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Development of Frequency Weighted Model Order Reduction Techniques for Discrete-Time One-Dimensional and Two-Dimensional Linear Systems With Error Bounds

MUHAMMAD IMRAN[1], MUHAMMAD IMRAN[1], AND MIAN ILYAS AHMAD[2]
[1]Department of Electrical Engineering, Military College of Signals (MCS), National University of Sciences and Technology (NUST), Islamabad 44000, Pakistan
[2]Research Centre for Modelling and Simulation (RCMS), Department of Computational Engineering, National University of Sciences and Technology (NUST), Islamabad 44000, Pakistan

Corresponding author: Muhammad Imran (m.imran@mcs.edu.pk)

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ABSTRACT Frequency weighted model reduction framework pretested by Enns yields an unstable reduced order model. Researchers demonstrated several stability preserving techniques to address this main shortcoming, ensuring the stability of one-dimensional and two-dimensional reduced-order systems; nevertheless, these approaches produce significant truncation errors. In this article, Gramians-based frequency weighted model order reduction frameworks have been presented for the discrete-time one-dimensional and two-dimensional systems. Proposed approaches overcome Enns' main shortcoming in reduced-order model instability. In comparison to the various stability-preserving approaches, proposed frameworks provide an easily measurable *a priori* error-bound expression. The simulation results show that proposed frameworks perform well in comparison to other existing stability-preserving strategies, demonstrating the efficacy of proposed frameworks.

INDEX TERMS Model reduction, minimal realization, Hankel-Singular values, optimal Hankel norm approximation, frequency response error, error bound.

ACRONYMS/ABBREVIATIONS AND ELEMENTARY OPERATORS

In this article, following acronyms/abbreviations are used:

MOR	Model order reduction
ROM	Reduced order model
ODEs	Ordinary differential equations
PDEs	Partial differential equations
m-D	Multi-dimensional
1-D	One-dimensional
2-D	Two-dimensional
BT	Balanced Truncation
HNA	Hankel norm approximation
CRSD	Causal recursive separable denominator
GJ	Gawronski & Juang
GA	Gugercin & Antoulas

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The Table 1 provides some basic terminologies and their corresponding operators used in this article.

WZ	Wang & Zilouchian
GS	Ghafoor & Sreeram
IG	Imran & Ghafoor
LTI	Linear Time In-Variant
MIMO	Multi-input and multi-output
CB	Campbell
VA	Varga & Anderson
IP	Image processing
SP	Signal Processing
SDP	Seismographic data processing
DSP	Digital signal processing
gKYP	generalized Kalman-Yakubovich-Popov

I. INTRODUCTION

A. MOTIVATION AND INCITEMENT

The MOR challenge aims to develop an alternate model for the original large-scale stable model that is simple to measure

TABLE 1. Elementary operators and terminologies.

Terminologies and Elementary Operators	
Elementary Operators	Terminology
$F_*[z]$	$D_* + C_*[zI - A_*]^{-1}B_* \Leftrightarrow$ realization $\{A_*, B_*, C_*, D_*\}$
ROM $F_{r_*}[z]$	$D_{r_*} + C_{r_*}[zI - A_{r_*}]^{-1}B_{r_*} \Leftrightarrow$ realization $\{A_{r_*}, B_{r_*}, C_{r_*}, D_{r_*}\}$
$F[z_1, z_2]$	$D + C[z_1 I_n \bigoplus z_2 I_m - A]^{-1}B \Leftrightarrow$ 2-D realization $\{A, B, C, D\}$
ROM $F_r[z_1, z_2]$	$D_r + C_r[z_1 I_{r_1} \oplus z_2 I_{r_2} - A_r]^{-1}B_r \Leftrightarrow$ 2-D realization $\{A_r, B_r, C_r, D_r\}$
Input-weightings $G_i[z_1, z_2]$	$D_i + C_i[z_1 I_{n_{1_i}} \oplus z_2 I_{n_{2_i}} - A_i]^{-1}B_i \Leftrightarrow$ 2-D realization $\{A_i, B_i, C_i, D_i\}$
Output-weightings $H_o[z_1, z_2]$	$D_o + C_o[z_1 I_{n_{1_o}} \oplus z_2 I_{n_{2_o}} - A_o]^{-1}B_o \Leftrightarrow$ 2-D realization $\{A_o, B_o, C_o, D_o\}$
Augmented-input $F_{ia}[z_1, z_2]$	$C_{ia}[z_1 I_{n_1+n_{1_i}} \oplus z_2 I_{n_2+n_{2_i}} - A_{ia}]^{-1}B_{ia} + D_{ia} \Leftrightarrow$ 2-D realization $\{A_i, B_i, C_i, D_i\}$
Augmented-output $F_{oa}[z_1, z_2]$	$C_{oa}[z_1 I_{n_1+n_{1_o}} \oplus z_2 I_{n_2+n_{2_o}} - A_{oa}]^{-1}B_{oa} + D_{oa} \Leftrightarrow$ 2-D realization $\{A_o, B_o, C_o, D_o\}$
$diag\{.\}$	Diagonal operator
$rank[.]$	Rank operator
$p \mapsto \infty \|x\|_\infty := max\|x_i\|$	Infinity- norm (maximum norm)
λ	Eigenvalues
ν	Eigenvector
$\bigoplus$	Direct sum
P_{c_*}	Controllability matrix
$Q^T_{o_*}$	Observability matrix
$\lambda_i[P_{c_*} Q_{o_*}]$	Hankel-Singular-values of matrix $\|P_{c_*} Q_{o_*}\|$
$\sum_{j=1}^{n} \bar{\sigma}_j$	$\bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3 + ... + \bar{\sigma}_n$

and has the same responses as the original. In ROMs, the MOR attempts to retain the key characteristics of the original large-scale model, such as stability/passivity and input-output response. The MOR has made a significant contribution to the control system, mainly in the simulation of complex systems such as large-scale complex integrated circuits, robotics systems, communication systems, and controller reduction, etc., A record of the user's interactions with the recommendation system [1]–[6].

A substantial amount of study in the MOR of large-scale systems has been done recently, and various methods to MOR have been proposed [7]–[12]. Mathematical modeling aims to analyze dynamical systems, which is an important part of control systems engineering. The need for more rigorous mathematical models is growing as models get more complicated. Simulation becomes computationally tedious large-scale systems containing lumped parameter systems, such as ODEs, distributed parameter systems, such as PDE, etc. Dealing with these conditions is made easier with MOR. The study of these complex and large-scale systems is a difficult task; as a result, ROMs are needed to make the analysis easier.

m-D systems are those models in control system theory where several independent variables occur (like time). The Roesser 2-D model is a sub-class of m-D systems, and it has vast applications in control systems theory. Due to their applications in various key areas such as IP, SP, SDP, water steam heating, DSP filters, etc., 2-D systems have been a continually increasing research interest area in recent years. The Roesser 2-D model contributes in the following various fields such as:

- Automated irrigation channels [13].
- Grid sensor network [14].
- Design of 2-D digital filters structures (i.e., FIR filter [15], IIR filter [16], Digital filter [17], H_∞ filter [18], [19]).
- Fault detection [20].
- Linear repetitive processes [21].
- Iterative learning control [22], [23].

The researcher focused on fundamental problems such as decomposition, factorization, stability, and model reduction, etc. As decomposition, factorization, stability, and model reduction are not straightforward extensions for 1-D models, the fundamental theorem of the algebra does not apply to m-D systems directly [24]–[27].

B. LITERATURE REVIEW

The most often used MOR approach is BT [28], while using BT [28] methods, it is necessary to balance the system, which is equivalent to determining the system's controllability and observability Gramians in a unique diagonal form. The Cholesky factors of these Gramians can be efficiently computed as dual Lyapunov equation solutions for systems with few inputs and outputs. BT [28] provides ROMs for the 1-D LTI continuous and the discrete-time systems that guarantee stability and yield error bounds. However, an entire frequency interim is used to execute MOR operations, while the particular frequency band is concerned only in practical applications, i.e., the controller reduction case. Similarly, Glover [29] used an optimal HNA to perform the MOR operation. The HNA is a model reduction method that offers the best Hankel semi-norm approximation. These promote the usage of frequency weights in MOR. Therefore, Enns [30], [31] provided the frequency weighted MOR approach for the 1-D LTI continuous and the discrete-time systems by inducing frequency weights (i.e., input, output and double-sided) in BT [28] approach. However, this approach [30] generates unstable ROMs in the case of double-sided weightings [32]. Similarly, the limited frequency interval [33] is of concern for some applications (i.e., controller and filter reduction). The frequency-limited intervals Gramians based MOR approach for the 1-D LTI continuous and the discrete-time systems were implemented by the GJ [34] and WZ [35], respectively; however, it does often result in unstable ROMs at certain frequency-intervals [36], [37] and there exist no *a priori* error bound expressions for these techniques [30], [34], [35].

Recently, a significant amount of research has been conducted on the MOR of large-scale systems, and a number of different MOR approaches have been developed. [7]–[12]. In [7], second-order dynamical systems using structure-preserving balanced truncation approaches are provided, which deals with first-order constrained balanced truncation approaches and apply them to second-order systems utilizing various second-order balanced truncation formulas. The work presented in [8] is based on a balanced truncation model order reduction for discrete-time systems that preserves stability after reduction. However, due to the iterative nature of this method, it becomes more complicated when the order of the original system increases. Correspondingly, [9] presents MOR based on cross Gramians; this method [9] uses Sylvester equations rather than Lyapunov equations as described by BT [28], Enns [30], GJ [34], and WZ [35]. Furthermore, this method is only applicable to bilinear systems that employ a truncated cross Gramian projection approach. A similar work based on interpolation is presented in [10]. It proposes adaptive techniques for computing time delay systems' reduced-order model. The algorithms use greedy iterations to choose expansion locations and interpolate the transfer function. Similarly, another interpolation-based approach is presented in [11]. It focuses on dominated and temporal moment retention. It condenses the large-scale complete order model into a lower order system, allowing approximate computation denominator by employing generalized pole clustering. The factors division procedure yields the approximate numerator, which results in the ROM. In [12], the MOR for 2-D discrete-time system MOR is presented. This method ensures the stability of the filtering error system and H_∞ performance when the noise frequency ranges are known beforehand. Using the gKYP lemma, Finsler's lemma, and some independent matrices yield fewer conservative findings. The research briefly discussed above are based on cross Gramians, interpolation, and Kalman filtering. Furthermore, to overcome the shortcomings as appeared in Enns [30], GJ [34], and WZ [35] substantial amount of research have been conducted over the couple of decades [36]–[46], which are briefly discussed as follows with their drawbacks.

To overcome the main drawback as appeared in [30], the Lin and Chiu [38] introduced strictly proper two-sided weights to ensure the stability of ROMs; however, this method cannot be used in controller reduction applications due to no pole-zero cancellation assumption required in the method. Later on, VA [39] introduced an alternative approach to ensure the stability of the ROM for the continuous-time frequency weighted systems. Since the main weakness of Lin and Chiu's [38] technique is the requirement that no pole-zero cancellation occurs when forming the augmented systems (input augmented and output augmented). This prevents the applicability of this method when solving controller reduction problems involving weights; however, this technique [39] is only valid for strictly proper original systems.

The instability problem in [30] is related to the indefiniteness of the corresponding input and output matrices; CB [40] provided the stability-preserving frequency weighted MOR method by ensuring the input and output matrices are positive/semi-positive definite. As a result, some eigenvalues have significant variations while others have slight variations. Dissimilar effects on each eigenvalue of the input and output matrices result in a significant approximation error in the ROM. The GS method [41] combines unweighted balanced and partial-fraction-based frequency weighted balanced reduction techniques, ensuring ROM stability but being parameterized. The GS [42] also proposed a MOR technique for 2-D discrete-time weighted systems. However, truncating negative eigenvalues causes a significant approximation error in 2-D ROM. The stability-preserving frequency-weighted MOR approach introduced by IG [43] involves varying the input and output matrices, but subtracting all eigenvalues from minor eigenvalues results in zeroing the last eigenvalue, resulting in an unequal effect to eigenvalues and a significant approximation error in the ROM.

Together with the use of positive/semi-positive definiteness of input and output matrices, GA [36] established stability preserving frequency limited Gramians based MOR approach. However, the asymmetrical impacts on all eigenvalues cause significant approximation error [36]. By using frequency-limited intervals, GS [41] developed ROM stability. GS's approach [41] produces a large approximation error due to the significant variation in the original system. In later work, IG [44] adjusted the eigenvalues matrix by subtracting the least dependent negative eigenvalue from all the eigenvalues; nonetheless, the modified eigenvalues cause significant changes to the original systems and large approximation error. Similarly, [45] offers three techniques to maintain ROM stability; however, [45] is iterative, which is inefficient when the original system's order rises.

Similarly, to overcome the main drawback as appeared in [35], GS [37] ensures the stability of the ROM by improvising the eigenvalues matrix; however, due to the truncation of negative eigenvalues and absolute of all the eigenvalues, it increases a distance from the eigenvalues matrix of the original systems, which leads to a large approximation in the ROM. Similarly, IG [46] also introduced frequency limited MOR approach for the discrete-time systems; however, this approach results in significant truncation errors in the desired discrete frequency intervals due to the significant variance from the original system and zeroing the effect of the last eigenvalue.

Recently, a significant amount of research has been conducted on the MOR of large-scale systems based on balanced approach [47]–[51]. In [47], weighted and limited interval discrete-time 1-D systems are provided. The frequency limited intervals for 1-D and 2-D systems are given in [48]. Similarly, frequency weighted and limited MOR approaches for power systems are given in [49]–[51].

The BT [28], Enns [30], GJ [34], and WZ [35] yield unstable ROM and do not provide *a priori* error-bound expressions. Further, their successive stability preserving approaches [36]–[46] ensure stability in some conditions and generate significant truncation error due to the substantial variation to the original systems (i.e., pole-zero cancellation, absolute of negative eigenvalues, truncation of all negative eigenvalues, zeroing the effect of the last eigenvalue, etc.).

C. MAIN CONTRIBUTION AND PAPER ORGANIZATION

A novel method for 1-D and 2-D discrete-time systems is proposed. For 1-D and 2-D discrete-time systems, the suggested method offers a new discrete frequency weighted strategy exhibiting small truncation error. The square root of all eigenvalues with similar effects prevents the zeroing of the last eigenvalues, provides an equal impact on all eigenvalues, and preserves the eigenvalues' structure of some input and output matrices. Compared to other stability-preserving model reduction frameworks based on frequency-weighted Gramians, the proposed method provides small variation to the original system.

The main contributions of this paper are as follows:

- Decomposition of the discrete-time 2-D CRSD model based on frequency weightings into two decomposed 1-D sub-models is attained by using the minimal rank-decomposition conditions.
- Modifications to associated input and output matrices are performed for 1-D models and corresponding decomposed 1-D sub-models to assure positive and semi-positive definiteness of associated input and output matrices.
- The controllability and observability Gramians for 1-D models and decomposed 1-D sub-models in the given frequency weights are computed, corresponding to modified input and output matrices.
- Stability of ROMs are ensured incase of 1-D and 2-D weighted systems.
- Frequency weighted *a priori* error bound formula for the 1-D and 2-D systems are derived based on balance truncation.
- Frequency weighted *a priori* error bound formula for the 1-D and 2-D systems are derived based on an optimal HNA.
- Comparison among different existing frequency weighted MOR techniques (including 1-D and 2-D) with proposed techniques are presented.

The MOR framework based on frequency weighted for linear time-invariant discrete-time 1-D and 2-D systems is presented in this paper. The 1-D and 2-D un-weighted and weighted models are discussed in Section II, and the 2-D model decomposition via minimal rank-decomposition conditions. The balance truncation approach, as well as frequency weighted MOR approaches, are discussed in Section III. The existing stability-preserving frequency weighted balancing related techniques for 1-D and 2-D discrete-time systems are also discussed in this part. Section IV lays out the proposed work for 1-D and 2-D discrete-time systems and the *a priori* error-bound expressions for 1-D and 2-D cases. In addition, the numerical simulation results are presented in section V, where a comparison is made between existing 1-D, and 2-D frequency weighted MOR techniques and proposed techniques, demonstrating the proposed techniques' efficacy.

II. PRELIMINARIES

This section presents the corresponding un-weighted and frequency weighted 1-D and 2-D state space systems.

A. 1-D STATE SPACE SYSTEM

Here we provide a brief overview of un-weighted, and frequency weighted 1-D state-space discrete-time systems.

1) UN-WEIGHTED 1-D STATE-SPACE SYSTEM

Consider a 1-D discrete time system be given as:

$$\begin{aligned} x[k+1] &= A_* x[k] + B_* u[k], \\ y[k] &= C_* x[k] + D_* u[k], \\ F_*[z] &= D_* + C_*[zI - A_*]^{-1} B_*, \end{aligned} \tag{1}$$

where $\{A_* \in \Re^{n\times n}, B_* \in \Re^{n\times m}, C_* \in \Re^{p\times n}, D_* \in \Re^{p\times m}\}$ is its n^{th} order minimal realization with m number of inputs and p number of outputs. The ROM is obtained as:

$$\begin{aligned} x_r[k+1] &= A_{r_*} x_r[k] + B_{r_*} u[k], \\ y_r[k] &= C_{r_*} x_r[k] + D_{r_*} u[k], \\ F_{r_*}[z] &= D_{r_*} + C_{r_*}[zI - A_{r_*}]^{-1} B_{r_*}, \end{aligned} \tag{2}$$

is achieved by truncating the large-scale stable original system [28] (i.e., in the entire-frequency intervals $[\omega_1, \omega_2] = [-\pi, \pi]$), where $\{A_{r_*} \in \Re^{r\times r}, B_{r_*} \in \Re^{r\times m}, C_{r_*} \in \Re^{p\times r}, D_{r_*} \in \Re^{p\times m}\}$ with $r \ll n$.

2) FREQUENCY WEIGHTED 1-D STATE-SPACE SYSTEM

Consider a transfer function form of a stable discrete-time input-weighting model be given as:

$$\begin{aligned} x_i[k+1] &= A_{iw} x_i[k] + B_{iw} u_i[k], \\ y_i[k] &= C_{iw} x_i[k] + D_{iw} u_i[k], \\ G_i[z] &= D_{iw} + C_{iw}[zI - A_{iw}]^{-1} B_{iw}, \end{aligned} \tag{3}$$

where $A_{iw} \in \Re^{(n_i)\times(n_i)}$, $B_{iw} \in \Re^{(n_i)\times(m_i)}$, $C_{iw} \in \Re^{(p_i)\times(n_i)}$, $D_{iw} \in \Re^{(p_i)\times(m_i)}$ and $\{A_{iw}, B_{iw}, C_{iw}, D_{iw}\}$ is its $n_i{}^{th}$ order minimal realization. Similarly, consider a transfer function form of a stable discrete-time output-weighting model

$$\begin{aligned} x_o[k+1] &= A_{ow} x_o[k] + B_{ow} u_o[k], \\ y_o[k] &= C_{ow} x_o[k] + D_{ow} u_o[k], \\ H_o[z] &= D_{ow} + C_{ow}[zI - A_{ow}]^{-1} B_{ow}, \end{aligned} \tag{4}$$

where $A_{ow} \in \Re^{(n_o)\times(n_o)}$, $B_{ow} \in \Re^{(n_o)\times(m_o)}$, $C_{ow} \in \Re^{(p_o)\times(n_o)}$, $D_{ow} \in \Re^{(p_o)\times(m_o)}$ and $\{A_{ow}, B_{ow}, C_{ow}, D_{ow}\}$ is its $n_o{}^{th}$ order minimal realization. The input-augmented and the output-augmented systems are given by:

$$F[z]G_i[z] = C_{ai}[zI - A_{ai}]^{-1} B_{ai} + D_{ai}, \tag{5}$$

$$H_o[z]F[z] = C_{ao}[zI - A_{ao}]^{-1} B_{ao} + D_{ao}, \tag{6}$$

where

$$\left[\begin{array}{c|c} A_{ai} & B_{ai} \\ \hline C_{ai} & D_{ai} \end{array}\right] = \left[\begin{array}{cc|c} A_* & B_* C_{iw} & B_* D_{iw} \\ 0 & A_{iw} & B_{iw} \\ \hline C & D_* C_{iw} & D_* D_{iw} \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{ao} & B_{ao} \\ \hline C_{ao} & D_{ao} \end{array}\right] = \left[\begin{array}{cc|c} A_{ow} & B_{ow}C_{*} & B_{ow}D_{*} \\ 0 & A_{*} & B_{*} \\ \hline C_{ow} & D_{ow}C_{*} & D_{ow}D_{*} \end{array}\right].$$

B. 2-D STATE SPACE SYSTEMS

Here we provide a brief overview of un-weighted and frequency weighted 2-D systems with its decomposition based on minimal rank-decomposition criteria and weighted 2-D state-space discrete-time systems.

1) UN-WEIGHTED 2-D STATE-SPACE SYSTEM

Consider a stable LTI MIMO, minimal separable denominator 2-D discrete-time Roesser's state-space model be given as [52]:

$$\dot{x}[i,j] = Ax[i,j] + Bu[i,j], \tag{7}$$

$$y[i,j] = Cx[i,j] + Du[i,j], \tag{8}$$

$$F[z_1,z_2] = D + C[z_1 I_n \bigoplus z_2 I_m - A]^{-1}B, \tag{9}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

$$x[i,j] = \begin{bmatrix} x_h[i,j] \\ x_v[i,j] \end{bmatrix}, \quad \dot{x}[i,j] = \begin{bmatrix} x_h[i+1,j] \\ x_v[i,j+1] \end{bmatrix}.$$

where i and j are vertical and horizontal coordinates respectively, $x_h(i,j) \in \Re^n$ and $x_v(i,j) \in \Re^m$ are the horizontal and vertical state vectors that convey horizontal and vertical information, respectively, $u(i,j) \in \Re^p$ and $y(i,j) \in \Re^q$ and $\{A \in \Re^{(n+m)\times(n+m)}, B \in \Re^{(n+m)\times(p)}, C \in \Re^{(q)\times(n+m)}, D \in \Re^{q\times p}\}$ is its $(n+m)^{th}$ order minimal realization with p number of inputs and q number of outputs. The MOR challenge is to determine

$$F_r[z_1,z_2] = D_r + C_r[z_1 I_{r_1} \oplus z_2 I_{r_2} - A_r]^{-1}B_r, \tag{10}$$

where $\{A_r \in \Re^{(r_1+r_2)\times(r_1+r_2)}, B_r \in \Re^{(r_1+r_2)\times(p)}, C_r \in \Re^{(q)\times(r_1+n_2)}, D_r \in \Re^{(q)\times(p)}\}$ with $r_1 \ll n_1$ and $r_2 \ll n_2$.

Let the minimal rank-decomposition of Roesser's state-space realization subject to $A_3 = 0$ be written as:

$$\begin{bmatrix} A_2 & B_1 \\ C_2 & D \end{bmatrix} = \begin{bmatrix} \bar{B}_{1_*} \\ \bar{D}_{1_*} \end{bmatrix} \begin{bmatrix} \bar{C}_{2_*} & \bar{D}_{2_*} \end{bmatrix}, \tag{11}$$

consequently, 2-D separable denominator state-space can be given as:

$$A = \begin{bmatrix} A_1 & \bar{B}_{1_*}\bar{C}_{2_*} \\ 0 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B}_{1_*}\bar{D}_{2_*} \\ B_2 \end{bmatrix}, \tag{12}$$

$$C = \begin{bmatrix} C_1 & \bar{D}_{1_*}\bar{C}_{2_*} \end{bmatrix}, \quad D = \bar{D}_{1_*}\bar{D}_{2_*}, \tag{13}$$

that results $F[z_1,z_2] = \bar{F}[z_1,z_2] = \bar{F}_1[z_1]\bar{F}_2[z_2]$, where

$$\bar{F}_1[z_1] = \bar{D}_{1_*} + C_1[z_1 I - A_1]^{-1}\bar{B}_{1_*}, \tag{14}$$

$$\bar{F}_2[z_2] = \bar{D}_{2_*} + \bar{C}_{2_*}[z_2 I - A_4]^{-1}B_2, \tag{15}$$

The decomposed 1-D system $\bar{F}_1[z_1]$ is a p-input/p_1-output system, and the decomposed 1-D system $\bar{F}_2[z_2]$ is a p_1-input/q-output system [52].

Similarly, the minimal rank-decomposition of Roesser's state-space realization subject to $A_2 = 0$ can be written as:

$$\begin{bmatrix} A_3 & B_2 \\ C_1 & D \end{bmatrix} = \begin{bmatrix} \hat{B}_{2_*} \\ \hat{D}_{2_*} \end{bmatrix} \begin{bmatrix} \hat{C}_{1_*} & \hat{D}_{1_*} \end{bmatrix}, \tag{16}$$

consequently, 2-D separable denominator state-space can be written as:

$$A = \begin{bmatrix} A_1 & 0 \\ \hat{B}_{2_*}\hat{C}_{1_*} & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \hat{B}_{2_*}\hat{D}_{1_*} \end{bmatrix}, \tag{17}$$

$$C = \begin{bmatrix} \hat{D}_{2_*}\hat{C}_{1_*} & C_2 \end{bmatrix}, \quad D = \hat{D}_{2_*}\hat{D}_{1_*} \tag{18}$$

that results $F[z_1,z_2] = \hat{F}[z_1,z_2] = \hat{F}_2[z_2]\hat{F}_1[z_1]$, where

$$\hat{F}_1[z_1] = \hat{D}_{1_*} + \hat{C}_{1_*}[z_1 I - A_1]^{-1}B_1, \tag{19}$$

$$\hat{F}_2[z_2] = \hat{D}_{2_*} + C_2[z_2 I - A_4]^{-1}\hat{B}_{2_*}, \tag{20}$$

Similarly, the decomposed 1-D system $\hat{F}_1[z_1]$ is a p-input/p_2-output system, and the decomposed 1-D system $\hat{F}_2[z_2]$ is a p_2-input/q-output system [52], where

$$rank \begin{bmatrix} A_2 & B_1 \\ C_2 & D \end{bmatrix} = p_1, \quad rank \begin{bmatrix} A_3 & B_2 \\ C_1 & D \end{bmatrix} = p_2.$$

Remark 1: The 2-D models, as in (9), generally don't exist in CRSD form; however, existing 1-D MOR schemes are only applicable to the 2-D systems when it exists in 2-D CRSD form. In addition, we need minimal rank-decomposition criteria to obtain decomposed 1-D sub-models as in (14)-(15) and (19)-(20).

Lemma 1 ([53]): Let the ROM for 2-D discrete-time systems be $F_r[z_1,z_2] = F_{1_r}[z_1]F_{2_r}[z_2]$ obtained by using 1-D BT [28], then the 2-D discrete-time ROM $F_r[z_1,z_2]$ is asymptotically stable. Moreover, the frequency response truncation error is bounded by:

$$\begin{aligned} &\|F[z_1,z_2] - F_r[z_1,z_2]\|_\infty \\ &\le 2(\|\bar{D}_{1_*}\| + 2\sum_{i=1}^{n} \bar{\rho}_i) \times 2 \sum_{i=m_r+1}^{m} \bar{\varphi}_i \\ &\quad + (\|\bar{D}_{2_*}\| + 2\sum_{i=1}^{m_r} \bar{\varphi}_i) \times 2 \sum_{i=n_r+1}^{n} \bar{\rho}_i. \end{aligned}$$

Alternatively,

$$\begin{aligned} &\|F[z_1,z_2] - F_r[z_1,z_2]\|_\infty \\ &\le 2(\|\bar{D}_{1_*}\| + 2\sum_{i=1}^{n_r} \bar{\rho}_i) \times 2 \sum_{i=m_r+1}^{m} \bar{\varphi}_i \\ &\quad + (\|\bar{D}_{2_*}\| + 2\sum_{i=1}^{m} \bar{\varphi}_i) \times 2 \sum_{i=n_r+1}^{n} \bar{\rho}_i. \end{aligned}$$

where $\bar{\rho}_i$ and $\bar{\varphi}_i$ are the Hankel Singular-values of the decomposed sub-systems $\bar{F}_1[z_1]$ and $\bar{F}_2[z_2]$, respectively.

Lemma 2 ([53]): Let the ROM for 2-D discrete-time systems be $F_{rh}[z_1,z_2] = F_{1_{rh}}[z_1]F_{2_{rh}}[z_2]$ obtained by using 1-D an optimal Hankel norm approximation [29], then the 2-D discrete-time ROM $F_{rh}[z_1,z_2]$ is asymptotically stable.

Moreover, the frequency response truncation error is bounded by:

$$\|F[z_1,z_2]-F_{rh}[z_1,z_2]\|_\infty \leq (\|\bar{D}_{1_*}\|+2\sum_{i=1}^{n}\bar{\rho}_i)\times 2\sum_{i=m_r+1}^{m}\bar{\varphi}_i + (\|\bar{D}_{2_*}\|+2\sum_{i=1}^{m_r}\bar{\varphi}_i+3\sum_{i=m_r+1}^{m}\bar{\varphi}_i)\times 2\sum_{i=n_r+1}^{n}\bar{\rho}_i.$$

Alternatively,

$$\|F[z_1,z_2]-F_{rh}[z_1,z_2]\|_\infty \leq (\|\bar{D}_{2_*}\|+2\sum_{i=1}^{m}\bar{\varphi}_i)\times 2\sum_{i=n_r+1}^{n}\bar{\rho}_i + (\|\bar{D}_{1_*}\|+2\sum_{i=1}^{n_r}\bar{\rho}_i+3\sum_{i=n_r+1}^{n}\bar{\rho}_i)\times 2\sum_{i=m_r+1}^{m}\bar{\varphi}_i.$$

where $\bar{\rho}_i$ and $\bar{\varphi}_i$ are the optimal Hankel Singular-values of the decomposed sub-systems $\bar{F}_1[z_1]$ and $\bar{F}_2[z_2]$, respectively.

2) FREQUENCY WEIGHTED 2-D STATE-SPACE SYSTEM

The 2-D weighted discrete-time systems arrangement is shown in Figure 1. Consider a transfer function stable 2-D linear time-invariant discrete-time input weighted system [42] be given as:

$$G_i[z_1,z_2]=D_i+C_i[z_1I_{n_{1_i}}\oplus z_2I_{n_{2_i}}-A_i]^{-1}B_i, \quad (21)$$

where

$$A_i=\begin{bmatrix}A_{1_i} & A_{2_i}\\ A_{3_i} & A_{4_i}\end{bmatrix},\quad B_i=\begin{bmatrix}B_{1_i}\\ B_{2_i}\end{bmatrix},\quad C_i=\begin{bmatrix}C_{1_i} & C_{2_i}\end{bmatrix},\quad D_i, \quad (22)$$

where $\{A_i \in \Re^{(n_{1_i}+n_{2_i})\times(n_{1_i}+n_{2_i})}, B_i \in \Re^{(n_{1_i}+n_{2_i})\times(p_i)}, C_i \in \Re^{(q_i)\times(n_{1_i}+n_{2_i})}, D_i \in \Re^{(q_i)\times(p_i)}\}$ is its $(n_{1_i}+n_{2_i})^{th}$ dimensional minimal realization with p_i number of inputs and q_i number of outputs. Similarly, consider a transfer function stable 2-D linear time-invariant discrete-time output weighted system [42] be given as:

$$H_o[z_1,z_2]=D_o+C_o[z_1I_{n_{1_o}}\oplus z_2I_{n_{2_o}}-A_o]^{-1}B_o, \quad (23)$$

where

$$A_o=\begin{bmatrix}A_{1_o} & A_{2_o}\\ A_{3_o} & A_{4_o}\end{bmatrix},\quad B_o=\begin{bmatrix}B_{1_o}\\ B_{2_o}\end{bmatrix},\quad C_o=\begin{bmatrix}C_{1_o} & C_{2_o}\end{bmatrix},\quad D_o, \quad (24)$$

where $\{A_o \in \Re^{(n_{1_o}+n_{2_o})\times(n_{1_o}+n_{2_o})}, B_o \in \Re^{(n_{1_o}+n_{2_o})\times(p_o)}, C_o \in \Re^{(q_o)\times(n_{1_o}+n_{2_o})}, D_o \in \Re^{(q_o)\times(p_o)}\}$ is its $(n_{1_o}+n_{2_o})^{th}$ dimensional minimal realization, p_o and q_o are the number of inputs and outputs respectively.

$$F_{ia}[z_1,z_2]=C_{ia}[z_1I_{n_1+n_{1_i}}\oplus z_2I_{n_2+n_{2_i}}-A_{ia}]^{-1}B_{ia}+D_{ia}, \quad (25)$$

$$F_{oa}[z_1,z_2]=C_{oa}[z_1I_{n_1+n_{1_o}}\oplus z_2I_{n_2+n_{2_o}}-A_{oa}]^{-1}B_{oa}+D_{oa}, \quad (26)$$

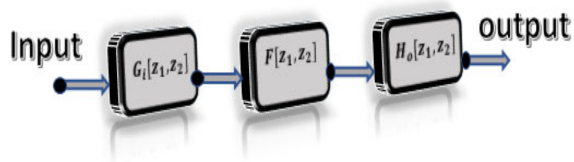

FIGURE 1. Input and output weighted 2-D discrete-time system.

Furthermore, realizations $\{A_{ia}, B_{ia}, C_{ia}, D_{ia}\}$ and $\{A_{oa}, B_{oa}, C_{oa}, D_{oa}\}$ are augmented-input (i.e., $F[z_1,z_2]G_i[z_1,z_2]$) and augmented-output (i.e., $H_o[z_1,z_2]F[z_1,z_2]$), respectively, (see Figure 2) [42].
where

$$A_{ia}=\left[\begin{array}{c|c}A_{ia_1} & A_{ia_2}\\ A_{ia_3} & A_{ia_4}\end{array}\right] = \left[\begin{array}{cc|cc}A_1 & B_1C_{1_i} & A_2 & B_1C_{2_i}\\ 0 & A_{1_i} & 0 & A_{2_i}\\ \hline 0 & B_2C_{1_i} & A_4 & B_2C_{2_i}\\ 0 & A_{3_i} & 0 & A_{4_i}\end{array}\right],$$

$$B_{ia}=\begin{bmatrix}B_{ia_1}\\ B_{ia_2}\end{bmatrix}=\left[\begin{array}{c}B_1D_i\\ B_{1_i}\\ \hline B_2D_i\\ B_{2_i}\end{array}\right],$$

$$C_{ia}=\begin{bmatrix}C_{ia_1} & C_{ia_2}\end{bmatrix}=\left[\begin{array}{cc|cc}I & 0 & 0 & 0\\ 0 & 0 & I & 0\end{array}\right],$$

$$A_{oa}=\left[\begin{array}{c|c}A_{oa_1} & A_{oa_2}\\ A_{oa_3} & A_{oa_4}\end{array}\right] = \left[\begin{array}{cc|cc}A_1 & 0 & A_3 & 0\\ B_{1_o}C_1 & A_{1_o} & B_{1_o}C_2 & A_{2_o}\\ \hline 0 & 0 & A_4 & 0\\ B_{2_o}C_1 & A_{3_o} & B_{2_o}C_2 & A_{4_o}\end{array}\right],$$

$$B_{oa}=\begin{bmatrix}B_{oa_1}\\ B_{oa_2}\end{bmatrix}=\left[\begin{array}{cc}I & 0\\ 0 & 0\\ \hline 0 & I\\ 0 & 0\end{array}\right],$$

$$C_{oa}=\begin{bmatrix}C_{oa_1} & C_{oa_2}\end{bmatrix}=\left[\begin{array}{cc|cc}D_oC_1 & C_{1_o} & D_oC_2 & C_{2_o}\end{array}\right].$$

III. 1-D MODEL REDUCTION TECHNIQUES

Here we provide a brief overview of un-weighted [28] and frequency weighted [30] model reduction techniques for the discrete-time 1-D systems.

A. UN-WEIGHTED 1-D MODEL REDUCTION TECHNIQUE

Let the controllability Gramians P_{c_*} and the observability Gramians Q_{o_*} for the entire frequency interim be given as [28]:

$$P_{c_*}=\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\psi}I-A_*)^{-1}B_*B_*^T(e^{-j\psi}I-A_*^T)^{-1}d\psi,$$

$$Q_{o_*}=\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\psi}I-A_*^T)^{-1}C_*^TC_*(e^{j\psi}I-A_*)^{-1}d\psi,$$

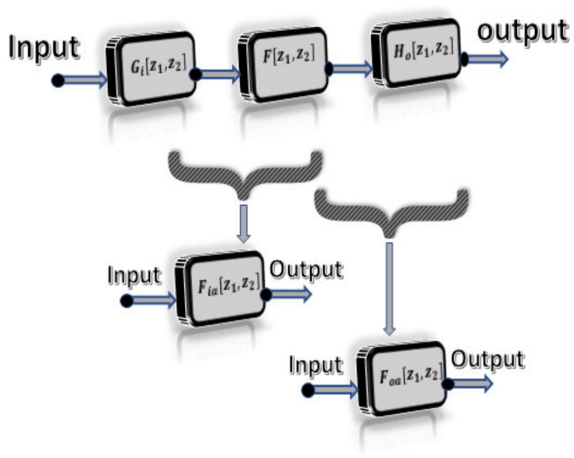

FIGURE 2. **Auxiliary Input and output weighted 2-D discrete-time system.**

that are the solution of the following Lyapunov equations:

$$A_* P_{c_*} A_*^T - P_{c_*} + B_* B_*^T = 0, \tag{27}$$
$$A_*^T Q_{o_*} A_* - Q_{o_*} + C_*^T C_* = 0, \tag{28}$$

Let a similarity transformation matrix T_b be given as:

$$T_b^T Q_{o_*} T_b = T_b^{-1} P_{c_*} T_b^{-T} = \Sigma_{co} = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \xi_n \end{bmatrix},$$

where $\Sigma_{co} = diag\{\Sigma_{co_1}, \Sigma_{co_2}\}$, $\xi_i \geq \xi_{i+1}$, $i = 1, 2, 3, \ldots, n-1$, $\xi_r > \xi_{r+1}$. The ROM is attained as [28], [29]:

$$T_b^{-1} A_* T_b = \begin{bmatrix} A_{r_*} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_b^{-1} B = \begin{bmatrix} B_{r_*} \\ B_2 \end{bmatrix}, \tag{29}$$
$$C T_b = \begin{bmatrix} C_{r_*} & C_2 \end{bmatrix}, \quad D = D_{r_*}. \tag{30}$$

Lemma 3 ([28]): The ROM (i.e., $F_{r_*}[z]$) obtained by using BT [28] is stable and the truncation error is bounded by:

$$\|F_*[z] - F_{r_*}[z]\|_\infty = 2 \sum_{i=r+1}^{n} \xi_i. \tag{31}$$

Lemma 4 ([29]): The ROM (i.e., $F_{rh_*}[z]$) obtained by using an optimal Hankel norm approximation [29] is stable and the truncation error is bounded by:

$$\|F_*[z] - F_{rh_*}[z]\|_\infty = \sum_{i=r+1}^{n} \xi_i. \tag{32}$$

B. FREQUENCY WEIGHTED 1-D MODEL REDUCTION TECHNIQUE

Let the controllability Gramians P_{ai} and the observability Gramians Q_{ao} for the corresponding input-augmented (5) and the output-augmented (6) realization respectively, that satisfy the following Lyapunov equations:

$$P_{ai} = \begin{bmatrix} P_E & P_{12} \\ P_{12}^T & P_G \end{bmatrix}, \quad Q_{ao} = \begin{bmatrix} Q_H & Q_{12}^T \\ Q_{12} & Q_E \end{bmatrix},$$

that satisfy the following Lyapunov equations:

$$A_{ai} P_{ai} A_{ai}^T - P_{ai} + B_{ai} B_{ai}^T = 0, \tag{33}$$
$$A_{ai}^T Q_{ai} A_{ai} - Q_{ai} + C_{ai}^T C_{ai} = 0, \tag{34}$$

Truncating 1^{st} and 4^{th} block of (33) and (34), respectively, we have the following Lyapunov equations:

$$A_* P_E A_*^T - P_E + X_E = 0, \tag{35}$$
$$A_*^T Q_E A_* - Q_E + Y_E = 0, \tag{36}$$

where

$$X_E = B_E B_E^T = B_* C_{iw} P_{12}^T A_*^T + A_* P_{12} C_{iw}^T B_*^T + B_* C_{iw} P_G C_{iw}^T B_*^T + B_* D_{iw} D_{iw}^T B_*^T, \tag{37}$$
$$Y_E = C_E^T C_E = C_*^T B_o^T Q_{12}^T A_* + A_*^T Q_{12} B_{ow} C_* + C_*^T B_{ow}^T Q_H C_* B_{ow} + C_*^T D_{ow}^T D_{ow} C_*. \tag{38}$$

By using the eigenvalues decomposition of X_E and Y_E we have the following:

$$X_E = U_E \begin{bmatrix} S_{E_1} & 0 \\ 0 & S_{E_2} \end{bmatrix} U_E^T, \tag{39}$$
$$B_E = U_E \begin{bmatrix} S_{E_1}{}^{1/2} & 0 \\ 0 & S_{E_2}{}^{1/2} \end{bmatrix} = U_E S_E{}^{1/2}, \tag{40}$$
$$Y_E = V_E \begin{bmatrix} R_{E_1} & 0 \\ 0 & R_{E_2} \end{bmatrix} V_E^T, \tag{41}$$
$$C_E = \begin{bmatrix} R_{E_1}{}^{1/2} & 0 \\ 0 & R_{E_2}{}^{1/2} \end{bmatrix} V_E^T = R_E{}^{1/2} V_E^T, \tag{42}$$

where

$$S_{E_1} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ 0 & 0 & \cdots & s_{l-1} \end{bmatrix},$$
$$S_{E_2} = \begin{bmatrix} s_l & 0 & \cdots & 0 \\ 0 & s_{l+1} & \cdots & 0 \\ 0 & 0 & \cdots & s_n \end{bmatrix},$$
$$R_{E_1} = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ 0 & 0 & \cdots & r_{k-1} \end{bmatrix},$$
$$R_{E_2} = \begin{bmatrix} r_k & 0 & \cdots & 0 \\ 0 & r_{k+1} & \cdots & 0 \\ 0 & 0 & \cdots & r_n \end{bmatrix},$$

S_{E_1} and R_{E_1} have $(l-1)$ and $(k-1)$ numbers of positive eigenvalues respectively; similarly, S_{E_2} and R_{E_2} have $(n-l)$ and $(n-k)$ numbers of negative eigenvalues respectively. Let T_E be the transformation matrix obtained as:

$$T_E^T Q_E T_E = T_E^{-1} P_E T_E^{-T} = diag\{\Xi_1, \Xi_2\} = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \xi_n \end{bmatrix},$$

where $\Xi_1 = diag\{\xi_1, \xi_2, \cdots, \xi_r\}$, $\Xi_2 = diag\{\xi_{r+1}, \xi_{r+2}, \cdots, \xi_n\}$, $\xi_j \geq \xi_{j+1}$, $j = 1, 2, \ldots, n-1$, $\xi_r > \xi_{r+1}$.

The transformation matrix T_E transforms the original stable large-scale system realization into a balanced realization. The ROM $F_{r_*}[z] = \hat{D}_{r_*} + \hat{C}_{r_*}[zI - \hat{A}_{r_*}]^{-1}\hat{B}_{r_*}$ is acquired by truncating the transformed balanced realization.

$$T_E^{-1}A_*T_E = \hat{A}_* = \begin{bmatrix} \hat{A}_{r_*} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad T_E^{-1}B_* = \hat{B}_* = \begin{bmatrix} \hat{B}_{r_*} \\ \hat{B}_2 \end{bmatrix}, \tag{43}$$

$$C_*T_E = \hat{C}_* = \begin{bmatrix} \hat{C}_{r_*} & \hat{C}_2 \end{bmatrix}, \quad D_* = \hat{D}_{r_*}, \tag{44}$$

Remark 2: This technique [30] provide unstable ROMs because input/output associated matrices X_E and Y_E respectively are indefinite (i.e., $X_E \leq 0$ and $Y_E \leq 0$) [32] when both-sided weights are used.

IV. EXISTING STABILITY PRESERVING FREQUENCY WEIGHTED MOR TECHNIQUES

Here we provide a brief overview of existing frequency weighted model reduction techniques for the 1-D [39], [40], [43] and 2-D [42] systems.

A. EXISTING 1-D STABILITY PRESERVING FREQUENCY WEIGHTED MOR TECHNIQUES

CB [40], VA [39], and IG [43] improvised Enns's [30] input and output associated matrices X_E and Y_E, respectively, to yield positive and positive-semi definiteness of these matrices, which consequently yield stability of the ROM. These techniques also offer an error bounds formula. The controllability and observability Gramians P_{ex} and Q_{ex}, respectively, satisfying the following Lyapunov equations:

$$A_*P_{ex}A_*^T - P_{ex} + B_{ex}B_{ex}^T = 0, \tag{45}$$
$$A_*^TQ_{ex}A_* - Q_{ex} + C_{ex}^TC_{ex} = 0, \tag{46}$$

The improvisation by CB [40], VA [39], and IG [43] introduced fictitious input and output associated matrices $B_{ex} \in \{B_{ex_1}$ [40], B_{ex_2}, [39], B_{ex_3} [43]} and $C_{ex} \in \{C_{ex_1}$ [40], C_{ex_2} [39], C_{ex_3} [43]}, respectively, can be computed as:

$$B_{ex_1} = U_{ex_1}S_{ex_1}U_{ex_1}^T = \begin{bmatrix} U_{ex_{1_1}} & U_{ex_{1_2}} \end{bmatrix} \begin{bmatrix} S_{E_1} & 0 \\ 0 & |S_{E_2}| \end{bmatrix} \times \begin{bmatrix} U_{ex_{1_1}}^T \\ U_{ex_{1_2}}^T \end{bmatrix},$$

$$B_{ex_2} = U_{ex_2}S_{ex_2}U_{ex_2}^T = \begin{bmatrix} U_{ex_{2_1}} & U_{ex_{2_2}} \end{bmatrix} \begin{bmatrix} S_{E_1} & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} U_{ex_{2_1}}^T \\ U_{ex_{2_2}}^T \end{bmatrix},$$

$$B_{ex_3} = U_{ex_3}S_{ex_3}U_{ex_3}^T = \begin{bmatrix} U_{ex_{3_1}} & U_{ex_{3_2}} \end{bmatrix} \times \begin{bmatrix} S_{E_1} - s_nI_{(l-1)(l-1)} & 0 \\ 0 & S_{E_2} - s_nI_{(n-l)(n-l)} \end{bmatrix} \times \begin{bmatrix} U_{ex_{3_1}}^T \\ U_{ex_{3_2}}^T \end{bmatrix},$$

$$C_{ex_1} = V_{ex_1}R_{ex_1}V_{ex_1}^T = \begin{bmatrix} V_{ex_{1_1}} & V_{ex_{1_2}} \end{bmatrix} \begin{bmatrix} R_{E_1} & 0 \\ 0 & |S_{E_2}| \end{bmatrix} \begin{bmatrix} V_{ex_{1_1}}^T \\ V_{ex_{1_2}}^T \end{bmatrix},$$

$$C_{ex_2} = V_{ex_2}R_{ex_2}V_{ex_2}^T = \begin{bmatrix} V_{ex_{2_1}} & V_{ex_{2_2}} \end{bmatrix} \begin{bmatrix} R_{E_1} & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} V_{ex_{2_1}}^T \\ V_{ex_{2_2}}^T \end{bmatrix},$$

$$C_{ex_3} = V_{ex_3}R_{ex_3}V_{ex_3}^T = \begin{bmatrix} V_{ex_{3_1}} & V_{ex_{3_2}} \end{bmatrix} \times \begin{bmatrix} R_{E_1} - r_nI_{(p-1)(p-1)} & 0 \\ 0 & R_{E_2} - r_nI_{(n-p)(n-p)} \end{bmatrix} \times \begin{bmatrix} V_{ex_{3_1}}^T \\ V_{ex_{3_2}}^T \end{bmatrix}.$$

Let $T_{ex} \in \{T_{ex_1}, T_{ex_2}, T_{ex_3}\}$ a transformation matrix be obtained as:

$$T_{ex}^TQ_{ex}T_{ex} = T_{ex}^{-1}P_{ex}T_{ex}^{-T} = \begin{bmatrix} \bar{\xi}_1 & 0 & \cdots & 0 \\ 0 & \bar{\xi}_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \bar{\xi}_n \end{bmatrix},$$

where $\bar{\xi}_j \geq \bar{\xi}_{j+1}$, $j = 1, 2, 3, \ldots, n-1$, $\bar{\xi}_r > \bar{\xi}_{r+1}$ where r is the order of the ROM. The ROM $F_{r_*}[z] = \bar{D}_{r_*} + \bar{C}_{r_*}[zI - \bar{A}_{r_*}]^{-1}\bar{B}_{r_*}$ is acquired as:

$$T_{ex}^{-1}A_*T_{ex} = \bar{A}_* = \begin{bmatrix} \bar{A}_{r_*} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad T_{ex}^{-1}B_* = \bar{B}_* = \begin{bmatrix} B_{r_*} \\ B_2 \end{bmatrix},$$
$$C_*T_{ex} = \bar{C}_* = \begin{bmatrix} \bar{C}_{r_*} & \bar{C}_2 \end{bmatrix}, \quad D_* = \bar{D}_{r_*}.$$

Remark 3: Since $X_E \leq B_{ex}B_{ex}^T$, $Y_E \leq C_{ex}^TC_{ex}$, $\{A_*, B_{ex}, C_{ex}\}$; consequently, yield minimal and stable ROMs. These techniques offer formula for the error bounds.

Remark 4: The following error-bound expression exists [40]:

$$\|F_*[z] - F_{r_*}[z]\|_\infty \leq 2\|L_{ex}\|_\infty\|K_{ex}\|_\infty \sum_{j=r+1}^{n} \bar{\xi}_j,$$

with the existence of the rank conditions [37] $rank\begin{bmatrix} B_{ex} & B_* \end{bmatrix} = rank\begin{bmatrix} B_{ex} \end{bmatrix}$ and $rank\begin{bmatrix} C_{ex} \\ C_* \end{bmatrix} = rank\begin{bmatrix} C_{ex} \end{bmatrix}$, where

$$L_{ex} = C_*V_{ex}diag(|r_1|^{-1/2}, |r_2|^{-1/2}, \cdots, |r_{li}|^{-1/2}, 0, \cdots 0),$$
$$K_{ex} = diag(|s_1|^{-1/2}, |s_2|^{-1/2}, \cdots, |s_{ko}|^{-1/2}, 0, \cdots, 0) \times U_{ex}^TB_*,$$

where $li = rank\,[X_E]$, $ko = rank\,[Y_E]$, $U_{ex} \in \{U_{ex_1}$ [40], U_{ex_2} [39], U_{ex_3} [43] } and $V_{ex} \in \{V_{ex_1}$ [40], V_{ex_2} [39], V_{ex_3} [43] }.

Remark 5: Since for each input related matrix $B_{ex} \in \{B_{ex_1}$ [40], B_{ex_2}, [39], B_{ex_3} [43]} and for each output related matrix $C_{ex} \in \{C_{ex_1}$ [40], C_{ex_2} [39], C_{ex_3} [43]} grant positive and positive-semi definite of the original system's input and the original system's output associated matrices, respectively; which results into the positive and positive-semi definite of the controllability matrices $P_{ex} \in \{P_{ex_1}$ [40], P_{ex_2}, [39], P_{ex_3} [43]} and the observability matrices $Q_{ex} \in \{Q_{ex_1}$ [40],

Q_{ex_2}, [39], Q_{ex_3} [43]} in a unique way. This leads to the existence of the different transformation matrices $T_{ex} \in \{T_{ex_1}$ [40], T_{ex_2}, [39], T_{ex_3} [43]}. As a consequence, three existing stability-preserving model order reduction techniques are established.

B. EXISTING 2-D STABILITY PRESERVING FREQUENCY WEIGHTED MOR TECHNIQUE

GS [42] modified Enns's [30] matrices X_E and Y_E and applied these matrices for 2-D MOR case (by using minimal rank-decomposition conditions) to grant positive and positive-semi definite of these input and output associated matrices, which consequently grant stable ROMs for the decomposed two 1-D systems and also yield error bounds. For decomposed systems $\bar{F}_1[z_1] = \bar{D}_{1_*} + C_1[z_1 I - A_1]^{-1}\bar{B}_{1_*}$ and $\bar{F}_2[z_2] = \bar{D}_{2_*} + \bar{C}_{2_*}[z_2 I - A_4]^{-1}B_2$ the controllability and observability Gramians $\hat{P}_{ex} \in \{\hat{P}_{ex_1}$ [42], $\hat{P}_{ex_2}$ [42] } and $\hat{Q}_{ex} \in \{\hat{Q}_{ex_1}$ [42], $\hat{Q}_{ex_2}$ [42] }, respectively, satisfying following Lyapunov equations:

$$A\hat{P}_{ex}A^T - \hat{P}_{ex} + \hat{B}_{ex}\hat{B}_{ex}^T = 0, \quad (47)$$
$$A^T\hat{Q}_{ex}A - \hat{Q}_{ex} + \hat{C}_{ex}^T\hat{C}_{ex} = 0. \quad (48)$$

For the systems $A \in \{A_1, A_4\}$ the input and output related matrices $\hat{B}_{ex} \in \{\hat{B}_{ex_1}$ [42], $\hat{B}_{ex_2}$, [42]} and $\hat{C}_{ex} \in \{\hat{C}_{ex_1}$ [42], $\hat{C}_{ex_2}$ [42]}, respectively, can be computed as:

$$\hat{B}_{ex_1} = \hat{U}_{ex_1}\hat{S}_{ex_1}\hat{U}_{ex_1}^T = \begin{bmatrix} \hat{U}_{ex_{1_1}} & \hat{U}_{ex_{1_2}} \end{bmatrix} \begin{bmatrix} S_{E_1} & 0 \\ 0 & |S_{E_2}| \end{bmatrix} \times \begin{bmatrix} \hat{U}_{ex_{1_1}}^T \\ \hat{U}_{ex_{1_2}}^T \end{bmatrix},$$

$$\hat{B}_{ex_2} = \hat{U}_{ex_2}\hat{S}_{ex_2}\hat{U}_{ex_2}^T = \begin{bmatrix} \hat{U}_{ex_{2_1}} & \hat{U}_{ex_{2_2}} \end{bmatrix} \begin{bmatrix} S_{E_1} & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} \hat{U}_{ex_{2_1}}^T \\ \hat{U}_{ex_{2_2}}^T \end{bmatrix},$$

$$\hat{C}_{ex_1} = \hat{V}_{ex_1}\hat{R}_{ex_1}\hat{V}_{ex_1}^T = \begin{bmatrix} \hat{V}_{ex_{1_1}} & \hat{V}_{ex_{1_2}} \end{bmatrix} \begin{bmatrix} R_{E_1} & 0 \\ 0 & |S_{E_2}| \end{bmatrix} \times \begin{bmatrix} \hat{V}_{ex_{1_1}}^T \\ \hat{V}_{ex_{1_2}}^T \end{bmatrix},$$

$$\hat{C}_{ex_2} = \hat{V}_{ex_2}\hat{R}_{ex_2}\hat{V}_{ex_2}^T = \begin{bmatrix} \hat{V}_{ex_{2_1}} & \hat{V}_{ex_{2_2}} \end{bmatrix} \begin{bmatrix} R_{E_1} & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} \hat{V}_{ex_{2_1}}^T \\ \hat{V}_{ex_{2_2}}^T \end{bmatrix}.$$

Let $\hat{T}_{ex} \in \{\hat{T}_{ex_1}$ [42], $\hat{T}_{ex_2}$, [42]} a transformation matrix be obtained as:

$$\hat{T}_{ex}^T\hat{Q}_{ex}\hat{T}_{ex} = \hat{T}_{ex}^{-1}\hat{P}_{ex}\hat{T}_{ex}^{-T} = \begin{bmatrix} \hat{\xi}_1 & 0 & \cdots & 0 \\ 0 & \hat{\xi}_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \hat{\xi}_n \end{bmatrix},$$

where $\hat{\xi}_j \geq \hat{\xi}_{j+1}$, $j = 1, 2, 3, \ldots, n-1$, $\hat{\xi}_r > \hat{\xi}_{r+1}$ where r is the order of the ROM. The ROM $\bar{F}_{1_r}[z_1] = C_{1_r}[zI - A_{1_r}]^{-1}\bar{B}_{1_{r*}} + \bar{D}_{1_{r*}}$ and $\bar{F}_{2_r}[z_2] = \bar{C}_{2_{r*}}[zI - A_{4_r}]^{-1}B_{2_r} + \bar{D}_{2_{r*}}$ can be acquired as:

$$\hat{T}_{ex_1}^{-1}A_1\hat{T}_{ex_1} = \begin{bmatrix} A_{1_r} & A_{1_{12}} \\ A_{1_{21}} & A_{1_{22}} \end{bmatrix}, \quad \hat{T}_{ex_1}^{-1}\bar{B}_{1_*} = \begin{bmatrix} \bar{B}_{1_{r*}} \\ \bar{B}_{1_{2*}} \end{bmatrix}, \quad (49)$$

$$C_1\hat{T}_{ex_1} = \begin{bmatrix} C_{1_{r*}} & C_{1_2} \end{bmatrix}, \quad \bar{D}_{1_*} = \bar{D}_{1_{r*}}, \quad (50)$$

similarly,

$$\hat{T}_{ex_2}^{-1}A_4\hat{T}_{ex_2} = \begin{bmatrix} A_{4_r} & A_{4_{12}} \\ A_{4_{21}} & A_{4_{22}} \end{bmatrix}, \quad \hat{T}_{ex_2}^{-1}\bar{B}_{1_*} = \begin{bmatrix} B_{2_{r*}} \\ B_{2_2} \end{bmatrix}, \quad (51)$$

$$\bar{C}_{2_*}\hat{T}_{ex_2} = \begin{bmatrix} \bar{C}_{2_{r*}} & \bar{C}_{2_{2*}} \end{bmatrix}, \quad \bar{D}_{2_*} = \bar{D}_{2_{r*}}, \quad (52)$$

For each decomposed original systems $\bar{F}_1[z_1] = \bar{D}_{1_*} + C_1[z_1 I - A_1]^{-1}\bar{B}_{1_*}$ and $\bar{F}_2[z_2] = \bar{D}_{2_*} + \bar{C}_{2_*}[z_2 I - A_4]^{-1}B_2$ the ROMs obtained $\{A_{1_{r*}}, \bar{B}_{1_{r*}}, C_{1_{r*}}, \bar{D}_{1_{r*}}\}$ and $\{A_{4_{r*}}, B_{2_r}, \bar{C}_{2_{r*}}, \bar{D}_{2_{r*}}\}$ respectively are minimal and stable.

Remark 6: Since $\bar{X}_E \leq \hat{B}_{ex}\hat{B}_{ex}^T$ and $\bar{Y}_E \leq \hat{C}_{ex}^T\hat{C}_{ex}$; consequently, the realizations $\{A_1, \hat{B}_{ex_1}, \hat{C}_{ex_1}\}$ and $\{A_4, \hat{B}_{ex_2}, \hat{C}_{ex_2}\}$ are minimal and stable respectively, moreover; yield minimal and stable ROMs. These techniques offer a formula for the error bounds.

Remark 7: Since for each input related matrix $\hat{B}_{ex} \in \{\hat{B}_{ex_1}$ [42], $\hat{B}_{ex_2}$, [42]} and for each output related matrix $\hat{C}_{ex} \in \{\hat{C}_{ex_1}$ [42], $\hat{C}_{ex_2}$ [42]} grant positive and positive-semi definite of decomposed original system's input and decomposed original system's output related matrices respectively; which results into the positive and positive-semi definite of the controllability matrices $\hat{P}_{ex} \in \{\hat{P}_{ex_1}$ [42], $\hat{P}_{ex_2}$ [42] } and the observability matrices $\hat{Q}_{ex} \in \{\hat{Q}_{ex_1}$ [42], $\hat{Q}_{ex_2}$ [42] } in a unique way. This leads to the existence of the different transformation matrices $\hat{T}_{ex} \in \{\hat{T}_{ex_1}$ [42], $\hat{T}_{ex_2}$, [42]} which subsequently results in ROMs $\{A_{1_{r*}}, \bar{B}_{1_{r*}}, C_{1_{r*}}, \bar{D}_{1_{r*}}\}$ and $\{A_{4_{r*}}, B_{2_{r*}}, \bar{C}_{2_{r*}}, \bar{D}_{2_{r*}}\}$ for the given decomposed systems $\bar{F}_1[z_1] = \bar{D}_{1_*} + C_1[z_1 I - A_1]^{-1}\bar{B}_{1_*}$ and $\bar{F}_2[z_2] = \bar{D}_{2_*} + \bar{C}_{2_*}[z_2 I - A_4]^{-1}B_2$, respectively. As a consequence, ROMs obtained are stable, and these techniques yield error bound formula [42].

Remark 8: Similarly, ROMs for decomposed systems $\hat{F}[z_1, z_2] = \hat{F}_2[z_2]\hat{F}_1[z_1]$ as in (19) and (20) are obtained in similar way as in (49-50) and (51-52) respectively. Moreover, ROMs obtained are stable and also yield error bound formula [42].

V. MAIN RESULTS

The stability preserving strategies for 1-D discrete-time systems proposed by CB [40], GS [37], and IG [46] modified X_E and Y_E to ensure the stability of the ROM by making positive and semi-positive definite of the associated input and the associated output matrices. However, these methods induce significant truncation errors in some distinct frequency weights due to significant variance form the original systems.

This paper presents a stability preserving frequency-weighted MOR technique for discrete-time 1-D and 2-D systems. For the 1-D and 2-D systems, the ROM's stability is ensured by inserting some fictitious input and output matrices. The fictitious matrices are created by square-rooting eigenvalues that have identical effects on each eigenvalue of 1-D and 2-D discrete-time input and output matrices to construct stable ROMs with low truncation errors at specified frequency weights. Decomposition is performed first for the discrete-time 2-D weighted system using the minimal rank-decomposition condition as illustrated in (11,16); then, the controllability and the observability Gramians are computed based on modified associated input and output matrices for decomposed 1-D sub-systems. The proposed scheme also provides an *a priori* error bound expressions by using the BT and an optimal Hankel norm approximation approaches, respectively, for the 1-D and 2-D discrete-time frequency weighted systems. A comparison among different existing frequency weighted MOR techniques (including 1-D and 2-D systems) with proposed techniques are presented, which show the efficacy of proposed methods.

A. 1-D FREQUENCY WEIGHTED MODEL REDUCTION TECHNIQUE FOR DISCRETE-TIME SYSTEMS

Let a new fictitious controllability Gramians matrix $\bar{P}_m$ and the observability Gramians matrix $\bar{Q}_m$ for 1-D discrete-time systems are computed as

$$A_*\bar{P}_mA_*^T - \bar{P}_m + \bar{X}_m = 0, \tag{53}$$

$$A_*^T\bar{Q}_mA_* - \bar{Q}_m + \bar{X}_m = 0, \tag{54}$$

where $\bar{X}_m = \bar{B}_m\bar{B}_m^T$ and $\bar{Y}_m = \bar{C}_m^T\bar{C}_m$. By eigenvalues decomposition of $\bar{X}_m$ and $\bar{Y}_m$ we have the following:

$$\bar{X}_m = \bar{U}_m\hat{S}_m\bar{U}_m^T, \tag{55}$$

$$\bar{Y}_m = \bar{V}_m\hat{R}_m\bar{V}_m^T, \tag{56}$$

The new fictitious $\bar{B}_m$ and $\bar{C}_m$ are given as input and output associated matrices respectively, where

$$\bar{B}_m = \begin{cases} \bar{U}_m\sqrt{\dfrac{(S_E - s_nI)^{1/2}}{s_n}} = \bar{U}_m\hat{S}_m^{1/2} & \text{for } s_n < 0 \\ U_ES_E^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{57}$$

$$\bar{C}_m = \begin{cases} \sqrt{\dfrac{(R_E - r_nI)^{1/2}}{r_n}}\bar{V}_m^T = \hat{R}_m^{1/2}\bar{V}_m^T & \text{for } r_n < 0 \\ R_E^{1/2}V_E^T & \text{for } r_n \geq 0 \end{cases} \tag{58}$$

Let the similarity transformation matrix $\bar{T}_m$ is calculated as:

$$\Sigma_m = \bar{T}_m^T\bar{Q}_m\bar{T}_m = \bar{T}_m^{-1}\bar{P}_m\bar{T}_m^{-T} = \begin{bmatrix} \bar{\rho}_1 & 0 & \cdots & 0 \\ 0 & \bar{\rho}_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \bar{\rho}_n \end{bmatrix},$$

where $\Sigma_m = diag\{\Sigma_{m_1}, \Sigma_{m_2}\}$, $\bar{\rho}_j \geq \bar{\rho}_{j+1}$ and $\bar{\rho}_r \geq \bar{\rho}_{r+1}$. The ROM $F_{r_*}[z] = \bar{D}_r + \bar{C}_{r_*}[zI - \bar{A}_{r_*}]^{-1}\bar{B}_{r_*}$ is obtained as

$$\bar{T}_m^{-1}A_*\bar{T}_m = \bar{A}_* = \begin{bmatrix} \bar{A}_{r_*} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{T}_m^{-1}B_* = \bar{B}_* = \begin{bmatrix} \bar{B}_r \\ \bar{B}_2 \end{bmatrix}, \tag{59}$$

$$C\bar{T}_m = \bar{C} = \begin{bmatrix} \bar{C}_{r_*} & \bar{C}_2 \end{bmatrix}, \quad D = \bar{D}_{r_*}. \tag{60}$$

The above MOR procedure can be viewed in the context of non-minimum phase systems.

Lemma 5 ([54]): If the n^{th} order square discrete-time 1-D minimal realization be given as:

$$F_*[z] \Leftrightarrow \left[\begin{array}{c|c} A_* & B_* \\ \hline C_* & D_* \end{array}\right] = \left[\begin{array}{cc|c} A_k & A_2 & B_1 \\ A_3 & A_4 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

and $H_o[z] = F_*^{-1}[z]$; then, $A_i = A_* - B_*D_*^{-1}C_*$ has k eigenvalues outside the unit circle. Let $\lambda_l[A_i]\bar{\lambda}_j[A_i] \neq 1 \; \forall \; l, j$; then, there exist a unique controllability and the observability matrices, P_{c_*} and Q_{o_*}, respectively, which are the solution to the Lyapunov equation as in (27) and $(A_i^TQ_{o_*}A_i - Q_{o_*} + C_*^T(D^{-1})^TDC_* = 0)$, respectively. Further, Q_{o_*} contain k and $n-k$ negative and positive eigenvalues, respectively.

Remark 9: The realization $F_*[z]$ can be decomposed into two sub-systems as:

$$F_*[z] = F_k[z] + F_{n-k}[z]$$

where

$$F_k[z] \Leftrightarrow \left[\begin{array}{c|c} A_k & B_1 \\ \hline C_1 & D \end{array}\right],$$

$$F_{n-k}[z] \Leftrightarrow \left[\begin{array}{c|c} A_4 & B_2 \\ \hline C_2 & D \end{array}\right].$$

The realization $F_k[z]$ has exactly k zeros outside of the unit disk; whereas, the rest of the zeros are inside the unit disc.

Similarly, the above MOR procedure can be viewed in the context of unstable minimum phase systems.

Lemma 6 ([54]): If the n^{th} order square discrete-time 1-D realization be given as:

$$F_*[z] \Leftrightarrow \left[\begin{array}{c|c} A_* & B_* \\ \hline C_* & D_* \end{array}\right] = \left[\begin{array}{cc|c} A_k & A_2 & B_1 \\ A_3 & A_4 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

with no eigenvalues on the unit circle and let $P_{c_*} = P_{c_*}^T$ be the solution to the Lyapunov equation as in (53) with $P_{c_*} = diag\{\Sigma_{c_1}, \Sigma_{c_2}\}$, where Σ_{c_1} is non-singular matrix and $\Sigma_{c_2} > 0$ is a diagonal matrix. Then, A_k and A_* have k unstable poles (eigenvalues) outside the unit circle; also, A_k has no eigenvalues inside the unit circle. Assume that $F_*[\pi] = D_*$ is a nonsingular and $\lambda_l[A_*]\bar{\lambda}_j[A_*] \neq 1 \; \forall \; l, j$. Further, P_{c_*} contain k and $n-k$ negative and positive eigenvalues, respectively.

Remark 10: The realization $F_{c_*}[z]$ can be decomposed into two sub-systems as:

$$F_{c_*}[z] = F_k[z] + F_{n-k}[z]$$

where

$$F_k[z] \Leftrightarrow \left[\begin{array}{c|c} A_k & B_1 \\ \hline C_1 & D \end{array}\right],$$
$$F_{n-k}[z] \Leftrightarrow \left[\begin{array}{c|c} A_4 & B_2 \\ \hline C_2 & D \end{array}\right].$$

The realization $F_k[z]$ has exactly k poles (eigenvalues) outside of the unit disk; whereas, the rest of the poles are inside the unit disc.

Furthermore, the proposed MOR procedure can be employed for the marginally stable systems by decomposing the original systems into sub-systems (i.e., asymptotically stable + marginally stable).

Lemma 7 ([55]): There exists a similarity transformation matrix T_{sm} that satisfies:

$$A_* = T_{sm} \left[\begin{array}{c|c} A_s & 0 \\ 0 & A_m \end{array}\right] T_{sm}^{-1},$$

such that the full-order-model as in (1) is marginally stable and matrix A_* has a full rank.

Remark 11: The decomposition as in Lemma. 7 enables each sub-system to be reduced in a manner that preserves its particular notion of stability. Further, the MOR for each sub-systems are obtained in a similar way as in (59)-(60).

Remark 12: Since $X_E \le \bar{B}_m \bar{B}_m^T \ge 0$, $Y_E \le \bar{C}_m^T \bar{C}_m \ge 0$, $\bar{P}_m > 0$ and $\bar{Q}_m > 0$. Therefore, the transformed-realization $\{A_*, \bar{B}_m, \bar{C}_m\}$ is minimal and the stability of the ROM is guaranteed.

Lemma 8: The fictitious input associated matrices $X_E \le \bar{B}_m \bar{B}_m^T \ge 0$ and the fictitious output associated matrices $Y_E \le \bar{C}_m^T \bar{C}_m \ge 0$, likewise, the controllability matrices $P_E < \bar{P}_m > 0$ and the observability matrices $Q_E < \bar{Q}_m > 0$. Therefore, the transformed-realization $\{A_*, \bar{B}_m, \bar{C}_m\}$ obtained is minimal and stable which also guaranteed the ROM's stability in the desired frequency-intervals.

Proof of Lemma 8: we will demonstrate that the realization $\{A_*, \bar{B}_m, \bar{C}_m\}$ is minimal (i.e., controllable and observable). Since the controllability Gramians matrix $\bar{P}_m$ and the observability Gramians matrix $\bar{Q}_m$ are solution of Lyapunov equations as in (53) and (54) respectively, so

$$\begin{aligned}
&\bar{B}_m \bar{B}_m^T - X_E \ge 0 \\
&\quad \bar{P}_m - P_E \ge \frac{1}{2\pi} \int_{\delta\omega} (e^{j\omega} I - A_*)^{-1} \bar{B}_m \bar{B}_m^T \\
&\qquad \times (e^{-j\omega} I - A_*^T)^{-1} d\omega \\
&\qquad - \frac{1}{2\pi} \int_{\delta\omega} (e^{j\omega} I - A_*)^{-1} B_E B_E^T \\
&\qquad \times (e^{-j\omega} I - A_*^T)^{-1} d\omega \\
&\quad = \frac{1}{2\pi} \int_{\delta\omega} (e^{j\omega} I - A_*)^{-1} (\bar{B}_m \bar{B}_m^T - X_E) \\
&\qquad \times (e^{-j\omega} I - A_*^T)^{-1} d\omega \\
&\quad \ge 0
\end{aligned}$$

Since for $P_E \ge 0$; consequently, $\bar{P}_m \ge 0$ [37]. Similarly, for $Q_E \ge 0$; consequently, $\bar{Q}_m \ge 0$. As a consequence, the original-system matrix A_* is stable. Resultantly, the pair $(A_*, \bar{B}_m)$ is controllable and the pair $(A_*, \bar{C}_m)$ is observable (i.e., $\{A_*, \bar{B}_m, \bar{C}_m\}$ is minimal).

Lemma 9: [56] Since the pair $(A, \bar{B}_m)$ satisfy the following Lyapunov equation (53),

$$A_* \bar{P}_m A_*^T - \bar{P}_m = -\bar{B}_m \bar{B}_m^T,$$

for $\bar{P}_m \ge 0$; then, the original large-scale system is asymptotically stable *iff* it is controllable. Suppose the original system is not asymptotically stable. In that case, eigenvalues of the original large-scale system (i.e., eig$|A_*|$) are outside of the unit circle, not on the inside of the unit circle.

Proof of Lemma 9: The first part is obvious. To proof the second part, let A_* and ν^* have eigenvalue λ and corresponding left eigenvector respectively; then, $\nu^* A_* = \nu^* \lambda$ and $A_*^T \nu = \bar{\lambda} \nu$. Appropriately pre- multiplying and post-multiplying the Lyapunov equation (53) by ν^* and ν respectively; consequently, gives

$$\nu^* A_* \bar{P}_m A_*^T \nu - \nu^* \bar{P}_m \nu = -\nu^* \bar{B}_m \bar{B}_m^T \nu = (\lambda\bar{\lambda} - 1) \nu^* \bar{P}_m \nu.$$

Since the matrix $\nu^* \bar{P}_m \nu \ge 0$ and the matrix $\nu^* \bar{B}_m \bar{B}_m^T \nu \ge 0$, this results $|\lambda\bar{\lambda}| \le 1$. Furthermore, if $Re|\lambda| \ne 0$; then, $\nu^* \bar{B}_m \bar{B}_m^T \nu \ne 0$; hence, $\nu^* \bar{B}_m \ne 0$ which results the transformed-realization $\{A_*, \bar{B}_m, \bar{C}_m\}$ is controllable and stable.

Theorem 1: The following error-bound expression exists:

$$\begin{aligned}
&\|H_o[z](F_*[z] - F_{r_*}[z]) G_i[z]\|_\infty \\
&\quad \le 2 \|H_o[z]\|_\infty \|L_m\|_\infty \|K_m\|_\infty \|G_i[z]\|_\infty \sum_{j=r+1}^{n} \bar{\rho}_j,
\end{aligned}$$

with the existence of the rank conditions $rank \begin{bmatrix} \bar{B}_m & B_* \end{bmatrix} = rank \begin{bmatrix} \bar{B}_m \end{bmatrix}$ and $rank \begin{bmatrix} \bar{C}_m \\ C_* \end{bmatrix} = rank \begin{bmatrix} \bar{C}_m \end{bmatrix}$, where

$$\bar{L}_m = \begin{cases} C_* \bar{V}_m \hat{R}_m^{-1/2} & \text{if } r_n < 0 \text{ exists} \\ C_* V_E R_E^{-1/2} & \text{otherwise} \end{cases}$$
$$\bar{K}_m = \begin{cases} \hat{S}_m^{-1/2} \bar{U}_m^T B_* & \text{if } s_n < 0 \text{ exists} \\ S_E^{-1/2} U_E^T B_* & \text{otherwise} \end{cases}$$

Proof of Theorem 1: Since $rank \begin{bmatrix} \bar{B}_m & B_* \end{bmatrix} = rank \begin{bmatrix} \bar{B}_m \end{bmatrix}$ and $rank \begin{bmatrix} \bar{C}_m \\ C_* \end{bmatrix} = rank \begin{bmatrix} \bar{C}_m \end{bmatrix}$, the relationships $B_* = \bar{B}_m \bar{K}_m$ and $C = \bar{L}_m \bar{C}_m$ holds: By partitioning $\bar{B}_m = \begin{bmatrix} \bar{B}_{m_1} \\ \bar{B}_{m_2} \end{bmatrix}$, $\bar{C}_m = \begin{bmatrix} \bar{C}_{m_1} & \bar{C}_{m_2} \end{bmatrix}$ and substituting $\bar{B}_{r_*} = \bar{B}_{m_1} \bar{K}_m$, $\bar{C}_{r_*} = \bar{L}_m \bar{C}_{m_1}$, respectively, yields:

$$\begin{aligned}
&\|H_o[z](F[z] - F_r[z]) G_i[z]\|_\infty \\
&\quad = \|H_o[z](C[zI - A]^{-1} B - \hat{C}_r [zI - \hat{A}_r]^{-1} \hat{B}_r) G_i[z]\|_\infty \\
&\quad = \|H_o[z](L_m C_m [zI - A]^{-1} B_m K_m \\
&\qquad - L_m C_{m_1} [zI \hat{A}_r]^{-1} B_{m_1} K_m) G_i[z]\|_\infty \\
&\quad = \|H_o[z] L_m (C_m [zI - A]^{-1} B_m \\
&\qquad - C_{m_1} [zI - \hat{A}_r]^{-1} B_{m_1}) K_m G_i[z]\|_\infty
\end{aligned}$$

$$= \|H_o[z]L_m\|_\infty \|(C_m[zI - A]^{-1}B_m - C_{m_1}[zI - \hat{A}_r]^{-1}B_{m_1})\|_\infty \|K_m G_i[z]\|_\infty.$$

If $\{\bar{A}_{r_*}, \bar{B}_{m_1}, \bar{C}_{m_1}\}$ is the ROM attained after reduction of the large-scale original transformed system $\{A_*, \bar{B}_m, \bar{C}_m\}$. Then,

$$\|(\bar{C}_m[zI - A_*]^{-1}\bar{B}_m - \bar{C}_{m_1}[zI - \bar{A}_{r_*}]^{-1}\bar{B}_{m_1})\|_\infty \leq 2\sum_{j=r+1}^{n} \bar{\rho}_j.$$

Therefore,

$$\|H_o[z](F_*[z] - F_{r_*}[z])G_i[z]\|_\infty \leq 2\|H_o[z]\|_\infty \|L_m\|_\infty \|K_m\|_\infty \|G_i[z]\|_\infty \sum_{j=r+1}^{n} \bar{\rho}_j.$$

Theorem 2: The following error-bound expression exists:

$$\|H_o[z](F_*[z] - F_{rh_*}[z])G_i[z]\|_\infty \leq \|H_o[z]\|_\infty \|L_{mh}\|_\infty \|K_{mh}\|_\infty \|G_i[z]\|_\infty \sum_{j=r+1}^{n} \bar{\rho}_j,$$

with the existence of the rank conditions $rank\begin{bmatrix}\bar{B}_{mh} & B_*\end{bmatrix} = rank\begin{bmatrix}\bar{B}_{mh}\end{bmatrix}$ and $rank\begin{bmatrix}\bar{C}_{mh} \\ C_*\end{bmatrix} = rank\begin{bmatrix}\bar{C}_{mh}\end{bmatrix}$, where

$$\bar{L}_{mh} = \begin{cases} C_* \bar{V}_{mh} \hat{R}_{mh}^{-1/2} & \text{if } r_n < 0 \text{ exists} \\ C_* V_E R_E^{-1/2} & \text{otherwise} \end{cases}$$

$$\bar{K}_{mh} = \begin{cases} \hat{S}_{mh}^{-1/2} \bar{U}_{mh}^T B_* & \text{if } s_n < 0 \text{ exists} \\ S_E^{-1/2} U_E^T B_* & \text{otherwise} \end{cases}$$

Proof of Theorem 2: The proof of above-mentioned Theorem 2 is similar to the proof of Theorem 1; hence, omitted for the brevity.

Corollary 1: Theorem 1 holds true subject to the following rank conditions: $rank\begin{bmatrix}\bar{B}_m & B_*\end{bmatrix} = rank\begin{bmatrix}\bar{B}_m\end{bmatrix}$ and $rank\begin{bmatrix}\bar{C}_m \\ C_*\end{bmatrix} = rank\begin{bmatrix}\bar{C}_m\end{bmatrix}$ (which follows from [57]) are satisfied.

Remark 13: When $X_E \geq 0$ and $Y_E \geq 0$; then, $P_E = P_{ex} = \bar{P}_m$ and $Q_E = Q_{ex} = \bar{Q}_m$; consequently, ROMs obtained by using [30], [39], [40], [43], and suggested technique are the equivalent. Otherwise $P_E < \bar{P}_m$ and $Q_E < \bar{Q}_m$. Furthermore, the frequency-weighted Hankel singular-values satisfy : $(\lambda_j[P_E Q_E])^{1/2} \leq (\bar{\lambda}_j[\bar{P}_m \bar{Q}_m])^{1/2}$.

Remark 14: When $X_E \geq 0$ and $Y_E \geq 0$; then, ROMs obtained using Enns [30] and suggested framework are the equivalent.

Remark 15: For the fictitious-input matrix $\bar{B}_m$ and the fictitious-output matrix $\bar{C}_m$ grant positive and positive-semi definite of the input associated matrix B_* and the output associated matrix C_* respectively; consequently, positive and positive-semi definite of the controllability Gramians matrix $\bar{P}_m$ and the observability Gramians matrix $\bar{Q}_m$. This corresponds to transformation matrix $\bar{T}_m$, resulting in the stability retention MOR algorithm. In addition, constants (i.e., $\bar{L}_m$ and $\bar{K}_m$) provides the relationship between the systems matrices (i.e., B_* and C_*) with the fictitious matrices (i.e., $\bar{B}_m$ and $\bar{C}_m$), resulting in the error-bound expression for the suggested framework.

Remark 16 ([58]): The ill-conditioning of the relevant discrete-time Lyapunov equations as in (53)-(54) causes difficulty in computing the ROM based on Gramians of sampled-data models for smaller sampling periods. The numerical results are distorted by errors up to a particular limit for the sampling step. To get over this limitation, an ''approximately'' balanced realization of the sampled-data system is obtained straight from its continuous-time counterpart's balanced realization. When the sample time is reduced to zero, this realization comes ''near'' to be exactly balanced for ''extremely small'' (i.e., $\delta[T] = T_2 - T_1 = \iota$) sample steps (i.e., considerably less than the systems' time constants), where T is sampling time, and ι is a very small number. Similarly, the error based on the Hankel singular values (i.e., $\bar{\rho}_j$) and frequency response error will be the same. It's also worth noting that the bilinear mapping (i.e., $z \longrightarrow (1+s)/(1-s)$) produces a balanced continuous-time equivalent system if the original discrete-time approach was similarly balanced [29].

Theorem 3: The following Lyapunov equation for the suggested framework holds:

$$A_* \bar{P}_{(ext)} A_*^T - \bar{P}_{(ext)} + \bar{B}_{(ext)} \bar{B}_{(ext)}^T = 0, \tag{61}$$

$$A_*^T \bar{Q}_{(ext)} A_* - \bar{Q}_{(ext)} + \bar{C}_{(ext)}^T \bar{C}_{(ext)} = 0. \tag{62}$$

Proof of Theorem 3: Using (39), (41), (57) and (58) we have the following:

$$\begin{aligned} S_E &= diag[S_{E_1},\ S_{E_2}] = diag[(s_1, .., s_{l-1}), (s_l, .., s_n)], \\ \hat{S}_m &= diag[\hat{S}_{m_1},\ \hat{S}_{m_2}] = diag[(\hat{s}_1, .., \hat{s}_{l-1}), (\hat{s}_l, .., \hat{s}_n)], \\ R_E &= diag[R_{E_1},\ R_{E_2}] = diag[(r_1, .., r_{p-1}), (r_p, .., r_n)], \\ \hat{R}_m &= diag[\hat{R}_{m_1}, \hat{R}_{m_2}] = diag[(\hat{r}_1, .., \hat{r}_{p-1}), (\hat{r}_p, .., \hat{r}_n)], \end{aligned}$$

$\bar{S}_{(ext)}$ and $\bar{R}_{(ext)}$ are obtained by $(\hat{S}_m - S_E)$ and $(\hat{R}_m - R_E)$, respectively.

$$\bar{S}_{(ext)} = \begin{bmatrix} \bar{S}_{(ext)_1} & 0 \\ 0 & \bar{S}_{(ext)_2} \end{bmatrix}, \quad \bar{R}_{(ext)} = \begin{bmatrix} \bar{R}_{(ext)_1} & 0 \\ 0 & \bar{R}_{(ext)_2} \end{bmatrix},$$

where matrices $\bar{B}_{(ext)}$ and $\bar{C}_{(ext)}$ are obtained by (57− 40) and (58 − 42), respectively.

$$\bar{B}_{(ext)} = \bar{U}_{(ext)} \begin{bmatrix} \bar{S}_{(ext)_1}^{1/2} & 0 \\ 0 & \bar{S}_{(ext)_2}^{1/2} \end{bmatrix} = \bar{U}_{(ext)} \bar{S}_{(ext)}^{1/2},$$

$$\bar{C}_{(ext)} = \begin{bmatrix} \bar{R}_{(ext)_1}^{1/2} & 0 \\ 0 & \bar{R}_{(ext)_2}^{1/2} \end{bmatrix} \bar{V}_{(ext)}^T = \bar{R}_{(ext)}^{1/2} \bar{V}_{(ext)}^T,$$

where $\bar{U}_{(ext)} = \bar{U}_m = U_E$ and $\bar{V}_{(ext)} = \bar{V}_m = V_E$. Since,

$$\begin{aligned} \bar{X}_{(ext)} &= \bar{B}_{(ext)} \bar{B}_{(ext)}^T = \bar{U} \bar{S}_{(ext)}^{1/2} \bar{S}_{(ext)}^{1/2} \bar{U}^T \\ &= \bar{U}_{(ext)} \bar{S}_{(ext)} \bar{U}_{(ext)}^T = \bar{U}_{(ext)} (\hat{S}_m - S_E) \bar{U}_{(ext)}^T \\ &= \bar{U}_m \hat{S}_m \bar{U}_m^T - U_E S_E U_E^T = \bar{X}_m - X_E, \end{aligned} \tag{63}$$

$$\begin{aligned} \bar{Y}_{(ext)} &= \bar{C}_{(ext)}^T \bar{C}_{(ext)} = \bar{V}_{(ext)} \bar{R}_{(ext)}^{1/2} \bar{R}_{(ext)}^{1/2} \bar{V}_{(ext)}^T \\ &= \bar{V}_{(ext)} \bar{R}_{(ext)} \bar{V}_{(ext)}^T = \bar{V}_{(ext)} (\hat{R}_m - R_E) \bar{V}_{(ext)}^T \end{aligned}$$

$$= \bar{V}_m \hat{R}_m \bar{V}_m^T - V_E R_E V_E^T = \bar{Y}_m - Y_E, \tag{64}$$

substitute (67 and 53) in (63) and (68 and 54) in (64) we have the following:

$$\begin{aligned}(A_*\bar{P}_mA_*^T - \bar{P}_m) - (A_*P_EA_*^T - P_E) &= -\bar{X}_{(ext)},\\ (A_*^T\bar{Q}_mA_* - \bar{Q}_m) - (A_*^TQ_EA_* - Q_E) &= -\bar{Y}_{(ext)},\\ A_*(\bar{P}_m - P_E)A_*^T - (\bar{P}_m - P_E) &= -\bar{X}_{(ext)},\\ A_*^T(\bar{Q}_m - Q_E)A_* - (\bar{Q}_m - Q_E) &= -\bar{Y}_{(ext)}.\end{aligned}$$

If the controllability Gramian matrix $\bar{P}_{(ext)} = \bar{P}_m - P_E$ and the observability Gramian matrix $\bar{Q}_{(ext)} = \bar{Q}_m - Q_E$. Then,

$$\begin{aligned}A_*\bar{P}_{(ext)}A_*^T - \bar{P}_{(ext)} + \bar{B}_{(ext)}\bar{B}_{(ext)}^T &= 0,\\ A_*^T\bar{Q}_{(ext)}A_* - \bar{Q}_{(ext)} + \bar{C}_{(ext)}^T\bar{C}_{(ext)} &= 0,\end{aligned}$$

Corollary 2: Theorem 3 holds true subject to the balanced-realization $\{A_*, \bar{B}_{(ext)}, \bar{C}_{(ext)}\}$ is minimal (i.e., controllable and observable) and stable.

Remark 17: For the balanced-realization $\{A_*, \bar{B}_{(ext)}, \bar{C}_{(ext)}\}$ to the following Lyapunov equation:

$$\begin{aligned}A_*\bar{P}_{(ext)}A_*^T - \bar{P}_{(ext)} + \bar{B}_{(ext)}\bar{B}_{(ext)}^T &= 0,\\ A_*^T\bar{Q}_{(ext)}A_* - \bar{Q}_{(ext)} + \bar{C}_{(ext)}^T\bar{C}_{(ext)} &= 0,\end{aligned}$$

where the matrix $\bar{B}_{(ext)} \geq 0$ and the matrix $\bar{C}_{(ext)} \geq 0$ grant positive and positive-semi definite of the matrix $\bar{B}_m$ and the matrix $\bar{C}_m$ respectively; consequently, positive and positive-semi definite of the controllability Gramians matrix $\bar{P}_{(ext)}$ and the observability Gramians matrix $\bar{Q}_{(ext)}$ in a way leads to the positive and positive-semi definite of the matrix $\bar{P}_m$ and the matrix $\bar{Q}_m$.

Remark 18: Note that by applying stability robustness theorem [59] to the frequency weighted model reduction problem, the combine weighted systems is stable if the following inequalities hold (see chapter 3 of [31] for more detail)

$$\begin{aligned}&(i) && \|H_o[z](F_*[z] - F_{r_*}[z])\|_\infty \leq 1.\\ &(ii) && \|(F_*[z] - F_{r_*}[z])G_i[z]\|_\infty \leq 1.\\ &(iii) && \|H_o[z](F_*[z] - F_{r_*}[z])G_i[z]\|_\infty \leq 1.\end{aligned}$$

The above inequalities also provide the criteria for the choice of weightings (i.e., input weightings and output weightings).

B. 2-D FREQUENCY WEIGHTED MODEL REDUCTION TECHNIQUE FOR DISCRETE-TIME SYSTEMS

Let the controllability Gramians P_{ia} and the observability Gramians Q_{oa} for the corresponding input-augmented (25) and the output-augmented (26) realization, respectively, be given as:

$$P_{ia} = \left[\begin{array}{c|c} P_{ia_1} & P_{ia_2} \\ \hline P_{ia_3} & P_{ia_4}\end{array}\right] = \left[\begin{array}{cc|cc} P_{ia_{11}} & P_{ia_{12}} & P_{ia_{21}} & P_{ia_{22}} \\ P_{ia_1}^T & P_{ia_{14}} & P_{ia_{23}} & P_{ia_{24}} \\ \hline P_{ia_{21}}^T & P_{ia_{23}}^T & P_{ia_{41}} & P_{ia_{42}} \\ P_{ia_{22}}^T & P_{ia_{24}}^T & P_{ia_{42}}^T & P_{ia_{44}}\end{array}\right],$$

$$Q_{oa} = \left[\begin{array}{c|c} Q_{oa_1} & Q_{oa_2} \\ \hline Q_{oa_3} & Q_{oa_4}\end{array}\right] = \left[\begin{array}{cc|cc} Q_{oa_{11}} & Q_{oa_{12}} & Q_{oa_{21}} & Q_{oa_{22}} \\ Q_{oa_{12}}^T & Q_{oa_{14}} & Q_{oa_{23}} & Q_{oa_{24}} \\ \hline Q_{oa_{21}}^T & Q_{oa_{23}}^T & Q_{oa_{41}} & Q_{oa_{42}} \\ Q_{oa_{22}}^T & Q_{oa_{24}}^T & Q_{oa_{42}}^T & Q_{oa_{44}}\end{array}\right],$$

that are the solution of the following Lyapunov equations:

$$A_{ia}P_{ai}A_{ia}^T - P_{ia} + B_{ia}B_{ia}^T = 0, \tag{65}$$

$$A_{oa}^TQ_{oa}A_{oa} - Q_{oa} + C_{oa}^TC_{oa} = 0, \tag{66}$$

Truncating (3, 3) and (1, 1) block of (65) and (66), respectively, we have the following Lyapunov equations:

$$A_4P_{ia_{41}}A_4^T - P_{ia_{41}} + X_{\epsilon_4} = 0, \tag{67}$$

$$A_1^TQ_{oa_{11}}A_1 - Q_{oa_{11}} + Y_{\epsilon_1} = 0, \tag{68}$$

where

$$\begin{aligned}Y_{\epsilon_1} &= A_1^TQ_{oa_{12}}B_{1_o}C_1 + A_1^TQ_{oa_{22}}B_{2_o}C_1\\ &\quad + (B_{1_o}C_1)^TQ_{oa_{12}}^TA_1 + (B_{1_o}C_1)^TQ_{oa_{14}}B_{1_o}C_1\\ &\quad + (B_{1_o}C_1)^TQ_{oa_{24}}B_{2_o}C_1 + (B_{2_o}C_1)^TQ_{oa_{23}}A_1\\ &\quad + (B_{2_o}C_1)^TQ_{oa_{24}}^TB_{1_o}C_1 + (B_{2_o}C_1)^TQ_{oa_{44}}B_{2_o}C_1\\ &\quad + (D_oC_1)^TD_oC_1, \end{aligned} \tag{69}$$

$$\begin{aligned}X_{\epsilon_2} &= B_2C_{1_i}P_{ia_{14}}(B_2C_{1_i})^T + B_2C_{1_i}P_{ia_{23}}A_4^T\\ &\quad + B_2C_{1_i}P_{ia_{14}}(B_2C_{2_i})^T + A_4P_{ia_{22}}(B_2C_{1_i})^T\\ &\quad + A_4P_{ia_{42}}(B_2C_{2_i})^T + (B_2C_{2_i})P_{ia_{24}}^T(B_2C_{1_i})^T\\ &\quad + B_2C_{2_i}P_{ia_{42}}^TA_4^T + B_2C_{2_i}P_{ia_{44}}(B_2C_{2_i})^T\\ &\quad + B_2D_i(B_2D_i)^T. \end{aligned} \tag{70}$$

The stability is ensured for 2-D discrete-time system by making the input $X_{\epsilon_2} = B_{\epsilon_2}\epsilon_2{}_2^T = U_{\epsilon_2}S_{\epsilon_2}U_{\epsilon_2}^T$ (70) and the output $Y_{\epsilon_1} = C_{\epsilon_1}^TC_{\epsilon_1} = V_{\epsilon_1}R_{\epsilon_1}V_{\epsilon_1}^T$ (69) associated matrices positive and positive semi definite. The fictitious matrices $\bar{B}_{m_{\epsilon_2}}$ and $\bar{C}_{m_{\epsilon_2}}$ are obtained by improvising $B_{\epsilon_2} = U_{\epsilon_2}S_{\epsilon_2}^{1/2}$ and $C_{\epsilon_1} = R_{\epsilon_1}^{1/2}V_{\epsilon_1}^T$, respectively.

$$\bar{B}_{m_{\epsilon_2}} = \begin{cases} \bar{U}_{m_{\epsilon_2}}\sqrt{\dfrac{(S_{\epsilon_2} - s_nI)^{1/2}}{s_n}} = \bar{U}_{m_{\epsilon_2}}\bar{S}_{m_{\epsilon_2}}^{1/2} & \text{for } s_n < 0\\ U_{\epsilon_2}S_{\epsilon_2}^{1/2} & \text{for } s_n \geq 0\end{cases} \tag{71}$$

$$\bar{C}_{m_{\epsilon_1}} = \begin{cases} \sqrt{\dfrac{(R_{\epsilon_1} - r_nI)^{1/2}}{r_n}}\bar{V}_{m_{\epsilon_1}}^T = \bar{R}_{m_{\epsilon_1}}^{1/2}\bar{V}_{m_{\epsilon_1}}^T & \text{for } r_n < 0\\ R_{\epsilon_1}^{1/2}V_{\epsilon_1}^T & \text{for } r_n \geq 0\end{cases} \tag{72}$$

Remark 19: When the following rank conditions holds:

$$rank\left[\bar{B}_{m_{\epsilon_2}} B_2\right] = rank\left[\bar{B}_{m_{\epsilon_2}}\right], \tag{73}$$

$$rank\begin{bmatrix}\bar{C}_{m_{\epsilon_1}}\\ C_1\end{bmatrix} = rank\left[\bar{C}_{m_{\epsilon_1}}\right]; \tag{74}$$

then, the following relationship holds for the fictitious input and the fictitious output matrices.

$$B_2 = \bar{B}_{m_{\epsilon_2}} \bar{K}_{m_{\epsilon_2}}, \quad (75)$$
$$C_1 = \bar{L}_{m_{\epsilon_1}} \bar{C}_{m_{\epsilon_1}}, \quad (76)$$

where

$$\bar{K}_{m_{\epsilon_2}} = \begin{cases} \bar{S}_{m_{\epsilon_2}}^{-1/2} \bar{U}_{m_{\epsilon_2}}^T B_2 & \text{if } \bar{s}_n < 0 \text{ exists} \\ S_{\epsilon_2}^{-1/2} U_{\epsilon_2}^T B_2 & \text{otherwise} \end{cases} \quad (77)$$

$$\bar{L}_{m_{\epsilon_1}} = \begin{cases} C_1 V_{m_{\epsilon_2}} \bar{R}_{m_{\epsilon_1}}^{-1/2} & \text{if } r_n < 0 \text{ exists} \\ C_1 V_{\epsilon_1} R_{\epsilon_2}^{-1/2} * & \text{otherwise} \end{cases} \quad (78)$$

Remark 20: It can be seen in [40] that (73) is always true. It can be seen that in (70), each terms are expressed as $B_2[*]$ or $[*]B_2^T$ or $B_2[*]B_2^T$, that is same as in [40], here terms $[*]$ are some matrices which doesn't affect our analysis. So (73) is always true. Similarly, it can also be seen in [40] that (74) is always true. It can also be seen that in (69), each terms are expressed as $C_1[*]$ or $[*]C_1^T$ or $C_1[*]C_1^T$, that is same as in [40], here terms $[*]$ are some matrices which doesn't affect our analysis. So (74) is always true.

Consider rank$[B_1\ B_2]$ = rank $[B_2]$ and rank $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ = rank $[C_1]$; then, there exists some constant matrices $\bar{K}_{m_{\epsilon_1}}$ and $\bar{L}_{m_{\epsilon_2}}$, such that

$$B_1 = \bar{K}_{m_{\epsilon_1}} B_2, \quad (79)$$
$$C_2 = C_1 \bar{L}_{m_{\epsilon_2}}, \quad (80)$$

Remark 21: Assumptions rank$[B_1\ B_2]$ = *rank* $[B_2]$ and rank $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ = *rank* $[C_1]$ will always be satisfied for B_2 and C_1 be full column rank and row rank, respectively.

Using (75), (76), (79) and (80), we can derive new matrices $\bar{B}_{m_{\epsilon_1}}$ and $\bar{C}_{m_{\epsilon_2}}$ as follows:

$$\bar{B}_{m_{\epsilon_1}} := \bar{K}_{m_{\epsilon_1}} \bar{B}_{m_{\epsilon_2}}, \quad (81)$$
$$\bar{C}_{m_{\epsilon_2}} := \bar{C}_{m_{\epsilon_1}} \bar{L}_{m_{\epsilon_2}}, \quad (82)$$

then,

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \bar{B}_{m_{\epsilon_1}} \\ \bar{B}_{m_{\epsilon_2}} \end{bmatrix} \bar{K}_{m_{\epsilon_2}} := \bar{B}_{m_\epsilon} \bar{K}_{m_{\epsilon_2}}, \quad (83)$$

$$[C_1\ C_2] = \bar{L}_{m_{\epsilon_1}} \begin{bmatrix} \bar{C}_{m_{\epsilon_1}} & \bar{C}_{m_{\epsilon_2}} \end{bmatrix} := \bar{L}_{m_{\epsilon_1}} \bar{C}_{m_\epsilon}. \quad (84)$$

Theorem 4: The following rank conditions are always hold:

$$a) rank \left[B\ \bar{B}_{m_\epsilon}\right] = rank \left[\bar{B}_{m_\epsilon}\right].$$
$$b) rank \begin{bmatrix} C \\ \bar{C}_{m_\epsilon} \end{bmatrix} = rank \left[\bar{C}_{m_\epsilon}\right].$$

Theorem 5: The realization $\{A, \bar{B}_{m_\epsilon}, \bar{C}_{m_\epsilon}, D\}$ is minimal, stable, and separable denominator.

Proof of Theorem 5: The proof of above Theorem 5 follows from the minimality, stability, and separability of the 2-D discrete-time system realization $\{A, B, C, D\}$.

The minimal rank-decomposition of new realization $\{A, \bar{B}_{m_\epsilon}, \bar{C}_{m_\epsilon}, D\}$ subject to $A_3 = 0$ can be written as:

$$\begin{bmatrix} A_2 & \bar{B}_{m_{\epsilon_1}} \\ \bar{C}_{m_{\epsilon_2}} & \bar{D}_{m_\epsilon} \end{bmatrix} = \begin{bmatrix} \bar{B}_{m_{\epsilon_1 *}} \\ \bar{D}_{m_{\epsilon_1 *}} \end{bmatrix} \begin{bmatrix} \bar{C}_{m_{\epsilon_2 *}} & \bar{D}_{m_{\epsilon_2 *}} \end{bmatrix} \quad (85)$$

that results $\bar{F}_{m_\epsilon}[z_1, z_2] = \bar{F}_{1_{m_\epsilon}}[z_1] \bar{F}_{2_{m_\epsilon}}[z_2]$, where

$$\bar{F}_{m_\epsilon}[z_1, z_2] = \bar{D}_{m_\epsilon} + \bar{C}_{m_\epsilon}[z_1 I_n \bigoplus z_2 I_m - A]^{-1} \bar{B}_{m_\epsilon},$$
$$\bar{F}_{1_{m_\epsilon}}[z_1] = \bar{D}_{m_{\epsilon_1 *}} + \bar{C}_{m_{\epsilon_1}}[z_1 I - A_1]^{-1} \bar{B}_{m_{\epsilon_1 *}},$$
$$\bar{F}_{2_{m_\epsilon}}[z_2] = \bar{D}_{m_{\epsilon_2 *}} + \bar{C}_{m_{\epsilon_2 *}}[z_2 I - A_4]^{-1} \bar{B}_{m_{\epsilon_2}}.$$
$$D = \bar{L}_{m_{\epsilon_1}} \bar{D}_{m_\epsilon} \bar{K}_{m_{\epsilon_4}} \quad (86)$$

Remark 22: The equation (86) can be solvable for $\bar{D}_{m_\epsilon}$ *iff* one of the following equivalent conditions holds [60]:

1) rank $[\bar{L}_{m_{\epsilon_1}}]$ = rank$[\bar{L}_{m_{\epsilon_1}}\ D]$ and rank $[\bar{K}_{m_{\epsilon_2}}]$ = $\begin{bmatrix} \bar{K}_{m_{\epsilon_2}} \\ D \end{bmatrix}$.

2) There exist some matrices Y_ϵ and Z_ϵ such that $D = \bar{L}_{m_{\epsilon_1}} Y_\epsilon$ and $D = Z_\epsilon \bar{K}_{m_{\epsilon_2}}$.

Remark 23: The requirements for the existence of (86) for strictly proper original systems is immediately met. This requirement will be met when the full row rank $\bar{L}_{m_{\epsilon_1}}$ and the full column rank $\bar{K}_{m_{\epsilon_2}}$ is exist. We notice that even by setting $\bar{D}_{m_\epsilon} = 0$ we can get rid of this assumption.

Remark 24: The realizations $\{A_1, \bar{B}_{m_{\epsilon_1 *}}, \bar{C}_{m_{\epsilon_1}}, \bar{D}_{m_{\epsilon_1 *}}\}$ and $\{A_4, \bar{B}_{m_{\epsilon_2}}, \bar{C}_{m_{\epsilon_2 *}}, \bar{D}_{m_{\epsilon_2 *}}\}$ are minimal and stable.

The new controllability $(\bar{P}_{m_{\epsilon_1}}, \bar{P}_{m_{\epsilon_2}})$ and the observability $(\bar{Q}_{m_{\epsilon_1}}, \bar{Q}_{m_{\epsilon_2}})$ Gramians correspond to the decomposed sub-system $(\bar{F}_{1_{m_\epsilon}}[z_1], \bar{F}_{2_{m_\epsilon}}[z_2])$, respectively, these Gramians satisfy the following corresponding Lyapunov equations i.e., for sub-system $\bar{F}_{1_{m_\epsilon}}[z_1] = \bar{D}_{m_{\epsilon_1 *}} + \bar{C}_{m_{\epsilon_1}}[z_1 I - A_1]^{-1} \bar{B}_{m_{\epsilon_1 *}}$

$$A_1 \bar{P}_{m_{\epsilon_1}} A_1^T - \bar{P}_{m_{\epsilon_1}} + \bar{X}_{m_{\epsilon_1}} = 0, \quad (87)$$
$$A_1^T \bar{Q}_{m_{\epsilon_1}} A_1 - \bar{Q}_{m_{\epsilon_1}} + \bar{Y}_{m_{\epsilon_1}} = 0, \quad (88)$$

where $\bar{X}_{m_{\epsilon_1}} = \bar{B}_{m_{\epsilon_1 *}} \bar{B}_{m_{\epsilon_1 *}}^T$ and $\bar{Y}_{m_{\epsilon_1}} = \bar{C}_{m_{\epsilon_1}}^T \bar{C}_{m_{\epsilon_1}}$. Let the similarity transformation matrix $\bar{T}_{m_{\epsilon_1}}$ is calculated as:

$$\bar{T}_{m_{\epsilon_1}}^T \bar{Q}_{m_{\epsilon_1}} \bar{T}_{m_{\epsilon_1}} = \bar{T}_{m_{\epsilon_1}}^{-1} \bar{P}_{m_{\epsilon_1}} \bar{T}_{m_{\epsilon_1}}^{-T} = \begin{bmatrix} \bar{\rho}_{\epsilon_1} & 0 & \cdots & 0 \\ 0 & \bar{\rho}_{\epsilon_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \bar{\rho}_{\epsilon_1} \end{bmatrix}, \quad (89)$$

where $\bar{\rho}_{\epsilon_j} \geq \bar{\rho}_{\epsilon_{j+1}}$ and $\bar{\rho}_{\epsilon_r} \geq \bar{\rho}_{\epsilon_{r+1}}$. The ROM $\bar{F}_{1_{m_{\epsilon_r}}}[z_1] = \hat{D}_{m_{\epsilon_{1_r}}} + \hat{C}_{m_{\epsilon_r}}[z_1 I - \hat{A}_{1_{\epsilon_r}}]^{-1} \hat{B}_{m_{\epsilon_{r*}}}$ is obtained as:

$$\bar{T}_{m_{\epsilon_1}}^{-1} A_1 \bar{T}_{m_{\epsilon_1}} = \hat{A}_{1_\epsilon} = \begin{bmatrix} \hat{A}_{1_{\epsilon_r}} & \hat{A}_{1_{\epsilon_{12}}} \\ \hat{A}_{1_{\epsilon_{21}}} & \hat{A}_{1_{\epsilon_{22}}} \end{bmatrix}, \quad (90)$$

$$\bar{T}_{m_{\epsilon_1}}^{-1} \bar{B}_{m_{\epsilon_1 *}} = \hat{B}_{m_{\epsilon_1 *}} = \begin{bmatrix} \hat{B}_{m_{\epsilon_r *}} \\ \hat{B}_{m_{\epsilon_{22} *}} \end{bmatrix}, \quad (91)$$

$$\bar{C}_{m_{\epsilon_1}} \bar{T}_{m_{\epsilon_1}} = \hat{C}_{m_{\epsilon_1}} = \begin{bmatrix} \hat{C}_{m_{\epsilon_r}} & \hat{C}_{m_{\epsilon_{22}}} \end{bmatrix}, \quad (92)$$

$$\bar{D}_{m_{\epsilon_1 *}} = \hat{D}_{m_{\epsilon_{1_r}}}, \tag{93}$$

i.e., for sub-system $\bar{F}_{2_{m_\epsilon}}[z_2] = \bar{D}_{m_{\epsilon_2 *}} + \bar{C}_{m_{\epsilon_2 *}}[z_2 I - A_4]^{-1}\bar{B}_{m_{\epsilon_2}}$

$$A_4 \bar{P}_{m_{\epsilon_2}} A_4^T - \bar{P}_{m_{\epsilon_2}} + \bar{X}_{m_{\epsilon_2}} = 0, \tag{94}$$

$$A_4^T \bar{Q}_{m_{\epsilon_2}} A_4 - \bar{Q}_{m_{\epsilon_2}} + \bar{Y}_{m_{\epsilon_2}} = 0, \tag{95}$$

where $\bar{X}_{m_{\epsilon_2}} = \bar{B}_{m_{\epsilon_2}} \bar{B}_{m_{\epsilon_2}}^T$ and $\bar{Y}_{m_{\epsilon_2}} = \bar{C}_{m_{\epsilon_2 *}}^T \bar{C}_{m_{\epsilon_2 *}}$. Let the similarity transformation matrix $\bar{T}_{m_{\epsilon_2}}$ is calculated as:

$$\bar{T}_{m_{\epsilon_2}}^T \bar{Q}_{m_{\epsilon_2}} \bar{T}_{m_{\epsilon_2}} = \bar{T}_{m_{\epsilon_2}}^{-1} \bar{P}_{m_{\epsilon_2}} \bar{T}_{m_{\epsilon_2}}^{-T} = \begin{bmatrix} \bar{\upsilon}_{\epsilon_1} & 0 & \cdots & 0 \\ 0 & \bar{\upsilon}_{\epsilon_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \bar{\upsilon}_{\epsilon_1} \end{bmatrix}, \tag{96}$$

where $\bar{\upsilon}_{\epsilon_j} \geq \bar{\upsilon}_{\epsilon_{j+1}}$ and $\bar{\upsilon}_{\epsilon_r} \geq \bar{\upsilon}_{\epsilon_{r+1}}$. The ROM $\bar{F}_{2_{m_{\epsilon_r}}}[z_2] = \hat{D}_{m_{\epsilon_{2_r}}} + \hat{C}_{m_{\epsilon_r *}}[z_1 I - \hat{A}_{4_{\epsilon_r}}]^{-1}\hat{B}_{m_{\epsilon_r}}$ is obtained as:

$$\bar{T}_{m_{\epsilon_2}}^{-1} A_4 \bar{T}_{m_{\epsilon_2}} = \hat{A}_{4_\epsilon} = \begin{bmatrix} \hat{A}_{4_{\epsilon_r}} & \hat{A}_{4_{\epsilon_{12}}} \\ \hat{A}_{4_{\epsilon_{21}}} & \hat{A}_{4_{\epsilon_{22}}} \end{bmatrix}, \tag{97}$$

$$\bar{T}_{m_{\epsilon_2}}^{-1} \bar{B}_{m_{\epsilon_2}} = \hat{B}_{m_{\epsilon_2}} = \begin{bmatrix} \hat{B}_{m_{\epsilon_r}} \\ \hat{B}_{m_{\epsilon_{22}}} \end{bmatrix}, \tag{98}$$

$$\bar{C}_{m_{\epsilon_2 *}} \bar{T}_{m_{\epsilon_2}} = \hat{C}_{m_{\epsilon_2 *}} = \begin{bmatrix} \hat{C}_{m_{\epsilon_r *}} & \hat{C}_{m_{\epsilon_{22*}}} \end{bmatrix}, \tag{99}$$

$$\bar{D}_{m_{\epsilon_2 *}} = \hat{D}_{m_{\epsilon_{2_r}}} \tag{100}$$

Remark 25: The realizations $\{\hat{A}_{1_{\epsilon_r}}, \hat{B}_{m_{\epsilon_r *}}, \hat{C}_{m_{\epsilon_r}}, \hat{D}_{m_{\epsilon_{1_r}}}\}$ and $\{\hat{A}_{4_{\epsilon_r}}, \hat{B}_{m_{\epsilon_r}}, \hat{C}_{m_{\epsilon_r *}}, \hat{D}_{m_{\epsilon_{2_r}}}\}$ are stable and minimal. Furthermore, the 2-D discrete-time weighted ROM $\bar{F}_{m_{\epsilon_r}}[z_1, z_2] = \bar{D}_{m_{\epsilon_r}} + \bar{C}_{m_{\epsilon_r}}[z_1 I_{n_r} \bigoplus z_2 I_{m_r} - \hat{A}_{\epsilon_r}]^{-1}\bar{B}_{m_{\epsilon_r}}$ of an original 2-D discrete-time $F[z_1, z_2] = D + C[z_1 I_n \bigoplus z_2 I_m - A]^{-1}B$, where

$$A_{m_{\epsilon_r}} = \begin{bmatrix} \hat{A}_{1_{\epsilon_r}} & \hat{B}_{m_{\epsilon_r *}} \hat{C}_{m_{\epsilon_r *}} \\ 0 & \hat{A}_{4_{\epsilon_r}} \end{bmatrix}, \tag{101}$$

$$B_{m_{\epsilon_r}} = \begin{bmatrix} \hat{B}_{m_{\epsilon_r *}} \hat{D}_{m_{\epsilon_{2_r}}} \\ \hat{B}_{m_{\epsilon_r}} \end{bmatrix} \bar{K}_{m_{\epsilon_2}} := \bar{B}_{m_{\epsilon_r}} \bar{K}_{m_{\epsilon_2}}, \tag{102}$$

$$C_{m_{\epsilon_r}} = \bar{L}_{m_{\epsilon_1}} \begin{bmatrix} \hat{C}_{m_{\epsilon_r}} & \hat{D}_{m_{\epsilon_{1_r}}} \hat{C}_{m_{\epsilon_r *}} \end{bmatrix} := \bar{L}_{m_{\epsilon_1}} \bar{C}_{m_{\epsilon_r}}, \tag{103}$$

$$D_{m_{\epsilon_r}} = \bar{L}_{m_{\epsilon_1}} \hat{D}_{m_{\epsilon_{1_r}}} \hat{D}_{m_{\epsilon_{2_r}}} \bar{K}_{m_{\epsilon_4}} = D. \tag{104}$$

Algorithm 1: Given a discrete time 2-D system $F[z_1, z_2]$ with input and output frequency weights $G_i[z_1, z_2]$ and $H_o[z_1, z_2]$. The ROM $\bar{F}_{m_{\epsilon_r}}[z_1, z_2]$ for 2-D discrete-time systems are obtained by using the following steps:

1) Compute the controllability Gramians P_{ia} and the observability Gramians Q_{oa} by using (65) and (66), respectively.
2) Compute Y_{ϵ_1} and X_{ϵ_4} by using (69) and (70), respectively.
3) Decompose Y_{ϵ_1} and X_{ϵ_4} by using singular-values decomposition as: $C_1^T C_1 = V_{\epsilon_1} R_{\epsilon_1} V_{\epsilon_1}^T$ and $B_2 B_2^T = U_{\epsilon_2} S_{\epsilon_2} U_{\epsilon_2}^T$, respectively, to compute $\bar{C}_{m_{\epsilon_1}} = \bar{R}_{m_{\epsilon_1}}^{1/2} \bar{V}_{m_{\epsilon_1}}^T$ and $\bar{B}_{m_{\epsilon_2}} = \bar{U}_{m_{\epsilon_2}} \bar{S}_{m_{\epsilon_2}}^{1/2}$ by using (72) and (71), respectively.
4) Compute constants $\bar{K}_{m_{\epsilon_2}}$, $\bar{L}_{m_{\epsilon_1}}$, $\bar{K}_{m_{\epsilon_1}}$, and $\bar{L}_{m_{\epsilon_2}}$ by using (77), (78), (79) and (80), respectively.
5) Compute $\bar{B}_{m_{\epsilon_1}}$ and $\bar{C}_{m_{\epsilon_2}}$ by using (81) and (82), respectively.
6) Compute $\bar{P}_{m_{\epsilon_1}}, \bar{Q}_{m_{\epsilon_1}}, \bar{P}_{m_{\epsilon_2}}$, and $\bar{Q}_{m_{\epsilon_2}}$ by using (87), (88), (94) and (95), respectively.
7) Compute the transformation matrices $\bar{T}_{m_{\epsilon_1}}$ and $\bar{T}_{m_{\epsilon_2}}$ to satisfy (89) and (96), respectively.
8) Compute the realizations $\{\hat{A}_{1_\epsilon}, \hat{B}_{m_{\epsilon_1 *}}, \hat{C}_{m_{\epsilon_1}}, \hat{D}_{m_{\epsilon_1 *}}\}$ and $\{\hat{A}_{4_\epsilon}, \hat{B}_{m_{\epsilon_2}}, \hat{C}_{m_{\epsilon_2 *}}, \hat{D}_{m_{\epsilon_2 *}}\}$ by using (90-93) and (97-100), respectively, to obtained corresponding ROMs $\bar{F}_{1_{m_{\epsilon_r}}}[z_1]$ and $\bar{F}_{2_{m_{\epsilon_r}}}[z_2]$.
9) Compute 2-D discrete-time systems ROMs by using (101-104):
where
$A_{m_{\epsilon_r}} \in \Re^{(n_r+m_r)\times(n_r+m_r)}$, $B_{m_{\epsilon_r}} \in \Re^{(n_r+m_r)\times p}$,
$C_{m_{\epsilon_r}} \in \Re^{p\times(n_r+m_r)}$, $D_{m_{\epsilon_r}} \in \Re^{(q\times p)}$,
and $n_r < n$, $m_r < m$

Remark 26: For the only input weighting, the realization based on frequency weighted becomes $\{A, \bar{B}_{m_\epsilon}, C, D\}$; consequently, C_2 replaces $\bar{C}_{m_{\epsilon_2}}$ in (85).

Remark 27: For the only output weighting, the realization based on frequency weighted becomes $\{A, B, \bar{C}_{m_\epsilon}, D\}$; consequently, B_1 replaces $\bar{B}_{m_{\epsilon_1}}$ in (85).

Remark 28: Notice that also in Remark (23) we can get rid of this assumption by setting $\bar{D}_{m_\epsilon} = 0$, then setting $D_{m_{\epsilon_r}} = D$ into (8) the appropriate dimensions. However, this comment might not be helpful if we use 1-D singular perturbation approximation for the method of 2-D MOR.

Remark 29: While it is expressly indicated for balanced truncation, the above algorithms can be easily expanded/defined for almost all 1-D reduction schemes, such as the Hankel norm approximation and singular perturbation approximation, etc.

Theorem 6: The 2-D ROM obtained with this procedure is stable.

Proof of Theorem 6: The proof follows directly from the stability of the un-weighted approximation and is thus excluded.

Remark 30: For the 2-D discrete-time 2-D system $\bar{F}[z_1, z_2] = \bar{F}_1[z_1]\bar{F}_2[z_2]$, the corresponding decomposed sub-systems $\bar{F}_{m_\epsilon}[z_1, z_2] = \bar{F}_{1_{m_\epsilon}}[z_1]\bar{F}_{2_{m_\epsilon}}[z_2]$ are formed, the matrices $X_{\epsilon_2} = B_2 B_2^T < \bar{B}_{m_{\epsilon_1 *}} \bar{B}_{m_{\epsilon_1 *}}^T$ and $Y_{\epsilon_1} = C_1^T C_1 < \bar{C}_{m_{\epsilon_1}}^T \bar{C}_{m_{\epsilon_1}}$; therefore, $\bar{B}_{m_{\epsilon_1 *}} \bar{B}_{m_{\epsilon_1 *}}^T \geq 0$, $\bar{C}_{m_{\epsilon_1}}^T \bar{C}_{m_{\epsilon_1}} \geq 0$; resultantly, $\bar{P}_{m_{\epsilon_1}} > 0, \bar{P}_{m_{\epsilon_2}} > 0$ and $\bar{Q}_{m_{\epsilon_1}} > 0, \bar{Q}_{m_{\epsilon_2}} > 0$. Realization $\{A_1, \bar{B}_{m_{\epsilon_1 *}}, \bar{C}_{m_{\epsilon_1}}\}$ and $\{A_4, \bar{B}_{m_{\epsilon_2}}, \bar{C}_{m_{\epsilon_2 *}}\}$ are minimal and the obtained ROMs are stable.

Remark 31: Similar to Remark (30), the 2-D discrete-time 2-D system $\hat{F}[z_1, z_2] = \hat{F}_2[z_2]\hat{F}_1[z_1]$, the corresponding decomposed sub-systems $\hat{F}_{m_\epsilon}[z_1, z_2] = \hat{F}_{2_{m_\epsilon}}[z_2]\hat{F}_{1_{m_\epsilon}}[z_1]$ can also be formed; consequently, their stability of ROMs are also ensured by making corresponding inputs (i.e., B_1 and B_2) and corresponding outputs (i.e., C_1 and C_2) matrices positive and positive semi definite (i.e., results in fictitious input and output matrices $\hat{B}_{m_{\epsilon_1*}}$, $\hat{B}_{m_{\epsilon_2}}$, $\hat{C}_{m_{\epsilon_1}}$ and $\hat{C}_{m_{\epsilon_2*}}$) in a similar way as given in equations (71,72), respectively, such that their corresponding controllability Gramians matrices (i.e., $\hat{P}_{m_{\epsilon_1}}$ and $\hat{P}_{m_{\epsilon_2}}$) and the observability Gramians matrices (i.e., $\hat{Q}_{m_{\epsilon_1}}$ and $\hat{Q}_{m_{\epsilon_2}}$) are positive and positive semi definite, which leads to two different transformation matrices (i.e. $\hat{T}_{m_{\epsilon_1}}$ and $\hat{T}_{m_{\epsilon_2}}$) for their corresponding sub-systems $\hat{F}_{1_{m_\epsilon}}[z_1]$ and $\hat{F}_{2_{m_\epsilon}}[z_2]$, respectively; subsequently, transformed-realization correspond to $\hat{F}_{1_{m_\epsilon}}[z_1]$ and $\hat{F}_{2_{m_\epsilon}}[z_2]$ are minimal and their ROMs are stable.

Theorem 7: Let ROMs be attained by using balanced truncation, then the frequency response approximation error is bounded by:

$$\begin{aligned}&\|H_o[z_1,z_2](F[z_1,z_2]-F_r[z_1,z_2])G_i[z_1,z_2]\|_\infty\\&\quad\le 2\kappa(\|\hat{D}_{m_{\epsilon_{2_r}}}\|+2\sum_{i=1}^{n}\bar{\upsilon}_{\epsilon_i})2\sum_{i=m_r+1}^{m}\bar{\rho}_{\epsilon_i}\\&\qquad+2\kappa(\|\hat{D}_{m_{\epsilon_{1_r}}}\|+2\sum_{i=1}^{m_r}\bar{\rho}_{\epsilon_i})2\sum_{i=n_r+1}^{n}\bar{\upsilon}_{\epsilon_i},\end{aligned}$$

where $\kappa = H_o[z_1,z_2]\bar{L}_{m_{\epsilon_1}}\bar{K}_{m_{\epsilon_2}}G_i[z_1,z_2]$, $\bar{\rho}_{\epsilon_i}$ and $\bar{\upsilon}_{\epsilon_i}$ are the Hankel singular-values of the realizations $\bar{F}_{1_{m_{\epsilon_r}}}[z_1]$ and $\bar{F}_{2_{m_{\epsilon_r}}}[z_2]$, respectively.

Proof of Theorem 7:

$$\begin{aligned}&\|H_o[z_1,z_2](F[z_1,z_2]-F_r[z_1,z_2])G_i[z_1,z_2]\|_\infty\\&=\|H_o[z_1,z_2]\ (C[z_1I_n\bigoplus z_2I_m-A]^{-1}B\\&\quad-C_r[z_1I_{n_r}\oplus z_2I_{m_r}-A_r]^{-1}B_r)\ G_i[z_1,z_2]\|_\infty\\&=\|H_o[z_1,z_2]\,(\bar{L}_{m_{\epsilon_1}}\bar{C}_{m_\epsilon}[z_1I_n\bigoplus z_2I_m-A]^{-1}\bar{B}_{m_\epsilon}\bar{K}_{m_{\epsilon_2}}\\&\quad-\bar{L}_{m_{\epsilon_1}}C_{m_{\epsilon_r}}[z_1I_{n_r}\oplus z_2I_{m_r}-A_{m_{\epsilon_r}}]^{-1}B_{m_{\epsilon_r}}\bar{K}_{m_{\epsilon_2}})\\&\quad\times G_i[z_1,z_2]\|_\infty\\&=\|H_o[z_1,z_2]\ \bar{L}_{m_{\epsilon_1}}\ (\bar{C}_{m_\epsilon}[z_1I_n\bigoplus z_2I_m-A]^{-1}\bar{B}_{m_\epsilon}\\&\quad-\bar{L}_{m_{\epsilon_1}}C_{m_{\epsilon_r}}[z_1I_{n_r}\oplus z_2I_{m_r}-A_{m_{\epsilon_r}}]^{-1}B_{m_{\epsilon_r}})\bar{K}_{m_{\epsilon_2}}\\&\quad\times G_i[z_1,z_2]\|_\infty\\&\le\|H_o[z_1,z_2]\,\bar{L}_{m_{\epsilon_1}}\|_\infty\|\|(\bar{C}_{m_\epsilon}[z_1I_n\bigoplus z_2I_m-A]^{-1}\bar{B}_{m_\epsilon}\\&\quad-C_{m_{\epsilon_r}}[z_1I_{n_r}\oplus z_2I_{m_r}-A_{m_{\epsilon_r}}]^{-1}B_{m_{\epsilon_r}})\|_\infty\|\bar{K}_{m_{\epsilon_2}}\\&\quad\times G_i[z_1,z_2]\|_\infty\end{aligned}$$

Since $\{A,\bar{B}_{m_\epsilon},\bar{C}_{m_\epsilon}\}$ is the balanced realization and $\{A_{m_{\epsilon_r}},B_{m_{\epsilon_r}},C_{m_{\epsilon_r}}\}$ is its ROM, using Lemma (2) we have the following:

$$\begin{aligned}&\|(\bar{C}_{m_\epsilon}[z_1I_n\bigoplus z_2I_m-A]^{-1}\bar{B}_{m_\epsilon}\\&\quad-C_{m_{\epsilon_r}}[z_1I_{n_r}\oplus z_2I_{m_r}-A_{m_{\epsilon_r}}]^{-1}B_{m_{\epsilon_r}})\|_\infty\\&\quad\le 2(\|\hat{D}_{m_{\epsilon_{2_r}}}\|+2\sum_{i=1}^{n}\bar{\upsilon}_{\epsilon_i})2\sum_{i=n_r+1}^{n}\bar{\rho}_{\epsilon_i}\\&\qquad+2(\|\hat{D}_{m_{\epsilon_{1_r}}}\|+2\sum_{i=m_r+1}^{m}\bar{\rho}_{\epsilon_i})2\sum_{i=m_r+1}^{m}\bar{\upsilon}_{\epsilon_i}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|H_o[z_1,z_2](F[z_1,z_2]-F_{rh}[z_1,z_2])G_i[z_1,z_2]\|_\infty\\&\quad\le 2\kappa(\|\hat{D}_{m_{\epsilon_{2_r}}}\|+2\sum_{i=1}^{n}\bar{\upsilon}_{\epsilon_i})2\sum_{i=n_r+1}^{n}\bar{\rho}_{\epsilon_i}\\&\qquad+2\kappa(\|\hat{D}_{m_{\epsilon_{1_r}}}\|+2\sum_{i=m_r+1}^{m}\bar{\rho}_{\epsilon_i})2\sum_{i=m_r+1}^{m}\bar{\upsilon}_{\epsilon_i}.\end{aligned}$$

Theorem 8: Let ROMs be attained by using optimal Hankel norm approximation, then the frequency response approximation error is bounded by:

$$\begin{aligned}&\|H_o[z_1,z_2](F[z_1,z_2]-F_{rh}[z_1,z_2])G_i[z_1,z_2]\|_\infty\\&\quad\le 2\kappa(\|\hat{D}_{m_{\epsilon_{2_r}}}\|+2\sum_{i=1}^{n_r}\bar{\upsilon}_{\epsilon_i}+3\sum_{i=n_r+1}^{n}\bar{\upsilon}_{\epsilon_i})2\sum_{i=m_r+1}^{m}\bar{\rho}_{\epsilon_i}\\&\qquad+2\kappa(\|\hat{D}_{m_{\epsilon_{1_r}}}\|+2\sum_{i=1}^{m_r}\bar{\rho}_{\epsilon_i})2\sum_{i=n_r+1}^{n}\bar{\upsilon}_{\epsilon_i}.\end{aligned}$$

where $\kappa = H_o[z_1,z_2]\bar{L}_{mh_{\epsilon_1}}\bar{K}_{mh_{\epsilon_2}}G_i[z_1,z_2]$, $\bar{\rho}_{\epsilon_i}$ and $\bar{\upsilon}_{\epsilon_i}$ are the optimal Hankel singular-values of the realizations $\bar{F}_{1_{m_{\epsilon_{rh}}}}[z_1]$ and $\bar{F}_{2_{m_{\epsilon_{rh}}}}[z_2]$, respectively.

Proof of Theorem 8: The proof of above-mentioned Theorem 8 is similar to the proof of Theorem 7; hence, omitted for the brevity.

Corollary 3: When the only input-weighting or the only output-weighting is present, then κ becomes $\|\bar{K}_{m_{\epsilon_2}}G_i[z_1,z_2]\|_\infty$ or $\|H_o[z_1,z_2]\bar{L}_{m_{\epsilon_1}}$, respectively. Furthermore, when there is no weighting (i.e., input and output) present, $\kappa = 1$.

Remark 32: For the decomposed sub-systems (i.e., $\bar{F}_{1_{m_\epsilon}}[z_1]$ and $\bar{F}_{2_{m_\epsilon}}[z_2]$), the fictitious-input matrices (i.e., $\bar{B}_{m_{\epsilon_1*}}$ and $\bar{B}_{m_{\epsilon_2}}$) and the fictitious-output matrices (i.e., $\bar{C}_{m_{\epsilon_1}}$ and $\bar{C}_{m_{\epsilon_2*}}$) grant positive and positive-semi definite of the input associated matrix (i.e., B_1 and B_2) and the output associated matrix (i.e., C_1, C_2), respectively; consequently,

$$\left[\begin{array}{c|c}A_{m_{\epsilon_r}} & B_{m_{\epsilon_r}}\\\hline C_{m_{\epsilon_r}} & D_{m_{\epsilon_r}}\end{array}\right]=\left[\begin{array}{cc|c}\hat{A}_{1_{\epsilon_r}} & \hat{B}_{m_{\epsilon_{r*}}}\hat{C}_{m_{\epsilon_{r*}}} & \hat{B}_{m_{\epsilon_{r*}}}\hat{D}_{m_{\epsilon_{2_r}}}\bar{K}_{m_{\epsilon_2}}\\0 & \hat{A}_{4_{\epsilon_r}} & \hat{B}_{m_{\epsilon_r}}\bar{K}_{m_{\epsilon_2}}\\\hline \bar{L}_{m_{\epsilon_1}}\hat{C}_{m_{\epsilon_r}} & \bar{L}_{m_{\epsilon_1}}\hat{D}_{m_{\epsilon_{1_r}}}\hat{C}_{m_{\epsilon_{r*}}} & \bar{L}_{m_{\epsilon_1}}\hat{D}_{m_{\epsilon_{1_r}}}\hat{D}_{m_{\epsilon_{2_r}}}\bar{K}_{m_{\epsilon_4}}\end{array}\right],$$

positive and positive-semi definite of the controllability Gramians matrices (i.e., $\bar{P}_{m_{\epsilon_1}}$ and $\bar{P}_{m_{\epsilon_2}}$) and the observability Gramians matrices (i.e., $\bar{Q}_{m_{\epsilon_1}}$ and $\bar{Q}_{m_{\epsilon_2}}$). This corresponds to transformation matrices (i.e. $\bar{T}_{m_{\epsilon_1}}$ and $\bar{T}_{m_{\epsilon_2}}$), resulting in stability retention MOR algorithm. In addition, constants (i.e., $\bar{K}_{m_{\epsilon_1}}$, $\bar{L}_{m_{\epsilon_1}}$, $\bar{K}_{m_{\epsilon_2}}$, and $\bar{L}_{m_{\epsilon_2}}$) provide the relationship between the systems matrices (i.e., B_1, B_2, C_1, and C_2) and the fictitious matrices (i.e., $\bar{B}_{m_{\epsilon_1 *}}$, $\bar{B}_{m_{\epsilon_2}}$, $\bar{C}_{m_{\epsilon_1}}$ and $\bar{C}_{m_{\epsilon_2 *}}$), resulting in the error-bound expression for the suggested algorithm.

Remark 33: Similarly, for the decomposed sub-systems (i.e., $\hat{F}_{1_{m_\epsilon}}[z_1]$ and $\hat{F}_{2_{m_\epsilon}}[z_2]$), the fictitious-input matrices (i.e., $\hat{B}_{m_{\epsilon_1 *}}$ and $\hat{B}_{m_{\epsilon_2}}$) and the fictitious-output matrices (i.e., $\hat{C}_{m_{\epsilon_1}}$ and $\hat{C}_{m_{\epsilon_2 *}}$) grant positive and positive-semi definite of the input associated matrices (i.e., B_1 and B_2) and the output associated matrix (i.e., C_1 and C_2), respectively; consequently, positive and positive-semi definite of the controllability Gramians matrices (i.e., $\hat{P}_{m_{\epsilon_1}}$ and $\hat{P}_{m_{\epsilon_2}}$) and the observability Gramians matrices (i.e., $\hat{Q}_{m_{\epsilon_1}}$ and $\hat{Q}_{m_{\epsilon_2}}$). This corresponds to transformation matrices (i.e. $\hat{T}_{m_{\epsilon_1}}$ and $\hat{T}_{m_{\epsilon_2}}$), resulting in stability retention MOR algorithm. In addition, constants (i.e., $\hat{K}_{m_{\epsilon_1}}$, $\hat{L}_{m_{\epsilon_1}}$, $\hat{K}_{m_{\epsilon_2}}$, and $\hat{L}_{m_{\epsilon_2}}$) provide the relationship between the systems matrices (i.e., B_1, B_2, C_1, and C_2) and the fictitious matrices (i.e., $\hat{B}_{m_{\epsilon_1 *}}$, $\hat{B}_{m_{\epsilon_2}}$, $\hat{C}_{m_{\epsilon_1}}$ and $\hat{C}_{m_{\epsilon_2 *}}$), resulting in the error-bound expression for the suggested algorithm.

Remark 34: Similar to the Remark 18, by applying the stability robustness theorem [59] to the frequency weighted model reduction problem for the discrete-time 2-D systems, the combine weighted systems is stable if the following inequalities hold:

$$(i) \quad \|H_o[z_1, z_2](F[z_1, z_2] - F_r[z_1, z_2])\|_\infty \leq 1.$$

$$(ii) \quad \|(F[z_1, z_2] - F_r[z_1, z_2])G_i[z_1, z_2]\|_\infty \leq 1.$$

$$(iii) \quad \|H_o[z_1, z_2](F[z_1, z_2] - F_r[z_1, z_2])G_i[z_1, z_2\|_\infty \leq 1.$$

VI. NUMERICAL SIMULATIONS

To highlight the comparison of existing frequency weighted models ([30], [39], [40], [43]), a numerical example of a multi-input multi-output doubly-fed induction generator (DFIG) based variable-speed wind turbine (double-cage induction generator) for the power system (current model) is presented in Example. 1. Furthermore, the 2-D discrete-time system is demonstrated in Example. 2. Figs. 3, 4, and 9 depicted the frequency response error for the entire frequency-weights of the approximated model obtained by using existing ([30], [39], [40], [43]) and suggested frameworks. In addition, Figs. 7 and 8 depict the original 2-D model, and ROMs acquired using the existing and suggested methods, in the specified frequency-weights, of the ROMs acquired through the use of different existing ([30], [42]) and suggested techniques. Whereas, Figs. 5 and 6 of 2^{nd} and 3^{rd} order ROMs represent the bode plot (phase and magnitude) comparison, respectively, obtained using existing ([30], [39], [40], [43]) and proposed methods.

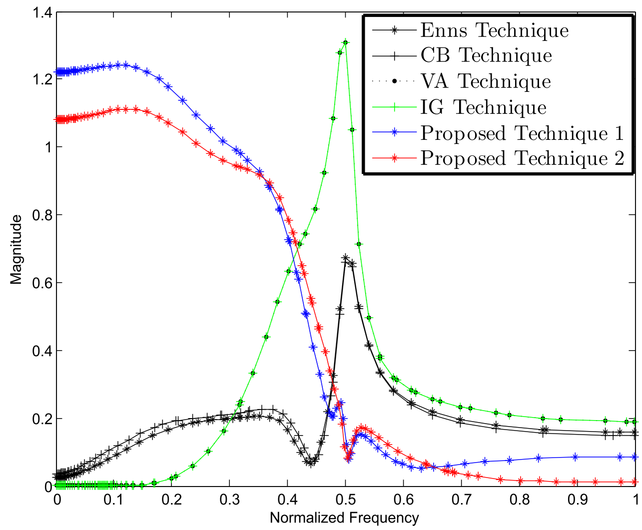

FIGURE 3. Frequency-response error comparison of 2^{nd} order ROM for Example. 1.

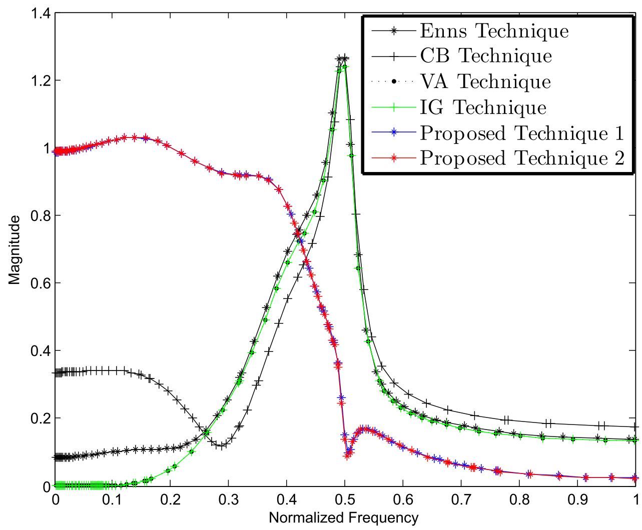

FIGURE 4. Frequency-response error comparison of 3^{rd} order ROM for Example. 1.

A. INDUCTION GENERATOR PARAMETERS

Base voltage $= 690V$, Base power $= 2MW$, Angular velocity $= 2\pi f_m$, $f_m = 50Hz$, Stator resistance $= 0.00488$ $p.u.$, Double-cage reactance $= 0.0453$ $p.u.$,, Stator leakage reactance $= 0.09241$ $p.u.$, Rotor resistance $= 0.00549$ $p.u.$, Rotor leakage reactance $= 0.09955$ $p.u.$, Rotor to double-cage mutual reactance $= 0.02$ $p.u.$, Magnetizing reactance $= 3.95279$ $p.u.$, Load inertia constant $= 3.5$, Double-cage resistance $= 0.2696$ $p.u.$.

B. DFIG CONTROL PARAMETERS

Speed limit=1800 r/min, Cut-in speed = 1000 r/min, Shutdown Speed=2000 r/min.

Example 1: Consider a stable LTI 6^{th} order DFIG model (current model) as given in [61], the discretized sampling time is $T_s = 0.001sec$, with the following input weights and the output weights:

The frequency-response error comparison is given in Figs. 3 and 4 of 2^{nd} and 3^{rd} order ROMs, respectively. Whereas, Figs. 5 and 6 of 2^{nd} and 3^{rd} order ROMs represent the bode plot (phase and magnitude) comparison, respectively, obtained using existing ([30], [39], [40], [43]) and proposed methods. The pole locations of existing ([30], [39], [40], [43]) and proposed techniques are provided in Table. 4, it can also be observed that [30] produces unstable 2^{nd} and 3^{rd} order ROMs along with the pole locations at $z = -1.12469 \pm 1.5327i$ and $z = 1.12133,\ 1.001579 \pm 1.002044i$, respectively. However, in the given frequency-weights, proposed techniques produce low frequency-response truncation error

$$A_{iw} = \begin{bmatrix} -0.75 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.75 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.75 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.75 \end{bmatrix},$$

$$B_{iw} = \begin{bmatrix} -0.33281 & -0.34895 \\ -0.15539 & -0.07846 \\ -0.02101 & -0.22836 \\ -0.30336 & -0.21175 \\ -0.22092 & -0.13554 \\ -0.12428 & -0.04977 \end{bmatrix},$$

$$C_{iw} = \begin{bmatrix} 0.00377 & 0.05555 & 0.02653 & 0.03357 & 0.02709 & 0.09909 \\ 0.06317 & 0.12623 & 0.14361 & 0.05603 & 0.00676 & 0.05758 \\ 0.02762 & 0.11013 & 0.03980 & 0.01313 & 0.10848 & 0.09410 \\ 0.10887 & 0.08565 & 0.13869 & 0.09602 & 0.05212 & 0.00325 \end{bmatrix},$$

$$D_{iw} = \begin{bmatrix} 0.09106 & 0.03833 \\ 0.08006 & 0.06173 \\ 0.07458 & 0.05755 \\ 0.08131 & 0.05301 \end{bmatrix},$$

$$A_{ow} = \begin{bmatrix} -0.95 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.95 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.95 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.95 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.95 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.95 \end{bmatrix},$$

$$B_o = \begin{bmatrix} -0.28072 & -0.22311 \\ -0.13737 & -0.03178 \\ -0.07214 & -0.20447 \\ -0.22917 & -0.13898 \\ -0.22780 & -0.06365 \\ -0.22219 & -0.02956 \end{bmatrix},$$

$$C_{ow} = \begin{bmatrix} 0.12354 & 0.10541 & 0.12138 & 0.05983 & 0.09805 & 0.06601 \\ 0.02625 & 0.02304 & 0.11229 & 0.06226 & 0.13989 & 0.03864 \\ 0.02454 & 0.14302 & 0.01803 & 0.02711 & 0.02453 & 0.11279 \\ 0.09990 & 0.08113 & 0.07876 & 0.03831 & 0.13816 & 0.03430 \\ 0.13416 & 0.10196 & 0.04888 & 0.00308 & 0.11920 & 0.00963 \\ 0.07748 & 0.00548 & 0.08197 & 0.13855 & 0.08661 & 0.11510 \end{bmatrix},$$

$$D_{ow} = \begin{bmatrix} 0.06712 & 0.03174 \\ 0.07152 & 0.08145 \\ 0.06421 & 0.07891 \\ 0.04190 & 0.08523 \\ 0.03908 & 0.05056 \\ 0.08161 & 0.06357 \end{bmatrix}.$$

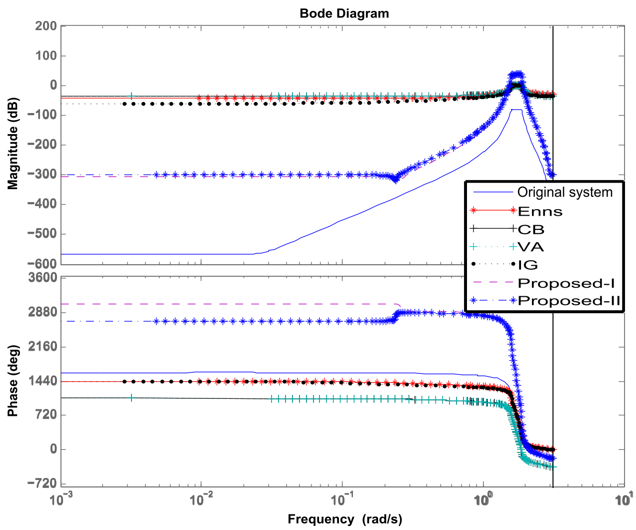

FIGURE 5. Bode plot (phase and magnitude) comparison of 2^{nd} order ROM for Example. 1.

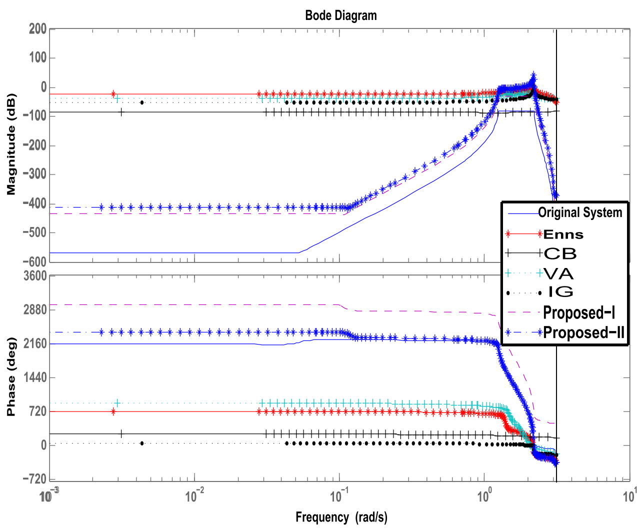

FIGURE 6. Bode plot (phase and magnitude) comparison of 3^{rd} order ROM for Example. 1.

with stable ROMs comparable to existing stability-preserving algorithms ([39], [40], [43]).

Example 2: Consider a 6^{th} order stable 2-D discrete-time system [37]: with the desired frequency-weights as given in [42]. Fig. 7 and 8 show the stable 2-D original and ROMs obtained using the existing [42] and proposed techniques, respectively. The frequency-response error comparison in the desired frequency-weights is given in Fig. 9. The pole locations of [30] and proposed techniques are provided in Table. 4, it can also be observed that [30] produce unstable 3^{rd} dimension ROMs along with the pole locations at $z_1 = 1.00889,\ 1.14789 \pm 0.00479i$ and $z_2 = 1.45781,\ -0.12147 \pm 0.12471i$, respectively. However, in the given frequency-weights, proposed techniques produce low frequency-response truncation error with stable ROMs comparable to existing stability-preserving algorithms ([42]).

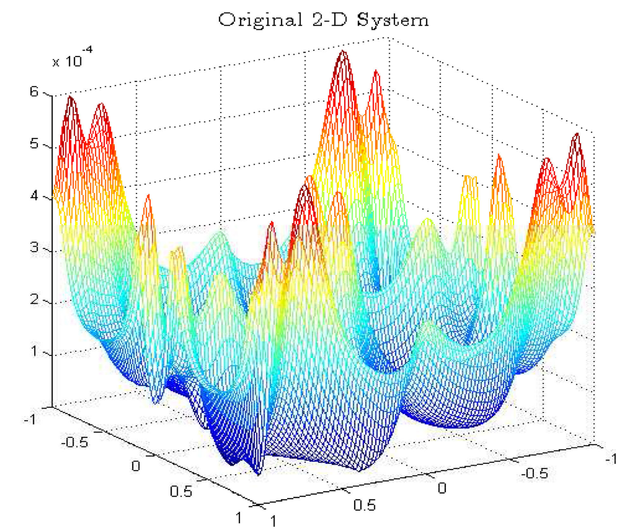

FIGURE 7. 2-D original discrete-time $F[z_1, z_2]$ for example-2.

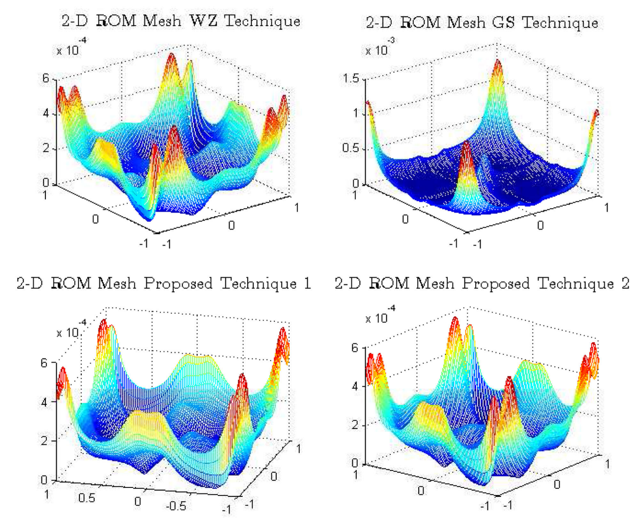

FIGURE 8. 4^{th} order ROM comparison for example-2.

VII. ANALYSIS AND DISCUSSION

Figs. 3, 4 and 9 indicate that ROMs attained with the Enns technique [30] provide a low-frequency response truncation error as compared to the other methods. However, this also yields unstable ROMs as seen in Table. 4. Whereas, Figs. 5 and 6 of 2^{nd} and 3^{rd} order ROMs represent the bode plot (phase and magnitude) comparison, respectively, obtained using existing ([30], [39], [40], [43]) and proposed methods. Proposed techniques, however, generates a low-frequency response truncation error with stable ROMs as compared with the existing stability preserving algorithms and provide closed proximity to the original system.

TABLE 2. Truncation error for example. 1

Examples	Order of ROMs	Frequency-Weighted Approximation Error and Error Bound								
		Error Value					Error Bound			
		[30]	[40]	[39]	[43]	Proposed	[40]	[39]	[43]	Proposed
Example-1	1	955.71	988.15	991.11	995.53	962.14	8891.8	9817.5	9957.7	7518.5
	2	918.34	934.18	951.24	943.76	920.05	7861.1	7219.3	7969.4	7114.3
	3	527.11	684.28	694.38	701.31	600.12	6911.9	6841.4	6687.7	6178.5
	4	387.59	448.79	479.18	497.07	418.05	3784.4	3917.2	3379.1	3179.8
	5	75.17	108.19	184.27	211.79	110.14	971.1	876.7	997.3	890.4

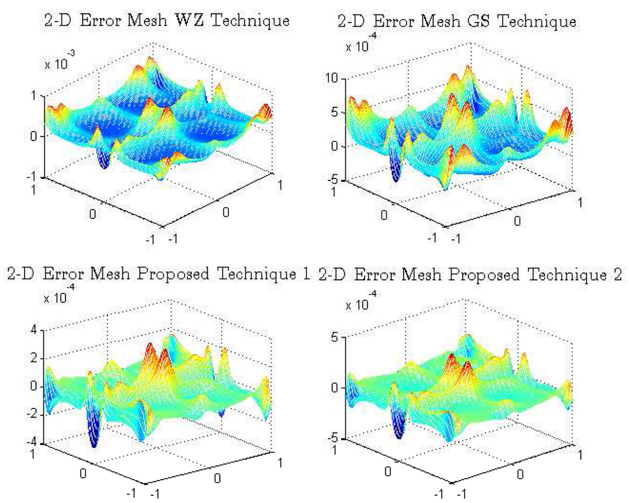

FIGURE 9. Frequency-response error comparison in the given frequency weights of 4^{th} order ROM for example-2.

TABLE 3. Truncation error for example. 2

Examples	Order of ROMs	Frequency-Weighted Approximation Error and Error Bound				
		Error-Values			Error-Bound	
		[30]	[42]	Proposed	[42]	Proposed
Example-1	1	6955.12	6988.45	6962.12	68891.77	67518.35
	2	4968.23	4914.65	4920.43	47871.45	47124.31
	3	3525.45	3685.12	3601.45	36912.65	36177.54
	4	3385.42	3442.56	3419.34	33783.45	33178.43
	5	1751.45	1105.32	193.12	1963.22	1881.33

TABLE 4. Poles locations of reduced order models.

Poles location obtained by using Enns [30] and proposed techniques				
Examples	ROMs	Enns [30]	Proposed Technique 1	Proposed Technique 2
Example-1	2^{nd} order	$z = -1.12469 \pm 1.5327i$	$z = 0.12119 \pm 0.5514i$	$z = 0.12166 \pm 0.5514i$
	3^{rd} order	$z = 1.12133, 1.001579 \pm 1.002044i$	$z = 0.1157, 0.3347 \pm 0.1254i$	$z = 0.1255, 0.5511 \pm 0.1255i$
Example-2	$\bar{F}_{1_{m_{\epsilon_r}}}[z_1] = 3^{rd}$ order	$z_1 = 1.00889, 1.14789 \pm 0.00479i$	$z_1 = 0.14551, 0.14555 \pm 0.00551i$	$z_1 = 0.18512, 0.12515 \pm 0.00117i$
	$\bar{F}_{2_{m_{\epsilon_r}}}[z_2] = 3^{rd}$ order	$z_2 = 1.45781, -0.12147 \pm 0.12471i$	$z_2 = 0.14785, -0.14785 \pm 0.114578i$	$z_2 = 0.3745, -0.12755 \pm 0.127512i$

VIII. CONCLUSION

In this work, the frequency-weighted model order reduction framework for the discrete-time one-dimensional and two-dimensional models is proposed by using balance truncation (proposed technique 1) and an optimal Hankel norm approximation (proposed technique 2) respectively. The suggested approach guarantees that some associated input matrices and associated output matrices for one-dimensional and two-dimensional discrete-time systems, which produce stable reduced-order models, are positive and semi-positive definite. The proposed algorithm also provides frequency-response error bound expression by using balance truncation and an optimal Hankel norm approximation, respectively, for the one-dimensional and two-dimensional discrete-time weighted systems. There are comparisons between existing frequency-weighted model order reduction methods with the proposed framework, indicating that the low-frequency response truncation errors with stable reduced-order models are obtained comparable to existing stability-preserving algorithms, which show the efficacy of the proposed framework. However, the proposed methodology only applies to linear time-invariant one-dimensional and causal recursive separable denominator two-dimensional discrete-time systems. Furthermore, more research is required to extend the proposed method to linear time-variant, descriptor, and bilinear systems. This method can also be used to analyze continuous systems in the time domain. Moreover, the proposed methodology may be expended for other variants of 2-D systems, such as positive 2-D continuous delayed systems based on L_1-gain control design.

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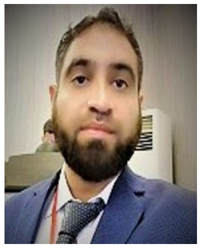
MUHAMMAD IMRAN received the B.S. degree in electrical engineering from the University of Engineering and Technology (UET), Taxila, Pakistan, in 2014, and the M.S. degree in electrical engineering control systems (model/controller order reduction) from the College of Electrical and Mechanical Engineering (CEME), National University of Sciences and Technology (NUST), Islamabad, Pakistan, in 2017, where he is currently pursuing the Ph.D. degree in electrical engineering control systems (model order reduction) with the Military College of Signals (MCS). His research interests include model order reduction, nonlinear control systems, fuzzy control, modeling, simulation and optimization of dynamical systems, designing and developing a real-time video-based detection and tracking system on FPGA for security applications, machine learning, nonlinear Bayesian estimation/filtering, multi-agent systems, unmanned autonomous vehicles, communication systems, and signal processing.



MUHAMMAD IMRAN received the master's and Ph.D. degrees in electrical engineering from the Military College of Signals, National University of Sciences and Technology, Islamabad, Pakistan, in 2011 and 2014, respectively. He is currently a Faculty Member at the Military College of Signals, National University of Sciences and Technology, as an Associate Professor and performing his duties as the Associate Head of Department. He is an Active Researcher and produced many publications in well-reputed journals and conferences. His research interests include model/controller order reduction, control systems theory, communication systems and signal processing, communication, and biomedical algorithms, unmanned autonomous vehicles, distributed decision making, map-based mobile application, and stochastic modeling.

MIAN ILYAS AHMAD received the B.Sc. degree in electrical engineering from the University of Engineering and Technology (UET), Peshawar, Pakistan, in 2006, and the Ph.D. degree in control systems from Imperial College London, U.K., in 2011. Then, he visited Imperial College London under NUST MS, leading to a Ph.D. Fellowship. In January 2013, he joined the Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany, under the Max Planck Postdoctoral Fellowship. After successfully completing three years of a postdoctoral fellowship, he resumed his position of an Assistant Professor at RCMS in December 2015. He has published in some of the top journals in electrical engineering and applied mathematics, including the *IET Control Theory and Applications* and *SIAM Journal on Matrix Analysis and Applications*. His research interests include large-scale system analysis and control, model order reduction, applied linear algebra, and nonlinear dynamics.