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Leaderless Consensus Control of Nonlinear PIDE-Type Multi-Agent Systems With Time Delays

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ABSTRACT This paper studies leaderless consensus of semi-linear parabolic partial integro-differential equations based multi-agent systems (PIDEMASs) with time delays. Making use of the information interaction and coordination among the neighboring agents, consensus control of the leaderless PIDEMAS is constructed. Consensus of the leaderless PIDEMAS is analyzed by using a Lyapunov approach. Dealing with time-invariant delays and time-varying delays, two sufficient conditions for consensus of the leaderless PIDEMAS are respectively obtained in terms of LMIs. Two examples illustrate the effectiveness of developed theoretical results.

INDEX TERMS Consensus, multi-agent systems, LMIs, Lyapunov, partial integro-differential equations.

I. INTRODUCTION

Multi-agent systems (MASs) have attracted a great deal of attention during the last few decades [1], [2]. They have been widely used in engineering fields, such as secure communication [3], [4], privacy-preserving [5], ship course-keeping [6], UAVs formation flying [7]–[9], and traffic flow [10].

In recent years, many important results have been obtained on consensus of MASs, whose goal is to enable agents to perform a designated task synchronously [11], [12]. Tian *et al.* proposed output consensus for second-order MASs [13]. Ji *et al.* studied adaptive learning fault-tolerant consensus for MASs [14]. Xiao *et al.* investigated variable impulsive control for consensus of stochastic perturbed MASs [15]. Yu *et al.* proposed a finite-horizon H_∞ consensus control method for multi-agent networks under the limited energy constraint [16]. In these results, time delays have not yet to be considered.

As well been known, time delays extensively exist in almost all sorts of systems. Therefore, it is desired to research consensus of MASs with time delays. Lu *et al.* investigated consensus of communication delayed MASs with antagonistic interactions [17]. Li *et al.* studied a dynamic gain obtained approach for consensus of delayed MASs [18]. Chen *et al.* studied H_∞ containment control for discrete time-varying linear MASs with multileaders [19]. Shahamatkhah and Tabatabaei proposed containment control of fractional-order MASs with time-delays [20].

As a whole, the mentioned literature have obtained important results, whereas they assumed the dynamics of agents relying on only time [21], [22]. Actually, dynamics of all processes rely on time and space in nature. Therefore, there is an importance to research MASs with spatio-temporal structures. Demetriou studied adaptive consensus and spatial SPID for partial differential equation-type MASs (PDEMASs) [23], [24]. Yang *et al.* proposed spatial boundary control of consensus for nonlinear PDEMASs [25] and boundary control for output consensus of nonlinear PDEMASs with input constraint [26]. Iterative learning for consensus of nonlinear PDEMASs was studied without time delays in [27] and with time delays in [28]. An adaptive unit-vector control method for consensus of uncertain PDEMASs was proposed in [29]. Qiu and Su studied distributed adaptive consensus of switching PDEMASs [30].

The papers [23]–[30] are modeled by PDEs, whereas there are few works considering models based on partial integro-differential equations(PIDEs). Numerical solutions of PIDEs have been studied in [31], [32]. PIDEs have applied to spread and traveling waves [33], pricing models [34], reaction–diffusion systems [35], biology [36], [37], pattern

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formation [38], [39], secure communication [40], medical science [41]. Many dynamical behaviors have been studied in [42]–[45]. However, there are still technical difficulties on consensus of PIDEs based MASs (PIDEMASs), like communication between agents and topology structure, which motives this paper.

This paper aims to research leaderless consensus control methods of a semi-linear parabolic PIDEMAS with time delays. The contribution of this paper contains: (1) A class of PIDEMAS models is built, considering time-invariant delays and time-varying delays, respectively; (2) A controller based on communication among agents is given; (3) The topology structure is analyzed among agents; (4) By choosing suitable Lyapunov functional, using Lyapunov direct method, two sufficient conditions for consensus of the leaderless PIDEMAS are respectively obtained in terms of LMIs.

Notations: I means the identity matrix with proper order, $P > 0 (P < 0)$ means symmetric positive definite (negative definite), and $\|\cdot\|$ denotes the 2-norm for vectors, or vector functions like $||y(\cdot,t)|| = \sqrt{\int_0^L y^T(\zeta,t)y(\zeta,t)d\zeta}$, $\lambda_{\max(\min)}(\cdot)$ is the maximum (minimum) eigenvalue. The superscript T is used for the transpose of a vector or a matrix, and the symbol $*$ is used as an ellipsis for terms in matrix expressions induced by the symmetry.

II. PROBLEM FORMULATION

This paper studies a class of semi-linear PIDEMASs with time delays as

$$\begin{aligned}\frac{\partial y_i(\zeta,t)}{\partial t} &= \Theta_1\frac{\partial^2 y_i(\zeta,t)}{\partial\zeta^2} + \Theta_2\frac{\partial y_i(\zeta,t)}{\partial\zeta}\\ &\quad +Ay_i(\zeta,t) + By_i(\zeta,t-\tau_1(t))\\ &\quad +f(y_i(\zeta,t-\tau_2(t)))\\ &\quad +C\int_0^\zeta y_i(s,t-\tau_3(t))ds + u_i(\zeta,t),\\ \frac{\partial y_i(0,t)}{\partial\zeta} &= 0,\ \frac{\partial y_i(L,t)}{\partial\zeta} = 0,\\ y_i(\zeta,t) &= y_i^0(\zeta,t),\quad (\zeta,t)\in[0,L]\times[-\tau,0],\end{aligned}\tag{1}$$

where $(\zeta,t)\in[0,L]\times[0,\infty)$ are space and time, respectively. $y_i(\zeta,t), u_i(\zeta,t)\in\mathbb{R}^n$ are state and control input, respectively. $0 < L\in\mathbb{R}$, $i\in\{1,2,\cdots,N\}$, $A, B, C, \Theta_2\in\mathbb{R}^{n\times n}$, $\Theta_1\in\mathbb{R}^{n\times n}$ is symmetric positive definite, $f(\cdot)$ is a time and spatial variable nonlinear function, $0\leqslant\dot\tau_1(t)\leqslant\mu_1$, $0\leqslant\dot\tau_2(t)\leqslant\mu_2$, and $0\leqslant\dot\tau_3(t)\leqslant\mu_3$.

Let consensus error to be $\epsilon_i(\zeta,t)\triangleq y_i(\zeta,t) - \frac{1}{N}\sum_{j=1}^N y_j(\zeta,t)$, and the controller is employed as

$$u_i(\zeta,t) = c\sum_{j=1}^N g_{ij}\Gamma(y_j(\zeta,t)-y_i(\zeta,t)),\tag{2}$$

where c is a control gain to be determined and Γ is symmetric positive definite. Assume that the topological structure $G = (g_{ij})_{N\times N}$ is defined as: $g_{ii} = 0$; $g_{ij} = g_{ji} > 0 (i\neq j)$ if the agent i connects to j, otherwise $g_{ij} = 0 (i\neq j)$.

Remark 1: The topological structure of the controller (2) is under undirected graph. It can make fully use of relative information among agents. By choosing suitable control gain c, the controller (2) drives the PIDEMAS (1) to consensus.

The error system of the PIDEMAS (1) can be obtained from (1) and (2) as

$$\begin{aligned}\frac{\partial\epsilon(\zeta,t)}{\partial t} &= (I_N\otimes\Theta_1)\frac{\partial^2\epsilon(\zeta,t)}{\partial\zeta^2} + (I_N\otimes\Theta_2)\frac{\partial\epsilon(\zeta,t)}{\partial\zeta}\\ &\quad +(I_N\otimes A)\epsilon(\zeta,t)\\ &\quad +(I_N\otimes B)\epsilon(\zeta,t-\tau_1(t))\\ &\quad +F(\epsilon(\zeta,t-\tau_2(t)))\\ &\quad +(I_N\otimes C)\int_0^\zeta \epsilon(s,t-\tau_3)ds\\ &\quad -c(\mathcal{L}\otimes\Gamma)\epsilon(\zeta,t),\\ \frac{\partial\epsilon(0,t)}{\partial\zeta} &= 0,\ \frac{\partial\epsilon(L,t)}{\partial\zeta} = 0,\\ \epsilon(\zeta,0) &= \epsilon^0(\zeta),\end{aligned}\tag{3}$$

where $\epsilon_i^0(\zeta)\triangleq y_i^0(\zeta) - \frac{1}{N}\sum_{j=1}^N y_j^0(\zeta)$, $\epsilon(\zeta,t)\triangleq[\epsilon_1^T(\zeta,t),\epsilon_2^T(\zeta,t),\cdots,\epsilon_N^T(\zeta,t)]^T$, $F(\epsilon_i(\zeta,t-\tau_2(t)))\triangleq f(y_i(\zeta,t-\tau_2(t))) - \frac{1}{N}\sum_{j=1}^N f(y_j(\zeta,t-\tau_2(t)))$, $F(\epsilon(\zeta,t-\tau_2(t)))\triangleq[F^T(\epsilon_1(\zeta,t-\tau_2(t))),F^T(\epsilon_2(\zeta,t-\tau_2(t))),\cdots,F^T(\epsilon_N(\zeta,t-\tau_2(t)))]^T$, $\mathcal{L} = D-G$, $D = diag\{d_1,d_2,\cdot,d_N\}$, $d_i = \sum_{j=1}^N g_{ij}$, and so $\mathcal{L}$ is a Laplace matrix.

This paper aims to use the controller (2) to reach consensus of the PIDEMAS (1). The following definition, assumption and Lemma are needed.

Definition 1: The PIDEMAS (1) reaches consensus, if

$$\lim_{t\to\infty}||y_i(\zeta,t) - \frac{1}{N}\sum_{j=1}^N y_j(\zeta,t)|| = 0,\quad i\in\{1,2,\cdots,N\}.\tag{4}$$

Assumption 1: For any a, b, assume there exists a scalar $\chi > 0$ satisfying

$$|f(a)-f(b)|\leqslant\chi|a-b|.\tag{5}$$

Lemma 1 [46]: For any square integrable vector ϵ with $\epsilon(0) = 0$ or $\epsilon(L) = 0$,

$$\int_0^L \epsilon^T(z)\epsilon(z)dz\leq 4L^2\pi^{-2}\int_0^L\dot\epsilon^T(z)\dot\epsilon(z)dz.\tag{6}$$

If Laplacian matrix $\mathcal{L}\in\mathbb{R}^{N\times N}$ is symmetric, then $0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L})\leq\mathcal{L}\leq\lambda_N(\mathcal{L})$. The smallest nonzero eigenvalue of $\lambda_2(\cdot)$ is known as algebraic connectivity of graphs [47].

Lemma 2 [48]: For Laplacian matrix $\mathcal{L}$, symmetric positive definite P and $x\in\mathbb{R}^{Nn}$ such that $1_{Nn}^T x = 0$, the following inequality is satisfied:

$$\lambda_2(\mathcal{L})x^T(I_N\otimes P)x\leq x^T(\mathcal{L}\otimes P)x.\tag{7}$$

III. CONSENSUS OF THE PIDEMAS WITH TIME-INVARIANT DELAYS

The error system of the PIDEMAS (1) can be obtained as

$$\begin{aligned}\frac{\partial\epsilon(\zeta,t)}{\partial t}&=\Theta_1\frac{\partial^2\epsilon(\zeta,t)}{\partial\zeta^2}+\Theta_2\frac{\partial\epsilon(\zeta,t)}{\partial\zeta}\\&\quad+(I_N\otimes A)\epsilon(\zeta,t)\\&\quad+(I_N\otimes B)\epsilon(\zeta,t-\tau_1)\\&\quad+F(\epsilon(\zeta,t-\tau_2))\\&\quad+(I_N\otimes C)\int_0^\zeta\epsilon(s,t-\tau_3(t))ds\\&\quad-c(\mathcal{L}\otimes\Gamma)\epsilon(\zeta,t),\\\frac{\partial\epsilon(0,t)}{\partial\zeta}&=0,\ \frac{\partial\epsilon(L,t)}{\partial\zeta}=0,\\\epsilon(\zeta,0)&=\epsilon^0(\zeta),\end{aligned}\tag{8}$$

where $F(\epsilon_i(\zeta,t-\tau_2))\triangleq f(y_i(\zeta,t-\tau_2))-\frac{1}{N}\sum_{j=1}^N f(y_j(\zeta,t-\tau_2))$ and $F(\epsilon(\zeta,t-\tau_2))\triangleq[F^T(\epsilon_1(\zeta,t-\tau_2)),F^T(\epsilon_2(\zeta,t-\tau_2)),\cdots,F^T(\epsilon_N(\zeta,t-\tau_2))]^T$.

Theorem 1: Suppose Assumption 1 holds and the communication graph G is strongly connected. The PIDEMAS (1) with time-invariant delays reaches consensus under the controller (2), if there exist scalars $c>0$ and $\alpha>0$ satisfying the following LMIs:

$$\Psi_1\triangleq\alpha\chi^2-1<0,\tag{9}$$

$$\Psi_2\triangleq 4L^2\pi^{-2}\alpha-1<0,\tag{10}$$

$$\Psi\triangleq\begin{bmatrix}\Psi_{11}&\Psi_{12}&\Psi_{13}&I&C\\ *&\Psi_{22}&0&0&0\\ *&*&-I&0&0\\ *&*&*&-\alpha I&0\\ *&*&*&*&-\alpha I\end{bmatrix}<0,\tag{11}$$

in which

$$\begin{aligned}\Psi_{11}&\triangleq[I_N\otimes A-c\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma)I+*]+3I,\\\Psi_{12}&\triangleq I_N\otimes\Theta_2,\\\Psi_{13}&\triangleq I_N\otimes B,\\\Psi_{22}&\triangleq-[I_N\otimes\Theta_1+*].\end{aligned}$$

Proof: Choose Lyapunov functional candidate as

$$\begin{aligned}V(t)&=\int_0^L\epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta\\&\quad+\int_0^L\int_{t-\tau_1}^t\epsilon^T(\zeta,\rho)\epsilon(\zeta,\rho)d\rho d\zeta\\&\quad+\int_0^L\int_{t-\tau_2}^t\epsilon^T(\zeta,\rho)\epsilon(\zeta,\rho)d\rho d\zeta\\&\quad+\int_0^L\int_{t-\tau_3}^t\epsilon^T(\zeta,\rho)\epsilon(\zeta,\rho)d\rho d\zeta.\end{aligned}\tag{12}$$

Taking the time derivative of $V(t)$, we get

$$\begin{aligned}\dot V(t)&=2\int_0^L\epsilon^T(\zeta,t)\frac{\partial\epsilon(\zeta,t)}{\partial t}d\zeta\\&=2\int_0^L\epsilon^T(\zeta,t)(I_N\otimes\Theta_1)\frac{\partial^2\epsilon(\zeta,t)}{\partial\zeta^2}d\zeta\\&\quad+2\int_0^L\epsilon^T(\zeta,t)(I_N\otimes\Theta_2)\frac{\partial\epsilon(\zeta,t)}{\partial\zeta}d\zeta\\&\quad+2\int_0^L\epsilon^T(\zeta,t)(I_N\otimes A-c\mathcal{L}\otimes\Gamma)\epsilon(\zeta,t)d\zeta\\&\quad+2\int_0^L\epsilon^T(\zeta,t)(I_N\otimes B)\epsilon(\zeta,t-\tau_1)d\zeta\\&\quad+2\int_0^L\epsilon^T(\zeta,t)F(\epsilon(\zeta,t-\tau_2))d\zeta\\&\quad+2\int_0^L\epsilon^T(\zeta,t)(I_N\otimes C)\int_0^\zeta\epsilon(s,t-\tau_3)dsd\zeta\\&\quad+3\int_0^L\epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta\\&\quad-\int_0^L\epsilon^T(\zeta,t-\tau_1)\epsilon(\zeta,t-\tau_1)d\zeta\\&\quad-\int_0^L\epsilon^T(\zeta,t-\tau_2)\epsilon(\zeta,t-\tau_2)d\zeta\\&\quad-\int_0^L\epsilon^T(\zeta,t-\tau_3)\epsilon(\zeta,t-\tau_3)d\zeta.\end{aligned}\tag{13}$$

Since $\mathcal{L}$ is a Laplace matrix and Γ is a symmetric positive definite matrix, using Lemma 2, one has

$$\begin{aligned}&-2c\int_0^L\epsilon^T(\zeta,t)(\mathcal{L}\otimes\Gamma)\epsilon(\zeta,t)d\zeta\\&\quad\leqslant-2c\lambda_2(\mathcal{L})\int_0^L\epsilon^T(\zeta,t)(I_N\otimes\Gamma)\epsilon(\zeta,t)d\zeta\\&\quad\leqslant-2c\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma)\int_0^L\epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta.\end{aligned}\tag{14}$$

For $\Theta_1>0$, employing integrating by parts, one has

$$\begin{aligned}&2\int_0^L\epsilon^T(\zeta,t)(I_N\otimes\Theta_1)\frac{\partial^2\epsilon(\zeta,t)}{\partial\zeta^2}d\zeta\\&\quad=-\int_0^L\frac{\partial\epsilon^T(\zeta,t)}{\partial\zeta}[I_N\otimes\Theta_1+*]\frac{\partial\epsilon(\zeta,t)}{\partial\zeta}d\zeta.\end{aligned}\tag{15}$$

Using Assumption 1 and Lemma 1, for any $\alpha>0$, one has

$$\begin{aligned}&2\int_0^L\epsilon^T(\zeta,t)F(\epsilon(\zeta,t-\tau_2))d\zeta\\&\quad\leqslant\alpha^{-1}\int_0^L\epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta\\&\qquad+\alpha\int_0^L F^T(\zeta,t-\tau_2)F(\epsilon(\zeta,t-\tau_2))d\zeta\\&\quad\leqslant\alpha^{-1}\int_0^L\epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta\\&\qquad+\alpha\chi^2\int_0^L\epsilon^T(\zeta,t-\tau_2)\epsilon(\zeta,t-\tau_2)d\zeta,\end{aligned}\tag{16}$$

and

$$2\int_0^L\epsilon^T(\zeta,t)(I_N\otimes C)\int_0^\zeta\epsilon(z,t-\tau_3)dzd\zeta$$

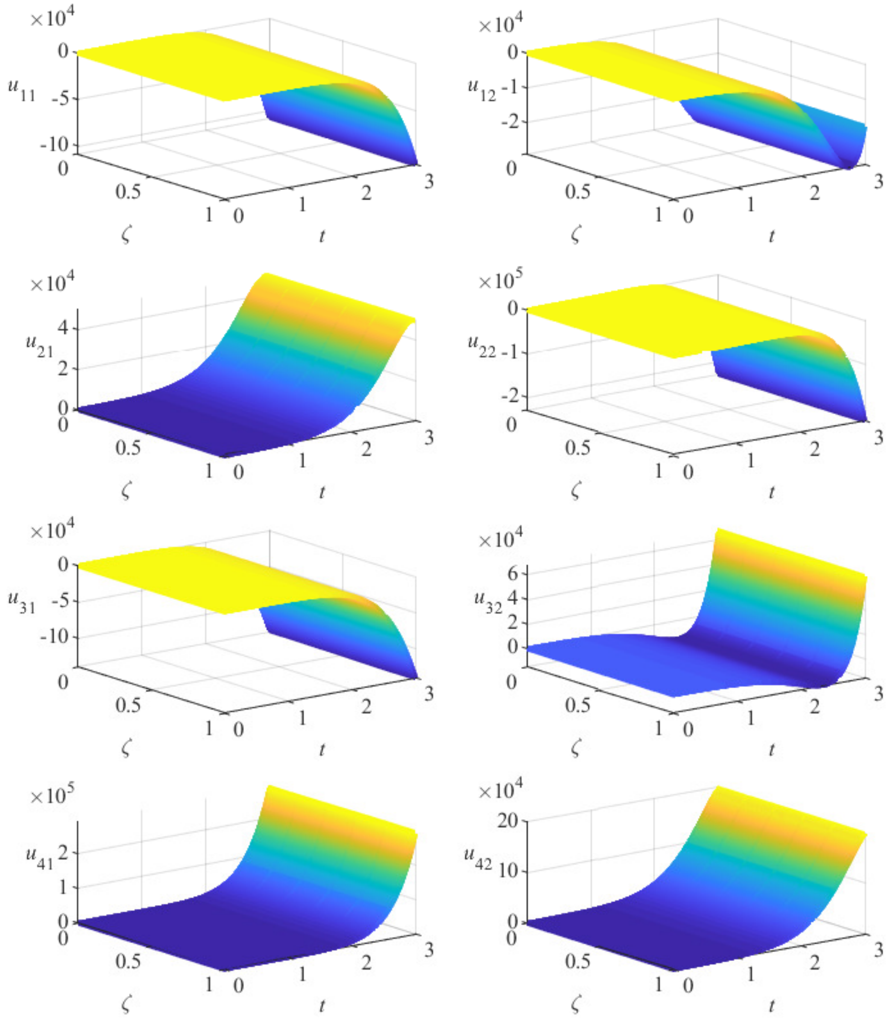

FIGURE 1. $\epsilon(\zeta, t)$ **of the PIDEMAS without control.**

$$
\begin{aligned}
&\leqslant \alpha^{-1}\int_0^L \epsilon^T(\zeta,t)(I_N\otimes CC^T)\epsilon(\zeta,t)d\zeta \\
&\quad +\alpha\int_0^L\int_0^\zeta \epsilon^T(z,t-\tau_3)dz\int_0^\zeta \epsilon(z,t-\tau_3)dzd\zeta \\
&\leqslant \alpha^{-1}\int_0^L \epsilon^T(\zeta,t)(I_N\otimes CC^T)\epsilon(\zeta,t)d\zeta \\
&\quad +4L^2\pi^{-2}\alpha\int_0^L \epsilon^T(\zeta,t-\tau_3)\epsilon(\zeta,t-\tau_3)d\zeta. \qquad (17)
\end{aligned}
$$

Substitution of (14)–(17) into (13) yields,

$$
\begin{aligned}
\dot V(t) \leqslant &\int_0^L \tilde\epsilon^T(\zeta,t)\bar\Psi\tilde\epsilon(\zeta,t)d\zeta \\
&+\int_0^L \epsilon^T(\zeta,t-\tau_2)\Psi_1\epsilon(\zeta,t-\tau_2)d\zeta \\
&+\int_0^L \epsilon^T(\zeta,t-\tau_3)\Psi_2\epsilon(\zeta,t-\tau_3)d\zeta, \qquad (18)
\end{aligned}
$$

where $\tilde\epsilon(\zeta,t)\triangleq[\epsilon^T(\zeta,t), \frac{\partial\epsilon^T(\zeta,t)}{\partial\zeta}, \epsilon^T(\zeta,t-\tau_1)]^T$, and

$$\bar\Psi_1 \triangleq (\alpha\chi^2-1)I, \qquad (19)$$

$$\bar\Psi_2 \triangleq (4L^2\pi^{-2}\alpha-1)I, \qquad (20)$$

$$\bar\Psi \triangleq \begin{bmatrix} \bar\Psi_{11} & I_N\otimes\Theta_2 & I_N\otimes B \\ * & -[I_N\otimes\Theta_1+*] & 0 \\ * & * & -I \end{bmatrix}, \qquad (21)$$

in which

$$
\begin{aligned}
\bar\Psi_{11} \triangleq [I_N\otimes A - c\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma)I+*]+\alpha^{-1}I \\
+\alpha^{-1}I_N\otimes CC^T+3\,I.
\end{aligned}
$$

Using Schur complement, (11) is equivalent to,

$$\bar\Psi < 0. \qquad (22)$$

Substitution of (9), (10) and (22) into (18), yields $\dot V(t)\leqslant -\lambda||\tilde\epsilon(\cdot,t)||\leqslant -\lambda||\epsilon(\cdot,t)||$, for all non-zero $\epsilon(\zeta,t)$, implying consensus of the PIDEMAS (1). □

Remark 2: Theorem 1 shows consensus conditions in terms of LMIs and a suitable gain is obtained. However,

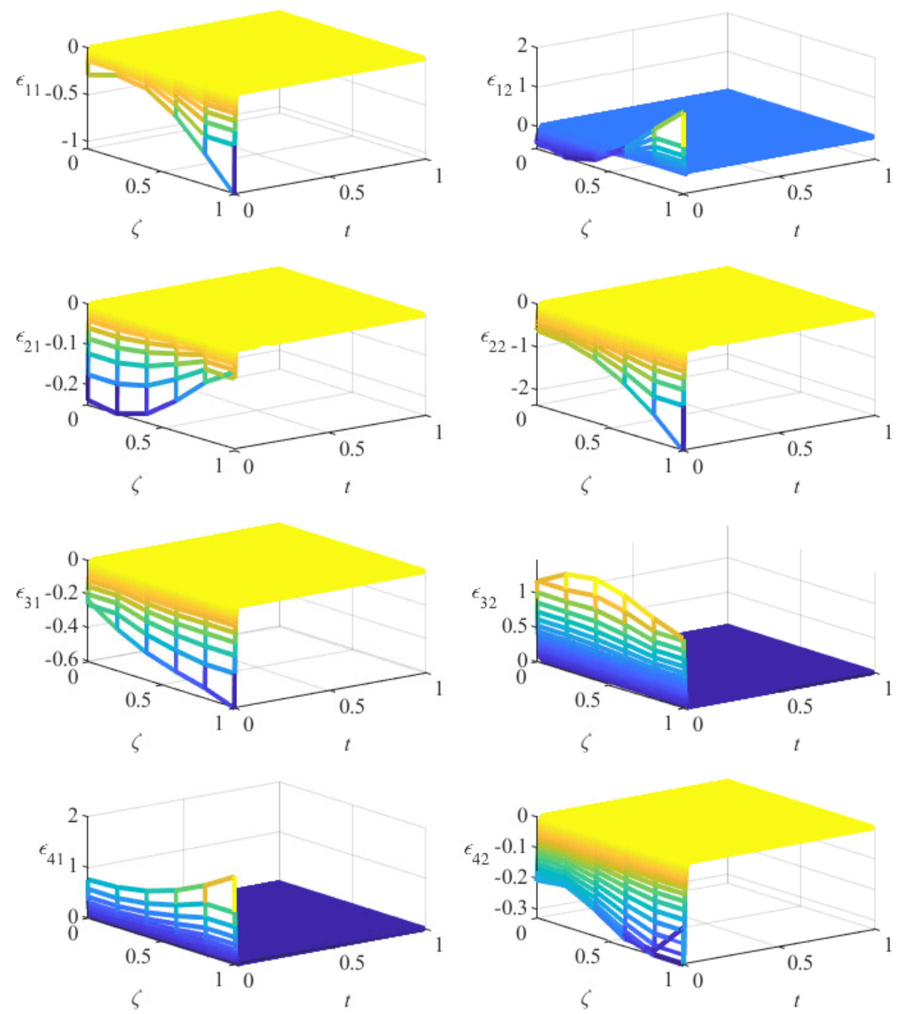

FIGURE 2. $\epsilon(\zeta, t)$ **of the PIDEMAS with control.**

the result is given in terms of the conditions (9)-(11) given in Theorem 1 is complex. Now, we show a simple result. According to (9) and (10), $0 < \alpha < \min\{\chi^{-2}, 0.25L^{-2}\pi^2\}$. Using Schur complement, (11) is equivalent to $[I_N \otimes A - c\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma)I + *] + \alpha^{-1}I + \alpha^{-1}I_N \otimes CC^T + 3I + +I_N \otimes BB^T + I_N \otimes \Theta_2\Theta_1^{-1}\Theta_2^T < 0$, and it is equivalent to $c > \frac{\lambda_{\max}(I_N\otimes(A+A^T)+\alpha^{-1}I+\alpha^{-1}I_N\otimes CC^T+3I+I_N\otimes BB^T+I_N\otimes\bar{\Theta})}{\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma)}$, where $\bar{\Theta} \triangleq \Theta_2\Theta_1^{-1}\Theta_2^T$.

IV. CONSENSUS OF THE PIDEMAS WITH TIME-VARYING DELAYS

This section will study the PIDEMAS (1) with time-varying delays via the controller (2).

Theorem 2: Suppose Assumption 1 holds and the communication graph G is strongly connected. The PIDEMAS (1) with time-varying delays reaches consensus under the controller (2), if there exist scalars $c > 0$ and $\alpha > 0$ satisfying the following LMIs:

$$\Xi_1 \triangleq \alpha\chi^2 - 1 + \mu_2 < 0, \tag{23}$$

$$\Xi_2 \triangleq 4L^2\pi^{-2}\alpha - 1 + \mu_3 < 0, \tag{24}$$

$$\Xi \triangleq \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & I & C \\ * & \Psi_{22} & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 \\ * & * & * & -\alpha I & 0 \\ * & * & * & * & -\alpha I \end{bmatrix} < 0, \tag{25}$$

in which

$$\begin{aligned} \Psi_{11} &\triangleq [I_N \otimes A - c\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma)I + *] + 3\,I, \\ \Psi_{12} &\triangleq I_N \otimes \Theta_2, \\ \Psi_{13} &\triangleq I_N \otimes B, \\ \Psi_{22} &\triangleq -[I_N \otimes \Theta_1 + *], \\ \Xi_{33} &\triangleq -(1 - \mu_1)I. \end{aligned}$$

Proof: Choose Lyapunov functional candidate as

$$V(t) = V_1(t) + V_2(t), \tag{26}$$

where

$$V_1(t) = \int_0^L \epsilon^T(\zeta, t)\epsilon(\zeta, t)d\zeta, \tag{27}$$

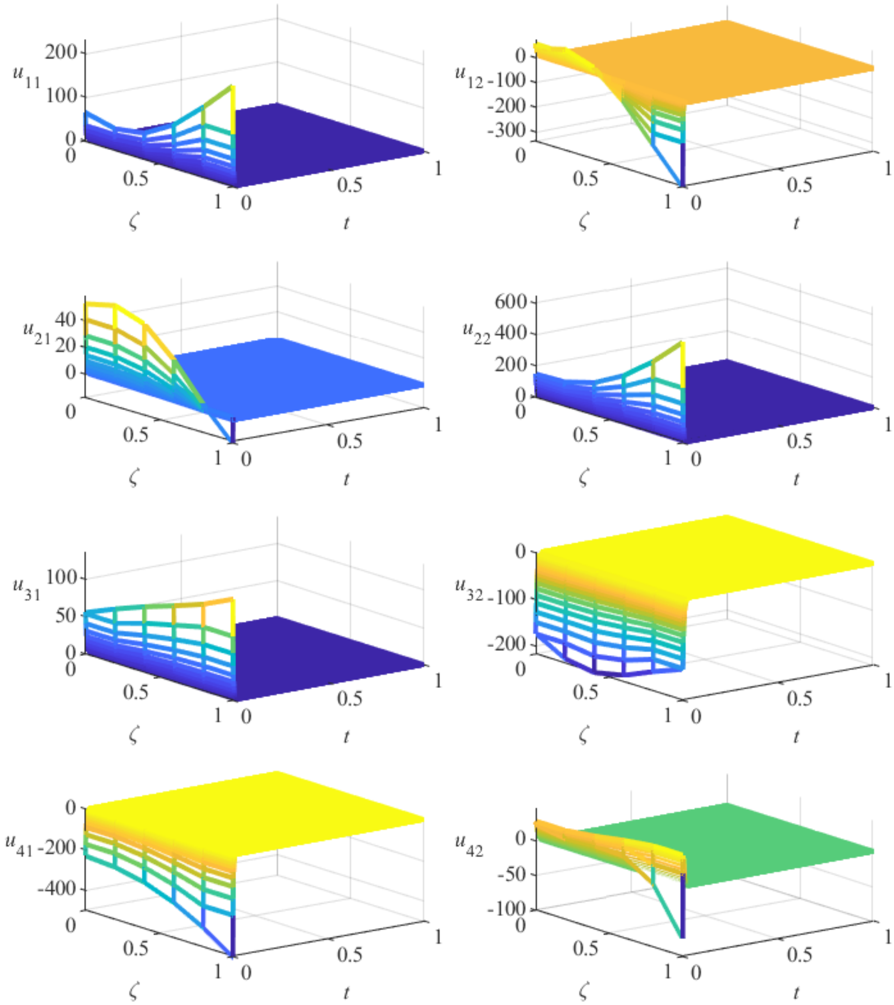

FIGURE 3. The control input of the PIDEMAS.

and

$$V_2(t) = \int_0^L \int_{t-\tau_1(t)}^t \epsilon^T(\zeta,\rho)\epsilon(\zeta,\rho)d\rho d\zeta + \int_0^L \int_{t-\tau_2(t)}^t \epsilon^T(\zeta,\rho)\epsilon(\zeta,\rho)d\rho d\zeta + \int_0^L \int_{t-\tau_3(t)}^t \epsilon^T(\zeta,\rho)\epsilon(\zeta,\rho)d\rho d\zeta. \quad (28)$$

Taking the time derivative of $V_1(t)$, we get

$$\begin{aligned}\dot{V}_1(t) &= 2\int_0^L \epsilon^T(\zeta,t)\frac{\partial e(\zeta,t)}{\partial t}d\zeta \\ &= 2\int_0^L \epsilon^T(\zeta,t)(I_N\otimes\Theta)\frac{\partial^2\epsilon(\zeta,t)}{\partial x^2}d\zeta \\ &\quad +2\int_0^L \epsilon^T(\zeta,t)(I_N\otimes A - c\mathcal{L}\otimes\Gamma)\epsilon(\zeta,t)d\zeta \\ &\quad +2\int_0^L \epsilon^T(\zeta,t)(I_N\otimes B)\epsilon(\zeta,t-\tau_1(t))d\zeta \\ &\quad +2\int_0^L \epsilon^T(\zeta,t)F(\epsilon(\zeta,t-\tau_2(t)))d\zeta \\ &\quad +2\int_0^L \epsilon^T(\zeta,t)(I_N\otimes C)\int_0^\zeta \epsilon(s,t-\tau_3(t))dsd\zeta \\ &\quad +3\int_0^L \epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta. \end{aligned} \quad (29)$$

Taking the time derivative of $V_2(t)$, one has

$$\begin{aligned}\dot{V}_2(t) &= 3\int_0^L \epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta \\ &\quad -(1-\dot{\tau}_1(t))\int_0^L \epsilon^T(\zeta,t-\tau_1(t))\epsilon(\zeta,t-\tau_1(t))d\zeta \\ &\quad -(1-\dot{\tau}_2(t))\int_0^L \epsilon^T(\zeta,t-\tau_2(t))\epsilon(\zeta,t-\tau_2(t))d\zeta \\ &\quad -(1-\dot{\tau}_3(t))\int_0^L \epsilon^T(\zeta,t-\tau_3(t))\epsilon(\zeta,t-\tau_3(t))d\zeta \\ &\le 3\int_0^L \epsilon^T(\zeta,t)\epsilon(\zeta,t)d\zeta \end{aligned}$$

$$
\begin{aligned}
&-(1-\mu_1)\int_0^L \epsilon^T(\zeta,t-\tau_1(t))\epsilon(\zeta,t-\tau_1(t))d\zeta \\
&-(1-\mu_2)\int_0^L \epsilon^T(\zeta,t-\tau_2(t))\epsilon(\zeta,t-\tau_2(t))d\zeta \\
&-(1-\mu_3)\int_0^L \epsilon^T(\zeta,t-\tau_3(t))\epsilon(\zeta,t-\tau_3(t))d\zeta.
\end{aligned} \tag{30}
$$

The later part of the proof is similar to that of Theorem 1, and so it is omitted. □

Remark 3: Theorem 2 shows consensus conditions in terms of LMIs and a suitable gain is obtained. However, the result is given in terms of the conditions (23)-(25) given in Theorem 2 is complex. Now, we show a simple result. According to (23) and (24), $0 < \alpha < \min\{\frac{1-\mu_2}{\chi^2}, \frac{1-\mu_3}{4L^2\pi^{-2}}\}$. Using Schur complement, (25) is equivalent to $[I_N \otimes A - c\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma) + *] + \alpha^{-1}I + \alpha^{-1}I_N \otimes CC^T + 3I + (1-\mu_1)I_N \otimes BB^T + I_N \otimes \Theta_2\Theta_1^{-1}\Theta_2^T < 0$, and it is equivalent to $c > \frac{\lambda_{\max}(I_N\otimes\bar{A}+\alpha^{-1}I_N\otimes(I+\bar{C})+3I+(1-\mu_1)I_N\otimes\bar{B}+I_N\otimes\bar{\Theta})}{\lambda_2(\mathcal{L})\lambda_{\min}(\Gamma)}$, where $\bar{A} = A + A^T$, $\bar{B} = BB^T$, $\bar{C} = CC^T$, and $\bar{\Theta} \triangleq \Theta_2\Theta_1^{-1}\Theta_2^T$.

Remark 4: Different from the control design for stability of PIDE systems in [49], [50], this paper deals with consensus of PIDEs based MASs by using communication between neighborhood agents.

Remark 5: There are many important results for PDE-MASs, for example [23]–[30] and the references herein, while this paper studies MASs based on PIDEs, as well as multiple time-invariant delays and time-varying delays being considered.

V. NUMERICAL SIMULATION

Example 1: In practice, there are many reaction-diffusion phenomena in nature and discipline fields [51]–[53]. Reaction-diffusion neural networks have been application to biology [36], [37], pattern formation [38], [39], secure communication [40], medical science [41]. This example considers a reaction–diffusion integro neural network, as one kind of the PIDEMAS (1), with the following parameters:

$$
\begin{aligned}
&\Theta_1 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \\
&A = \begin{bmatrix} 2 & 0.6 \\ -1.5 & 2.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \\
&C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
&L = 1, g_{ij} = 1, \quad \text{for } i, j = 1, 2, 3, 4 \text{ and } i \neq j,
\end{aligned} \tag{31}
$$

and with random initial conditions.

From Figure 1, it can be seen that the PIDEMAS (1) cannot achieve synchronization without control. According to Theorem 1, solving LMIs (9)-(11) by Matlab, $c = 28.4556$ and $\alpha = 0.8723$ are obtained. It can be shown in Figure 2 that the PIDEMAS (1) achieves cluster consensus. The control input (2) with the feedback gain $c = 8.3221$ is shown in Figure 3.

Remark 6: Different from difference, bifurcation, and solution of PIDEs or integro-differential reaction-diffusion systems [54]–[58], this paper proposed consensus of MASs basd on PIDEs via constructing a communication based controller.

VI. CONCLUSION

This paper has studied leaderless consensus of a nonlinear PIDEMAS with time delays, modeled by semi-linear parabolic PIDEs. Making use of the information interaction and coordination among the neighboring agents, leaderless consensus control of the PIDEMAS was constructed. Dealing with the PIDEMAS with time-invariant delays, a Lyapunov approach was used and one sufficient condition for consensus was obtained in terms of LMIs. Then, it was extended to the PIDEMAS with time-varying delays. An example illustrated the effectiveness of developed theoretical results. Because there are lots of factors may influence the dynamic behavior of PIDEMASs, in future work, containment control, event-triggered control, stochastic disturbance and many other factors will be studied.

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