z$, SSSC(x, y, z), is a solid Latin cube which can be divided into m^2 solid subcubes of order $x \times y \times z$, and each element appears exactly once in each of these subcubes.

Definition 2 describes a standard solid Sudoku cube, SSSC(x, y, z), of order m. Hence an SSSC(x, y, z) of order m, where $x \cdot y \cdot z = m$, is an $m \times m \times m$ cube that has m^2 subcubes having entries filled with m different numbers. For simplicity, throughout this paper we define the indexes and their ranges as $0 \le i, r \le x-1$, $0 \le j, s \le y-1$, $0 \le k \le z-1$, and $0 \le l \le m-1$. We also define $x \cdot y = b$, and $0 \le t \le b-1$. Note that the symbol $\times$ is used when discussing dimension and $\cdot$ for multiplication.

Since every cube has six faces, dividing a cube into subcubes of order $x \times y \times z$ should be defined according to which face and direction it is divided. A cube has top, front, left, right, back, and bottom faces. Thus, without loss of generality, this division is done along the X, Y, and Z axes, respectively, so when a cube is divided into subcubes of order $x \times y \times z$, the division is done along the axes. Figure 4 illustrates this division.

III. SSSC CONSTRUCTION

To construct a standard solid sudoku cube (SSSC) of order m with subcubes of order $x \times y \times z$, namely SSSC(x, y, z), consider $\mathbf{Z}_n$ and q such that $q^z = 1 \bmod n$, where z is the least positive integer that satisfies this congruence. Therefore, $C_{a_0} = C_1 = \{1, q, q^2, \ldots, q^{z-1}\}$, and for $1 \le t \le b-1$ there is a number a_t which is the least number that has not appeared in a previous cyclotomoic coset and satisfies the properties in Section II. Then $C_{a_t} = \{a_t, a_t \cdot q, a_t \cdot q^2, \ldots, a_t \cdot q^{z-1}\}$, and if we consider C_{a_t} as a vector, i.e.

$$C_{a_t} = [a_t \ \ a_t \cdot q \ \ a_t \cdot q^2 \ \ \ldots \ \ a_t \cdot q^{z-1}] \quad (2)$$

then it contains z different numbers. We define the vector Z by cyclotomic cosets

$$Z = [C_1 \ \ C_{a_1} \ \ C_{a_2} \ \ \ldots \ \ C_{a_{b-1}}] \quad (3)$$

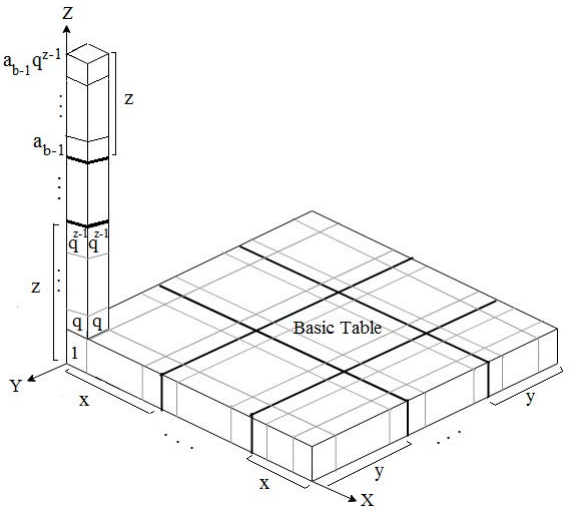

FIGURE 4. A basic table and vector Z which shows how an SSSC(x, y, z) is divided.

which contains b different vectors of order z, $b \cdot z = m$, so Z has length m.

For $0 \le l \le m-1$, if $l = z \cdot t + k$ where $0 \le k \le z-1$ and $0 \le t \le b-1$, then the lth element in the vector Z is $Z(l) = Z(z \cdot t + k) = a_t \cdot q^k \bmod n$. For $0 \le i \le x-1$, the vector u_i of length y is defined as

$$\begin{aligned} u_0 &= [1 \ \ a_1 \ \ a_2 \ \ \ldots \ \ a_{y-1}] \\ u_1 &= [a_y \ \ a_{y+1} \ \ \ldots \ \ a_{2y-1}] \\ u_2 &= [a_{2y} \ \ a_{2y+1} \ \ \ldots \ \ a_{3y-1}] \\ &\vdots \qquad\qquad \vdots \\ u_{x-1} &= [a_{b-y} \ \ a_{b-y+1} \ \ \ldots \ \ a_{b-1}]. \end{aligned} \quad (4)$$

For $0 \le j \le y-1$, the vector $u_i^{(j)}$ of length y is obtained from the vector u_i by a j cyclic shift to the left. Then $u_i^{(0)} = u_i$ and for $j = 1$, the vectors $u_i^{(1)}$ are

$$\begin{aligned} u_0^{(1)} &= [a_1 \ \ a_2 \ \ \ldots \ \ a_{y-1} \ \ 1] \\ u_1^{(1)} &= [a_{y+1} \ \ \ldots \ \ a_{2y-1} \ \ a_y] \\ u_2^{(1)} &= [a_{2y+1} \ \ \ldots \ \ a_{3y-1} \ \ a_{2y}] \\ &\vdots \qquad\qquad \vdots \\ u_{x-1}^{(1)} &= [a_{b-y+1} \ \ \ldots \ \ a_{b-1} \ \ a_{b-y}]. \end{aligned} \quad (5)$$

For $1 < j \le y-1$, the vectors $u_0^{(j)}$, $u_1^{(j)}$, …, and $u_{x-1}^{(j)}$ are obtained in the same way. For $0 \le j, s \le y-1$, and $0 \le i \le x-1$ the sth element of $u_i^{(j)}$ is $u_i^{(j)}(s) = a_{y \cdot i + j + s}$. Therefore, the table B of order $b \times b$ constructed from these

vectors is

$$B = \begin{bmatrix} u_0 & u_1 & u_2 & \dots & u_{x-1} \\ u_1 & u_2 & u_3 & \dots & u_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{x-1} & u_0 & u_1 & \dots & u_{x-2} \\ u_0^{(1)} & u_1^{(1)} & u_2^{(1)} & \dots & u_{x-1}^{(1)} \\ u_1^{(1)} & u_2^{(1)} & u_3^{(1)} & \dots & u_0^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{x-1}^{(1)} & u_0^{(1)} & u_1^{(1)} & \dots & u_{x-2}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_0^{(y-1)} & u_1^{(y-1)} & u_2^{(y-1)} & \dots & u_{x-1}^{(y-1)} \\ u_1^{(y-1)} & u_2^{(y-1)} & u_3^{(y-1)} & \dots & u_0^{(y-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{x-1}^{(y-1)} & u_0^{(y-1)} & u_1^{(y-1)} & \dots & u_{x-2}^{(y-1)} \end{bmatrix} \quad (6)$$

with entries consisting of the coset leaders a_t, $0 \le t \le b-1$. Each entry of B is the sth element of the vector $u_{i+r}^{(j)}$, $0 \le s \le y-1$

$$B(j \cdot x + r, i \cdot y + s) = u_{i+r}^{(j)}(s) = a_{y\cdot(i+r)+s+j} = a_t \quad (7)$$

where $t = y \cdot (i + r) + s + j \bmod b$.

The basic table T of order $m \times m$ is

$$T = \begin{bmatrix} B & q \cdot B & q^2 \cdot B & \dots & q^{z-1} \cdot B \\ q \cdot B & q^2 \cdot B & q^3 \cdot B & \dots & B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q^{z-1} \cdot B & B & q \cdot B & \dots & q^{z-2} \cdot B \end{bmatrix} \quad (8)$$

where the entry $q^k \cdot B$ is obtained by multiplying all entries of B by q^k. Then, for $0 \le t \le b-1$ and $0 \le k \le z-1$, the table $T_{z\cdot t+k}$ of order $m \times m$ contains m different numbers where

$$T_{z\cdot t+k} = a_t \cdot q^k \cdot T \mod n. \quad (9)$$

By considering these m SMD tables as Solid tables and stacking them in order, a standard solid Sudoku cube SSSC(x, y, z), denoted Q, is obtained.

Theorem 1: Table B in (6) is a Sudoku table of order b with $x \times y$ subtables which contain all coset leaders of the cyclotomic cosets.

Proof: From the construction of B and $u_i^{(j)}$, which is a vector of length y, it is clear that B is a Latin square. Thus, it is sufficient to show that any subtable of order $x \times y$ contains b different numbers and any two distinct entries contain different numbers. To achieve this, for $0 \le i_1, i_2 \le x-1$ and $0 \le s_1, s_2 \le y-1$, we show that

$$B(x \cdot j + r, y \cdot i_1 + s_1) \ne B(x \cdot j + r, y \cdot i_2 + s_2).$$

From the construction of B and (7), it is sufficient to show that $u_{i_1+r}^{(j)}(s_1) \ne u_{i_2+r}^{(j)}(s_2)$. Then, based on the vectors $u_i^{(j)}$, we have

$$a_{y\cdot(i_1+r)+j+s_1} \ne a_{y\cdot(i_2+r)+j+s_2}$$

so we should show that

$$y \cdot (i_1 + r) + j + s_1 \ne y \cdot (i_2 + r) + j + s_2 \bmod b.$$

Simplifying this equation, we have $y \cdot (i_2 - i_1) + s_2 - s_1 \ne 0 \bmod b$. Since $i_1 \ne i_2$ or $s_1 \ne s_2$, then

$$0 < y \cdot (i_2 - i_1) + s_2 - s_1 < y \cdot (x-1) + y - 1 < b. \quad (10)$$

As $x \cdot y = b$, (10) holds, which completes the proof. □

Theorem 2: The basic table T in (8) is a Latin square.

Proof: It is sufficient to show that any row or column contains m different numbers. In a row of the table, for $0 \le k_1, k_2 \le z-1$, $0 \le i, r \le x-1$ and $0 \le s_1, s_2 \le y-1$ where $k_1 \ne k_2$ or $s_1 \ne s_2$, we show that

$$q^{k_1} \cdot B(x \cdot j + r, y \cdot i + s_1) \ne q^{k_2} \cdot B(x \cdot j + r, y \cdot i + s_2).$$

In other words, the following should hold

$$q^{k_1} \cdot a_{y\cdot(i+r)+j+s_1} \ne q^{k_2} a_{y\cdot(i+r)+j+s_2} \bmod n.$$

If $k_1 = k_2$ and $s_1 \ne s_2$, then the proof is the same as that of Theorem 1. If $s_1 = s_2$ and $k_1 \ne k_2$, say $y \cdot (i + r) + j + s = t \bmod b$, then it is clear that $a_t \cdot q^{k_1} \ne a_t \cdot q^{k_2} \bmod n$. For the case $k_1 \ne k_2$ and $s_1 \ne s_2$, we must show that

$$q^{k_1} \cdot a_{y\cdot(i+r)+j+s_1} \ne q^{k_2} \cdot a_{y\cdot(i+r)+j+s_2} \bmod n$$

and simplifying gives

$$a_{t_1} \cdot q^{k_1} \ne a_{t_2} \cdot q^{k_2} \bmod n$$

where $y \cdot (i + r) + j + s_1 = t_1 \bmod b$, and $y \cdot (i + r) + j + s_2 = t_2 \bmod b$. Since any two different coset leaders belong to two different cyclotomic cosets, then

$$a_{t_1} \ne a_{t_2} \cdot q^{k_2-k_1} \bmod n.$$

□

Corollary 1: From Theorem 2, it can be deduced that each table $T_{z\cdot t+k}$ in (9) is a Latin square.

Proposition 1: For $0 \le t \le b-1$ and $0 \le k \le z-1$, the m tables $T_{z\cdot t+k}$ in (9) make a perfect set of SMD Latin squares of order m.

Proof: Since each table is a Latin square, then based on their construction, the proof of Theorem 2 and Lemma 1, these m tables make a perfect set of SMD Latin squares of order m. □

Theorem 3: The cube Q constructed by stacking the solid tables $T_{z\cdot t+k}$ in (9) is an SSSC(x, y, z) of order m.

Proof: It is sufficient to show that any subcube of order $x \times y \times z$ contains m different numbers, i.e. any two entries of a subcube contain different numbers. Thus, for two elements of Z, i.e. $a_t \cdot q^{k_1}$ and $a_t \cdot q^{k_2}$, and also two different entries of a subtable of T, i.e. $q^{k'} \cdot B(j \cdot x + r_1, i \cdot y + s_1)$ and $q^{k'} \cdot B(j \cdot x + r_2, i \cdot y + s_2)$, we show that $e_1 \ne e_2$ where

e_1 and e_2 are values in two distinct entries of a subcube from the SSSC(x, y, z)

$$e_1 = a_t \cdot q^{k_1} \cdot q^{k'} \cdot B(j \cdot x + r_1, i \cdot y + s_1) \quad (11)$$
$$e_2 = a_t \cdot q^{k_2} \cdot q^{k'} \cdot B(j \cdot x + r_2, i \cdot y + s_2). \quad (12)$$

Removing the common terms from (11) and (12) and simplifying, we obtain

$$q^{k_1} \cdot u^{(j)}_{i+r_1}(s_1) \neq q^{k_2} \cdot u^{(j)}_{i+r_2}(s_2). \quad (13)$$

Based on the design of the vectors $u^{(j)}_i$ where $u^{(j)}_{i+r_1}(s) = a_{y\cdot(i+r_1)+j+s}$, and (13), we have

$$q^{k_1} \cdot a_{y\cdot(i+r_1)+j+s_1} \neq q^{k_2} \cdot a_{y\cdot(i+r_2)+j+s_2} \bmod n \quad (14)$$

where $k_1 \neq k_2$ or $r_1 \neq r_2$ or $s_1 \neq s_2$. If $k_1 = k_2$, and $r_1 \neq r_2$ or $s_1 \neq s_2$, the proof follows from Theorem 1. If $k_1 \neq k_2$, there are three possibilities which must be considered.

(i) If $r_1 = r_2$ and $s_1 = s_2$, then it is clear that (14) holds, i.e. $a_t \cdot q^{k_1} \neq a_t \cdot q^{k_2} \bmod n$ because two elements of any cyclotomic coset are different.

(ii) If $r_1 = r_2$ and $s_1 \neq s_2$, from (14) we obtain that $q^{k_1} \cdot a_{s_1} \neq q^{k_2} \cdot a_{s_2} \bmod n$, i.e. $a_{s_1} \neq a_{s_2} \cdot q^{k_2-k_1} \bmod n$. Since any coset leader is unique and does not belong to another cyclotomic coset, it is clear that (14) holds.

(iii) If $s_1 = s_2$ and $r_1 \neq r_2$, the proof is the same as (ii).

□

Table 1 gives the parameters to construct an SSSC(x, y, z) of order m with $x \times y \times z$ subcubes. This shows that the number of cyclotomic cosets required is $x \cdot y = b$.

TABLE 1. Parameters to construct an SSSC(x, y, z) of order m.

m	x	y	z	$\mathbf{Z}_n$	q	*Cyclotomic Cosets*
2	1	1	2	Z_3	2	C_1
3	1	1	3	Z_7	2	C_1
4	1	1	4	Z_5	3	C_1
4	1	2	2	Z_5	4	C_1, C_2
5	1	1	5	Z_{11}	3	C_1
6	1	1	6	Z_7	3	C_1
6	1	2	3	Z_7	2	C_1, C_3
7	1	1	7	Z_{29}	16	C_1
8	1	1	8	Z_{17}	2	C_1
8	1	2	4	Z_{17}	4	C_1, C_2
8	2	2	2	Z_{17}	16	C_1, C_2, C_4, C_8
9	1	1	9	Z_{19}	5	C_1
9	1	3	3	Z_{19}	7	C_1, C_4, C_5
10	1	1	10	Z_{11}	2	C_1
10	1	2	5	Z_{11}	4	C_1, C_2
11	1	1	11	Z_{23}	2	C_1
12	1	1	12	Z_{13}	2	C_1
12	1	2	6	Z_{13}	4	C_1, C_2
12	1	3	4	Z_{13}	5	C_1, C_2, C_4
12	2	2	3	Z_{13}	3	C_1, C_2, C_4, C_7
14	1	2	7	Z_{29}	16	C_1, C_4
15	1	3	5	Z_{31}	2	C_1, C_5, C_7
16	1	2	8	Z_{17}	2	C_1, C_3
16	1	4	4	Z_{17}	4	C_1, C_2, C_3, C_6
16	2	2	4	Z_{17}	3	C_1, C_2, C_3, C_6
18	1	2	9	Z_{19}	4	C_1, C_2
18	1	3	6	Z_{19}	8	C_1, C_2, C_4
18	2	3	3	Z_{19}	7	$C_1, C_2, C_4, C_5, C_8, C_{10}$
20	1	2	10	Z_{41}	4	C_1, C_2
20	1	4	5	Z_{41}	7	C_1, C_2, C_4, C_5
20	2	2	5	Z_{41}	7	C_1, C_2, C_4, C_5
21	1	1	21	Z_{43}	9	C_1
21	1	3	7	Z_{43}	4	C_1, C_6, C_9
22	1	2	11	Z_{23}	2	C_1, C_5
24	1	2	12	Z_{73}	3	C_1, C_7
24	1	3	8	Z_{73}	10	C_1, C_3, C_9
24	1	4	6	Z_{73}	9	C_1, C_3, C_7, C_{21}
24	2	2	6	Z_{73}	9	C_1, C_3, C_7, C_{21}
24	2	3	4	Z_{73}	27	$C_1, C_3, C_7, C_9, C_{10}, C_{17}$
25	1	5	5	Z_{101}	36	$C_1, C_5, C_{19}, C_{24}, C_{25}$

IV. STRONGLY MUTUALLY DISTINCT SSSCs

Let Q be an SSSC(x, y, z) constructed from Theorem 3. For any $0 \leq l \leq m-1$, let $Z(l) = Z(z \cdot t + k) = a_t \cdot q^k$ be the lth entry of the vector Z. Then for any l, we define the cube Q_l as

$$Q_l = Z(l) \cdot Q \bmod n \quad (15)$$

which is a standard solid Sudoku cube of order m. These cubes are made by multiplying the entries of the standard solid Sudoku cube Q by $Z(l)$, which is the lth number in the vector Z. Theorem 4 shows that (15) makes a perfect set of SMDSSSC(x, y, z).

Theorem 4: For $0 \leq l \leq m-1$, Q_l in (15) is a standard solid Sudoku cube, and they make a perfect set of SMDSSSC(x, y, z) of order m.

Proof: For $0 \leq l \leq m-1$, since $Z(l) = Z(z \cdot t + k) = a_t \cdot q^k$ is a number from $\hat{C}$, from Lemma 1 it is clear that Q_l is an SSSC(x, y, z). For $0 \leq l_1, l_2 \leq m-1$, since $Z(l_1) \neq Z(l_2)$, assume that an entry in Q contains $c \in \hat{C}$ so that the entries in Q_{l_1} and Q_{l_2} contain values $c \cdot Z(l_1)$ and $c \cdot Z(l_2)$, respectively. Then from Lemma 1, $c \cdot Z(l_1) \neq c \cdot Z(l_2) \bmod n$, so Q_{l_1} and Q_{l_2} are SMDSSSC(x, y, z) of order m. □

Corollary 2: The m SMD standard solid Sudoku cubes in (15) are m different Latin cubes, so they can be considered as Latin squares in four dimensions, i.e. Latin quads of order m. Then, a Latin quad is m different Latin cubes such that the same entries of these m cubes contain m different numbers, i.e. they are SMD Latin cubes.

Example 1: In this example, we construct an SSSC$(1, 2, 2)$ of order 4 where $x = 1$, $y = 2$, and $z = 2$. Thus, we choose $\mathbf{Z}_n = \mathbf{Z}_5$ and $q = 4$ so the cyclotomic cosets are $C_1 = \{1, 4\}$ and $C_2 = \{2, 3\}$ and $Z = [C_1\ C_2] = [1\ 4\ 2\ 3]$. Based on the construction, $u = [a_0\ a_1] = [1\ 2]$ and $u^{(1)} = [2\ 1]$. Then, the table B of order 2×2 from u and $u^{(1)}$ is

$$B = \begin{bmatrix} u \\ u^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad (16)$$

and the basic table T of order 4×4 is

$$T = \begin{bmatrix} B & 4 \cdot B \\ 4 \cdot B & B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{bmatrix}. \quad (17)$$

For $0 \leq t, k \leq 1$, the 4 tables $T_{2\cdot t+k}$ of order 4×4 are obtained by multiplying $Z(2 \cdot t + k) = a_t \cdot 4^k$ by the entries

of T, so $T_0 = T$ and the other three tables are

$$T_1 = 4 \cdot T \bmod 5 = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix} \tag{18}$$

$$T_2 = 2 \cdot T \bmod 5 = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 2 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix} \tag{19}$$

$$T_3 = 3 \cdot T \bmod 5 = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & 3 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & 1 & 3 \end{bmatrix}. \tag{20}$$

By stacking these tables in order, i.e. T_0, T_1, T_2, T_3, an SSSC(1, 2, 2) of order $4 \times 4 \times 4$ is obtained in which each subcube of order $1 \times 2 \times 2$ contains 4 different numbers. One of the interesting properties of these SSSCs which can be seen in this example is that the 64 entries of the solid cube have only 4 different numbers.

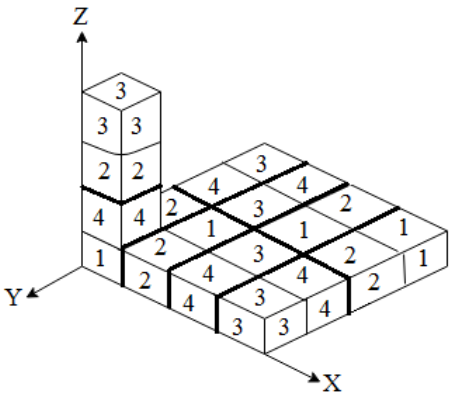

FIGURE 5. The basic table T and the corresponding vector Z related to SSSC(1, 2, 2).

Figure 5 gives the basic table T and the corresponding column vector Z of SSSC(1, 2, 2). This shows that T contains 4 different numbers and is a Latin square. Figure 6 presents the SSSC(1, 2, 2) and shows that it contains subcubes of order $1 \times 2 \times 2$ along the X, Y, and Z axes. For $0 \le t, k \le 1$, the 4 SMDSSSC(1, 2, 2), i.e. Q_0, Q_1, Q_2, Q_3, of order $4 \times 4 \times 4$, which are a perfect set of SMDSSSC(1, 2, 2), are obtained by multiplying $Z(2 \cdot t + k) = a_t \cdot 4^k$ by the entries of Q. Therefore, $Q_0 = Q$, $Q_1 = 4 \cdot Q \bmod 5$, $Q_2 = 2 \cdot Q \bmod 5$, and $Q_3 = 3 \cdot Q \bmod 5$.

Remark 2: For the given construction of SSSCs with $x = 1$, it is obvious that $y \cdot z = m$. Thus if we divide SSSC(1, y, z) into subcubes of order $1 \times m \times m$, from Example 1 it can be concluded that the corresponding $1 \times m \times m$ subcubes are solid Sudoku tables of order m with subtables of order $y \times z$.

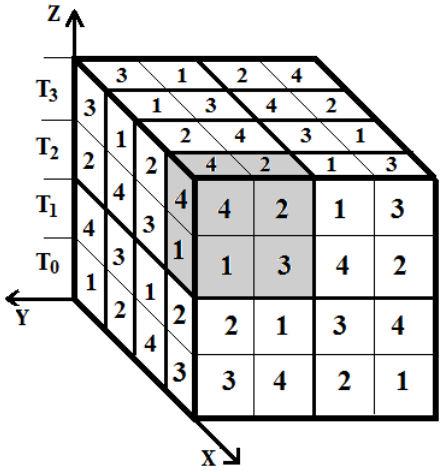

FIGURE 6. The SSSC(1, 2, 2) with subcubes of order $1 \times 2 \times 2$.

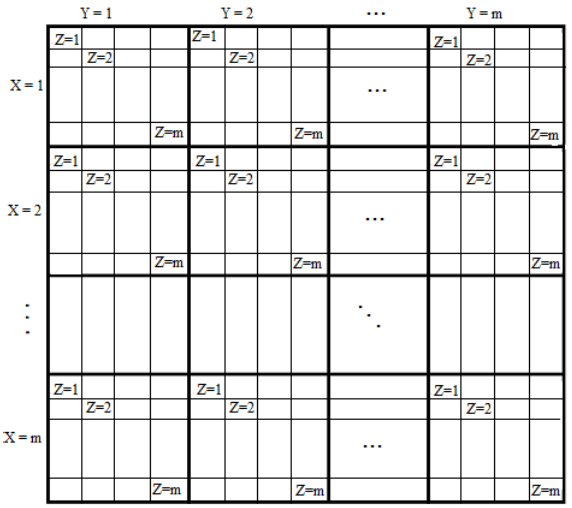

FIGURE 7. Two-dimensional representation of an SSSC(x, y, z).

V. TWO-DIMENSIONAL REPRESENTATION OF SSSCs

Since it can be difficult to visualize the structure and internal components of SSSCs, it is helpful to provide a way to illustrate this class of Sudoku cubes. In this section, we present the 2D representation for SSSC(x, y, z) shown in Figure 7.

The 2D representation shown in Figure 7 is an $m^2 \times m^2$ table that we call a diagonal table with m^2 subtables of order $m \times m$. The m different numbers in a column of an SSSC(x, y, z), i.e. in the X, Y, or Z direction, are placed on the diagonal of an $m \times m$ subtable. To do so, we can embed a column into one of the 2D planes, i.e. the XY-plane, XZ-plane, or ZY-plane. If we embed an SSSC(x, y, z) into the XY-plane, then the entries on the diagonal of the subtables contain m different numbers from a column in the Z direction.

The diagonal table for the SSSC(1,2,2) in Figure 6 is shown in Figure 8. The first entries on the diagonal of a subtable contain numbers from T_0, the second entries contain numbers from T_1, the third entries are from T_2, and the fourth

entries are from T_3. In general, the entries on the diagonal of each subtable are as follows. The first entry is the number in the basic table, i.e. T_0, the second is the number in the second table, i.e. T_1, the third is the number in the third table, i.e. T_2, and this continues until the mth entry is the number in the mth table, i.e. T_{m-1}. Therefore, with this representation all m different numbers occur as diagonal entries of the $m \times m$ subtables of the table. In this table, each element occurs exactly once in each row, column, and diagonal of a subtable. Further, any z consecutive entries starting from the first position on the diagonal of any $x \times y$ block of subtables divided from top to the bottom and left to right will contain m different numbers.

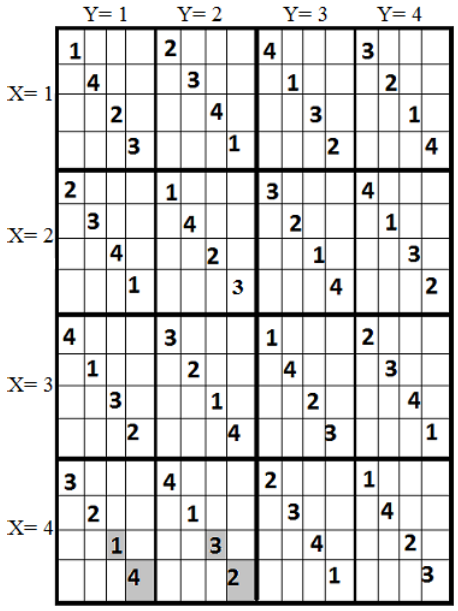

FIGURE 8. The 2D representation of the SSSC(1, 2, 2) in Figure 6, which is obtained by embedding the numbers of SSSC(1, 2, 2) in the entries of the diagonal table in the XY-plane.

Example 2: To illustrate the two dimensional representation of an SSSC(x, y, z), the SSSC(1, 2, 2) presented in Example 1 is investigated. Figure 8 shows the 2D representation of this SSSC. Denote this table as D. It is obvious that each element occurs exactly once in each row, column, and diagonal of the subtables of D. Table D corresponds to the standard solid Sudoku cube Q_0 in Figure 6 from Example 1. Therefore, to obtain tables D_1, D_2, and D_3 corresponding to Q_1, Q_2, and Q_3, we multiply D by 4, 2, and 3, respectively. Thus, $D_0 = 1 \cdot D$, $D_1 = 4 \cdot D \bmod 5$, $D_2 = 2 \cdot D \bmod 5$, and $D_3 = 3 \cdot D \bmod 5$. The shaded entries in Figure 8 correspond to the shaded entries of SSSC(1, 2, 2) in Figure 6.

In Example 3, we show the construction of an SSSC(x, y, z) of order 8 where $x = y = z = 2$.

Example 3: To construct a standard solid Sudoku cube of order $m = 8$ with subcubes of order $2\times2\times2$ ($x = y = z = 2$), we consider $\mathbf{Z}_{17}$ and $q = 16$ so the cyclotomic cosets are $C_1 = \{1, 16\}$, $C_2 = \{2, 15\}$, $C_4 = \{4, 13\}$, and $C_8 = \{8, 9\}$. The vector $Z = [1\ 16\ 2\ 15\ 4\ 13\ 8\ 9]$ of length 8 contains the numbers in $\hat{C}$. For $0 \le i, j \le 1$, the vectors $u_i^{(j)}$, are obtained using (4), i.e. $u_0 = [1\ \ 2]$ and $u_1 = [4\ \ 8]$ so that $u_0^{(1)} = [2\ \ 1]$ and $u_1^{(1)} = [8\ \ 4]$. In fact, $u_0^{(1)}$ and $u_1^{(1)}$ are formed by a cyclic shift to the left of u_0 and u_1, respectively. Then the table B of order 4×4 is constructed from the vectors u_0, u_1, $u_0^{(1)}$, and $u_1^{(1)}$ as

$$B = \begin{bmatrix} u_0 & u_1 \\ u_1 & u_0 \\ u_0^{(1)} & u_1^{(1)} \\ u_1^{(1)} & u_0^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 4 & 8 & 1 & 2 \\ 2 & 1 & 8 & 4 \\ 8 & 4 & 2 & 1 \end{bmatrix}. \tag{21}$$

For $0 \le i, j, r, s \le 1$, the $(2 \cdot j + r, 2 \cdot i + s)$th entry of B is the sth entry of $u_{i+r}^{(j)}$. In other words, $B(2 \cdot j + r, 2 \cdot i + s) = u_{i+r}^{(j)}(s) = a_{2\cdot(i+r)+j+s} = a_t$ where $t = 2\cdot(i+r)+j+s \bmod 4$. The basic table T of order 8×8 from B is

$$T = \begin{bmatrix} B & q \cdot B \\ q \cdot B & B \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} 1 & 2 & 4 & 8 & 16 & 15 & 13 & 9 \\ 4 & 8 & 1 & 2 & 13 & 9 & 16 & 15 \\ 2 & 1 & 8 & 4 & 15 & 16 & 9 & 13 \\ 8 & 4 & 2 & 1 & 9 & 13 & 15 & 16 \\ 16 & 15 & 13 & 9 & 1 & 2 & 4 & 8 \\ 13 & 9 & 16 & 15 & 4 & 8 & 1 & 2 \\ 15 & 16 & 9 & 13 & 2 & 1 & 8 & 4 \\ 9 & 13 & 15 & 16 & 8 & 4 & 2 & 1 \end{bmatrix}. \tag{23}$$

Then for $0 \le l \le 7$, the 8 strongly mutually distinct tables T_l obtained from T are given by

$$T_l = T_{2\cdot t+k} = Z(2 \cdot t + k) \cdot T = a_t \cdot q^k \cdot T \bmod 17 \tag{24}$$

where for $0 \le t \le 3$ and $0 \le k \le 1$, the entries of T are multiplied by $a_t \cdot q^k$. By stacking these 8 strongly mutually distinct solid tables in order $T_0, T_1, \ldots, T_7$, the SSSC(2, 2, 2) is obtained. We replace these eight different numbers with the consecutive numbers 1 to 8 so that $1 \to 1$, $2 \to 2$, $13 \to 3$, $4 \to 4$, $15 \to 5$, $16 \to 6$, $9 \to 7$, $8 \to 8$.

Figure 9 gives the 2D representation of SSSC(2, 2, 2). It is clear that any of the 8 different numbers appears exactly once in each row, column, and diagonal of any subtable of this table. In addition, any $z = 2$ consecutive entries starting from the beginning of the diagonal of any block of 2×2 subtables divided from top to the bottom and left to right contain 8 different numbers. The two pairs of rectangles and parallelograms shown in Figure 9 at the corners of the 2×2 block of subtables represent two $2 \times 2 \times 2$ subcubes of the SSSC(2, 2, 2). To obtain other subcubes of this SSSC, one can follow the same pattern on the table. In other words, any 2 consecutive entries on the diagonals of a 2×2 block of subtables, corresponding to a subcube in the SSSC(2, 2, 2), contains 8 different numbers.

In Section VI, this representation method is used to show how to create and also represent 3D Sudoku puzzles. This representation not only provides us with a method of

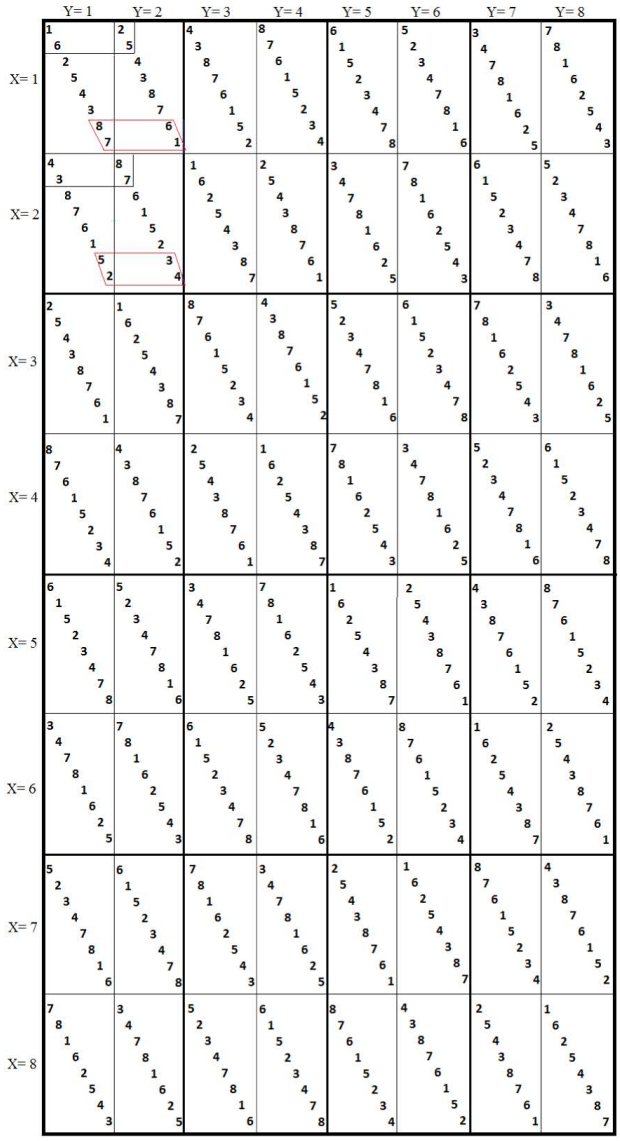

FIGURE 9. 2D representation of the SSSC(2, 2, 2) in Example 3. Any two consecutive entries on the diagonals of any 2 × 2 block of subtables divided from top to bottom and left to right contain 8 different numbers. The two rectangles represent a subcube in the SSSC(2, 2, 2), and the two parallelograms show another subcube.

understanding these SSSCs, but also creates a new type of Sudoku table which is interesting.

VI. SUDOKU PUZZLES FROM SSSCs

Sudoku puzzles in 3D are challenging problems. In [18], the only known 3D Sudoku puzzles were defined as six 2D Sudoku puzzles on the six faces of a cube. In [6], a method to construct Sudoku tables was presented as well as the construction of 2D Sudoku puzzles from these tables. In other words, it was shown that a 2D Sudoku puzzle can be created by removing some entries of a Sudoku table. This method guarantees that these puzzles are solvable, i.e. they have at least one solution because they are obtained from a Sudoku

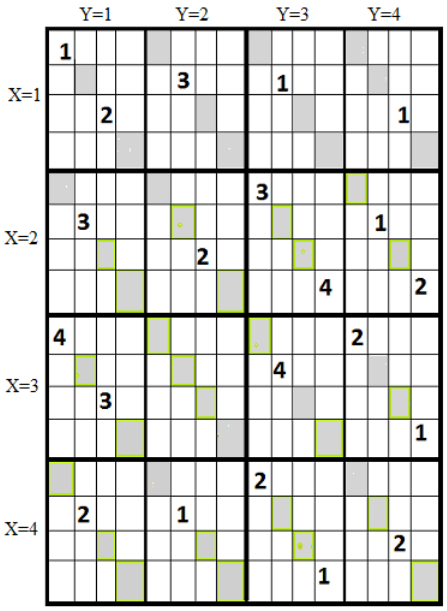

FIGURE 10. Two-dimensional representation of an SSSP(1, 2, 2).

table. In [7], twin Sudoku puzzles were introduced. These puzzles are obtained from the twin Sudoku tables constructed in [7] by removing some entries from the tables.

Whenever a new type of Sudoku table is designed, a new class of Sudoku puzzles can be created by removing some entries from the tables. Here, we obtain a new class of 3D Sudoku puzzles by remove some entries in an SSSC(x, y, z) to obtain a standard solid Sudoku puzzle (SSSP(x, y, z)). Since it is hard to visualize the elements of an SSSP(x, y, z), we use the 2D representation presented in Section V. Then, these puzzles should be solved such that the m different numbers appear exactly once in each row, column, and diagonal of the subtables divided from top to bottom and left to right of the puzzle. Further, any z consecutive entries starting from the first position on the diagonals of any $x \times y$ block of subtables in the table, divided from top to bottom and left to right, should contain m different numbers. The last property comes from the fact that any number should occur exactly once in each $x \times y \times z$ subcube of the SSSP(x, y, z).

Example 4: Figure 10 gives the 2D representation of an SSSP(1, 2, 2) corresponding to the SSSC(1, 2, 2) in Example 1. To solve this puzzle, the shaded entries should be filled with the four different numbers such that any number appears exactly once in each row, column, and diagonal of any subtable. It should also satisfy the property that any $z = 2$ consecutive entries starting from the first position on the diagonals of any 1×2 block of subtables divided from top to bottom and left to right of the Sudoku puzzle contain the four different numbers.

VII. CONCLUSION

A method to design a new class of solid Sudoku cubes called standard solid Sudoku cubes (SSSC) was created. In this

construction, algebraic tools such as cyclotomic cosets in a group $\mathbf{Z}_n$ are used to create an SSSC of order m. That is, for any number m and any factorization of m, $x \cdot y \cdot z = m$, not only an SSSC(x, y, z) but also m strongly mutually distinct SSSCs(x, y, z) are created. These m SMD SSSCs(x, y, z) can also be considered as an extended version of a Latin square in 4D as a Latin quad. Based on a 2D representation of SSSC(x, y, z), a new class of 3D Sudoku puzzles, namely SSSP(x, y, z), was introduced. These 3D Sudoku puzzles are interesting not only for game designers but also for people who are interested in solving Sudoku puzzles.

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