

# Finite Horizon $H_\infty$ Control of Nonlinear Time-Varying Systems Under WTOD Protocol

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**ABSTRACT** The finite horizon  $H_\infty$  control problem is investigated for a class of discrete nonlinear time-varying systems subject to Weighted Try-Once-Discard communication protocol. The equations of state and output under investigation involve both deterministic and stochastic nonlinearities. By resorting to the Taylor series expansion formula, Lyapunov stability theory and cross-amplification lemma, sufficient conditions are established for the existence of the desired observer-based  $H_\infty$  controller. The gain matrices of the controller and observer are obtained by solving a set of recursive linear matrix inequalities.

**INDEX TERMS**  $H_\infty$  control, finite horizon, nonlinear time-varying networked control systems, weighted try-once-discard communication protocols.

## I. INTRODUCTION

In recent years, more and more scholars have developed a strong interest in the control and filtering of nonlinear systems [1]–[3]. Literature [4] studied the sliding mode control (SMC) problem for Markovian jump systems with probabilistic denial-of-service (DoS) attacks. In Literature [5], an event-triggered nonfragile distributed  $H_\infty$  control problem was considered for uncertain nonlinear networked control systems over sensor networks with random communication packet dropouts and redundant channels. Literature [6] treated the problem of periodic tracking control of fractional-order IT-2 fuzzy systems with state and external disturbance, where an observer-based fractional-order modified repetitive controller has been designed to make the resulting closed-loop fuzzy control system asymptotically stable together with a prescribed  $H_\infty$  disturbance attenuation level. In literature [7], the envelope constraint and filtering problem of the system were considered simultaneously. The nonlinearities here include deterministic nonlinearities and stochastic nonlinearities. Literature [8] investigated the recursive filtering problem for nonlinear time-varying stochastic systems with measurement outliers. Literature [9] assumed that the spatial distance between the filter and the sensor is relatively long, and the measured value may be lost in the transmission

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process from the sensor to the filter due to insufficient energy stored in the sensor. Under such a condition, the finite horizon filtering problem of a class of nonlinear time-delay systems with energy acquisition sensors is studied.

It is worth noting that the most current literature on the study of nonlinear system is not enough in-depth, only the existence of nonlinear term in the system is considered in the literature. And the parameters of the system is steady, the control problem is not considered when both the state equation and the output equation of the system have only nonlinear terms with time-varying parameters.

Networked control system (NCS) is a closed-loop control system formed by the real-time network of sensors, controllers, actuators and other system related components [10], [11]. In a network system, components transmit data over a communication network. In the process of data transmission, communication resources are often limited, which inevitably leads to communication constraints such as data conflicts, network congestion and channel competition, etc. Therefore, communication networks need to add communication protocols to allocate permissions for each node to use the network [12], [13]. In networked systems with multiple network nodes, the Weighted Try-Once-Discard (WTOD) protocol provides a good solution to the communication privilege allocation problem of each sensor node. Literature [14] first proposed WTOD protocol, and elaborated the scheduling principle and implementation method of WTOD protocol.

For the tracking control problem of nonlinear continuous time system, literature [15] gave the maximum allowable sampling interval of tracking control under the constraint of WTOD protocol. In literature [16], the influence of WTOD protocol was considered in the study of synovial control algorithm of automobile electronic valve system, and good stability effect was achieved.

Based on the above discussion, this paper investigates the finite horizon control method for networked time-varying nonlinear systems under the weighted try-once-discard communication protocol. Compared with the existing research, the main contributions of this paper are as follows: (1) Based on the Lyapunov stability theory, cross amplification lemma technology and recursive linear matrix inequality technique, an observer-based finite horizon controller is designed. (2) The influence of WTOD communication protocol is considered in the finite time domain  $H_\infty$  control problem for networked time-varying nonlinear systems. (3) The proposed controller can ensure both the stability and the prescribed  $H_\infty$  performance index of the closed-loop system over a given finite horizon.

## II. NONLINEAR CONTROL SYSTEM MODEL

Take the nonlinear time-varying system in finite horizon as the research object, and the controlled object is described as

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k) + \mathbf{f}(\mathbf{x}_k) + \mathbf{B}_k \mathbf{u}_k + \mathbf{D}_k \mathbf{w}_k \\ \mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{E}_k \mathbf{v}_k \\ \mathbf{z}_k = \mathbf{L}_k \mathbf{x}_k \end{cases} \quad (1)$$

where  $\mathbf{x}_k \in \mathbb{R}^{n_x}$ ,  $\mathbf{u}_k \in \mathbb{R}^{n_u}$ ,  $\mathbf{z}_k \in \mathbb{R}^{n_z}$  and  $\mathbf{y}_k \in \mathbb{R}^{n_y}$  respectively represent system state, control input, controlled output and measurement output vector without network transmission.  $\mathbf{w}_k \in \mathcal{K}l_{2[0,N]}$  and  $\mathbf{v}_k \in l_{2[0,N]}$  are the process noise and measurement interference of the system respectively. The matrices  $\mathbf{B}_k$ ,  $\mathbf{D}_k$ ,  $\mathbf{E}_k$  and  $\mathbf{L}_k$  are time-varying parameter matrices with known appropriate dimensions.  $\mathbf{g}(\mathbf{x}_k)$  and  $\mathbf{h}(\mathbf{x}_k)$  are known nonlinear functions which can be analyzable everywhere in the finite horizon  $k \in [0, N]$ .  $\mathbf{f}(\mathbf{x}_k)$  is the random nonlinear function and the first moment and covariance of the random nonlinear function  $\mathbf{f}(\mathbf{x}_k)$  satisfy

$$\mathbb{E}\{\mathbf{f}(\mathbf{x}_k) | \mathbf{x}_k\} = 0 \quad (2)$$

$$\mathbb{E}\{\mathbf{f}(\mathbf{x}_k) \mathbf{f}^T(\mathbf{x}_j) | \mathbf{x}_k\} = 0, \quad k \neq j \quad (3)$$

$$\mathbb{E}\{\mathbf{f}(\mathbf{x}_k) \mathbf{f}^T(\mathbf{x}_k) | \mathbf{x}_k\} = \sum_{n=1}^q \mathbf{K}_{n,k} \mathbf{K}_{n,k}^T \Gamma_{n,k} \mathbf{x}_k \quad (4)$$

where  $q$  is known as a non-negative integer,  $\ell_{n,k}$  and  $\Gamma_{n,k}$  ( $n = 1, 2, \dots, q$ ) are known as the appropriate dimensional matrix.

Consider the communication network between the sensor of the controlled object and the remote controller using a WTOD protocol. Dynamic scheduling protocol is a scheduling method in which network nodes obtain communication network access permissions according to some given rules. The Weighted try-once-Discard protocol is a kind of dynamic scheduling protocol which determines the node receiving

network communication permissions at instant  $k$  based on the difference in size between the data to be sent by each transfer node at time  $k$  and the data sent last time. The greater the difference is, the higher the transmission demand of this node is, and the sensor node with the highest transmission demand will have the priority to obtain the permission to use the communication network. The sensor node defined to obtain access to the network at the instant  $k$  of sampling is  $\xi_k \in \mathcal{S} @ \{1, 2, \dots, n_y\}$ , and the value of  $\xi_k$  is calculated by the following formula:

$$\xi_k = \arg \max_{i \in \mathcal{S}} \left( \mathbf{y}_{i,k} - \bar{\mathbf{y}}_{i,k-1} \right)^T \mathbf{Q}_i \left( \mathbf{y}_{i,k} - \bar{\mathbf{y}}_{i,k-1} \right) \quad (5)$$

where  $\bar{\mathbf{y}}_{i,k-1}$  refers to the data last sent by sensor node  $i$  before moment  $k$  (excluding  $k$ ), and  $\mathbf{Q}_i$  refers to the known positive determination matrix, which represents the weight matrix of transfer node  $i$  under Weighted try-once-discard protocol schedule.

Equation (5) can be further converted into

$$\xi_k = \arg \max_{i \in \mathcal{S}} \left( \mathbf{y}_k - \bar{\mathbf{y}}_{k-1} \right)^T \bar{\mathbf{Q}} \Xi_i \left( \mathbf{y}_k - \bar{\mathbf{y}}_{k-1} \right) \quad (6)$$

where, under the dispatching action of WTOD protocol, the measured output after transmission from the sensor to the controller network is  $\bar{\mathbf{y}}_k = \left[ \bar{\mathbf{y}}_{1,k}^T \bar{\mathbf{y}}_{2,k}^T \dots \bar{\mathbf{y}}_{n_y,k}^T \right]^T$ ,  $\bar{\mathbf{Q}} = \text{diag} \{ \mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_{n_y} \}$ ,  $\Xi_i = \text{diag} \{ \delta(i-1)\mathbf{I}, \delta(i-2)\mathbf{I}, \dots, \delta(i-n_y)\mathbf{I} \}$ , and  $\delta(\cdot) \in \{0, 1\}$  are Kronecker-delta functions. Consider equations (1) and (3), we can get:

$$\bar{\mathbf{y}}_k = \Xi_{\xi_k} \mathbf{y}_k + (\mathbf{I} - \Xi_{\xi_k}) \bar{\mathbf{y}}_{k-1} \quad (7)$$

*Remark 1:* In a networked control system with multiple sensor nodes, the WTOD protocol can solve the problem of distribution of communication authority of each sensor node. In the network where the WTOD protocol exists, sensor nodes obtain communication rights through ‘‘competition’’. The rules of this ‘‘competition’’ are based on the difference between the amount of data sent by the node at the previous moment and the amount of pre-sent data at the current moment. The greater the difference, the higher the transmission demand, so the sensor with the highest demand will have priority in obtaining the communication authority.

For system (1), an observer-based state feedback controller is designed in the following form:

$$\begin{cases} \hat{\mathbf{x}}_{k+1} = \mathbf{G}_k \hat{\mathbf{x}}_k + \mathbf{H}_k \bar{\mathbf{y}}_k + \mathbf{B}_k \mathbf{u}_k \\ \mathbf{u}(k) = \mathbf{K}_k \hat{\mathbf{x}}_k \end{cases} \quad (8)$$

where  $\hat{\mathbf{x}}_k \in \mathbb{R}^{n_x}$  is the state estimation of system (1),  $\hat{\mathbf{x}}_0 = 0$ ,  $\mathbf{G}_k \in \mathbb{R}^{n_x \times n_x}$ ,  $\mathbf{H}_k \in \mathbb{R}^{n_x \times n_y}$ ,  $\mathbf{K}_k \in \mathbb{R}^{n_u \times n_x}$  are the gain matrix of the observer and controller to be solved.

In the finite horizon  $k \in [0, N]$ , the nonlinear function  $\mathbf{g}(\mathbf{x}_k)$  and  $\mathbf{h}(\mathbf{x}_k)$  can be linearized according to the Taylor series expansion formula, and the Taylor expansion formula of  $\mathbf{g}(\mathbf{x}_k)$  and  $\mathbf{h}(\mathbf{x}_k)$  at the system state estimation  $\hat{\mathbf{x}}_k$  is:

$$\mathbf{g}(\mathbf{x}_k) = \mathbf{g}(\hat{\mathbf{x}}_k) + \Phi_k (\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{L}_1 \Delta_1 (\mathbf{x}_k - \hat{\mathbf{x}}_k) \quad (9)$$

$$h(x_k) = h(\hat{x}_k) + \Psi_k(x_k - \hat{x}_k) + L_2 \Delta_2(x_k - \hat{x}_k) \quad (10)$$

where  $\Phi_k = \left. \frac{\partial g(x)}{\partial x} \right|_{x=\hat{x}_k}$ ,  $\Psi_k = \left. \frac{\partial h(x)}{\partial x} \right|_{x=\hat{x}_k}$ ,  $L_1 \in \mathbb{R}^{n_x \times n_{l_1}}$  and  $L_2 \in \mathbb{R}^{n_y \times n_{l_2}}$  are known scaling matrices,  $\Delta_1 \in \mathbb{R}^{n_{l_1} \times n_x}$  and  $\Delta_2 \in \mathbb{R}^{n_{l_2} \times n_x}$  is unknown matrix, which satisfies  $\|\Delta_1\| \leq 1$ ,  $\|\Delta_2\| \leq 1$ .

The estimation error  $e_k = x_k - \hat{x}_k$  and the augmented vector  $\eta(k) = [x^T(k) \bar{y}_{k-1}^T e^T(k)]^T$  of the system are defined. Combined with equations (7)-(10), the closed-loop nonlinear system (1) can be transformed into the following closed-loop augmented networked time-varying system:

$$\begin{cases} \eta_{k+1} = A_k \eta_k + Ff(T\eta_k) + \bar{D}_k \bar{w}_k \\ z_k = \bar{L}_k \eta_k \end{cases} \quad (11)$$

where  $A_k$ ,  $F$ ,  $T$ ,  $\bar{D}_k$ ,  $\bar{w}_k$ , and  $\bar{L}_k$ , as shown at the bottom of the page.

The main research purpose of this paper is on the finite domain  $k \in [0, N]$ , design a kind of controller based on observer such as type (8), by solving the controller and observer parameter  $K_k$ ,  $G_k$ ,  $H_k$  for the pre-given disturbance suppression level  $\gamma$ , positive definite matrix  $S$  and initial state  $x_0$  of the system, the controlled output  $z_k$  of the system (11) satisfies the following  $H_\infty$  performance:

$$J @ E \left\{ \sum_{k=0}^N (\|z_k\|^2 - \gamma^2 \|\bar{w}_k\|^2) - \gamma^2 \eta_0^T S \eta_0 \right\} < 0, \quad \forall \bar{w}_k \neq 0 \quad (12)$$

*Remark 2:* Taylor expansion technique will unavoidably lead to conservatism, cause the higher order terms are ignored. Furthermore, the definition of finite horizon  $H_\infty$  performance index (12) of networked time-varying systems refers to literatures [17], [18]. The physical meaning of Equation (12) refers to that considering the influence of the initial state  $\eta_0$  of the system, on a given finite horizon  $[0, N]$ , the energy gain of external interference  $\bar{w}_k$  to the controlled output  $z_k$  of the system is less than the given disturbance suppression level  $\gamma$ .

### III. PERFORMANCE ANALYSIS OF NONLINEAR SYSTEM CONSTRAINED BY WTOD PROTOCOL

*Lemma 1:* For given the scalar  $\varepsilon > 0$ , the inequality (13) is always true for any matrix  $X$  and  $Y$  with appropriate dimensions.

$$\varepsilon XX^T + \varepsilon^{-1} Y^T Y \geq XY + Y^T X \quad (13)$$

*Proof:*

Known  $\left(\sqrt{\varepsilon} X^T - \frac{1}{\sqrt{\varepsilon}} Y\right)^T \left(\sqrt{\varepsilon} X^T - \frac{1}{\sqrt{\varepsilon}} Y\right) \geq 0$ , then  $\varepsilon XX^T - XY - Y^T X + \varepsilon^{-1} Y^T Y \geq 0$ , that is  $\varepsilon XX^T + \varepsilon^{-1} Y^T Y \geq XY + Y^T X$ , prove completion.

*Theorem 1:* For given  $H_\infty$  performance indexes  $\gamma > 0$  and positive definite matrix  $S$ , the gain matrix  $K_k$  of the controller and the gain matrix  $G_k$ ,  $H_k$  of the observer are known. If there is a symmetric positive definite matrix  $P_k$ , which satisfies matrix inequality (14) and (15), as shown at the bottom of the page, then the closed-loop augmented networked control system (11) satisfies  $H_\infty$  performance requirements.

*Proof:* Construct Lyapunov function as:

$$V_k = \eta_k P_k \eta_k \quad (16)$$

According to equations (11) and (2)-(4), the difference of the preceding term for can be obtained

$$\begin{aligned} & E \{V_{k+1} - V_k\} \\ &= E \{ \eta_{k+1} P_{k+1} \eta_{k+1} - \eta_k P_k \eta_k \} \\ &= E \left\{ (A_k \eta_k + Ff(T\eta_k) + \bar{D}_k \bar{w}_k)^T P_{k+1} \right. \\ &\quad \times (A_k \eta_k + Ff(T\eta_k) + \bar{D}_k \bar{w}_k) - \eta_k P_k \eta_k \left. \right\} \\ &= E \left\{ \eta_k^T A_k^T P_{k+1} A_k \eta_k + 2 \eta_k^T A_k^T P_{k+1} \bar{D}_k \bar{w}_k \right. \\ &\quad + f^T(T\eta_k) F^T P_{k+1} F f(T\eta_k) \\ &\quad \left. + \bar{w}_k^T \bar{D}_k^T P_{k+1} \bar{D}_k \bar{w}_k - \eta_k P_k \eta_k \right\} \\ &= E \left\{ \eta_k^T A_k^T P_{k+1} A_k \eta_k + 2 \eta_k^T A_k^T P_{k+1} \bar{D}_k \bar{w}_k \right. \end{aligned}$$

$$A_k = \begin{bmatrix} g(\hat{x}_k) + B_k K_k \hat{x}_k & 0 & \Phi_k + L_1 \Delta_1 \\ \Xi_{\xi_k} h(\hat{x}_k) & I - \Xi_{\xi_k} & \Xi_{\xi_k} \Psi_k + \Xi_{\xi_k} L_2 \Delta_2 \\ g(\hat{x}_k) - G_k \hat{x}_k - H_k \Xi_{\xi_k} h(\hat{x}_k) & -H_k (I - \Xi_{\xi_k}) & \Phi_k + L_1 \Delta_1 - H_k \Xi_{\xi_k} \Psi_k - H_k \Xi_{\xi_k} L_2 \Delta_2 \end{bmatrix},$$

$$F = [I \ 0 \ I]^T, \quad T = [I \ 0 \ 0],$$

$$\bar{D}_k = \begin{bmatrix} D_k & 0 \\ 0 & \Xi_{\xi_k} E_k \\ D_k & -H_k \Xi_{\xi_k} E_k \end{bmatrix}, \quad \bar{w}_k = [w_k^T \ v_k^T]^T,$$

$$\bar{L}_k = [L_k \ 0 \ 0].$$

$$\Omega_k = \begin{bmatrix} A_k^T P_{k+1} A_k + \bar{L}_k^T \bar{L}_k + \sum_{n=1}^q T^T \Gamma_{n,k} T \ell_{n,k}^T F^T P_{k+1} F \ell_{n,k} \eta_k - P_k & A_k^T P_{k+1} \bar{D}_k \\ \bar{D}_k^T P_{k+1} A_k & \bar{D}_k^T P_{k+1} \bar{D}_k - \gamma^2 I \end{bmatrix} \leq 0 \quad (14)$$

$$P_0 - \gamma^2 S \leq 0 \quad (15)$$

$$\begin{aligned}
 & + \sum_{n=1}^q \eta_k^T T^T \Gamma_{n,k} T \ell_{n,k}^T F^T P_{k+1} F \ell_{n,k} \eta_k \\
 & + \bar{w}_k^T \bar{D}_k^T P_{k+1} \bar{D}_k \bar{w}_k - \eta_k P_k \eta_k \} \quad (17)
 \end{aligned}$$

Add zero term  $\|z_k\|^2 - \gamma^2 \|\bar{w}_k\|^2 + \gamma^2 \|\bar{w}_k\| - \|z_k\|^2$  to the right side of equation (17) and define  $\bar{\eta}_k = [\eta_k^T \bar{w}_k^T]^T$ , we can get:

$$\Delta V_k = E \{V_{k+1} - V_k\} E @ \left\{ \bar{\eta}_k^T \Omega_k \bar{\eta}_k - \|z_k\|^2 + \gamma^2 \|\bar{w}_k\| \right\} \quad (18)$$

When  $\bar{w}_k = 0$ , if matrix inequality (14) is true, then  $\Delta V_k < 0$ ; For any non-zero external disturbance  $\bar{w}_k \in l_2[0, \infty)$ , if matrix inequality (14) is true, then

$$\Delta V_k = E \{V_{k+1} - V_k\} \leq E \left\{ \gamma^2 \|\bar{w}_k\|^2 - \|z_k\|^2 \right\} \quad (19)$$

Add the sum from 0 to N on both sides of inequality (19), and you can get

$$E \{V_{N+1} - V_0\} \leq E \left\{ \sum_{k=0}^N \left( \gamma^2 \|\bar{w}_k\|^2 - \|z_k\|^2 \right) \right\} \quad (20)$$

Due to  $V_{N+1} > 0$ , if matrix inequality (15) was set up, then

$$J \leq V_0 - V_{N+1} - \gamma^2 \eta_0^T S \eta_0 \leq V_0 - \gamma^2 \eta_0^T S \eta_0 \leq 0 \quad (21)$$

Therefore, the system (11) meets the required  $H_\infty$  performance. The proof is completed.

#### IV. DESIGN OF FINITE HORIZON $H_\infty$ CONTROLLER

*Theorem 2:* For given  $H_\infty$  performance index  $\gamma > 0$ , positive definite matrix  $S$ . If there is scalar  $-\vartheta_{n,k} > 0$ ,  $\varepsilon_{1,k} > 0$ ,  $\varepsilon_{2,k} > 0$  and the symmetric positive definite matrix  $P_k$  satisfies matrix inequalities (15), (22) and (23), then the closed-loop augmented networked control system (11) meets the  $H_\infty$

performance requirements.

$$\begin{bmatrix} -\vartheta_{n,k} & \ell_{n,k}^T T^T \\ T \ell_{n,k} & -P_{k+1}^{-1} \end{bmatrix} \leq 0 \quad (22)$$

$$\begin{bmatrix} \tilde{\Omega}_k & \tilde{L} \\ \tilde{L}^T & -J \end{bmatrix} \leq 0 \quad (23)$$

where  $\tilde{\Omega}_k, \bar{\tilde{\Omega}}_k, \bar{\tilde{\Omega}}_k, \bar{A}_k, \bar{A}_k, \tilde{L}, \tilde{L}_1, \tilde{L}_2, \tilde{I}$ , and  $J$ , as shown at the bottom of the page.

*Proof:* according to Schur complement lemma, matrix inequality (22) is equivalent to

$$K_{n,k}^T F^T P_{k+1} F K_{n,k} \leq \vartheta_{n,k} \quad (24)$$

Then by using the property of matrix trace, we can get

$$tr \left[ T K_{n,k} K_{n,k}^T T^T P_{k+1} \right] \leq \vartheta_{n,k} \quad (25)$$

Similarly, according to Schur's complement lemma, matrix inequality (23) is equivalent to equation (26).

$$\begin{bmatrix} \bar{\tilde{\Omega}}_k & \bar{A}_k^{-T} \\ \bar{A}_k & -P_{k+1}^{-1} \end{bmatrix} + \tilde{L} J^{-1} \tilde{L}^T \leq 0 \quad (26)$$

Define  $A_k = [A_k \bar{D}_k]$ , it can be obtained from the matrix inequality (14) that

$$\Omega_k = \bar{\tilde{\Omega}}_k + A_k^T P_{k+1} A_k \leq 0 \quad (27)$$

According to Schur complement lemma, equation (27) is equivalent to

$$\begin{bmatrix} \bar{\tilde{\Omega}}_k & A_k^T \\ A_k & -P_{k+1}^{-1} \end{bmatrix} \leq 0 \quad (28)$$

$$\begin{aligned}
 \tilde{\Omega}_k &= \begin{bmatrix} \bar{\tilde{\Omega}}_k & \bar{A}_k^{-T} \\ \bar{A}_k & -P_{k+1}^{-1} \end{bmatrix}, \\
 \bar{\tilde{\Omega}}_k &= \begin{bmatrix} \bar{L}_k^T \bar{L}_k + \sum_{n=1}^q T^T \Gamma_{n,k} T \vartheta_{n,k} - P_k & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \\
 \bar{\tilde{\Omega}}_k &= \begin{bmatrix} \bar{L}_k^T \bar{L}_k + \sum_{n=1}^q T^T \Gamma_{n,k} T \ell_{n,k}^T F^T P_{k+1} F \ell_{n,k} - P_k & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \\
 \bar{A}_k &= [\bar{A}_k \quad \bar{D}_k], \\
 \bar{A}_k &= \begin{bmatrix} g(\hat{x}_k) + B_k K_k \hat{x}_k & 0 & \Phi_k \\ \Xi_{\xi_k} h(\hat{x}_k) & I - \Xi_{\xi_k} & \Xi_{\xi_k} \Psi_k \\ g(\hat{x}_k) - G_k \hat{x}_k - H_k \Xi_{\xi_k} h(\hat{x}_k) & -H_k (I - \Xi_{\xi_k}) & \Phi_k - H_k \Xi_{\xi_k} \Psi_k \end{bmatrix}, \\
 \tilde{L} &= [\varepsilon_{1,k} \tilde{L}_1 \quad \varepsilon_{2,k} \tilde{L}_2 \quad \tilde{I}^T \quad \tilde{I}^T], \\
 \tilde{L}_1 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad L_1^T \quad 0 \quad L_1^T]^T, \\
 \tilde{L}_2 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad L_2^T \Xi_{\xi_k} \quad -L_2^T \Xi_{\xi_k} H_k^T]^T, \\
 \tilde{I} &= [0 \quad 0 \quad 0 \quad I \quad 0 \quad 0 \quad 0 \quad 0], \\
 J &= diag \{ \varepsilon_{1,k} I, \varepsilon_{2,k} I, \varepsilon_{1,k} I, \varepsilon_{2,k} I \}.
 \end{aligned}$$

Further converting equation (28) can be obtained

$$\begin{aligned} & \begin{bmatrix} \bar{\Omega}_k & \mathbf{A}_k^T \\ \mathbf{A}_k & -\mathbf{P}_{k+1}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\Omega}_k & \mathbf{A}_k^{-T} \\ \mathbf{A}_k & -\mathbf{P}_{k+1}^{-1} \end{bmatrix} \\ &+ [0 \ 0 \ 0 \ 0 \ 0 \ \Delta_1^T \mathbf{L}_1^T \ 0 \ \Delta_1^T \mathbf{L}_1^T]^T \tilde{\mathbf{I}} \\ &+ [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T]^T \tilde{\mathbf{I}} \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ \Delta_1^T \mathbf{L}_1^T \ 0 \ \Delta_1^T \mathbf{L}_1^T] \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T] \leq 0 \end{aligned} \quad (29)$$

According to lemma 1,

$$\begin{aligned} & [0 \ 0 \ 0 \ 0 \ 0 \ \Delta_1^T \mathbf{L}_1^T \ 0 \ \Delta_1^T \mathbf{L}_1^T]^T \tilde{\mathbf{I}} \\ &+ [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T]^T \tilde{\mathbf{I}} \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ \Delta_1^T \mathbf{L}_1^T \ 0 \ \Delta_1^T \mathbf{L}_1^T] \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T] \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T] \\ &\leq \varepsilon_{1,k} [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T]^T \\ &\quad \times [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T] \\ &+ \varepsilon_{2,k} [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T]^T \\ &\quad \times [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T] \\ &+ \varepsilon_{1,k}^{-1} \tilde{\mathbf{I}}^T \tilde{\mathbf{I}} + \varepsilon_{2,k}^{-1} \tilde{\mathbf{I}}^T \tilde{\mathbf{I}} \end{aligned} \quad (30)$$

For given  $\|\Delta_1\| \leq 1$  and  $\|\Delta_2\| \leq 1$ , it is not difficult to figure out that (31), as shown at the bottom of the page.

Combined with matrix inequality (25), equation (29) and equation (31), it can be obtained

$$\begin{bmatrix} \bar{\Omega}_k & \mathbf{A}_k^T \\ \mathbf{A}_k & -\mathbf{P}_{k+1}^{-1} \end{bmatrix} \leq \begin{bmatrix} \bar{\Omega}_k & \mathbf{A}^{-T} \\ \mathbf{A}_k & -\mathbf{P}_{k+1}^{-1} \end{bmatrix} + \tilde{\mathbf{L}}\mathbf{J}^{-1}\tilde{\mathbf{L}}^T \quad (32)$$

Further, it is not difficult to conclude that if  $\begin{bmatrix} \bar{\Omega}_k & \tilde{\mathbf{L}} \\ \tilde{\mathbf{L}}^T & -\mathbf{J} \end{bmatrix} \leq 0$ ,  $\Omega_k \leq 0$  is true, that is, theorem 2 is true to ensure that theorem 1 is true, to ensure that the closed-loop augmented system (11) meets the  $H_\infty$  performance requirements. According

to Theorem 2, the observer gain matrix  $\mathbf{K}_k$  and controller gain matrices  $\mathbf{G}_k$  and  $\mathbf{H}_k$  can be obtained by solving the recursive time-varying matrix inequalities (22)-(23) using LMI toolbox. The proof is completed.

### V. THE SIMULATION RESULTS

In order to verify the effectiveness of the method proposed in this paper, the non-stationary growth system model proposed in literature [7] and literature [19] are taken into account, and its specific parameters are as follows:

$$\begin{cases} \mathbf{x}_{k+1} = 0.5\mathbf{x}_k + 25 \frac{\mathbf{x}_k}{1 + \mathbf{x}_k^2} + 0.06\text{sign}(\mathbf{x}_k) \mathbf{x}_k \Gamma_k \\ \quad + \mathbf{B}_k \mathbf{u}_k + 8 \cos(1.2(k+1)) + 0.5e^{-0.2k} \\ \mathbf{y}_{1,k} = \frac{\mathbf{x}_k^2}{20} + 5 \frac{\cos(2k)}{k+1} \\ \mathbf{y}_{2,k} = \frac{e^{-0.2k} \mathbf{x}_k^2}{10} + 2 \frac{\sin(2k)}{1+k} \\ \mathbf{z}_k = 0.2\mathbf{x}_k \end{cases} \quad (33)$$

where  $\mathbf{B}_k = 0.1$ ,  $\mathbf{f}(\mathbf{x}_k) = 0.06\text{sign}(\mathbf{x}_k) \mathbf{x}_k \Gamma_k$ ,  $\Gamma_k$  is gaussian white noise matrix. It's not difficult to figure out

$$\begin{aligned} & E\{\mathbf{f}(\mathbf{x}_k) | \mathbf{x}_k\} = 0, \\ & E\{\mathbf{f}(\mathbf{x}_k) \mathbf{f}^T(\mathbf{x}_k) | \mathbf{x}_k\} = 0.0036 \mathbf{x}_k^T \mathbf{x}_k. \end{aligned}$$

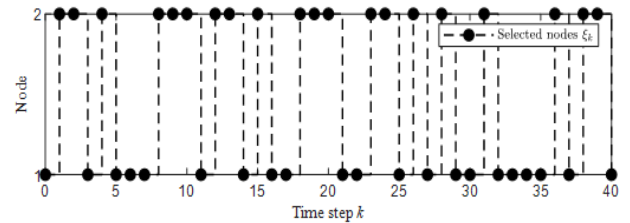


FIGURE 1. Changes of the selected sensor nodes.

Given the initial state of the system  $\mathbf{x}_0 = 0$ ,  $\mathbf{L}_1$  and  $\mathbf{L}_2$  are set as  $0.05\mathbf{I}$ ,  $\mathbf{Q}_1 = 0.92$ ,  $\mathbf{Q}_2 = 1.15$ . Given finite horizon  $k \in [0, 40]$ . On the premise that the proposed method has a solution and the  $H_\infty$  performance index is as small as possible,  $\gamma = 2.25$  and positive definite matrix  $\mathbf{S} = 0.5$ ,  $\mathbf{P}_0 = 0.5\mathbf{I}$  are set. According to Theorem 2, recursive time-varying matrix inequality (22) and equation (23) can

$$\begin{aligned} & [0 \ 0 \ 0 \ 0 \ 0 \ \Delta_1^T \mathbf{L}_1^T \ 0 \ \Delta_1^T \mathbf{L}_1^T]^T \tilde{\mathbf{I}} \\ &+ [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T]^T \tilde{\mathbf{I}} \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ \Delta_1^T \mathbf{L}_1^T \ 0 \ \Delta_1^T \mathbf{L}_1^T] \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T] \\ &+ \tilde{\mathbf{I}}^T [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \ -\Delta_2^T \mathbf{L}_2^T \Xi_{\xi_k} \mathbf{H}_k^T] \\ &\leq \varepsilon_{1,k} \tilde{\mathbf{L}}_1 \tilde{\mathbf{L}}_1^T + \varepsilon_{2,k} \tilde{\mathbf{L}}_2 \tilde{\mathbf{L}}_2^T + \varepsilon_{1,k} \tilde{\mathbf{I}}^T \tilde{\mathbf{I}} + \varepsilon_{2,k} \tilde{\mathbf{I}}^T \tilde{\mathbf{I}} \\ &= \tilde{\mathbf{L}}\mathbf{J}^{-1}\tilde{\mathbf{L}}^T \end{aligned} \quad (31)$$

be solved by LMI toolbox. The corresponding simulation results of the system are shown in Figure 1, Figure 2 and Figure 3. Figure 1 depicts the changes of sensor nodes that obtain network permissions under the influence of W TOD communication protocol. FIG. 2 and FIG. 3 show the controlled output images of the open-loop system and closed-loop system respectively. It can be seen from Figure 3 that, on a given finite horizon AA, the controlled output curve of the closed-loop system oscillates and converges. It can be calculated from Figure 3 that when there is disturbance, the  $H_\infty$  performance index formula (12) in the finite horizon is established. The simulation results demonstrate the effectiveness of the finite horizon  $H_\infty$  controller design method.

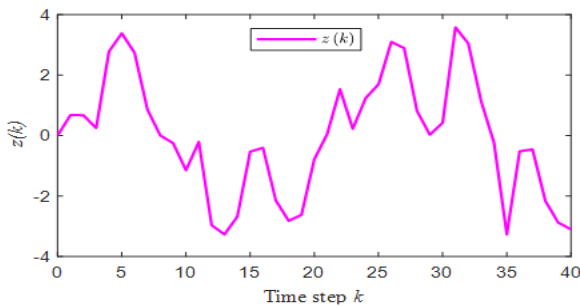


FIGURE 2. The state trajectories of the open-loop system.

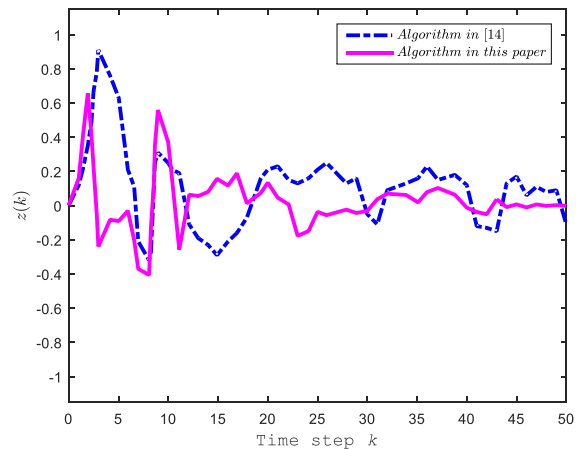


FIGURE 3. The state trajectories of the closed-loop system.

## VI. CONCLUSION

In this paper, under the influence of W TOD protocol, the finite horizon  $H_\infty$  control problem with both nonlinear and random nonlinear time-varying systems that can be analyzed everywhere in the state equation and the output equation is studied. First, a nonlinear time-varying system model is constructed. Then, in the sensor-controller transmission network, the W TOD communication protocol is added to avoid data conflicts. Then, the nonlinear terms determined by Taylor expansion technique are processed, the stochastic nonlinear terms are processed by statistical method, and the

sufficient conditions for the system to meet the  $H_\infty$  performance requirements are obtained by Lyapunov stability theory and cross amplification lemma. On this basis, the controller design algorithm based on observer is given. Finally, simulation shows the effectiveness of the proposed method. In addition, extending the proposed finite horizon  $H_\infty$  control method to nonlinear systems under Round-Robin protocol and Random Access protocol is a problem that requires further research.

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