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On the Algebraic Attributes of (α, β)-Pythagorean Fuzzy Subrings and (α, β)-Pythagorean Fuzzy Ideals of Rings

SUPRIYA BHUNIA [1], GANESH GHORAI [1], QIN XIN [2], AND MUHAMMAD GULZAR [3]

[1]Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India
[2]Faculty of Science and Technology, University of the Faroe Islands, Torshavn 100, Faroe Islands
[3]Department of Mathematics, Government College University Faisalabad, Faisalabad 38000, Pakistan

Corresponding author: Muhammad Gulzar (98kohly@gmail.com)

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ABSTRACT (α, β)-Pythagorean fuzzy set is a very efficient way of dealing with uncertainty. In this article, we have introduced the notions of (α, β)-Pythagorean fuzzy subring and (α, β)-Pythagorean fuzzy ideal of a ring. Further, we have briefly described various results related to it. Also, we have discussed the level subring of an (α, β)-Pythagorean fuzzy subring. Moreover, we have studied the direct product and ring homomorphism of (α, β)-Pythagorean fuzzy subrings.

INDEX TERMS (α, β)-PFS, (α, β)-PFSR, (α, β)-PFID, (α, β)-PFLSR.

I. INTRODUCTION

In classical ring theory, the concepts of subring and ideal are extremely important. Uncertainty is an unavoidable element of our lives. This universe isn't built on assumptions or precise measures. It is not always feasible to make a straight forward decision. We face a significant problem in dealing with errors in decision-making situations. In 1965, Zadeh [18] established the concept of a fuzzy set to deal with ambiguity in real-world situations, breaking the usual conception of yes or no. Any mapping from a universal set to $[0, 1]$ is a fuzzy set. As a result, an element's membership value lies in $[0, 1]$. In 1971, Rosenfeld [15] was the first to investigate the concept of ideal and fuzzy subgroup. Liu [12], [13] investigated various properties of fuzzy ideals. Ren [16] looked at fuzzy ideals and quotient fuzzy rings. Dixit *et al.* [8] studied various aspects of fuzzy rings. In 2021, Alghazzawi *et al.* [1] studied ω-Q-fuzzy subrings. Gulzar *et al.* [9] characterized Q-complex fuzzy subrings. Kausar *et al.* [11] discussed anti-fuzzy bi-ideals in 2020.

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When it comes to decision-making, assigning membership values isn't always adequate. In 1986, Atanassov [2] established intuitionistic fuzzy set by assigning non-membership degree with membership degree. Hur *et al.* [10] proposed the idea of an intuitionistic fuzzy ring. Banerjee and Basnet [3] did more work on intuitionistic fuzzy subrings and intuitionistic fuzzy ideals. Yager [17] defined Pythagorean fuzzy set in 2013. In comparison to intuitionistic fuzzy sets, Pythagorean fuzzy set presents a cutting-edge method for modelling ambiguity and uncertainty with great precision and accuracy. Consider a point with membership grades $(a, 0.8, 0.6)$. Here $0.8^2 + 0.6^2 = 1$, so it is a Pythagorean membership grade. However $0.8 + 0.6 = 1.4$, then it is not an intuitionistic membership grade. The collection of Pythagorean membership grades is bigger than intuitionistic membership grades, which is very important in decision-making problems. Bhunia *et al.* [5] proposed Pythagorean fuzzy subgroups in 2021. several results related to Pythagorean fuzzy sets and Pythagorean fuzzy subgroups were provided by [7], [14].

In 2021, Bhunia [4] and Ghorai began studying (α, β)-Pythagorean fuzzy sets. Imposing the constraints α and β we can make a non-Pythagorean fuzzy set to a

Pythagorean fuzzy set. α and β gives more flexibility to collect data. When both intuitionistic fuzzy set and Pythagorean fuzzy set fails then (α, β)-Pythagorean fuzzy sets come into play. They explained that (α, β)-Pythagorean fuzzy sets are more precise than intuitionistic fuzzy sets and Pythagorean fuzzy sets. They established the concept of an (α, β)-Pythagorean fuzzy subgroup and demonstrated several properties of it. In 2021, Lagrange's theorem is also proved by Bhunia *et al.* [6] in (α, β)-Pythagorean fuzzy subgroup.

The benefits of (α, β)-Pythagorean fuzzy sets and the intention to explore fuzzy rings in (α, β)-Pythagorean fuzzy sets is the main motive of this research. The following are the objectives of this manuscript:

1) To define the notion of an (α, β)-Pythagorean fuzzy subring and (α, β)-Pythagorean fuzzy ideal of a ring
2) To investigate certain fundamental properties of (α, β)-Pythagorean fuzzy subrings and (α, β)-Pythagorean fuzzy ideals
3) To describe (α, β)-Pythagorean fuzzy level subring of a ring
4) To discuss the direct product and ring homomorphism of (α, β)-Pythagorean fuzzy subring.

The following is a summary of the contribution of this paper: Section III review some key definitions and ideas. We develop the idea of (α, β)-Pythagorean fuzzy subrings and (α, β)-Pythagorean fuzzy ideals in Section IV. Section V deals with (α, β)-Pythagorean fuzzy level subring and its properties. In Section VI, we describe the direct product and ring homomorphism of (α, β)-Pythagorean fuzzy subring. In Section VII, we come to a conclusion.

II. LIST OF ABBREVIATIONS

IFS	- Intuitionistic fuzzy set.
PFS	- Pythagorean fuzzy set.
(α, β)-PFS	- (α, β)-Pythagorean fuzzy set.
IFSR	- Intuitionistic fuzzy subring.
IFID	- Intuitionistic fuzzy ideal.
(α, β)-PFS	- (α, β)-Pythagorean fuzzy set.
(α, β)-PFSR	- (α, β)-Pythagorean fuzzy subring.
(α, β)-PFID	- (α, β)-Pythagorean fuzzy ideal.
(α, β)-PFLS	- (α, β)-Pythagorean fuzzy level subset.
(α, β)-PFLSR	- (α, β)-Pythagorean fuzzy level subring.
(α, β)-PFLID	- (α, β)-Pythagorean fuzzy level ideal.

III. PRELIMINARIES

This section introduces several key terminology and concepts.

Definition 1 [2]: An intuitionistic fuzzy set (IFS) I of a universal set W is of the form $I = \{(w, \mu(w), \nu(w)) | w \in W\}$, *where* $0 \le \mu(w) + \nu(w) \le 1$. *Here,* $\mu(w), \nu(w) \in [0, 1]$ *are membership degree and non-membership degree of* $w \in W$ *respectively.*

Definition 2 [3]: Assume a ring $(W, +, \cdot)$ *have an IFS* $I = \{(w, \mu(w), \nu(w)) | w \in W\}$, *I is referred as intuitionistic fuzzy subring (IFSR) of W if*

1) $\mu(w_1 - w_2) \ge \mu(w_1) \wedge \mu(w_2)$ *and* $\nu(w_1 - w_2) \le \nu(w_1) \vee \nu(w_2)\ \forall w_1, w_2 \in W$
2) $\mu(w_1 \cdot w_2) \ge \mu(w_1) \wedge \mu(w_2)$ *and* $\nu(w_1 \cdot w_2) \le \nu(w_1) \vee \nu(w_2)\ \forall w_1, w_2 \in W$.

Definition 3 [17]: A Pythagorean fuzzy set (PFS) ψ *of an universal set W is of the form* $\psi = \{(w, \mu(w), \nu(w)) | w \in W\}$, *where* $0 \le \mu^2(w) + \nu^2(w) \le 1$.

Definition 4 [4]: An (α, β)*-Pythagorean fuzzy set (PFS)* ψ^* *of an universal set W is of the form* $\psi^* = \{(w, \mu^\alpha(w), \nu^\beta(w)) | w \in W\}$, *where* $\mu^\alpha(w) = \mu(w) \wedge \alpha$, $\nu^\beta(w) = \nu(w) \vee \beta$ *and* $0 \le (\mu^\alpha(w))^2 + \left(\nu^\beta(w)\right)^2 \le 1$. *Here,* $\alpha, \beta \in [0, 1]$ *with* $0 \le \alpha^2 + \beta^2 \le 1$.

Proposition 1 [4]: Let $\psi_1^* = \{(w, \mu_1^\alpha(w), \nu_1^\beta(w)) | w \in W\}$ *and* $\psi_2^* = \{(w, \mu_2^\alpha(w), \nu_2^\beta(w)) | w \in W\}$ *be two* (α, β)*-PFSs in W. Then*

1) $\psi_1^* \cup \psi_2^* = \{(w, \mu_1^\alpha(w) \vee \mu_2^\alpha(w), \nu_1^\beta(w) \wedge \nu_2^\beta(w)) | w \in W\}$
2) $\psi_1^* \cap \psi_2^* = \{(w, \mu_1^\alpha(w) \wedge \mu_2^\alpha(w), \nu_1^\beta(w) \vee \nu_2^\beta(w)) | w \in W\}$
3) $\psi_1^* \subseteq \psi_2^*$ *if* $\mu_1^\alpha(w) \le \mu_2^\alpha(w)$ *and* $\nu_1^\beta(w) \ge \nu_2^\beta(w)$ *for all* $w \in W$
4) $\psi_1^* = \psi_2^*$ *if* $\mu_1^\alpha(w) = \mu_2^\alpha(w)$ *and* $\nu_1^\beta(w) = \nu_2^\beta(w)$ *for all* $w \in W$.

Definition 5 [4]: Let $\psi^* = (\mu^\alpha, \nu^\beta)$ *be an* (α, β)*-PFS of an universal set W. Then* $\psi^*_{(\theta,\tau)} = \{w \in W | \mu^\alpha(w) \ge \theta$ *and* $\nu^\beta(w) \le \tau\}$ *is called an* (α, β)*-Pythagorean fuzzy level subset (PFLS) of* ψ^*, *where* $\theta, \tau \in [0, 1]$.

Proposition 2 [4]: Let $\psi_1^* = (\mu_1^\alpha, \nu_1^\beta)$ *and* $\psi_2^* = (\mu_2^\alpha, \nu_2^\beta)$ *be two* (α, β)*-PFSs of a set W. Then for* ϵ, τ, θ *and* $\delta \in [0, 1]$,

1) $\epsilon \le \theta, \tau \le \delta \Rightarrow \psi^*_{(\theta,\tau)} \subseteq \psi^*_{(\epsilon,\delta)}$
2) $\psi_1^* \subseteq \psi_2^* \Rightarrow \psi^*_{1(\theta,\tau)} \subseteq \psi^*_{2(\theta,\tau)}$.

Proposition 3 [6]: Let $\psi_1^* = (\mu_1^\alpha, \nu_1^\beta)$ *and* $\psi_2^* = (\mu_2^\alpha, \nu_2^\beta)$ *be two* (α, β)*-PFSs on* W_1 *and* W_2 *respectively. Let r be a mapping from* W_1 *to* W_2. *Then* $r(\psi_1^*)$ *is an* (α, β)*-PFS on* W_2 *and defined by* $r(\psi_1^*)(w_2) = (r(\mu_1^\alpha)(w_2), r(\nu_1^\beta)(w_2))$ *for all* $w_2 \in W_2$, *where*

$$r(\mu_1^\alpha)(w_2) = \begin{cases} \vee \left\{\mu_1^\alpha(w_1) | w_1 \in W_1 \text{ and } r(w_1) = w_2\right\}, \\ \text{when } r^{-1}(w_2) \ne \varnothing \\ 0, \text{ elsewhere} \end{cases}$$

and

$$r(\nu_1^\beta)(w_2) = \begin{cases} \wedge \left\{\nu_1^\beta(w_1) | w_1 \in W_1 \text{ and } r(w_1) = w_2\right\}, \\ \text{when } r^{-1}(w_2) \ne \varnothing \\ 1, \text{ elsewhere}. \end{cases}$$

Also, $r^{-1}(\psi_2^*)$ *is an* (α, β)*-PFS on* W_1 *and defined by* $r^{-1}(\psi_2^*)(w_1) = (r^{-1}(\mu_2^\alpha)(w_1), r^{-1}(\nu_2^\beta)(w_1))$ *for all* $w_1 \in W_1$, *where* $(r^{-1}(\mu_2^\alpha))(w_1) = (\mu_2^\alpha(r(w_1))$ *and* $(r^{-1}(\nu_2^\beta))(w_1) = (\nu_2^\beta(r(w_1))$.

IV. (α, β)-PFSR AND (α, β)-PFID

Now, (α, β)-PFSR and (α, β)-PFID of rings will be discussed.

Definition 6: Assume $(W, +, \cdot)$ *is a ring and* $\psi^* = (\mu^\alpha, \nu^\beta)$ *is an* (α, β)*-PFS of W. The ring* $(W, +, \cdot)$ *is then* *said to have an (α, β)-Pythagorean fuzzy subring (PFSR) ψ^* if*

1) $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$
2) $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$.

Example 1: Take the ring $(\mathbb{Z}, +, \cdot)$. Consider, $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFS of the ring $(\mathbb{Z}, +, \cdot)$, where $\psi^* = (\mu^\alpha, \nu^\beta)$ is defined by

$$\mu^\alpha(z) = \begin{cases} 0.91, & when \;\; z = 0 \\ 0.63, & when \;\; z \in 2\mathbb{Z} - \{0\} \\ 0.82, & elsewhere \end{cases}$$

and

$$\nu^\beta(z) = \begin{cases} 0.16, & when \;\; z = \{0\} \\ 0.32, & when \;\; z \in 2\mathbb{Z} - \{0\} \\ 0.27, & elsewhere. \end{cases}$$

Clearly, the ring $(\mathbb{Z}, +, \cdot)$ have an (α, β)-PFSR $\psi^* = (\mu^\alpha, \nu^\beta)$.

Definition 7: Assume $(W, +, \cdot)$ is a ring and $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFS of W. The ring $(W, +, \cdot)$ is then said to have an (α, β)-Pythagorean fuzzy ideal (PFID) ψ^* if

1) $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$
2) $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$.

Example 2: Take the ring $(\mathbb{Z}_9, +_9, \cdot_9)$. Consider, $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFS of the ring $(\mathbb{Z}_9, +_9, \cdot_9)$, where $\psi^* = (\mu^\alpha, \nu^\beta)$ is defined by

$$\mu^\alpha(w) = \begin{cases} 0.93, & when \;\; w = 0 \\ 0.56, & when \;\; w \in \{3, 6\} \\ 0.22, & elsewhere \end{cases}$$

and

$$\nu^\beta(w) = \begin{cases} 0.12, & when \;\; w = \{0\} \\ 0.39, & when \;\; w \in \{3, 6\} \\ 0.67, & elsewhere. \end{cases}$$

Clearly, we can check that $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFID of the ring $(\mathbb{Z}_9, +_9, \cdot_9)$.

In a ring, every ideal is a subring of that ring, however the opposite may not be true. As for the example, for the ring $(\mathbb{Q}, +, \cdot)$, $(\mathbb{Z}, +, \cdot)$ is a subring of $\mathbb{Q}$ but not an ideal of $\mathbb{Q}$.

Now, we will establish a relation between (α, β)-PFID and (α, β)-PFSR of a ring.

Theorem 1: Every (α, β)-PFID of a ring is an (α, β)-PFSR of that ring.

Proof: Assume $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFID of a ring $(W, +, \cdot)$. Then $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$, $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ and $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2)$, $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

To prove $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFSR of the ring $(W, +, \cdot)$, we need to show that $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$, $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ and $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$, $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

The first two condition of (α, β)-PFSR is automatically satisfied. Now, many case will arise for last two condition. We will study some cases

Case 1: Assume $\mu^\alpha(w_1) > \mu^\alpha(w_2)$ and $\nu^\beta(w_1) > \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

Then $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2) = \mu^\alpha(w_1) > \mu^\alpha(w_2) = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$.

Therefore $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ for all w_1, $w_2 \in W$.

Also, $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2) = \nu^\beta(w_2) < \nu^\beta(w_1) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.

Therefore $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all w_1, $w_2 \in W$.

Case 2: Assume $\mu^\alpha(w_1) < \mu^\alpha(w_2)$ and $\nu^\beta(w_1) < \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

Then $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2) = \mu^\alpha(w_2) > \mu^\alpha(w_1) = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$.

Therefore $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ for all w_1, $w_2 \in W$.

Also, $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2) = \nu^\beta(w_1) < \nu^\beta(w_2) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.

Therefore $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all w_1, $w_2 \in W$.

Case 3: Assume $\mu^\alpha(w_1) = \mu^\alpha(w_2)$ and $\nu^\beta(w_1) = \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

Then $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2) = \mu^\alpha(w_2) = \mu^\alpha(w_1) = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$.

Therefore $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ for all w_1, $w_2 \in W$.

Also, $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2) = \nu^\beta(w_1) = \nu^\beta(w_2) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.

Therefore $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all w_1, $w_2 \in W$.

Considering all the possibilities and using the same technique, we can simply verify that $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ and $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all $w_1, w_2 \in W$. Thus $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFSR of the ring $(W, +, \cdot)$. Hence every (α, β)-PFID of a ring is an (α, β)-PFSR of that ring. □

Example 3: In Example 1, take the (α, β)-PFSR ψ^* of the ring $(\mathbb{Z}, +, \cdot)$.

Now, we take two element $z_1 = 2$ and $z_2 = 3$. Then $\mu^\alpha(2) = 0.63$, $\nu^\beta(2) = 0.32$, $\mu^\alpha(3) = 0.82$ and $\nu^\beta(3) = 0.27$.

Therefore $\mu^\alpha(2) \wedge \mu^\alpha(3) = 0.63$, $\mu^\alpha(2) \vee \mu^\alpha(3) = 0.82$, $\nu^\beta(2) \wedge \nu^\beta(3) = 0.27$ and $\nu^\beta(2) \vee \nu^\beta(3) = 0.32$.

So, $\mu^\alpha(3 - 2) = \mu^\alpha(1) = 0.82 > 0.63 = \mu^\alpha(2) \wedge \mu^\alpha(3)$, $\nu^\beta(3 - 2) = \nu^\beta(1) = 0.27 < 0.32 = \nu^\beta(2) \vee \nu^\beta(3)$. But, $\mu^\alpha(3 \cdot 2) = \mu^\alpha(6) = 0.63 \ngeq 0.82 = \mu^\alpha(2) \vee \mu^\alpha(3)$, $\nu^\beta(3 \cdot 2) = \nu^\beta(6) = 0.32 \nleq 0.27 = \nu^\beta(2) \wedge \nu^\beta(3)$. This shows that, ψ^* violate the condition of (α, β)-PFID of a ring.

Therefore, $\psi^* = (\mu^\alpha, \nu^\beta)$ is not an (α, β)-PFID of the ring $(\mathbb{Z}, +, \cdot)$.

Remark 1: *Every (α, β)-PFID of a ring is an (α, β)-PFSR of that ring, however the opposite statement is not true.*

Proposition 4: *Assume $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFSR of a ring $(W, +, \cdot)$. Then*

1) $\mu^\alpha(0) \geq \mu^\alpha(w_1)$ *and* $\nu^\beta(0) \leq \nu^\beta(w_1)$ *for all* $w_1 \in W$
2) $\mu^\alpha(-w_1) = \mu^\alpha(w_1)$ *and* $\nu^\beta(-w_1) = \nu^\beta(w_1)$ *for all* $w_1 \in W$
3) $\mu^\alpha(w_1 + w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1 + w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$.

Proof: Since, the ring $(W, +, \cdot)$ have an (α, β)-PFSR $\psi^* = (\mu^\alpha, \nu^\beta)$, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ and $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

1) $\mu^\alpha(0) = \mu^\alpha(w_1 - w_1) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_1) = \mu^\alpha(w_1)$. So, $\mu^\alpha(0) \geq \mu^\alpha(w_1)$ for all $w_1 \in W$.
   Also, $\nu^\beta(0) = \nu^\beta(w_1 - w_1) \leq \nu^\beta(w_1) \vee \nu^\beta(w_1) = \nu^\beta(w_1)$. Therefore $\nu^\beta(0) \leq \nu^\beta(w_1)$ for all $w_1 \in W_1$, where 0 is the additive identity of W.
2) $\mu^\alpha(-w_1) = \mu^\alpha(0 - w_1) \geq \mu^\alpha(0) \wedge \mu^\alpha(w_1) = \mu^\alpha(w_1)$ and $\nu^\beta(-w_1) = \nu^\beta(0 - w_1) \leq \nu^\beta(0) \vee \nu^\beta(w_1) = \nu^\beta(w_1)$. Therefore $\mu^\alpha(-w_1) \geq \mu^\alpha(w_1)$ and $\nu^\beta(-w_1) \leq \nu^\beta(w_1)$.
   Again, $\mu^\alpha(w_1) = \mu^\alpha(-(-w_1)) \geq \mu^\alpha(-w_1)$ and $\nu^\beta(w_1) = \nu^\beta(-(-w_1)) \leq \nu^\beta(-w_1)$. Hence $\mu^\alpha(-w_1) = \mu^\alpha(w_1)$ and $\nu^\beta(-w_1) = \nu^\beta(w_1)$ for all $w_1 \in W$, where $(-w_1)$ is the additive inverse of w_1 in W.
3) $\mu^\alpha(w_1 + w_2) = \mu^\alpha(w_1 - (-w_2)) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(-w_2) = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ and $\nu^\beta(w_1 + w_2) = \nu^\beta(w_1 - (-w_2)) \leq \nu^\beta(w_1) \vee \nu^\beta(-w_2) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.
   Therefore $\mu^\alpha(w_1 + w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ and $\nu^\beta(w_1 + w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

□

Remark 2: *If $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFID of a ring $(W, +, \cdot)$, all the properties of an (α, β)-PFSR in Proposition* 4.1 *also hold for the (α, β)-PFID $\psi^* = (\mu^\alpha, \nu^\beta)$.*

Proposition 5: *Assume $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFSR of a ring $(W, +, \cdot)$. Then*

1) $\mu^\alpha(kw_1) \geq \mu^\alpha(w_1)$ *and* $\nu^\beta(kw_1) \leq \nu^\beta(w_1)$ *for all* $w_1 \in W$, $k \in \mathbb{Z}$
2) $\mu^\alpha(w_1^k) \geq \mu^\alpha(w_1)$ *and* $\nu^\beta(w_1^k) \leq \nu^\beta(w_1)$ *for all* $w_1 \in W$, $k \in \mathbb{Z}$.

Proof: Since, the ring $(W, +, \cdot)$ have an (α, β)-PFSR $\psi^* = (\mu^\alpha, \nu^\beta)$, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$, $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$, $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ and $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

1) By Proposition 4, we have $\mu^\alpha(kw_1) = \mu^\alpha(w_1 + w_1 + \cdots + w_1)$ (k *times*)
   $\geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_1) \wedge \cdots \wedge \mu^\alpha(w_1)$ (k *times*)
   $= \mu^\alpha(w_1)$.
   Also, $\nu^\beta(kw_1) = \nu^\beta(w_1 + w_1 + \cdots + w_1)$ (k *times*)
   $\leq \nu^\beta(w_1) \vee \nu^\beta(w_1) \vee \cdots \vee \nu^\beta(w_1)$ (k *times*)
   $= \nu^\beta(w_1)$.
   Therefore $\mu^\alpha(kw_1) \geq \mu^\alpha(w_1)$ and $\nu^\beta(kw_1) \leq \nu^\beta(w_1)$ for all $w_1 \in W$, $k \in \mathbb{Z}$.
2) Now, $\mu^\alpha(w_1^k) = \mu^\alpha(w_1 \cdot w_1 \cdot \ldots \cdot w_1)$ (k *times*)
   $\geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_1) \wedge \cdots \wedge \mu^\alpha(w_1)$ (k *times*)
   $= \mu^\alpha(w_1)$.
   Also, $\nu^\beta(w_1^k) = \nu^\beta(w_1 \cdot w_1 \cdot \ldots \cdot w_1)$ (k *times*)
   $\leq \nu^\beta(w_1) \vee \nu^\beta(w_1) \vee \cdots \vee \nu^\beta(w_1)$ (k *times*)
   $= \nu^\beta(w_1)$.
   Therefore $\mu^\alpha(w_1^k) \geq \mu^\alpha(w_1)$ and $\nu^\beta(w_1^k) \leq \nu^\beta(w_1)$ for all $w_1 \in W$, $k \in \mathbb{Z}$.

□

Proposition 6: *Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be an (α, β)-PFSR of a ring $(W, +, \cdot)$. If $\mu^\alpha(w_1 - w_2) = \mu^\alpha(0)$ and $\nu^\beta(w_1 - w_2) = \nu^\beta(0)$, then $\mu^\alpha(w_1) = \mu^\alpha(w_2)$ and $\nu^\beta(w_1) = \nu^\beta(w_2)$ respectively.*

Proof: Since the ring $(W, +, \cdot)$ have an (α, β)-PFSR $\psi^* = (\mu^\alpha, \nu^\beta)$, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$, $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$, $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ and $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.

Now, by proposition 4, we have

$$\begin{aligned}\mu^\alpha(w_1) &= \mu^\alpha(w_1 - w_2 + w_2)\\ &\geq \mu^\alpha(w_1 - w_2) \wedge \mu^\alpha(w_2)\\ &= \mu^\alpha(0) \wedge \mu^\alpha(w_2)\\ &= \mu^\alpha(w_2)\end{aligned}$$

By replacing w_1 with w_2 in above relation, we get $\mu^\alpha(w_2) \geq \mu^\alpha(w_1)$. Since w_1, w_2 are arbitrary, $\mu^\alpha(w_1) = \mu^\alpha(w_2)$ for all $w_1, w_2 \in W$.

Again, by proposition 4, we have

$$\begin{aligned}\nu^\beta(w_1) &= \nu^\beta(w_1 - w_2 + w_2)\\ &\leq \nu^\beta(w_1 - w_2) \vee \nu^\beta(w_2)\\ &= \nu^\alpha(0) \vee \nu^\beta(w_2)\\ &= \nu^\beta(w_2)\end{aligned}$$

Similarly, we can show that $\nu^\beta(w_2) \leq \nu^\beta(w_1)$. As w_1, w_2 are arbitrary, $\nu^\beta(w_1) = \nu^\beta(w_2)$ for all $w_1, w_2 \in W$. □

Proposition 7: *Let a commutative ring with unity (CRU) $(W, +, \cdot)$ have an (α, β)-PFID $\psi^* = (\mu^\alpha, \nu^\beta)$. Then*

1) $\mu^\alpha(1) \leq \mu^\alpha(w_1)$ *and* $\nu^\beta(1) \geq \nu^\beta(w_1)$ *for all* $w_1 \in W$, *where* 1 *is the multiplicative identity of* W
2) $\mu^\alpha(w_1) = \mu^\alpha(w_1^{-1}) = \mu^\alpha(1)$ *and* $\nu^\beta(w_1) = \nu^\beta(w_1^{-1}) = \nu^\beta(1)$ *for all* $w_1 \in W$, *where* w_1^{-1} *is the multiplicative inverse of* w_1 *in* W.

Proof:

1) Since the CRU $(W, +, \cdot)$ have an (α, β)-PFID $\psi^* = (\mu^\alpha, \nu^\beta)$, $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2)$ and $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2)$ for all $w_1, w_2 \in W$.
   Therefore $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1)$ and $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_2)$. Also, $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1)$ and $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_2)$.
   So, $\mu^\alpha(w_1) = \mu^\alpha(w_1 \cdot 1) \geq \mu^\alpha(1)$, $\nu^\beta(w_1) = \nu^\beta(w_1 \cdot 1) \leq \nu^\beta(1)$ $\forall w_1 \in W$, where 1 is the multiplicative identity of W.
2) Now, $\mu^\alpha(1) = \mu^\alpha(w_1 \cdot w_1^{-1}) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_1^{-1})$. This shows that, $\mu^\alpha(1) \geq \mu^\alpha(w_1)$ and $\mu^\alpha(1) \geq \mu^\alpha(w_1^{-1})$.

Also, by previous result we get $\mu^{\alpha}(w_1) \geq \mu^{\alpha}(1)$ for all $w_1 \in W$.
Thus $\mu^{\alpha}(w_1) = \mu^{\alpha}(w_1^{-1}) = \mu^{\alpha}(1)\ \forall w_1 \in W$.
Again, $\nu^{\beta}(1) = \nu^{\beta}(w_1 \cdot w_1^{-1}) \leq \nu^{\beta}(w_1) \wedge \nu^{\beta}(w_1^{-1})$. This present that $\nu^{\beta}(1) \leq \nu^{\beta}(w_1)$ and $\nu^{\beta}(1) \leq \nu^{\beta}(w_1^{-1})$.
Also, we have $\nu^{\beta}(w_1) \leq \nu^{\beta}(1)$ for all $w_1 \in W$.
Therefore $\nu^{\beta}(w_1) = \nu^{\beta}(w_1^{-1}) = \nu^{\beta}(1)$ for all $w_1 \in W$, where w_1^{-1} is the multiplicative inverse of w_1 in W. □

Theorem 2: If $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is an (α, β)-PFID of a ring $(W, +, \cdot)$, then $P = \{w \in W | \mu^{\alpha}(w) = \mu^{\alpha}(0), \nu^{\beta}(w) = \nu^{\beta}(0)\}$ is an ideal of the ring $(W, +, \cdot)$.

Proof: Clearly, P is a non-empty subset of W as $0 \in P$.

Let w_1, w_2 be two elements of P. Then $\mu^{\alpha}(w_1) = \mu^{\alpha}(0)$, $\nu^{\beta}(w_1) = \nu^{\beta}(0)$, $\mu^{\alpha}(w_2) = \mu^{\alpha}(0)$, and $\nu^{\beta}(w_2) = \nu^{\beta}(0)$.

Now, $\mu^{\alpha}(w_1 - w_2) \geq \mu^{\alpha}(w_1) \wedge \mu^{\alpha}(w_2) = \mu^{\alpha}(0) \wedge \mu^{\alpha}(0) = \mu^{\alpha}(0)$. That is $\mu^{\alpha}(w_1 - w_2) \geq \mu^{\alpha}(0)$.

Again, $\nu^{\beta}(w_1 - w_2) \leq \nu^{\beta}(w_1) \vee \nu^{\beta}(w_2) = \nu^{\beta}(0) \vee \nu^{\beta}(0) = \nu^{\beta}(0)$. So, $\nu^{\beta}(w_1 - w_2) \leq \nu^{\beta}(0)$.

By Proposition 4, we have $\mu^{\alpha}(0) \geq \mu^{\alpha}(w_1 - w_2)$ and $\nu^{\beta}(w_1 - w_2) \geq \nu^{\beta}(0)$.

Thus $\mu^{\alpha}(w_1 - w_2) = \mu^{\alpha}(0)$ and $\nu^{\beta}(w_1 - w_2) = \nu^{\beta}(0)$. Therefore $w_1 - w_2 \in P$.

Let $w \in W, w_1 \in P$.

Then $\mu^{\alpha}(w \cdot w_1) \geq \mu^{\alpha}(w) \vee \mu^{\alpha}(w_1)$. Therefore $\mu^{\alpha}(w \cdot w_1) \geq \mu^{\alpha}(w_1) = \mu^{\alpha}(0)$. Similarly, we can verify that $\nu^{\beta}(w \cdot w_1) \leq \nu^{\beta}(0)$.

By Proposition 4, we have $\mu^{\alpha}(w \cdot w_1) \leq \mu^{\alpha}(0)$ and $\nu^{\beta}(w \cdot w_1) \geq \nu^{\beta}(0)$. Therefore $\mu^{\alpha}(w \cdot w_1) = \mu^{\alpha}(0)$ and $\nu^{\beta}(w \cdot w_1) = \nu^{\beta}(0)$. So, $w.w_1 \in P$. Similarly, we can verify that $w_1.w \in P$.

Hence P is an ideal of the ring $(W, +, \cdot)$. □

Theorem 3: Intersection of any two (α, β)-PFSR of a ring is an (α, β)-PFSR of that ring.

Proof: Let $\psi_1^* = (\mu_1^{\alpha}, \nu_1^{\beta})$ and $\psi_2^* = (\mu_2^{\alpha}, \nu_2^{\beta})$ be two (α, β)-PFSR of a ring $(W, +, \cdot)$. Assume $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is the intersection of ψ_1^* and ψ_2^*. That is $\psi^* = \psi_1^* \cap \psi_2^*$. Therefore $\mu^{\alpha}(w_1) = \mu_1^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_1)$ and $\nu^{\beta}(w_1) = \nu_1^{\beta}(w_1) \vee \nu_2^{\beta}(w_1)$ for all $w_1 \in W$.

Let w_1, w_2 be any two elements of W. Now,

$$\begin{aligned}\mu^{\alpha}(w_1 - w_2) &= \mu_1^{\alpha}(w_1 - w_2) \wedge \mu_2^{\alpha}(w_1 - w_2)\\ &\geq (\mu_1^{\alpha}(w_1) \wedge \mu_1^{\alpha}(w_2)) \wedge (\mu_2^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_2))\\ &= (\mu_1^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_1)) \wedge (\mu_1^{\alpha}(w_2) \wedge \mu_2^{\alpha}(w_2))\\ &= \mu^{\alpha}(w_1) \wedge \mu^{\alpha}(w_2)\end{aligned}$$

Also,

$$\begin{aligned}\nu^{\beta}(w_1 - w_2) &= \nu_1^{\beta}(w_1 - w_2) \vee \nu_2^{\beta}(w_1 - w_2)\\ &\leq (\nu_1^{\beta}(w_1) \vee \nu_1^{\beta}(w_2)) \vee (\nu_2^{\beta}(w_1) \vee \nu_2^{\beta}(w_2))\\ &= (\nu_1^{\beta}(w_1) \vee \nu_2^{\beta}(w_1)) \vee (\nu_1^{\beta}(w_2) \vee \nu_2^{\beta}(w_2))\\ &= \nu^{\beta}(w_1) \vee \nu^{\beta}(w_2)\end{aligned}$$

Therefore $\mu^{\alpha}(w_1 - w_2) \geq \mu^{\alpha}(w_1) \wedge \mu^{\alpha}(w_2)$ and $\nu^{\beta}(w_1 - w_2) \leq \nu^{\beta}(w_1) \vee \nu^{\beta}(w_2)$.

Again, $\mu^{\alpha}(w_1 \cdot w_2) = \mu_1^{\alpha}(w_1 \cdot w_2) \wedge \mu_2^{\alpha}(w_1 \cdot w_2) \geq (\mu_1^{\alpha}(w_1) \wedge \mu_1^{\alpha}(w_2)) \wedge (\mu_2^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_2)) = (\mu_1^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_1)) \wedge (\mu_1^{\alpha}(w_2) \wedge \mu_2^{\alpha}(w_2)) = \mu^{\alpha}(w_1) \wedge \mu^{\alpha}(w_2)$.

Also, $\nu^{\beta}(w_1 \cdot w_2) = \nu_1^{\beta}(w_1 \cdot w_2) \vee \nu_2^{\beta}(w_1 \cdot w_2) \leq (\nu_1^{\beta}(w_1) \vee \nu_1^{\beta}(w_2)) \vee (\nu_2^{\beta}(w_1) \vee \nu_2^{\beta}(w_2)) = (\nu_1^{\beta}(w_1) \vee \nu_2^{\beta}(w_1)) \vee (\nu_1^{\beta}(w_2) \vee \nu_2^{\beta}(w_2)) = \nu^{\beta}(w_1) \vee \nu^{\beta}(w_2)$.

Thus $\mu^{\alpha}(w_1 \cdot w_2) \geq \mu^{\alpha}(w_1) \wedge \mu^{\alpha}(w_2)$ and $\nu^{\beta}(w_1 \cdot w_2) \leq \nu^{\beta}(w_1) \vee \nu^{\beta}(w_2)$.

Therefore $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is an (α, β)-PFSR of the ring $(W, +, \cdot)$. □

Corollary 1: Intersection of the family of (α, β)-PFSR of a ring is an (α, β)-PFSR of that ring.

Theorem 4: Let $\psi_1^* = (\mu_1^{\alpha}, \nu_1^{\beta})$ and $\psi_2^* = (\mu_2^{\alpha}, \nu_2^{\beta})$ be two (α, β)-PFID of a ring $(W, +, \cdot)$. Then $\psi_1^* \cap \psi_2^*$ is an (α, β)-PFID of that ring.

Proof: Let $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ be the intersection of $\psi_1^* = (\mu_1^{\alpha}, \nu_1^{\beta})$ and $\psi_2^* = (\mu_2^{\alpha}, \nu_2^{\beta})$. Then $\mu^{\alpha}(w) = \mu_1^{\alpha}(w) \wedge \mu_2^{\alpha}(w)$ and $\nu^{\beta}(w) = \nu_1^{\beta}(w) \vee \nu_2^{\beta}(w)$ for all $w \in W$.

Let w_1, w_2 be two arbitrary elements of W. Then,

$$\begin{aligned}\mu^{\alpha}(w_1 - w_2) &= \mu_1^{\alpha}(w_1 - w_2) \wedge \mu_2^{\alpha}(w_1 - w_2)\\ &\geq (\mu_1^{\alpha}(w_1) \wedge \mu_1^{\alpha}(w_2)) \wedge (\mu_2^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_2))\\ &= (\mu_1^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_1)) \wedge (\mu_1^{\alpha}(w_2) \wedge \mu_2^{\alpha}(w_2))\\ &= \mu^{\alpha}(w_1) \wedge \mu^{\alpha}(w_2)\end{aligned}$$

Also,

$$\begin{aligned}\nu^{\beta}(w_1 - w_2) &= \nu_1^{\beta}(w_1 - w_2) \vee \nu_2^{\beta}(w_1 - w_2)\\ &\leq (\nu_1^{\beta}(w_1) \vee \nu_1^{\beta}(w_2)) \vee (\nu_2^{\beta}(w_1) \vee \nu_2^{\beta}(w_2))\\ &= (\nu_1^{\beta}(w_1) \vee \nu_2^{\beta}(w_1)) \vee (\nu_1^{\beta}(w_2) \vee \nu_2^{\beta}(w_2))\\ &= \nu^{\beta}(w_1) \vee \nu^{\beta}(w_2)\end{aligned}$$

Therefore $\mu^{\alpha}(w_1 - w_2) \geq \mu^{\alpha}(w_1) \wedge \mu^{\alpha}(w_2)$ and $\nu^{\beta}(w_1 - w_2) \leq \nu^{\beta}(w_1) \vee \nu^{\beta}(w_2)$.

Again,

$$\begin{aligned}\mu^{\alpha}(w_1 \cdot w_2) &= \mu_1^{\alpha}(w_1 \cdot w_2) \wedge \mu_2^{\alpha}(w_1 \cdot w_2)\\ &\geq (\mu_1^{\alpha}(w_1) \vee \mu_1^{\alpha}(w_2)) \wedge (\mu_2^{\alpha}(w_1) \vee \mu_2^{\alpha}(w_2))\\ &= (\mu_1^{\alpha}(w_1) \wedge \mu_2^{\alpha}(w_1)) \vee (\mu_1^{\alpha}(w_2) \wedge \mu_2^{\alpha}(w_2))\\ &= \mu^{\alpha}(w_1) \vee \mu^{\alpha}(w_2)\end{aligned}$$

Also,

$$\begin{aligned}\nu^{\beta}(w_1 \cdot w_2) &= \nu_1^{\beta}(w_1 \cdot w_2) \vee \nu_2^{\beta}(w_1 \cdot w_2)\\ &\leq (\nu_1^{\beta}(w_1) \wedge \nu_1^{\beta}(w_2)) \vee (\nu_2^{\beta}(w_1) \wedge \nu_2^{\beta}(w_2))\\ &= (\nu_1^{\beta}(w_1) \vee \nu_2^{\beta}(w_1)) \wedge (\nu_1^{\beta}(w_2) \vee \nu_2^{\beta}(w_2))\\ &= \nu^{\beta}(w_1) \wedge \nu^{\beta}(w_2)\end{aligned}$$

Therefore $\mu^{\alpha}(w_1 \cdot w_2) \geq \mu^{\alpha}(w_1) \vee \mu^{\alpha}(w_2)$ and $\nu^{\beta}(w_1 \cdot w_2) \leq \nu^{\beta}(w_1) \wedge \nu^{\beta}(w_2)$ for all $w_1, w_2 \in W$.

Hence $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is an (α, β)-PFID of the ring $(W, +, \cdot)$. □

Theorem 5: If a ring $(W, +, \cdot)$ have an IFSR $I = (\mu, \nu)$, $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is an (α, β)-PFSR of the ring $(W, +, \cdot)$.

Proof: Since the ring $(W, +, \cdot)$ have an IFSR $I = (\mu, \nu)$, $\mu(w_1 - w_2) \geq \mu(w_1) \wedge \mu(w_2)$, $\nu(w_1 - w_2) \leq \nu(w_1) \vee \nu(w_2)$, $\mu(w_1 \cdot w_2) \geq \mu(w_1) \wedge \mu(w_2)$ and $\nu(w_1 \cdot w_2) \leq \nu(w_1) \vee \nu(w_2)$ for all $w_1, w_2 \in W$.

We will prove this theorem by studying several cases.

1) Let $\mu(w_1) > \mu(w_2)$ and $\nu(w_1) > \nu(w_2)$ for $w_1, w_2 \in W$. Then $\mu^\alpha(w_1) \geq \mu^\alpha(w_2)$ and $\nu^\beta(w_1) \geq \nu^\beta(w_2)$, where $\alpha, \beta \in [0, 1]$.
   Now, $\mu(w_1 - w_2) \geq \mu(w_1) \wedge \mu(w_2) = \mu(w_2)$. So, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_2) = \mu^\alpha w_1) \wedge \mu^\alpha(w_2)$.
   Also, $\mu(w_1 \cdot w_2) \geq \mu(w_1) \wedge \mu(w_2) = \mu(W_2)$. Therefore $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_2) = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$.
   Again, $\nu(w_1 - w_2) \leq \nu(w_1) \vee \nu(w_2) = \nu(w_1)$. This implies that $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.
   Also, $\nu(w_1 \cdot w_2) \leq \nu(w_1) \vee \nu(w_2) = \nu(w_1)$. Therefore $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.
2) Let $\mu(w_1) < \mu(w_2)$ and $\nu(w_1) < \nu(w_2)$ for $w_1, w_2 \in W$. So, $\mu^\alpha(w_1) \leq \mu^\alpha(w_2)$ and $\nu^\beta(w_1) \leq \nu^\beta(w_2)$, where $\alpha, \beta \in [0, 1]$.
   Now, $\mu(w_1 - w_2) \geq \mu(w_1) \wedge \mu(w_2) = \mu(w_1)$. So, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) = \mu^\alpha w_1) \wedge \mu^\alpha(w_2)$.
   Also, $\mu(w_1 \cdot w_2) \geq \mu(w_1) \wedge \mu(w_2) = \mu(w_1)$. Therefore $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$.
   Again, $\nu(w_1 - w_2) \leq \nu(w_1) \vee \nu(w_2) = \nu(w_2)$. This implies that $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_2) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.
   Also, $\nu(w_1 \cdot w_2) \leq \nu(w_1) \vee \nu(w_2) = \nu(w_2)$. Therefore $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_2) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.
3) Let $\mu(w_1) = \mu(w_2)$ and $\nu(w_1) = \nu(w_2)$ for $w_1, w_2 \in W$. So, $\mu^\alpha(w_1) = \mu^\alpha(w_2)$ and $\nu^\beta(w_1) = \nu^\beta(w_2)$, where $\alpha, \beta \in [0, 1]$.
   Now, $\mu(w_1 - w_2) \geq \mu(w_1) \wedge \mu(w_2) = \mu(w_1) = \mu(w_2)$. So, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) = \mu^\alpha(w_2) = \mu^\alpha w_1) \wedge \mu^\alpha(w_2)$.
   Also, $\mu(w_1 \cdot w_2) \geq \mu(w_1) \wedge \mu(w_2) = \mu(w_1) = \mu(w_2)$. Therefore $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) = \mu^\alpha(w_2) = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$.
   Again, $\nu(w_1 - w_2) \leq \nu(w_1) \vee \nu(w_2) = \nu(w_1) = \nu(w_2)$. This implies that $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_2) = \nu^\beta(w_1) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.
   Also, $\nu(w_1 \cdot w_2) \leq \nu(w_1) \vee \nu(w_2) = \nu(w_1) = \nu(w_2)$. Therefore $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) = \nu^\beta(w_2) = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.

Proceeding in the similar way by considering all the cases, we get $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$, $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$, $\mu^\alpha(w_1 \cdot w_2) \geq \mu(w_1) \wedge \mu(w_2)$ and $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ for all $w_1, w_2 \in W$. Hence the ring $(W, +, \cdot)$ have an (α, β)-PFSR $\psi^* = (\mu^\alpha, \nu^\beta)$. □

V. LEVEL SUBRING OF (α, β)-PFSR

This section will elaborate (α, β)-PFLSR and it's properties.

Theorem 6: Assume a ring $(W, +, \cdot)$ have an (α, β)-PFSR $\psi^* = (\mu^\alpha, \nu^\beta)$, the (α, β)-PFLS $\psi^*_{(\theta,\tau)}$ forms a subring of $(W, +, \cdot)$, where $\theta \leq \mu^\alpha(0)$ and $\tau \geq \nu^\beta(0)$.

Proof: We have, $\psi^*_{(\theta,\tau)} = \{w_1 \in W | \mu^\alpha(w_1) \geq \theta$ and $\nu^\beta(w_1) \leq \tau\}$.

As $0 \in \psi^*_{(\theta,\tau)}$, clearly $\psi^*_{(\theta,\tau)}$ is non empty.

To show, $\psi^*_{(\theta,\tau)}$ is a subring of $(W, +, \cdot)$, we need to verify that for $w_1, w_2 \in \psi^*_{(\theta,\tau)}$, $w_1 - w_2 \in \psi^*_{(\theta,\tau)}$ and $w_1 \cdot w_2 \in \psi^*_{(\theta,\tau)}$.

Let us take $w_1, w_2 \in \psi^*_{(\theta,\tau)}$. Then $\mu^\alpha(w_1) \geq \theta$, $\nu^\beta(w_1) \leq \tau$, $\mu^\alpha(w_2) \geq \theta$, $\nu^\beta(w_2) \leq \tau$.

Since the ring $(W, +, \cdot)$ have an (α, β)-PFSR $\psi^* = (\mu^\alpha, \nu^\beta)$, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)) \geq \theta \wedge \theta = \theta$ and $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2) \leq \tau \vee \tau = \tau$.

Therefore, $\mu^\alpha(w_1 - w_2) \geq \theta$ and $\nu^\beta(w_1 - w_2) \leq \tau$. So, $w_1 - w_2 \in \psi^*_{(\theta,\tau)}$.

Also, $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)) \geq \theta \wedge \theta = \theta$ and $\nu^\beta(w_1 \cdot w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2) \leq \tau \vee \tau = \tau$.

Thus $\mu^\alpha(w_1 \cdot w_2) \geq \theta$ and $\nu^\beta(w_1 \cdot w_2) \leq \tau$. So, $w_1 \cdot w_2 \in W$. Hence $\psi^*_{(\theta,\tau)}$ is a subring of the ring $(W, +, \cdot)$. □

Definition 8: The subring $\psi^*_{(\theta,\tau)}$ of the ring $(W, +, \cdot)$ is called (α, β)-Pythagorean fuzzy level subring (PFLSR) of ψ^*.

Theorem 7: Assume a ring $(W, +, \cdot)$ have an (α, β)-PFID $\psi^* = (\mu^\alpha, \nu^\beta)$, the (α, β)-PFLS $\psi^*_{(\theta,\tau)}$ forms an ideal of the ring $(W, +, \cdot)$, where $\theta \leq \mu^\alpha(0)$ and $\tau \geq \nu^\beta(0)$.

Proof: Here $\psi^*_{(\theta,\tau)} = \{w_1 \in W | \mu^\alpha(w_1) \geq \theta$ and $\nu^\beta(w_1) \leq \tau\}$. Clearly $\psi^*_{(\theta,\tau)}$ is non empty, as $0 \in \psi^*_{(\theta,\tau)}$.

To show, the ring $(W, +, \cdot)$ have an ideal $\psi^*_{(\theta,\tau)}$, we need to present that $w_1 - w_2 \in \psi^*_{(\theta,\tau)}$ for $w_1, w_2 \in \psi^*_{(\theta,\tau)}$ and $w_1 \cdot w_2 \in \psi^*_{(\theta,\tau)}$ for $w_1 \in \psi^*_{(\theta,\tau)}$, $w_2 \in W$.

Assume $w_1, w_2 \in \psi^*_{(\theta,\tau)}$, $\mu^\alpha(w_1) \geq \theta$, $\nu^\beta(w_1) \leq \tau$, $\mu^\alpha(w_2) \geq \theta$, $\nu^\beta(w_2) \leq \tau$.

Since, the ring $(W, +, \cdot)$ have an (α, β)-PFID $\psi^* = (\mu^\alpha, \nu^\beta)$, $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)) \geq \theta \wedge \theta = \theta$ and $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2) \leq \tau \vee \tau = \tau$.

Therefore, $\mu^\alpha(w_1 - w_2) \geq \theta$ and $\nu^\beta(w_1 - w_2) \leq \tau$. So, $w_1 - w_2 \in \psi^*_{(\theta,\tau)}$.

Now assume that, $w_1 \in \psi^*_{(\theta,\tau)}$ and $w_2 \in W$.

Then $\mu^\alpha(w_1 \cdot w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2)) \geq \mu^\alpha(w_1) \geq \theta$ and $\nu^\beta(w_1.w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2) \leq \nu^\beta(w_1) \leq \tau$.

Thus $\mu^\alpha(w_1 \cdot w_2) \geq \theta$ and $\nu^\beta(w_1 \cdot w_2) \leq \tau$. Consequently, $w_1 \cdot w_2 \in W$.

Hence the ring $(W, +, \cdot)$ have an ideal $\psi^*_{(\theta,\tau)}$. □

Definition 9: The ideal $\psi^*_{(\theta,\tau)}$ of $(W, +, \cdot)$ is called (α, β)-Pythagorean fuzzy level ideal (PFLID) of the (α, β)-PFID ψ^*.

Theorem 8: Assume a ring $(W, +, \cdot)$ have an (α, β)-PFS $\psi^* = (\mu^\alpha, \nu^\beta)$, $\psi^* = (\mu^\alpha, \nu^\beta)$ is (α, β)-PFSR of $(W, +, \cdot)$ if all (α, β)-PFLS $\psi^*_{(\theta,\tau)}$ forms a subring of $(W, +, \cdot)$, where $\theta \leq \mu^\alpha(0)$, $\tau \geq \nu^\beta(0)$.

Proof: Here $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β)-PFS of W.

Let $w_1, w_2 \in W$. Also assume that $\mu^\alpha(w_1) = \theta_1$, $\mu^\alpha(w_2) = \theta_2$ with $\theta_1 < \theta_2$ and $\nu^\beta(w_1) = \tau_1$, $\nu^\beta(w_2) = \tau_2$ with $\tau_1 > \tau_2$.

Therefore, $w_1 \in \psi^*_{(\theta_1,\tau_1)}$ and $w_2 \in \psi^*_{(\theta_2,\tau_2)}$.

As $\theta_1 < \theta_2$ and $\tau_1 > \tau_2$, then by Proposition 2, we have $\psi^*_{(\theta_2,\tau_2)} \subseteq \psi^*_{(\theta_1,\tau_1)}$. Thus $w_2 \in \psi^*_{(\theta_1,\tau_1)}$.

Now, $w_1 \in \psi^*_{(\theta_1,\tau_1)}$ and $w_2 \in \psi^*_{(\theta_1,\tau_1)}$.

Since, $\psi^*_{(\theta_1,\tau_1)}$ is a subring of $(W, +, \cdot)$, $w_1 - w_2 \in \psi^*_{(\theta_1,\tau_1)}$ and $w_1 \cdot w_2 \in \psi^*_{(\theta_1,\tau_1)}$.

Therefore, $\mu^\alpha(w_1 - w_2) \geq \theta_1 = \theta_1 \wedge \theta_2 = \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ and $\nu^\beta(w_1 - w_2) \leq \tau_1 = \tau_1 \vee \tau_2 = \nu^\beta(w_1) \vee \nu^\beta(w_2)$.

So, $\mu^\alpha(w_1-w_2)\geq\mu^\alpha(w_1)\wedge\mu^\alpha(w_2)$ and $\nu^\beta(w_1-w_2)\leq\nu^\beta(w_1)\vee\nu^\beta(w_2)$.

Again, $\mu^\alpha(w_1\cdot w_2)\geq\theta_1=\theta_1\wedge\theta_2=\mu^\alpha(w_1)\wedge\mu^\alpha(w_2)$ and $\nu^\beta(w_1\cdot w_2)\leq\tau_1=\tau_1\vee\tau_2=\nu^\beta(w_1)\vee\nu^\beta(w_2)$.

Thus $\mu^\alpha(w_1\cdot w_2)\geq\mu^\alpha(w_1)\wedge\mu^\alpha(w_2)$ and $\nu^\beta(w_1\cdot w_2)\leq\nu^\beta(w_1)\vee\nu^\beta(w_2)$.

Since, w_1,w_2 are random elements of W, $\mu^\alpha(w_1-w_2)\geq\mu^\alpha(w_1)\wedge\mu^\alpha(w_2)$, $\nu^\beta(w_1-w_2)\leq\nu^\beta(w_1)\vee\nu^\beta(w_2)$ and $\mu^\alpha(w_1\cdot w_2)\geq\mu^\alpha(w_1)\wedge\mu^\alpha(w_2)$, $\nu^\beta(w_1\cdot w_2)\leq\nu^\beta(w_1)\vee\nu^\beta(w_2)$ for all $w_1,w_2\in W$.

Hence the ring $(W,+,\cdot)$ have an (α,β)-PFSR ψ^*. □

Corollary 2: If all (α,β)-PFLS $\psi^*_{(\theta,\tau)}$ of an (α,β)-PFS $\psi^*=(\mu^\alpha,\nu^\beta)$ is an ideal of the ring $(W,+,\cdot)$, then $\psi^*=(\mu^\alpha,\nu^\beta)$ is an (α,β)-PFID of the ring $(W,+,\cdot)$, where $\theta\leq\mu^\alpha(0)$ and $\tau\geq\nu^\beta(0)$.

VI. DIRECT PRODUCT AND RING HOMOMORPHISM OF (α,β)-PFSR

Definition 10: Let $\psi_1^*=(\mu_1^\alpha,\nu_1^\beta)$ and $\psi_2^*=(\mu_2^\alpha,\nu_2^\beta)$ be two (α,β)-PFS of rings $(W_1,+,\cdot)$ and $(W_2,+,\cdot)$ respectively. The direct product of ψ_1^* and ψ_2^* is referred as $\psi_1^*\bigotimes\psi_2^*$ and presented by $(\psi_1^*\bigotimes\psi_2^*)(w_1,w_2)=\{\mu_1^\alpha(w_1)\wedge\mu_2^\alpha(w_2),\nu_1^\beta(w_1)\vee\nu_2^\beta(w_2)\}$, where $(w_1,w_2)\in W_1\times W_2$.

Theorem 9: Let $\psi_1^*=(\mu_1^\alpha,\nu_1^\beta)$ and $\psi_2^*=(\mu_2^\alpha,\nu_2^\beta)$ be two (α,β)-PFSR of rings $(W_1,+,\cdot)$ and $(W_2,+,\cdot)$ respectively. Then the direct product $\psi_1^*\bigotimes\psi_2^*$ is an (α,β)-PFSR of the ring $W_1\times W_2$.

Proof: Let $\psi^*=(\mu^\alpha,\nu^\beta)$ be the direct product of $\psi_1^*=(\mu_1^\alpha,\nu_1^\beta)$ and $\psi_2^*=(\mu_2^\alpha,\nu_2^\beta)$. That is $\psi^*=\psi_1^*\bigotimes\psi_2^*$.

We have, $(\psi_1^*\bigotimes\psi_2^*)(w_1,w_2)=\{\mu_1^\alpha(w_1)\wedge\mu_2^\alpha(w_2),\nu_1^\beta(w_1)\vee\nu_2^\beta(w_2)\}$, where $(w_1,w_2)\in W_1\times W_2$. Therefore $\mu^\alpha(w_1,w_2)=\mu_1^\alpha(w_1)\wedge\mu_2^\alpha(w_2)$ and $\nu^\beta(w_1,w_2)=\nu_1^\beta(w_1)\vee\nu_2^\beta(w_2)$.

Let (w_1,w_2) and (w_3,w_4) be any two elements of $W_1\times W_2$. Then,

$$\begin{aligned}&\mu^\alpha((w_1,w_2)\text{-}(w_3,w_4))\\&=\mu^\alpha((w_1-w_3),(w_2-w_4))\\&=\mu_1^\alpha(w_1-w_3)\wedge\mu_2^\alpha(w_2-w_4)\\&\geq(\mu_1^\alpha(w_1)\wedge\mu_1^\alpha(w_3))\wedge(\mu_2^\alpha(w_2)\wedge\mu_2^\alpha(w_4))\\&=(\mu_1^\alpha(w_1)\wedge\mu_2^\alpha(w_2))\wedge(\mu_1^\alpha(w_3)\wedge\mu_2^\alpha(w_4))\\&=\mu^\alpha(w_1,w_2)\wedge\mu^\alpha(w_3,w_4).\end{aligned}$$

Also,

$$\begin{aligned}&\nu^\beta((w_1,w_2)\text{-}(w_3,w_4))\\&=\nu^\beta((w_1-w_3),(w_2-w_4))\\&=\nu_1^\beta(w_1-w_3)\vee\nu_2^\beta(w_2-w_4)\\&\leq(\nu_1^\beta(w_1)\vee\nu_1^\beta(w_3))\vee(\nu_2^\beta(w_2)\vee\nu_2^\beta(w_4))\\&=(\nu_1^\beta(w_1)\vee\nu_2^\beta(w_2))\vee(\nu_1^\beta(w_3)\vee\nu_2^\beta(w_4))\\&=\nu^\beta(w_1,w_2)\vee\nu^\beta(w_3,w_4).\end{aligned}$$

Therefore $\mu^\alpha((w_1,w_2)\text{-}(w_3,w_4))\geq\mu^\alpha(w_1,w_2)\wedge\mu^\alpha(w_3,w_4)$ and $\nu^\beta((w_1,w_2)\text{-}(w_3,w_4))\leq\nu^\beta(w_1,w_2)\vee\nu^\beta(w_3,w_4)$.

Again,

$$\begin{aligned}&\mu^\alpha((w_1,w_2)\cdot(w_3,w_4))\\&=\mu^\alpha((w_1\cdot w_3),(w_2\cdot w_4))\\&=\mu_1^\alpha(w_1\cdot w_3)\wedge\mu_2^\alpha(w_2\cdot w_4)\\&\geq(\mu_1^\alpha(w_1)\wedge\mu_1^\alpha(w_3))\wedge(\mu_2^\alpha(w_2)\wedge\mu_2^\alpha(w_4))\\&=(\mu_1^\alpha(w_1)\wedge\mu_2^\alpha(w_2))\wedge(\mu_1^\alpha(w_3)\wedge\mu_2^\alpha(w_4))\\&=\mu^\alpha(w_1,w_2)\wedge\mu^\alpha(w_3,w_4).\end{aligned}$$

Also,

$$\begin{aligned}&\nu^\beta((w_1,w_2).(w_3,w_4))\\&=\nu^\beta((w_1\cdot w_3),(w_2\cdot w_4))\\&=\nu_1^\beta(w_1\cdot w_3)\vee\nu_2^\beta(w_2\cdot w_4)\\&\leq(\nu_1^\beta(w_1)\vee\nu_1^\beta(w_3))\vee(\nu_2^\beta(w_2)\vee\nu_2^\beta(w_4))\\&=(\nu_1^\beta(w_1)\vee\nu_2^\beta(w_2))\vee(\nu_1^\beta(w_3)\vee\nu_2^\beta(w_4))\\&=\nu^\beta(w_1,w_2)\vee\nu^\beta(w_3,w_4).\end{aligned}$$

Thus $\mu^\alpha((w_1,w_2)\cdot(w_3,w_4))\geq\mu^\alpha(w_1,w_2)\wedge\mu^\alpha(w_3,w_4)$ and $\nu^\beta((w_1,w_2)\cdot(w_3,w_4))\leq\nu^\beta(w_1,w_2)\vee\nu^\beta(w_3,w_4)$.

Hence the direct product $\psi_1^*\bigotimes\psi_2^*$ is an (α,β)-PFSR of the ring $S_1\times S_2$. □

Corollary 3: Let $\psi_1^*=(\mu_1^\alpha,\nu_1^\beta)$ and $\psi_2^*=(\mu_2^\alpha,\nu_2^\beta)$ be two (α,β)-PFID of rings $(W_1,+,\cdot)$ and $(W_2,+,\cdot)$ respectively. Then the direct product $\psi_1^*\bigotimes\psi_2^*$ is an (α,β)-PFID of the ring $W_1\times W_2$.

Theorem 10: Let $(W_1,+,\cdot)$ and $(W_2,+,\cdot)$ be two rings and $\psi^*=(\mu^\alpha,\nu^\beta)$ is an (α,β)-PFSR of the ring $(W_1,+,\cdot)$. If $r:W_1\to W_2$ is a surjective ring homomorphism, $r(\psi^*)$ is an (α,β)-PFSR of the ring $(W_2,+,\cdot)$.

Proof: Here $r(\psi^*)(w)=(r(\mu^\alpha)(w),r(\nu^\beta)(w))$ for all $w\in W_2$.

Since $r:W_1\to W_2$ is a surjective ring homomorphism, $r(W_1)=W_2$.

Let $w_3,w_4\in W_2$. Then $w_3=r(w_1)$ and $w_4=r(w_2)$ for some $w_1,w_2\in W_1$.

To show, $r(\psi^*)$ is an (α,β)-PFSR of the ring $(W_2,+,\cdot)$, we need to show that $r(\mu^\alpha)(w_3-w_4)\geq r(\mu^\alpha)(w_3)\wedge r(\mu^\alpha)(w_4)$, $r(\nu^\beta)(w_3-w_4)\leq r(\nu^\beta)(w_3)\vee r(\nu^\beta)(w_4)$ and $r(\mu^\alpha)(w_3\cdot w_4)\geq r(\mu^\alpha)(w_3)\wedge r(\mu^\alpha)(w_4)$, $r(\nu^\beta)(w_3\cdot w_4)\leq r(\nu^\beta)(w_3)\vee r(\nu^\beta)(w_4)$.

Now,

$$\begin{aligned}&r(\mu^\alpha)(w_3-w_4)\\&=\vee\left\{\mu^\alpha(w_1-w_2)|w_1,w_2\in W_1,r(w_1)=w_3,r(w_2)=w_4\right\}\\&\geq\vee\left\{\mu^\alpha(w_1)\wedge\mu^\alpha(w_2)|r(w_1)=w_3,r(w_2)=w_4\right\}\\&=\left(\vee\left\{\mu^\alpha(w_1)|w_1\in W_1\text{ and }r(w_1)=w_3\right\}\right)\\&\wedge\left(\vee\left\{\mu^\alpha(w_2)|w_2\in W_1\text{ and }r(w_2)=w_4\right\}\right)\\&=r(\mu^\alpha)(w_3)\wedge r(\mu^\alpha)(w_4).\end{aligned}$$

Also,

$$\begin{aligned}&r(\nu^\beta)(w_2-w_4)\\&\quad=\wedge\left\{\nu^\beta(w_1-w_2)|w_1,w_2\in W_1\text{ and }r(w_1)\right.\end{aligned}$$

$$= w_3, r(w_2) = w_4\}$$
$$\leq \wedge \left\{\nu^{\beta}(w_1) \vee \nu^{\beta}(w_2) | w_1, w_2 \in W_1 \text{ and } r(w_1) = w_3, r(w_2) = w_4\right\}$$
$$= \left(\wedge \left\{\mu^{\alpha}(w_1) | w_1 \in W_1 \text{ and } r(w_1) = w_3\right\}\right) \vee \left(\wedge \left\{\mu^{\alpha}(w_2) | w_2 \in W_1 \text{ and } r(w_2) = w_4\right\}\right)$$
$$= r(\nu^{\beta})(w_3) \vee r(\nu^{\beta})(w_4).$$

Therefore, $r(\mu^{\alpha})(w_3 - w_4) \geq r(\mu^{\alpha})(w_3) \wedge r(\mu^{\alpha})(w_4)$ and $r(\nu^{\beta})(w_3 - w_4) \leq r(\nu^{\beta})(w_3) \vee r(\nu^{\beta})(w_4)$.

Similarly, we can prove that $r(\mu^{\alpha})(w_3 \cdot w_4) \geq r(\mu^{\alpha})(w_3) \wedge r(\mu^{\alpha})(w_4)$ and $r(\nu^{\beta})(w_3 \cdot w_4) \leq r(\nu^{\beta})(w_3) \vee r(\nu^{\beta})(w_4)$.

Hence $r(\psi^*)$ is an (α, β)-PFSR of the ring $(W_2, +, \cdot)$. □

Corollary 4: Assume $(W_1, +, \cdot)$ and $(W_2, +, \cdot)$ are two rings and $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is an (α, β)-PFID of the ring $(W_1, +, \cdot)$. If $r : W_1 \to W_2$ is a surjective ring homomorphism, $r(\psi^*)$ is an (α, β)-PFID of the ring $(W_2, +, \cdot)$.

Theorem 11: Let $(W_1, +, \cdot)$ and $(W_2, +, \cdot)$ be two rings and $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is an (α, β)-PFSR of the ring $(W_2, +, \cdot)$. If $r : W_1 \to W_2$ is a bijective ring homomorphism, $r^{-1}(\psi^*)$ is an (α, β)-PFSR of the ring $(W_1, +, \cdot)$.

Proof: Here $r^{-1}(\psi^*)(w_1) = (r^{-1}(\mu^{\alpha})(w_1), r^{-1}(\nu^{\beta})(w_1))$ for all $w_1 \in W_1$. Let $w_1, w_2 \in W_1$.

Now,

$$\begin{aligned} r^{-1}(\mu^{\alpha})(w_1 - w_2) &= \mu^{\alpha}\left(r(w_1 - w_2)\right) \\ &= \mu^{\alpha}\left(r(w_1) - r(w_2)\right) \\ &\geq \mu^{\alpha}(r(w_1)) \wedge \mu^{\alpha}(r(w_2)) \\ &= r^{-1}(\mu^{\alpha})(w_1) \wedge r^{-1}(\mu^{\alpha})(w_2). \end{aligned}$$

Therefore $r^{-1}(\mu^{\alpha})(w_1 - w_2) \geq r^{-1}(\mu^{\alpha})(w_1) \wedge r^{-1}(\mu^{\alpha})(w_2)$ for all $w_1, w_2 \in W_1$.

Similarly, one can prove that $r^{-1}(\nu^{\beta})(w_1 - w_2) \leq r^{-1}(\nu^{\beta})(w_1) \vee r^{-1}(\nu^{\beta})(w_2)$ $\forall w_1, w_2 \in W_1$.

Again,

$$\begin{aligned} r^{-1}(\mu^{\alpha})(w_1 \cdot w_2) &= \mu^{\alpha}\left(r(w_1 \cdot w_2)\right) \\ &= \mu^{\alpha}\left(r(w_1) \cdot r(w_2)\right) \\ &\geq \mu^{\alpha}(r(w_1)) \wedge \mu^{\alpha}(r(w_2)) \\ &= r^{-1}(\mu^{\alpha})(w_1) \wedge r^{-1}(\mu^{\alpha})(w_2). \end{aligned}$$

Therefore $r^{-1}(\mu^{\alpha})(w_1 \cdot w_2) \geq r^{-1}(\mu^{\alpha})(w_1) \wedge r^{-1}(\mu^{\alpha})(w_2)$ $\forall w_1, w_2 \in W_1$.

Similarly, one can prove that $r^{-1}(\nu^{\beta})(w_1 \cdot w_2) \leq r^{-1}(\nu^{\beta})(w_1) \vee r^{-1}(\nu^{\beta})(w_2)$ for all $w_1, w_2 \in W_1$.

Hence $r^{-1}(\psi^*)$ is an (α, β)-PFSR of the ring $(W_1, +, \cdot)$. □

Corollary 5: Assume $(W_1, +, \cdot)$ and $(W_2, +, \cdot)$ are two rings and $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is an (α, β)-PFID of the ring $(W_2, +, \cdot)$. If $r : W_1 \to W_2$ is a bijective ring homomorphism, $r^{-1}(\psi^*)$ is an (α, β)-PFID of the ring $(W_1, +, \cdot)$.

VII. CONCLUSION

This paper initiated the study of (α, β)-PFSRs and (α, β)-PFIDs of any ring. A relationship between (α, β)-PFSR and (α, β)-PFID has been established. We have proved that every IFSR of a ring is an (α, β)-PFSR of that ring. We have briefly described the concept of level subring of (α, β)-PFSR. In addition, we have elaborated the (α, β)-PFSR's direct product. Furthermore, we looked into the impact of ring homomorphism on (α, β)-PFSR. We will continue to work on the classification of (α, β)-PFID's of a ring in the future.

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SUPRIYA BHUNIA received the M.Sc. degree in mathematics from the Indian Institute of Technology, Kharagpur, India. He is currently a Senior Research Fellow with the Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, India. He has received a CSIR Research Fellowship for his Ph.D. studies. He has published research articles in international scientific journals with good impact factor. His research interests include applications of fuzzy set, fuzzy group theory, and pythagorean fuzzy algebraic structures.

GANESH GHORAI received the M.Sc. degree in mathematics from IIT Bombay, in 2011, and the Ph.D. degree in mathematics from Vidyasagar University, India, in 2017, in the area of fuzzy graph theory and their generalization. He is currently an Assistant Professor with the Department of Applied Mathematics, Vidyasagar University. He has published more than 40 research papers in international peer-reviewed journals on fuzzy mathematics, fuzzy graph, bipolar fuzzy graph, vague graph, and m-polar fuzzy graphs and their applications in the real world. He is also guiding four research scholars for Ph.D. degree. He completed one research project funded by UGC. He is also the Deputy Coordinator of one of the ongoing project funded by DST-SERB. He has been nominated for the Commonwealth Fellowship/Scholarship, U.K., in 2014. He is also a member of the Calcutta Mathematical Society.

QIN XIN received the Ph.D. degree from the Department of Computer Science, University of Liverpool, U.K., in December 2004. He is currently working as a Full Professor in computer science with the Faculty of Science and Technology, University of the Faroe Islands (UoFI), Faroe Islands, Denmark. Prior to joining UoFI, he had held variant research positions in world leading universities and research laboratory, including a Senior Research Fellowship with the Universite Catholique de Louvain, Belgium. He is also a Research Scientist/a Postdoctoral Research Fellowship with the Simula Research Laboratory, Norway, and a Postdoctoral Research Fellowship with the University of Bergen, Norway. His research interests include design and analysis of sequential, parallel and distributed algorithms for various communication and optimization problems in wireless communication networks, cryptography, and digital currencies, including quantum money. Moreover, he also investigates the combinatorial optimization problems with applications in bioinformatics, data mining, and space research. He is currently serving on management committee board of Denmark for several EU ICT projects. He has produced more than 80 peer-reviewed scientific papers. His works have been published in leading international conferences and journals, such as ICALP, ACM PODC, SWAT, IEEE MASS, ISAAC, SIROCCO, IEEE ICC, Algorithmica, *Theoretical Computer Science*, *Distributed Computing*, IEEE Transactions on Computers, *Journal of Parallel and Distributed Computing*, IEEE Transactions on Dielectrics and Electrical Insulation, IEEE Transactions on Sustainable Computing, and *Advances in Space Research*. He has been very actively involved in the services for the community in terms of acting (or acted) on various positions (e.g., the Session Chair, a member of Technical Program Committee, a Symposium Organizer, and the Local Organization Co-Chair) for numerous international leading conferences in the fields of distributed computing, wireless communications, and ubiquitous intelligence and computing, including IEEE MASS, IEEE LCN, ACM SAC, IEEE ICC, IEEE GLOBECOM, IEEE WCNC, IEEE VTC, IFIP NPC, and IEEE Sarnoff. He is the Organizing Committee Chair for the 17th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2020, Torshavn, Faroe Islands). He is also serves on the editorial board for more than ten international journals.

MUHAMMAD GULZAR was born in Gojra, Pakistan. He received the M.Phil. degree in fuzzy group theory from Government College University Faisalabad, Faisalabad, Pakistan, in 2018. He is the Pioneer of complex intuitionistic subgroups and complex fuzzy subfields. He has published a number of research articles in reputable journals. His research interests include applications of fuzzy sets in group theory, ring theory, graph theory, decision making, and various intuitionistic fuzzy algebraic varieties.