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# $H\infty$ Control Observer-Based for Discrete-Time Singularly Perturbed Systems With Nonlinear Disturbances

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**ABSTRACT** This paper focuses on the $H\infty$ observer-based control problem for discrete-time singularly perturbed systems (DTSPSs) with nonlinear disturbances. The main contributions of this paper include three aspects: First, a proper observer is constructed. A sufficient condition in terms of linear matrix inequality (LMI) and Lyapunov function is proposed such that the resulting observer error system is asymptotically stable with a prescribed $H_\infty$ norm bound for sufficiently small values of the perturbation parameter. Second, based on the input-to-state stability (ISS) property, an observer-based feedback controller is designed such that the resulting closed-loop system is ISS with respect to the observer error. Meanwhile, the $H\infty$ performance index is also satisfied. Then, a workable way for solving the exact upper bound is also given. Finally, two numerical examples are given to demonstrate the validity of the developed method.

**INDEX TERMS** Discrete-time singularly perturbed system, observer, state feedback, H∞ control, linear matrix inequality (LMI).

## I. INTRODUCTION

Singularly perturbed systems, as a special class of systems with small parameters, are widely used in the engineering application, network control and complex system, see [1]–[6]. In addition, among various methods developed to control this kind of systems, the linear matrix inequality (LMI) technique have been extensively adopted, which can effectively avoid the computational difficulty when solving state equations of systems. Meanwhile, many remarkable results have been obtained [7]–[13]. For example, the robust stability of singularly perturbed systems is investigated based on fixed-point principle and LMI in [11]. Recently, because of the development of computer technology, a new challenging task is aroused, that is, the control problem of DTSPSs has attracted more and more attention, and many important results have been obtained [14]–[19]. In terms of the dynamic output feedback controller, fast and slow subsystems were discussed separately based on reduced technique and LMI in [17]. Reference [19] investigates the state estimation problem for a class of discrete-time singularly perturbed systems with distributed time-delays, in which the ultimate bound of the error dynamics is estimated. With the progress network communication technology, networked control systems have entered a stage of rapid development and many methods have been developed [20]–[24]. It is shown in [20] that new stability criteria and stabilization methods on networked control systems with short time-varying delay are proposed, in which the conservatism of the stability condition caused by short time-varying delay can be reduced. Reference [24] develops a new metric to measure the significance of a network's community structure, the result shows that the proposed method can yield good performance in terms of accuracy as well as stability. Recently, the relevant results have been extended to singularly perturbed system. For example, the sliding mode control issue of the networked singularly perturbed systems under slow sampling is considered in [25], a novel sliding function is constructed the sufficient conditions are derived to ensure the asymptotic stability of the sliding mode dynamics. The details on the recent development of the sliding mode control can be found from the survey paper [26] and the references therein. Moreover,

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as Markovian jump systems can give better descriptions of practical systems with stochastic abrupt structural changes, the relevant results have also been extended to DTSPSs [27]. However, the research of DTSPSs is not comprehensive, especially in the presence of nonlinear disturbance. To make up for this deficiency, more efforts need to be paid on this topic.

In past years, the $H\infty$ control has been an important research area in singularly perturbed systems and widely applied in network control, satellite etc., [28]–[35]. In particular, the LMI technique has also been proposed to solve the $H\infty$ problem for different kinds of DTSPSs, which effectively eliminates the regularity restrictions attached to the Riccati-based solutions. It is shown in [36] and [37] that several inequalities are used in the derivation of the solvability conditions, which would lead to some conservatism. For this, Xu and Feng in [38] present a new sufficient condition to make the closed-loop system asymptotically stable with a prescribed $H\infty$ performance. The result is proven to be less conservative. In [31], the $H\infty$ control problem for a class of nonlinear DTSPSs is addressed, the sufficient conditions for asymptotic stability of the closed-loop system are given via LMI and Lyapunov function.

However, most of the aforementioned results are based on the assumption that the state variables of systems are available for direct measurements. The fact is that, in many control systems and applications, not all the state variables can be measured or we may choose not to measure some of them due to technical or economic reasons. In this case, it is necessary to design a state observer or filter used to reconstruct the states of a dynamic system. Over the past years, various approaches of the observer design for different control systems have been proposed [40]–[46]. For example, a continuous-time nonlinear system with input and output quantization is discussed in [42], the stability and $H\infty$ performance index are guaranteed by constructing an observer-based output feedback controller. A proper observer-based feedback controller is also designed in [43], and a criterion is revealed to guarantee that the Lur'e singularly perturbed system is absolutely stable. However, it is noticed that there are few works on the observer design for DTSPSs. It is known that applying the routine design methods for normal systems to singularly perturbed systems usually leads to ill-conditioned numerical problems. Therefore, many difficult and efficient observer design issues still need to be addressed. This will be one of our main concerns.

In spired by the above results, the problem of the $H\infty$ observer-based control for DTSPSs with nonlinear disturbances is studied in this paper. The purpose of the paper is to reconstruct the system state, such that the controlled system can obtain the desired property. First, a proper Luenberger-like full-order observer is constructed, then a sufficient condition expressed in terms of LMI is derived such that the resulting observer error system is asymptotically stable with the prescribed disturbance attenuation level $\gamma$. Then for the observer-based $H\infty$ control, based on input-to-state stability (ISS) property, an appropriate observer-based state feedback control law is designed to guarantee the ISS for the resulting closed-loop system. Meantime, the $H\infty$ performance index is also satisfied. Moreover, the upper bound of the small parameter and the minimum of $H\infty$ performance index $\gamma$ can be obtained by a feasible approach.

Compared with the existing literature [36]–[38], [41], [42], the advantages of this paper are roughly summarized as follows: 1) a more general class of systems is addressed, in which the external disturbance of the system has not only the linear part, but also the nonlinear part. 2) Not only the minimum of $H\infty$ performance index $\gamma$ and the upper bound of perturbation parameter in the closed loop system are derived, but also the minimum of disturbance attenuation level $\gamma$ and the upper bound of the small parameter in the error system are given. 3) the controller law can be obtained easily by solving LMIs, in which there is no equality constraint in [12] involved when using our approach.

*Notation:* The symbol 'T' that appears throughout the article stands for matrix transposition; matrix inequality $P > 0$ indicates that $P$ is a positive definite matrix; Symbol $||\cdot||$ represents the Euclidean vector norm or the induced Euclidean matrix norm; the element '$*$'under the main diagonal of the symmetric matrix stands for an ellipsis for terms that are induced by symmetry.

## II. PROBLEM FORMULATION

Consider the following DTSPSs with nonlinear perturbation described by

$$x(k+1) = A_\varepsilon x(k) + H_\varepsilon f(x(k)) + B_{u\varepsilon} u(k) + B_{w\varepsilon} w(k), \quad (1)$$

$$y(k) = Cx(k), \quad (2)$$

where $x = (x_1^T, x_2^T)^T \in R^n$ is the system state with slow state $x_1 \in R^{n_1}$ and fast state $x_2 \in R^{n_2}$ $(n_1 + n_2 = n)$; $x(0) = x_0$ is the initial condition; $u \in R^q$ is the control input; $w \in R^p$ is the disturbance input; $y \in R^m$ is the system output; $\varepsilon > 0$ is a singularly perturbation parameter which is small and positive but may be unknown; $f(x)$ is a vector-valued nonlinear function with $f(0) = 0$, which is assumed to satisfy the following Lipschitz condition for all $x, \tilde{x} \in R^n$:

$$\|f(x) - f(\tilde{x})\| \le \|F(x - \tilde{x})\|, \quad (3)$$

where $F$ is a known Lipschitz constant matrix with appropriate dimensions; $A_\varepsilon, H_\varepsilon, B_{u\varepsilon}, B_{w\varepsilon}, C$ are known and satisfy the following definition:

$$A_\varepsilon = E_0 + E_\varepsilon A, \quad B_{u\varepsilon} = E_\varepsilon B_u,\ B_{w\varepsilon} = E_\varepsilon B_w,\ H_\varepsilon = E_\varepsilon H,$$

where

$$E_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad E_\varepsilon = \begin{pmatrix} \varepsilon I & 0 \\ 0 & I \end{pmatrix},$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B_u = \begin{pmatrix} B_{u1} \\ B_{u2} \end{pmatrix},$$

$$B_w = \begin{pmatrix} B_{w1} \\ B_{w2} \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix},$$
$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}.$$

*Remark 1:* The structure in the form of (3) has been widely considered [11], [12], [15], [23], at least since the classical paper by Kalman and Bertram. Subsequently, there has been a long chain of papers dealing with this problem. The system (1)-(2) can be used to represent many important physical systems, such as power networks, transportation, aerospace, water resources. It is worth mentioning that the matched condition can be regarded as a special case of (3). For example, consider the following linear uncertain discrete-time singularly perturbed system

$$\begin{cases} x_1(k+1) = (I + \varepsilon(A_{11} + \Delta A_{11}))x_1(k) \\ \qquad\qquad + \varepsilon(A_{12} + \Delta A_{12})x_2(k), \\ x_2(k+1) = (A_{21} + \Delta A_{21})x_1(k) \\ \qquad\qquad + (A_{22} + \Delta A_{22})x_2(k), \end{cases}$$

where the uncertain matrices $\Delta A_{11}$, $\Delta A_{12}$, $\Delta A_{21}$ and $\Delta A_{22}$ satisfy the matched condition

$$\begin{pmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \Delta(k) \begin{pmatrix} E_1 & E_2 \end{pmatrix},$$

and $\Delta(t)$ is a time-varying uncertainty with appropriate dimension satisfying

$$\Delta^T(k)\Delta(k) \leq I.$$

Define

$$\Delta A = \begin{pmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & E_2 \end{pmatrix}.$$

Then the system can be rewritten in a compact form

$$x(k+1) = (A_\varepsilon + \Delta A_\varepsilon)x(k),$$

where $\Delta A_\varepsilon = E_\varepsilon \Delta A$. Let $\varphi(k, x) = \Delta(k)Ex$, then it is easy to find that $\varphi(k, x)$ satisfy the constraint (3), which implies that the matched condition is a special case of this paper.

*Remark 2:* A widely used constraint on $f(x)$ is of the form

$$\|f(x) - f(\tilde{x})\| \leq l\|x - \tilde{x}\|,$$

in which all components of $f(x)$ are weighted equally. In this paper, we assume instead that $f(x)$ satisfy the condition (3), which is capable of describing the structure of the uncertain term more accurately in many practical situation, because each component of $f(x)$ is weighted differently.

*Assumption*: Matrix $B_u$ is of full column remark.

In this paper, a Luenberger-like full-order observer is considered with the following form:

$$\hat{x}(k+1) = A_\varepsilon\hat{x}(k) + B_{u\varepsilon}u(k) + L\left(y - C\hat{x}\right), \quad (4)$$
$$\hat{y}(k) = C\hat{x}(k), \quad (5)$$

where $\hat{x} = (\hat{x}_1^T, \hat{x}_2^T)^T \in R^n$ is the reconstructed state of system state $x$. $\hat{y}$ is the observer output. $L = \begin{pmatrix} L_1^T & L_2^T \end{pmatrix}^T$ is an observer gain matrix, which needs to be solved later.

The observer error system between reconstructed state $\hat{x}$ and original system $x$ can be obtained. Next, for simplicity, let $e(k) = x(k) - \hat{x}(k)$, $z(k) = y(k) - \hat{y}(k)$, where $e$ is the observer error, $z$ is output of the observer error system. Then the observer error system has the following form:

$$e(k+1) = (A_\varepsilon - LC)e(k) + H_\varepsilon f(x) + B_{w\varepsilon}w(k), \quad (6)$$
$$z(k) = Ce(k). \quad (7)$$

An observer-based feedback controller is designed as follows:

$$u(k) = -Kx(k), \quad (8)$$

where $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ is the control gain matrix.

*Remark 3:* The values of unknown and non-unique variable matrices $K$ and $L$ are related to the coefficient matrices and the constructed Lyapunov function, the complex relationship between them can be obtained in a later calculation. The core of our work is to find a proper control gain matrix $K$ and an observer gain matrix $L$ by using some methods and skill such that the closed-loop system is ISS and the $H\infty$ performance can be satisfied.

For the given DTSPSs with nonlinear disturbance (1)-(2), the $H\infty$ control problem can be summarized as follows: a proper Luenberger-like full-order observer is designed such that the observer error system is asymptotically stable with sufficient small disturbance attenuation level $\gamma$. In other words, within the allowed error range, the original state $x$ is replaced by the reconstructed state $\hat{x}$. Then the state feedback controller $u(k) = -K\hat{x}(k)$ is constructed to guarantee the ISS of the closed-loop system and the achievement of $H\infty$ performance index. The design criteria of state observer and observer-based feedback controller require that the error system and the closed-loop system satisfy the following requirements:

1) The systems are asymptotically stable or ISS when $w(k) = 0$.
2) The output $z(k)$ of system satisfies

$$\sum_{k=0}^{\infty} ||z(k)|| < \gamma^2 \sum_{k=0}^{\infty} ||w(k)|| \quad (9)$$

for all nonzero vector $w(k) \in L_2[0, \infty]$.

Some basic definitions and lemmas are given before further analysis and discussion.

*Definition 1 [47]:* Consider the system

$$x(k+1) = f(x(k), u(k)), \quad (10)$$

where $x \in R^n$ is the system state, the input $u \in R^q$ is a piecewise continuous, bounded function, $f : R^n \times R^m \to R^n$ is continuous and locally Lipschitz in $x$ and $u$. The system (10) is said to be input-to-state stable (ISS), if there exist a class $\mathcal{KL}$ function $\beta$ and a class $\mathcal{K}$ function $\gamma$ such that the system

$x$ satisfy the following inequalities for any initial conditions $\xi$ and any $k \geq 0$:

$$\|x(k)\| \leq \beta(\|\xi\|, k) + \gamma \left( \sup_{0 \leq \tau \leq k} \|u(k)\| \right),$$

*Lemma 1 [47]: Let* $V : R^n \to R$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$V(x(k+1)) - V(x(k)) \leq -W(x(k)),$$
$$\forall\ \|x\| \geq \rho(\|u\|) > 0,$$

where $\alpha_1, \alpha_2$ are class $\mathcal{K}_\infty$ functions, $\rho$ is a class $\mathcal{K}$ function, and $W(x)$ is a continuous positive definite functions on $R^n$. Then system (10) is ISS.

*Lemma 2: [48] (Schur's Complement)* Let $S$ is a partitioned square matrix as follows:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix},$$

where $S_{11}$ and $S_{22}$ are symmetric matrices. Then, the following there statements are equivalent:

1) $S < 0$;
2) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;
3) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

## III. STABILITY ANALYSIS OF OBSERVER ERROR

In this subsection, the sufficient condition of the asymptotic stability for the error system with the Luenberger-like full-order observer is investigated. Meanwhile, the method how to determine the minimum of the disturbance attenuation level $\gamma$ is also given.

*Theorem 1:* There exist a scalar $\tilde{\varepsilon} > 0$ such that the observer error system (6)-(7) is asymptotically stable with disturbance attenuation level $\gamma$ for all $\varepsilon \in (0, \tilde{\varepsilon}]$, if there exist matrices $Q_{11} > 0$, $Q_{22} > 0$, a matrix $Y$, and a positive scalar $\mu_1$ satisfying the following condition, (23), as shown at the bottom of the next page,

$$\Phi = \begin{pmatrix} \Theta_{11} & 0 & 0 & \Theta_{14} & Q_1^T C^T & Q_1^T F^T \\ * & -\mu_1^{-1} I & 0 & \mu_1^{-1} \bar{H}^T & 0 & 0 \\ * & * & -\gamma^2 I & \bar{B}_w^T & 0 & 0 \\ * & * & * & -Q_1 & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\mu_1^{-1} I \end{pmatrix} < 0, \quad (11)$$

where

$$\Theta_{11} = \left(Y\bar{A}_1\right)^T C + C^T\left(Y\bar{A}_1\right) - A_1^T Q_1 A_1 - Q_2,$$
$$\Theta_{14} = Q_1 A^T - C^T Y, \quad Q_1 = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix},$$
$$Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix},$$
$$\bar{H} = \begin{pmatrix} 0 & 0 \\ 0 & H_2 \end{pmatrix}, \quad \bar{B}_w = \begin{pmatrix} 0 \\ B_{w2} \end{pmatrix},$$
$$A = \begin{pmatrix} A_{11} & A_{22} \\ 0 & 0 \end{pmatrix}$$
$$\bar{A}_1 = A_1 - E_0 \begin{pmatrix} A_{11} - I & A_{12} \\ 0 & 0 \end{pmatrix}.$$

The observer gain matrix can be chosen as $L = Q_1^{-T} Y^T$.

*Proof:* The following inequality is obtained by substituting $L$ into (11) and using the Schur's Complement. Obviously, the inequality (11) is equivalent to (12).

$$\begin{pmatrix} \bar{\Theta}_{11} & 0 & 0 & Q_1(A - LC)^T \\ * & -\mu_1^{-1} I & 0 & \mu_1^{-1} \bar{H}^T \\ * & * & -\gamma^2 I & \bar{B}_w^T \\ * & * & * & -Q_1 \end{pmatrix} < 0, \quad (12)$$

where

$$\bar{\Theta}_{11} = \bar{A}_1^T Q_1^T (LC) + (LC)^T Q_1 \bar{A}_1 + \mu_1 Q_1^T F^T F Q_1 + Q_1^T C^T C Q_1 - Q_2 - A_1^T Q_1 A_1.$$

Pre-multiplying inequality (12) by $diag\left(Q_1^{-T}, \ \mu_1 I, \ I, \ I\right)$ and Post- multiplying (12) $diag\left(Q_1^{-1}, \ \mu_1 I, \ I, \ I\right)$, respectively, then let

$$Q_1^{-1} = P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}, \quad Q_2^{-1} = P_2 = \begin{pmatrix} 0 & 0 \\ 0 & P_{22} \end{pmatrix},$$

The inequality (13) is obtained by using the Schur's Complement again,

$$\Phi_0 = \begin{pmatrix} \bar{\bar{\Theta}}_{11} & 0 & 0 \\ * & -\mu_1 I & 0 \\ * & * & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} (A - LC)^T \\ \bar{H}^T \\ \bar{B}_w^T \end{pmatrix} \times P \begin{pmatrix} (A - LC)^T \\ \bar{H}^T \\ \bar{B}_w^T \end{pmatrix}^T < 0, \quad (13)$$

where

$$\bar{\bar{\Theta}}_{11} = \bar{A}_1^T P(LC) + (LC)^T P \bar{A}_1 - A_1^T P A_1 + \mu_1 F^T F + C^T C - P_2.$$

Since $P_{11}$ and $P_{22}$ are positive definite matrices, there exists a scalar $\eta = (\pm\varepsilon_1)^2 > 0$ such that

$$P_{11} - \eta P_{21}^T P_{22}^{-1} P_{21} = P_{11} - \varepsilon_1^2 P_{21}^T P_{22}^{-1} P_{21} > 0,$$

therefore, there exists a scalar $\varepsilon_1 > 0$ such that $P_{11} - \varepsilon_1^2 P_{21}^T P_{22}^{-1} P_{21} > 0$. We choose the Lyapunov function candidate as follows:

$$V(e) = e^T P_\varepsilon e, \quad (14)$$

where $P_\varepsilon = \begin{pmatrix} P_{11} & \varepsilon P_{21}^T \\ \varepsilon P_{21} & P_{22} \end{pmatrix} > 0$. Let $w(k) = 0$, then for all scalar $\mu_1 > 0$, we have

$$\Delta V(e) = \begin{pmatrix} 2 & 1 \end{pmatrix} = e(k+1)^T P_\varepsilon e(k+1) - e(k)^T P_\varepsilon e(k)$$
$$\leq \left(e^T (A_\varepsilon - LC)^T + f^T H_\varepsilon^T\right) P_\varepsilon \left((A_\varepsilon - LC) e + H_\varepsilon f\right)$$

$$-e^TP_\varepsilon e+\mu_1\left(e^TF^TFe-f^Tf\right)$$
$$=\left(e^T\ f^T\right)\left(\bar{\Phi}_0+\varepsilon\bar{\Phi}_1+\varepsilon^2\bar{\Phi}_2\right)\left(e^T\ f^T\right)^T, \quad (15)$$

where

$$\bar{\Phi}_0=\begin{pmatrix}\tilde{\Theta}_{11} & 0\\ * & -\mu_1^{-1}I\end{pmatrix}+\begin{pmatrix}(A-LC)^T\\ \bar{H}^T\end{pmatrix}P\begin{pmatrix}(A-LC)^T\\ \bar{H}^T\end{pmatrix}^T,$$
$$\tilde{\Theta}_{11}=\bar{A}_1^TP(LC)+(LC)^TP\bar{A}_1-A_1^TPA_1-P_2+\mu_1F^TF.$$
$$\bar{\Phi}_1=\begin{pmatrix}(E_0+A_2-LC)^T\\ 0\end{pmatrix}P_1\begin{pmatrix}\left(A-\bar{L}_2C\right)^T\\ H^T\end{pmatrix}^T+\begin{pmatrix}\left(A-\bar{L}_2C\right)^T\\ H^T\end{pmatrix}P_1\begin{pmatrix}(E_0+A_2-LC)^T\\ 0\end{pmatrix}^T,$$
$$A_2=\begin{pmatrix}0 & 0\\ A_{21} & A_{22}\end{pmatrix},$$
$$\bar{\Phi}_2=\begin{pmatrix}A^T-\left(\bar{L}_2C\right)^T\\ H^T\end{pmatrix}P_3\begin{pmatrix}A^T-\left(\bar{L}_2C\right)^T\\ H^T\end{pmatrix}^T,$$
$$\bar{L}_2=\begin{pmatrix}0\\ L_2\end{pmatrix}, P_1=\begin{pmatrix}P_{11} & P_{21}^T\\ 0 & 0\end{pmatrix},$$
$$P_3=\begin{pmatrix}P_{11} & P_{21}^T\\ P_{21} & 0\end{pmatrix}.$$

$\bar{\Phi}<0$, which is implied in inequality (13), then there exist a sufficiently small positive scalar $\varepsilon_2$ such that $\left(\bar{\Phi}+\varepsilon\bar{\Phi}_1+\varepsilon^2\bar{\Phi}_2\right)<0$ for any $\varepsilon\in(0,\varepsilon_2]$. Thus, the observer error system (6)-(7) is asymptotically stable.

Furthermore, the asymptotic stability of the observer error system with disturbance attenuation level $\gamma$ is considered. The following inequality is derived via referring to [33],

$$V[e(k+1)]-V[e(k)]+z^T(k)z(k)-\gamma^2w^T(k)w(k)$$
$$\leq\left(e^T(A_\varepsilon-LC)^T+f^TH_\varepsilon^T+w^TB_{w\varepsilon}^T\right)P_\varepsilon((A_\varepsilon-LC)e+H_\varepsilon f+B_{w\varepsilon}w)-e^TP_\varepsilon e+\mu_1\left(e^TF^TFe-f^Tf\right)+z^T(k)z(k)-\gamma^2w^T(k)w(k)$$
$$=\left(e^T\ f^T\ w^T\right)\left(\Phi_0+\varepsilon\Phi_1+\varepsilon^2\Phi_2\right)\left(e^T\ f^T\ w^T\right)^T, \quad (16)$$

where

$$\Phi_1=\begin{pmatrix}(E_0+A_2-LC)^T\\ 0\\ 0\end{pmatrix}P_1\begin{pmatrix}A^T-\left(\bar{L}_2C\right)^T\\ H^T\\ B_w^T\end{pmatrix}^T+\begin{pmatrix}A^T-\left(\bar{L}_2C\right)^T\\ H^T\\ B_w^T\end{pmatrix}P_1\begin{pmatrix}(E_0+A_2-LC)^T\\ 0\\ 0\end{pmatrix}^T,$$
$$\Phi_2=\left(A-\bar{L}_2C\ \ H\ \ B_w\right)^TP_3\left(A-\bar{L}_2C\ \ H\ \ B_w\right).$$

According to (11), there exist a sufficiently small positive scalar $\varepsilon_3$ such that $\left(\Phi_0+\varepsilon\Phi_1+\varepsilon^2\Phi_2\right)<0$ for any $\varepsilon\in(0,\varepsilon_3]$. Denote $\tilde{\varepsilon}=\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$. Then, the following inequality holds:

$$V(e(k+1))-V(e(k))+z^T(k)z(k)-\gamma^2w^T(k)w(k)<0. \quad (17)$$

Under zero initial conditions, every term in inequality (18) is summed from 0 to $\infty$, we get

$$V(e(\infty))-V(e(0))+\sum_{k=0}^{\infty}z^T(k)z(k)-\sum_{k=0}^{\infty}\gamma^2w^T(k)w(k)<0, \quad (18)$$

thus, we have

$$\sum_{k=0}^{\infty}z^T(k)z(k)\leq\sum_{k=0}^{\infty}\gamma^2w^T(k)w(k). \quad (19)$$

The observer error system is asymptotically stable with disturbance attenuation level $\gamma$ for any $\varepsilon\in(0,\tilde{\varepsilon}]$ after careful calculation. This ends the proof. ■

An appropriate observer is designed in Theorem 1. Meanwhile, the sufficient condition for the asymptotic stability of the error system with disturbance attenuation level $\gamma$ is obtained by using the LMI technique, the concrete structure of Lyapunov function is also given clearly.

Next, the minimum value of the disturbance attenuation level $\gamma$ is available. One feasible method is to convert the problem of minimum $\gamma$ to the optimization problem, thus the concrete results can be solved by MATLAB.

The upper bound of perturbation parameter has become an interesting topic due to the non-uniqueness of the upper bound $\tilde{\varepsilon}$. A common way is to assume the upper bound of perturbation parameter, then verify the selected value, further adjustments will be made if the prescribed value doesn't satisfy the obtained inequalities. Now, the result is given in the following theorem.

*Theorem 2:* The observer error system (6)-(7) is asymptotically stable with disturbance attenuation level $\gamma$ for any $\varepsilon\in(0,\tilde{\varepsilon}]$. If there exist a scalar $\tilde{\varepsilon}>0$, matrices $P_{11}>0$, $P_{22}>0$ satisfying the following conditions:

$$\Phi_0<0,\quad \Phi_0+\varepsilon\Phi_1<0,\ \Phi_0+\varepsilon\Phi_1+\varepsilon^2\Phi_2<0. \quad (20)$$

*Remark 4:* The upper bound $\tilde{\varepsilon}$ can be found by a bisectional search algorithm, a detailed discussion can be found in the literature [13].

## IV. H∞ CONTROL OF CLOSED-LOOP SYSTEM

In this subsection, we will design an observer-based feedback controller of the following form:

$$u(k)=-K\hat{x}(k) \quad (21)$$

to render the closed-loop system ISS with $H\infty$ performance index less than $\gamma$ for any $\varepsilon\in(0,\bar{\varepsilon}]$. Meanwhile, the method for deriving the minimum values of the $H\infty$ performance index $\gamma$ and the upper bound are proposed.

The closed-loop system is given in the following form:

$$x(k+1)=(A_\varepsilon-B_{u\varepsilon}K)x(k)+H_\varepsilon f(x(k))+B_{u\varepsilon}Ke+B_{w\varepsilon}w(k), \quad (22a)$$
$$y(k)=Cx(k). \quad (22b)$$

*Theorem 3:* The closed-loop system (22) is ISS with $H\infty$ performance index less than $\gamma$ for any $\varepsilon \in (0, \bar{\varepsilon}]$. If there exist matrices $X_{11} > 0$, $X_{22} > 0$, a matrix $G$, two positive scalars $\mu_2$ and $\bar{\varepsilon}$ satisfying the following condition:

$$\Psi = \begin{pmatrix} \Lambda_{11} & -\mu_2^{-1}H & \mu_2^{-1}B_w & X^T\bar{A}_2^T - G^TB_{u2}^T & X^TC^T & X^TF^T \\ * & -\mu_2^{-1}I & 0 & \mu_2^{-1}\bar{H}_2^T & 0 & 0 \\ * & * & -\gamma^2 I & B_{w2}^T & 0 & 0 \\ * & * & * & -X_{22} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\mu_2^{-1}I \end{pmatrix} < 0, \quad (23)$$

where

$$\Lambda_{11} = \bar{A}X + X^T\bar{A}^T - B_uG - G^TB_u^T, X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix},$$

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - I \end{pmatrix}, \bar{A}_2 = \begin{pmatrix} A_{21} & A_{22} - I \end{pmatrix},$$

$$\bar{H}_2 = \begin{pmatrix} 0 & H_2 \end{pmatrix}.$$

In addition, the feedback gain matrix can be chosen as $K = GX^{-1}$.

*Proof:* Substituting $K = GX^{-1}$ into (23) and using the Schur's Complement, we obtain the following equivalent form:

$$\begin{pmatrix} \bar{\Lambda}_{11} & -\mu_2^{-1}H & \mu_2^{-1}B_w & X^T\bar{A}_2 - X^TK^TB_{u2}^T \\ * & -\mu_2^{-1}I & 0 & \mu_2^{-1}\bar{H}_2^T \\ * & * & -\gamma^2 I & B_{w2}^T \\ * & * & * & -X_{22} \end{pmatrix} < 0, \quad (24)$$

where

$$\bar{\Lambda}_{11} = \bar{A}X + X^TA - B_uKX - X^TK^TB_u^T + \mu_2 X^TF^TFX + X^TC^TCX.$$

Pre-multiplying (23) by $diag\left(X^{-T}, \mu_2 I, I, I\right)$ and Post-multiplying (23) by $diag\left(X^{-1}, \mu_2 I, I, I\right)$, respectively, let $X^{-1} = P_4 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix}$, then $X_{22}^{-1} = P_{22}$, we get

$$\begin{pmatrix} \tilde{\Lambda}_{11} & P_4^TH & P_4^TB_w & \left(\bar{A}_2 - B_{u2}K\right)^T \\ * & -\mu_2 I & 0 & \bar{H}_2^T \\ * & * & -\gamma^2 I & B_{w2}^T \\ * & * & * & -X_{22} \end{pmatrix} < 0, \quad (25)$$

where

$$\tilde{\Lambda}_{11} = P_4^T\left(\bar{A} - B_uK\right) + \left(\bar{A} - B_uK\right)^T P_4 + \mu_2 F^TF + C^TC.$$

Using the Schur's Complement again, we have

$$\begin{pmatrix} \tilde{\Lambda}_{11} & P_4^TH & P_4^TB_w \\ * & -\mu_2 I & 0 \\ * & * & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} \left(\bar{A}_2 - B_{u2}K\right)^T \\ \bar{H}_2^T \\ B_{w2}^T \end{pmatrix} \times P_{22} \begin{pmatrix} \left(\bar{A}_2 - B_{u2}K\right)^T \\ \bar{H}_2^T \\ B_{w2}^T \end{pmatrix}^T < 0$$

which is equivalent to

$$\begin{pmatrix} \tilde{\Lambda}_{11} & P_4^TH & P_4^TB_w \\ * & -\mu_2 I & 0 \\ * & * & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} \left(\bar{A} - B_uK\right)^T \\ H^T \\ B_w^T \end{pmatrix} \times P_2 \begin{pmatrix} \left(\bar{A} - B_uK\right)^T \\ H^T \\ B_w^T \end{pmatrix}^T = \Pi_0 + \Pi_1 < 0, \quad (26)$$

where $P_2 = diag(O, P_{22})$. For the closed-loop system, we choose the Lyapunov function candidate as follows:

$$V(x) = x^T\tilde{P}_\varepsilon x, \quad (27)$$

where

$$\tilde{P}_\varepsilon = \begin{pmatrix} \varepsilon^{-1}P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix}, \quad \varepsilon \in (0, \varepsilon_4] \quad (28)$$

Let $w(k) = 0$

$$\begin{aligned} \Delta V(x) &= x(k+1)^T\tilde{P}_\varepsilon x(k+1) - x(k)^T\tilde{P}_\varepsilon x(k) \\ &\le x(k+1)^T\tilde{P}_\varepsilon x(k+1) - x^T\tilde{P}_\varepsilon x \\ &\quad + \mu_2\left(x^TF^TFx - f^Tf\right) \\ &= \left(x^T \ f^T\right)\left(\bar{\Pi}_0 + \bar{\Pi}_1 + \varepsilon\bar{\Pi}_2\right)\left(x^T \ f^T\right)^T \\ &\quad + e^T\left(B_{u\varepsilon}K\right)^T\tilde{P}_\varepsilon\left(B_{u\varepsilon}K\right)e + x^T\left(A_\varepsilon - B_{u\varepsilon}K\right)^T \\ &\quad \times \tilde{P}_\varepsilon\left(B_{u\varepsilon}K\right)e \\ &\quad + f^TH_\varepsilon^T\tilde{P}_\varepsilon\left(B_{u\varepsilon}K\right)e. \end{aligned} \quad (29)$$

where

$$\bar{\Pi}_0 = \begin{pmatrix} P_4^T\left(\bar{A} - B_uK\right) + \left(\bar{A} - B_uK\right)^T P_4 + \mu_2 F^TF & P_4^TH \\ * & -\mu_2 I \end{pmatrix},$$

$$\bar{\Pi}_1 = \left(\bar{A} - B_uK \quad H\right)^T P_2 \left(\bar{A} - B_uK \quad H\right),$$

$$\bar{\Pi}_2 = \left(A - B_uK \quad H\right)^T P_3 \left(A - B_uK \quad H\right),$$

Obviously, it can be concluded from (26) that $\bar{\Pi}_0 + \bar{\Pi}_1 < 0$. Thus, there exist a scalar $\varepsilon_5 > 0$, such that $\bar{\Pi}_0 + \bar{\Pi}_1 + \varepsilon\bar{\Pi}_2 < 0$ for any $\varepsilon \in (0, \varepsilon_5]$. Let $b = \lambda_{min}\left(-\bar{\Pi}_0 - \bar{\Pi}_1 - \varepsilon\bar{\Pi}_2\right)$, then

$b > 0$ for all $\varepsilon \in (0, \varepsilon_5]$. Thus

$$\Delta V(x) \le -b\|x\|^2 + \tau\|x\|\,\|e\| + \varsigma\|e\|^2 \le -b(1-\theta)\|x\|^2,$$
$$\forall \|x\| \ge \frac{\tau + \sqrt{\tau^2 + 4\varsigma b\theta}}{2b\theta}\|e\|,$$

where

$$0 < \theta < 1, \quad \tau = 2\sup_{\varepsilon\in(0,\varepsilon_4)}\left\{||(A_\varepsilon - B_{u\varepsilon}K)^T\tilde{P}_\varepsilon(B_{u\varepsilon}K)|| + ||H_\varepsilon^T\tilde{P}_\varepsilon(B_{u\varepsilon}K)||\right\},$$
$$\varsigma = ||(B_{u\varepsilon}K)^T(B_{u\varepsilon}K)||.$$

Denote $\bar{\varepsilon} = \min\{\varepsilon_4, \varepsilon_5\}$. Next, we have

$$\begin{aligned} &V(x(k+1)) - V(x(k)) + y^T(k)y(k) - \gamma^2 w^T(k)w(k) \\ &\le x^T(k+1)\tilde{P}_\varepsilon x(k+1) - x^T(k)\tilde{P}_\varepsilon x(k) + y^T(k)y(k) \\ &\quad - \gamma^2 w^T(k)w(k) + \mu_2\left(x^T F^T F x - f^T f\right) \\ &= \begin{pmatrix} x^T & f^T & w^T \end{pmatrix}(\Psi_0 + \Psi_1 + \varepsilon\Psi_2)\begin{pmatrix} x^T & f^T & w^T \end{pmatrix}^T, \end{aligned} \tag{30}$$

where, $\Psi_0$, $\Psi_1$, and $\Psi_2$, as shown at the bottom of the next page. It is known by condition (25), there exists a scalar $\bar{\varepsilon} > 0$, such that $\Psi_0 + \Psi_1 + \varepsilon\Psi_2 < 0$ for any given $\varepsilon \in (0, \bar{\varepsilon}]$. Similar to the proof of Theorem 1, it can be concluded that the closed-loop system (22) is ISS with $H\infty$ performance index less than $\gamma$ for any $\varepsilon \in (0, \bar{\varepsilon}]$, the detail is omitted here.

*Remark 5:* By the proof of theorems 1 and 3, it is found that the sufficient condition of the asymptotic stability or ISS is hidden in the sufficient condition of $H\infty$ performance index. Specifically, $\Psi_0 + \Psi_1 + \varepsilon\Psi_2 < 0$ is a sufficient condition for $\left(\bar{\Pi}_0 + \bar{\Pi}_1 + \varepsilon\bar{\Pi}_2\right) < -\Gamma$, where $\Gamma$ is a positive definite matrix. Next, for the discussion of $H\infty$ control problem, we just need to find the sufficient condition to ensure the $H\infty$ performance index. Therefore, the proof of asymptotic stability can be ignored.

*Theorem 4:* Under the condition that the observer-based feedback gain matrix $K$ have been obtained, the system (22) is ISS with an $H\infty$ performance index less than $\gamma$ for any $\varepsilon \in (0, \bar{\varepsilon}]$, where $\bar{\varepsilon} = \beta^{-1}$. If there exist matrices $P_{11} > 0$, $P_{22} > 0$, $\Xi$ and a positive scalar $\beta$ satisfying the following conditions:

$$\begin{pmatrix} \Xi & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \quad \Xi < \beta P_{11},$$
$$\Psi_0 + \Psi_1 < 0, \quad \Psi_2 < -\beta(\Psi_0 + \Psi_1), \tag{31}$$

where $\Psi_0$, $\Psi_1$ and $\Psi_2$ are mentioned in (30).

*Proof:* By analyzing (31), we get

$$\begin{pmatrix} \beta P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \quad \Psi_1 + \beta\Psi < 0,$$

Let $\bar{\varepsilon} = \frac{1}{\beta}$, we have

$$\begin{pmatrix} \bar{\varepsilon}^{-1}P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0,$$

for all $\varepsilon \in (0, \bar{\varepsilon}]$. This completes the proof. ■

The value of $\bar{\varepsilon}$ depends on the value $\beta$, which can be obtained by solving the inequality (31). Therefore, the solution for the upper bound of the perturbation parameter can be transformed as minimization problem. The calculation of this inequality is completed by MATLAB.

*A Special Case*

The corresponding conclusions for some special singularly perturbed systems are given directly, such as linear singularly perturbed systems. Take the following system for example.

$$x(k+1) = A_\varepsilon x(k) + B_{u\varepsilon}u(k) + B_{w\varepsilon}w(k), \tag{32}$$
$$y(k) = Cx(k). \tag{33}$$

*Corollary 1:* There exist an $\tilde{\varepsilon} > 0$ such that the observer error system is asymptotically stable with disturbance attenuation level $\gamma$ for $\forall\varepsilon \in (0, \tilde{\varepsilon}]$, if there exist matrices $Q_{11} > 0$, $Q_{22} > 0$, and a matrix $Y$ satisfying the following condition hold:

$$\begin{pmatrix} \Theta_{11} & 0 & \Theta_{14} & Q^T C^T \\ * & -\gamma^2 I & \bar{B}_w & 0 \\ * & * & -Q & 0 \\ * & * & * & -I \end{pmatrix} < 0, \tag{34}$$

where

$$\Theta_{11} = \left(Y\bar{A}_1\right)^T C + C^T\left(Y\bar{A}_1\right) - A_1^T Q A_1 - Q_2,$$
$$\Theta_{14} = QA^T - C^T Y,$$
$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix}, \quad \bar{B}_w = \begin{pmatrix} 0 \\ B_{w2} \end{pmatrix},$$
$$Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix},$$
$$A_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}, \quad \bar{A}_1 = \begin{pmatrix} A_{11} - I & A_{12} \\ 0 & 0 \end{pmatrix},$$

the observer gain matrix is chosen as $L = Q^{-T}Y^T$.

The minimum value of the disturbance attenuation level $\gamma$ can be obtained by solving the LMI (34), and the practical operation can be completed by the LMI Toolbox.

*Corollary 2:* The observer error system is asymptotically stable with disturbance attenuation level $\gamma$ for any $\varepsilon \in (0, \tilde{\varepsilon}]$. If there exist a scalar $\tilde{\varepsilon} > 0$, matrices $P_{11} > 0$, $P_{22} > 0$ satisfying the following conditions:

$$\tilde{\Phi} < 0, \quad \tilde{\Phi}_0 + \tilde{\varepsilon}\tilde{\Phi}_1 < 0, \quad \tilde{\Phi}_0 + \tilde{\varepsilon}\tilde{\Phi}_1 + \tilde{\varepsilon}^2\tilde{\Phi}_2 < 0, \tag{35}$$

where

$$\tilde{\Phi}_0 = \begin{pmatrix} \tilde{\Theta}_{11} & 0 \\ * & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} (A-LC)^T \\ \bar{B}_w^T \end{pmatrix} P \begin{pmatrix} (A-LC)^T \\ \bar{B}_w^T \end{pmatrix}^T,$$
$$\tilde{\Theta}_{11} = \bar{A}_1^T P(LC) + (LC)^T P\bar{A}_1 - A_1^T P A_1 - P_2 + C^T C,$$
$$\tilde{\Phi}_1 = \begin{pmatrix} (E_0 + A_2 - LC)^T \\ 0 \end{pmatrix} P_1 \begin{pmatrix} \left(A - \bar{L}_2 C\right)^T \\ B_w^T \end{pmatrix}^T + \begin{pmatrix} \left(A - \bar{L}_2 C\right)^T \\ B_w^T \end{pmatrix} P_1 \begin{pmatrix} (E_0 + A_2 - LC)^T \\ 0 \end{pmatrix}^T,$$

$$\tilde{\Phi}_2 = \begin{pmatrix} (A-\bar{L}_2C)^T \\ B_w^T \end{pmatrix} P_3 \begin{pmatrix} (A-\bar{L}_2C)^T \\ B_w^T \end{pmatrix}^T,$$

$$P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix},$$

$$P_1 = \begin{pmatrix} P_{11} & P_{21}^T \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & P_{22} \end{pmatrix},$$

$$P_3 = \begin{pmatrix} P_{11} & P_{21}^T \\ P_{21} & 0 \end{pmatrix}, \quad \bar{L}_2 = \begin{pmatrix} 0 \\ L_2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}.$$

*Corollary 3:* The closed-loop system is ISS with an $H\infty$ performance index less than $\gamma$ for any $\varepsilon \in (0, \bar{\varepsilon}]$. If there exist matrices $X_{11} > 0$, $X_{22} > 0$, a matrix $G$, and a positive scalar $\mu_2$, such that the following condition holds:

$$\begin{pmatrix} \Lambda_{11} & B_w & X^T\bar{A}_2^T - G^TB_{u2}^T & X^TC^T \\ * & -\gamma^2 I & B_{w2}^T & 0 \\ * & * & -X_{22} & 0 \\ * & * & * & -I \end{pmatrix} < 0, \quad (36)$$

where

$$\Lambda_{11} = \bar{A}X + X^T\bar{A}^T - B_uG - G^TB_u^T, \quad X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix},$$

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}-I \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} A_{21} & A_{22}-I \end{pmatrix}.$$

Furthermore, the control gain matrix $K = GX^{-1}$ is obtained.

In addition, the minimum of $\gamma$ can be obtained via solving the optimization problem (36).

*Corollary 4:* The upper bound $\bar{\varepsilon} = \beta^{-1}$ can be obtained based on the feedback control gain matrix $K$. If there exist matrices $\Xi$, $\bar{P}_{11}$, $\bar{P}_{22}$, matrix $\bar{P}_{21}$ and a positive scalar $\beta$ satisfying the following linear inequalities:

$$\begin{pmatrix} \Xi & \bar{P}_{21}^T \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix} > 0, \quad \Xi < \beta\bar{P}_{11}, \ \bar{\Psi} < 0, \ \bar{\Psi}_1 < -\beta\bar{\Psi}, \quad (37)$$

where

$$\bar{\Psi} = \begin{pmatrix} \bar{\bar{\Lambda}}_{11} & P_4^TB_u \\ * & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} (A-B_uK)^T \\ B_w^T \end{pmatrix} \times P_2 \begin{pmatrix} (A-B_uK)^T \\ B_w^T \end{pmatrix}^T,$$

$$\bar{\bar{\Lambda}}_{11} = P_4^T(\bar{A}-B_uK) + (\bar{A}-B_uK)^T P_4 + C^TC,$$

$$\bar{\Psi}_1 = \begin{pmatrix} (A-B_uK)^T \\ B_w^T \end{pmatrix} P_3 \begin{pmatrix} (A-B_uK)^T \\ B_w^T \end{pmatrix}^T,$$

$$P_4 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix}.$$

## V. NUMERICAL EXAMPLES

In this section, two examples are given to show the validity of the results in this paper.

*Example 1:* Consider the fast sampling linear discrete-time singularly perturbed system in [37] and [38] with the following parameter:

$$A = \begin{pmatrix} -0.3417 & 0.3417 \\ 0.2733 & 0.7267 \end{pmatrix}, \quad B_u = \begin{pmatrix} 9.0021 \\ 42.7983 \end{pmatrix},$$

$$B_w = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$D_u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad D_w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The example contrasts the corollaries 1-4 with the existing conclusion [36]–[38].Applying Corollary 3 and the methods in [36]–[38], the obtained the minimum H$\infty$ norm and the corresponding controller gains are given in Table 1. It can be seen form this example that the obtained result is the same as that in [38].

In Table 2, the minimum of disturbance attenuation level and the upper bound of the perturbation parameter with respect to observer error system and closed-loop system are shown in Table 2. Furthermore, to show that our method can provide a tighter upper bound than the existing methods, a comparison of the derived upper bounds by Corollary 3 with those calculated by the methods in [37] and [38] is given in Table 3.

From Table 3, it is easy to see that our new criterion does provide an improved estimation over the available corresponding criteria in the literature. Thus, Corollary 3 is less conservative than those in [37] and [38] in the sense that Corollary 3 can lead to a larger upper bound of the singular perturbation parameter.

$$L = \begin{pmatrix} 0.6224 & 0.1036 & 0 \\ 0.2470 & 0.7262 & 0 \end{pmatrix}.$$

$$\Psi_0 = \begin{pmatrix} P_4^T(\bar{A}-B_uK) + (\bar{A}-B_uK)^T P_4 + \mu_2F^TF + C^TC & P_4^TH & P_4^TB_w \\ * & -\mu_2 I & 0 \\ * & * & -\gamma^2 I \end{pmatrix},$$

$$\Psi_1 = \begin{pmatrix} \bar{A}-B_uK & H & B_w \end{pmatrix}^T P_2 \begin{pmatrix} \bar{A}-B_uK & H & B_w \end{pmatrix},$$

$$\Psi_2 = \begin{pmatrix} A-B_uK & H & B_w \end{pmatrix}^T P_3 \begin{pmatrix} A-B_uK & H & B_w \end{pmatrix}.$$

**TABLE 1.** Comparison for the minimum of $H\infty$ norm and controller gain $K$.

| | $\gamma$ | $K$ |
|---|---|---|
| Corollary 3 | 0.2001 | (-0.0138 -0.0170) |
| Method in [38] | 0.2001 | (-0.0138 -0.0170) |
| Method in [37] | 0.2049 | (-0.0095 -0.0172) |
| Method in [36] | 0.4317 | (-0.0100 -0.0170) |

**TABLE 2.** The minimum of $\gamma$ and the upper bound of $\varepsilon$.

| | $\gamma_{\min}$ | upper bound |
|---|---|---|
| Closed-loop systems | 0.2001 | 0.4326 |

**TABLE 3.** Comparison for the upper bound of the perturbation parameter.

| $\gamma$ | $\bar{\varepsilon}$ | Method in[38] | Method in [37] |
|---|---|---|---|
| 0.207 | 0.8133 | 0.8042 | 0.49 |
| 0.301 | 3.5906 | 3.1407 | 0.99 |

*Example 2:* The nuclear reaction model is an example based on actual engineering, the concrete form of the model is as follows:

$$\dot{x}_1 = -\lambda x_1 + \lambda x_2,$$
$$\dot{x}_2 = \frac{\beta}{v}x_1 + \frac{\beta}{v}x_2 + \frac{\rho}{v},$$

where $x_1, x_2$ represent the concentration and neutron density of the normalized precursor, $\lambda$, $\beta$, $v$ and $\rho$ represent respectively decay constant of precursor, delayed-neutron yield, neutron generation time and reactivity. Let $\rho = u + f(x_1, x_2)$, where $u$ is linear input, $f$ is non-linear input. When the corresponding parameters are $\lambda = 0.001$, $\beta = 0.0064$ and $v = 0.08$, the sampling period is $T = 0.05$s, then the parameters of the nonlinear discrete-time singularly perturbed system are obtained:

$$A = \begin{pmatrix} -0.3417 & 0.3417 \\ 0.2733 & 0.7267 \end{pmatrix}, \quad B_u = \begin{pmatrix} 9.0021 \\ 42.7983 \end{pmatrix},$$
$$B_w = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix},$$
$$H = \begin{pmatrix} 9.0021 & 0 \\ 0 & 42.7983 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \end{pmatrix},$$

Let the nonlinear function

$$f = \left( \left( \frac{x_1 |x_2 - 1|}{3 + 8(x_2 - 1)^2} \right)^T \left( \frac{x_2 |x_1 - 2|}{5 + 8(x_1 - 2)^2} \right)^T \right)^T,$$

we might as well note

$$F = \begin{pmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{pmatrix}.$$

Then the following results are obtained:

$$Q = \begin{pmatrix} 0.1912 & 0 \\ 0 & 0.3375 \end{pmatrix}, \quad Y = \begin{pmatrix} 0.0883 \\ 0.0613 \end{pmatrix}^T.$$

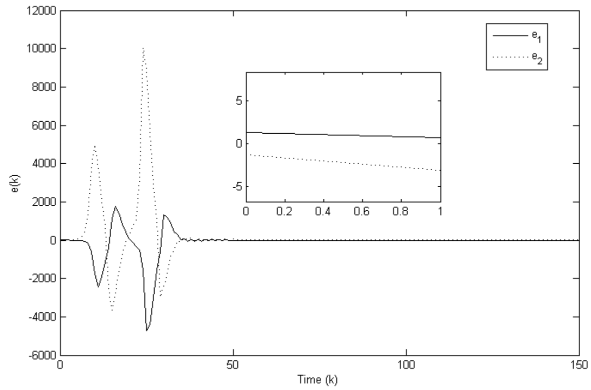


**FIGURE 1.** States of the observer error system.

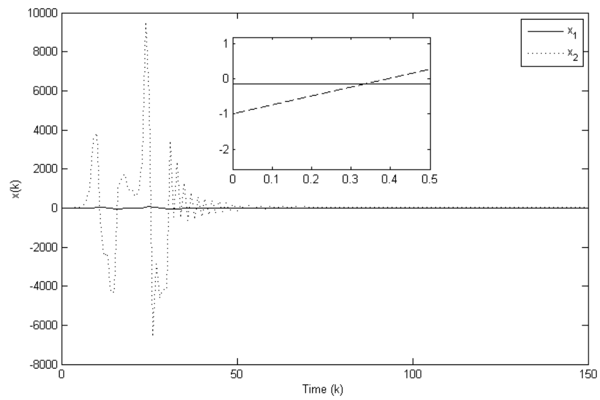


**FIGURE 2.** States of the closed-loop system.

So the observer gain matrix is given by

$$L = Q^{-T}Y^T = \begin{pmatrix} 0.4616 & 0.1818 \end{pmatrix}^T.$$

Meanwhile, the minimum of the disturbance attenuation level $\gamma_{\min} = 0.5023$ and the upper bound $\tilde{\varepsilon} = 4.0386$ are presented. Next, the state feedback gain matrix of the nuclear reaction system is obtained,

$$K = GX^{-1} = \begin{pmatrix} 0.0767 & 0.0327 \end{pmatrix},$$

where

$$X = \begin{pmatrix} 0.4861 & 0 \\ -1.6196 & 2.1416 \end{pmatrix}, \quad G = \begin{pmatrix} -0.0157 \\ 0.0700 \end{pmatrix}^T.$$

The minimum of the $H\infty$ performance index $\gamma_{\min} = 0.3207$ and the upper bound $\bar{\varepsilon} = 0.8562$ are also obtained.

To facilitate simulation, we take the small parameter as $\varepsilon = 0.02$, given the initial conditions $x(0) = (-0.15 \ -1)^T$, $\hat{x}(0) = (-1.5 \ \ 0.3)^T$ and the disturbance $w(k) = 1/(1+t^2)$. The simulation of the observer error system is shown in Fig.1. As shown in the simulation, the observer error can be close to zero. Fig.2 is a simulation for the closed-loop system, which shows that ISS can be achieved under the designed control law.

From the above numerical studies, it is noticed that the controllers are derived in this paper without the equality constraint involved in [12], which can be obtained easily by solving the LMI. Thus, the method here is much simpler and easier to compute.

## VI. CONCLUSION

This paper has investigated the $H\infty$ observer-based control for DTSPSs with nonlinear disturbances. By using Lyapunov function and LMI technique, a proper observer has been constructed to ensure that the error system is asymptotically stable with sufficient small disturbance attenuation level $\gamma$ for all $\varepsilon \in (0, \tilde{\varepsilon}]$. Then based on ISS property, a sufficient condition has been presented such that the ISS of the closed-loop system can be guaranteed. Meanwhile, the $H\infty$ performance index can be satisfied. In addition, the minimum $\gamma$ and the upper bound of the perturbation parameters have been solved by using MATLAB Toolbox. Finally, two numerical examples have been given to verify the validity of the proposed methods.

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