Received December 26, 2021, accepted January 14, 2022, date of publication January 18, 2022, date of current version January 27, 2022.

*Digital Object Identifier 10.1109/ACCESS.2022.3144486*

# Anti-Periodic Synchronization of Clifford-Valued Neutral-Type Recurrent Neural Networks With *D* Operator

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This work was supported in part by the Science Research Fund of Education Department of Yunnan Province of China under Grant 2018JS517, and in part by the Mathematics and Applied Mathematics Teaching Team of Puer University under Grant 2020JXTD018.

**ABSTRACT** In this paper, a class of Clifford-valued neutral-type recurrent neural networks with *D* operator is explored. By using non-decomposition method and the Banach fixed point theorem, we obtain several sufficient conditions for the existence of anti-periodic solutions for Clifford-valued neutral-type recurrent neural networks with *D* operator. By using the proof by contradiction and inequality techniques, we obtain the global exponential synchronization of anti-periodic solutions for Clifford-valued neutral-type recurrent neural networks with *D* operator. Finally, we give one example to illustrate the feasibility and effectiveness of main results.

**INDEX TERMS** Clifford algebra, recurrent neural networks, synchronization, anti-periodic solutions, *D* operator.

## I. INTRODUCTION

As we well know, a neural networks model, which is recurrent neural networks model, was extensively explored by many scholars and has been widely applied in many fields, such as image processing perception, pattern recognition, image processing, etc. In the past decades, the dynamics of recurrent neural networks have been extensively researched (see [1]–[7]). In recent years, the existence and stability of periodic and anti-periodic solutions for recurrent neural networks have been discussed (see [8]–[10]). Recurrent neural network is a kind of neural network for memory function, there are some practical application backgrounds for the network model, for instance, generate image description (see [11]–[14]), speech recognition (see [15]–[18]), video tagging (see [19], [20]).

The Radial basis function neural network has been widely studied by some authors, in the existing results such as Fault-Estimation-Based Output-Feedback Adaptive FTC for Uncertain Nonlinear Systems With Actuator Faults (see [21]). However, radial basis function neural network is different from recurrent neural network, that is, recurrent neural network (RNN) is a kind of neural network with short-term memory ability; radial basis function (RBF) neural network is a kind of feedforward network. The difference between the recurrent neural network with the Radial basis function neural network: (1) For RBF neural network, RBF is used as the activation function of the hidden layer unit to map the input data to the high-dimensional hidden space without weight connection. The transmission of information is one-way. Although this limitation makes the network easier to learn, it also weakens the ability of neural network model to some extent. RBF neural network can be regarded as a complex function, each input is independent, that is, the output of the network only depends on the current input. However, in many realistic tasks, the output of the network is not only related to the input at the current moment, but also related to its output in the past period of time. (2) For recurrent neural network, neurons can not only receive information from other neurons, but also receive information from themselves, forming a network structure with loops. Compared with RBF neural network, recurrent neural network is more consistent with the structure of biological neural network. Because recurrent neural networks have short-term memory ability, which is equivalent to storage devices, their computational power is very strong. Recurrent neural networks can process

The associate editor coordinating the review of this manuscript and approving it for publication was Qi Zhou.

arbitrary length of time series data by using self-feedback neurons.

Time delays are inevitable in implementation of neural networks, since the finite switching speed of neurons and amplifiers. In many practical applications for delayed neural networks, especially neutral-type neural networks, which is described as non-operator-based neutral neural networks and $D$-operator-based neutral neural networks. However, neutral neural networks with $D$ operator have more general and more realistic significance than non-operator-based ones, thus it is received many scholars favor. There are many good results about periodic, anti-periodic, almost periodic, pseudo almost periodic, almost automorphic solutions for neutral-type neural networks with $D$ operator (see [22]–[35]).

As all know, the one neural network is Clifford-valued neural network, which represents a generalization of the real-valued, complex-valued and quaternion-valued neural networks. Although the multiplication of Clifford algebras does not satisfy the commutativity, it is not necessary to decompose the Clifford-valued neural networks into real-valued neural networks, thus it reduces the complexity of the calculation. Recently, there are a number of research results about the Clifford-valued neural networks (see [36]–[41]).

In practical applications for the synchronization of neural networks, particularly the anti-periodic synchronization, which has attracted the research interest of many scholars. The anti-periodic synchronization has played an key role in the research of neural network. In recent years, there's been a lot of research about the synchronization by many authors (see [42]–[49]). Some authors have explored the anti-periodic synchronization (see [50]).

With the inspiration from the previous research, in order to fill the gap in the research field of Clifford-valued neutral-type recurrent neural networks, the work of this article comes from three main motivations. (1) Recently, neutral-type neural networks with $D$ operator have been discussed by many authors. However, there is little research about Clifford-valued neutral-type recurrent neural networks with $D$ operator. (2) Many authors have discussed the synchronization for neural networks, but there are few research results on anti-periodic synchronization for neural networks. (3) Up to now, in practical applications for neural networks, there has been no paper about anti-periodic synchronization for Clifford-valued neutral-type recurrent neural networks with $D$ operator. Therefore, in this paper, we will study anti-periodic synchronization of Clifford-valued neutral-type neural networks with $D$ operator by using non-decomposition method, Banach fixed point theorem and the proof by contradiction.

Compared with the previous literatures, the main contributions of this paper are listed as follows. (1) Firstly, the introduction of the Clifford-valued neutral-type recurrent neural networks with $D$ operator, for the first time in the literature, to the best of our knowledge. (2) Secondly, in [50], some authors have studied the anti-periodic synchronization by

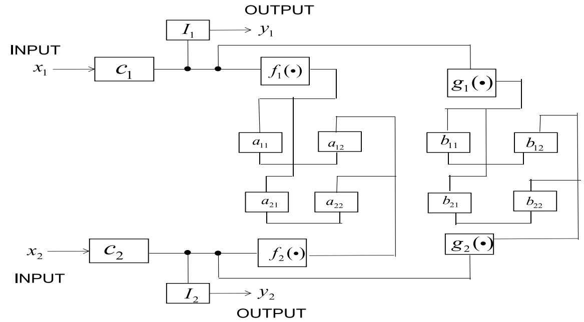


**FIGURE 1. The structure diagram of the network model.**

using decomposition method. By contrast, without separating the Clifford-valued neural networks into real-valued neural networks, our methods of this paper reduces the complexity of the calculation. (3) Thirdly, this is the first time to study the anti-periodic synchronization of Clifford-valued neutral-type neural networks with $D$ operator. (4) Fourthly, our method of this paper can be used to discuss the synchronization for other types of Clifford-valued neural networks with $D$ operator (or without $D$ operator). (5) Finally, we give one example to verify the effectiveness of the conclusion.

Inspired by the above ideas, we will study the Clifford-valued neutral-type recurrent neural networks with delays and $D$ operator:

$$\begin{aligned}[x_i(t) - r_i(t)x_i(t-\tau_i(t))]' &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j\big(x_j(t)\big) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t)g_j\big(x_j(t-\gamma_{ij}(t))\big) + I_i(t), \end{aligned} \tag{1.1}$$

where $i = 1, 2, \cdots, n$, $x_i(t) \in \mathcal{A}$ is the state vector of the $i$th unit at time $t$, $c_i(t) > 0$ represents the rate with which the $i$th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}, b_{ij} \in \mathcal{A}$ denote the strength of connectivity, the activation functions $f_j, g_j \in \mathcal{A}$ show how the $j$th neuron reacts to input, delay factors satisfy that $\tau_i(t), \gamma_{ij}(t) \in \mathbb{R}_+$, $I_i \in \mathcal{A}$ denotes the $i$th component of an external input source introduced from outside the network to the unit $i$ at time $t$, $r_i(t)$ is a continuous function with respect to $t$.

The initial value of system (1.1) is the following

$$x_i(s) = \varphi_i(s), \quad s \in [-\theta, 0], \tag{1.2}$$

where $\varphi_i(s) \in C\big([-\theta, 0], \mathcal{A}\big)$, $i = 1, 2, \cdots, n$, $\theta = \max\{\tau, \gamma\}$, $\tau = \max_{1\le i\le n}\Big\{\sup_{t\in[0,\frac{\omega}{2}]} \tau_i(t)\Big\}$, $\gamma = \max_{1\le i,j\le n}\Big\{\sup_{t\in[0,\frac{\omega}{2}]} \gamma_{ij}(t)\Big\}$.

This paper is organized as follows: In Section 2, we introduce some definitions and preliminary lemmas. In Section 3, we establish some sufficient conditions for the existence anti-periodic solutions of system (1.1), global exponential synchronization for system (1.1) and system (3.4). In Section 4, some numerical examples are provided to verify the effectiveness of the theoretical results. Finally, we draw a conclusion in Section 5.

*Notations:* $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$ denotes the set of non-negative real numbers, $\mathcal{A}$ denotes the set of Clifford numbers, $\mathcal{A}^n$ denotes the $n$ dimensional Clifford numbers, $\|\cdot\|_{\mathcal{A}}$ represents the vector Clifford norm. For $x = \sum_A x^A e_A \in \mathcal{A}$, we define $\|x\|_{\mathcal{A}} = \max_A \left\{|x^A|\right\}$ and for $x = (x_1, x_2, \cdots, x_n)^T \in \mathcal{A}^n$, we define $\|x\|_{\mathcal{A}^n} = \max_{1\le i\le n} \left\{\|x_i\|_{\mathcal{A}}\right\}$.

## II. PRELIMINARIES

The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$\mathcal{A} = \left\{ \sum_{A\in\{1,2,\cdots,m\}} u^A e_A, u^A \in \mathbb{R}\right\},$$

where $e_A = e_{h_1}\cdots e_{h_\nu}$ with $A = \{h_1 \cdots h_\nu\}$, $1 \le h_1 < h_2 < \cdots < h_\nu \le m$ and $1 \le \nu \le m$. Moreover, $e_\emptyset = e_0 = 1$ and $e_h, h = 1, 2, \cdots, m$ are said to be Clifford generators and satisfy $e_p^2 = -1, p = 1, 2, \cdots, m$, and $e_pe_q + e_qe_p = 0$, $p \neq q, p, q = 1, 2, \cdots, m$. Let $Q = \{\emptyset, 1, 2, \ldots, A, \ldots, 12\cdots m\}$, then it is easy to see that $\mathcal{A} = \{\sum_A u^A e_A, u^A \in \mathbb{R}\}$, where $\sum_A$ is short for $\sum_{A\in Q}$ and $\dim \mathcal{A} = 2^m$.

In order to study the existence of $\frac{\omega}{2}$-anti-periodic solution of system (1.1), we need the following assumptions:

$(H_1)$ For $i,j = 1, 2, \cdots, n$, $r_i, c_i, \tau_i, \gamma_{ij} \in \mathbb{R}_+$, $a_{ij}, b_{ij}, f_j, g_j, I_i \in \mathcal{A}$, there exists $\omega > 0$ such that $c_i(t + \frac{\omega}{2}) = c_i(t)$, $r_i(t + \frac{\omega}{2}) = r_i(t)$, $I_i(t + \frac{\omega}{2}) = -I_i(t)$, $\tau_i(t + \frac{\omega}{2}) = \tau_i(t)$, $\gamma_{ij}(t + \frac{\omega}{2}) = \gamma_{ij}(t)$, $a_{ij}(t + \frac{\omega}{2})f_j(u) = -a_{ij}(t)f_j(-u)$, $b_{ij}(t + \frac{\omega}{2})g_j(u) = -b_{ij}(t)g_j(-u)$;

$(H_2)$ For $j = 1, 2, \cdots, n$, there exist positive constants $L_f$, $L_g$ such that

$$\|f_j(u) - f_j(v)\|_{\mathcal{A}} \le L_f\|u - v\|_{\mathcal{A}},$$
$$\|g_j(u) - g_j(v)\|_{\mathcal{A}} \le L_g\|u - v\|_{\mathcal{A}};$$

$(H_3)$

$$\delta := \frac{1}{c_i^-}\left[c_i^- r_i^+ + c_i^+ r_i^+ + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij}^+ L_g\right] < 1,$$

where

$$r_i^- = \inf_{[0,\frac{\omega}{2}]} r_i(t), \quad r_i^+ = \sup_{[0,\frac{\omega}{2}]} r_i(t), \; c_i^- = \inf_{[0,\frac{\omega}{2}]} c_i(t),$$

$$c_i^+ = \sup_{[0,\frac{\omega}{2}]} c_i(t), \; a_{ij}^+ = \max_{1\le i,j\le n} \|a_{ij}(t)\|_{\mathcal{A}},$$
$$b_{ij}^+ = \max_{1\le i,j\le n} \|b_{ij}(t)\|_{\mathcal{A}}, \quad I_i = \max_{1\le i\le n} \|I_i(t)\|_{\mathcal{A}}.$$

*Lemma 2.1: [51] Let $X$ be a Banach spaces, $E \subset X$ is a closed subset, mapping $T : E \longrightarrow E$ be a contraction, i.e. there exists a constant $\theta \in (0, 1)$ such that*

$$\|Tx - Ty\| \le \theta\|x - y\|, \;\; \forall x, \;\; y \in E.$$

*Then $T$ has at least one fixed point $\bar{x}$.*

*Definition 2.1: A continuous function $x = \big(x_1, x_2 \cdots, x_n\big)^T : [0, +\infty] \to \mathcal{A}^n$ is said to be a solution of system (1.1), if*

*(i) $x_i(s) = \varphi_i(s)$, for $s \in [-\theta, 0]$, $\varphi_i^x \in C([-\theta, 0], \mathcal{A})$, $i = 1, 2, \cdots, n$;*

*(ii) $x(t)$ satisfies system (1.1) for $t \ge 0$.*

*Definition 2.2: A solution $x$ of system (1.1) is said to be $\frac{\omega}{2}$-anti-periodic solution of system (1.1), if there exists $\omega > 0$ such that*

$$x(t + \frac{\omega}{2}) = -x(t).$$

## III. MAIN RESULTS

In this section, we will investigate the existence and global exponential synchronization of anti-periodic solutions of Clifford-valued neutral-type recurrent neural networks (1.1), based on Banach fixed point theorem and the proof by contradiction.

Denote

$$\mathbb{X} = \left\{x \in C\Big([0, \frac{\omega}{2}], \mathcal{A}^n\Big) : x\Big(t + \frac{\omega}{2}\Big) = -x(t), t \in \mathbb{R}\right\}$$

be a Banach spaces equipped with the norm

$$\|x\|_{\mathbb{X}} = \|x\|_{\mathcal{A}^n}.$$

Let $E \subset \mathbb{X}$ is a closed subset, $E = \Big\{x : x(t) \in C(\mathbb{R}, \mathcal{A}^n), x(t + \frac{\omega}{2}) = -x(t), \|x\|_{\mathbb{X}} \le \xi\Big\}$, where

$$\xi := \frac{1}{c_i^-}\left[c_i^- r_i^+ + c_i^+ r_i^+ + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij} L_g\right]\eta_1 + \sum_{j=1}^n \frac{a_{ij}^+\eta_2}{c_i^-} + \sum_{j=1}^n \frac{b_{ij}^+\eta_3}{c_i^-} + \frac{I_i}{c_i^-},$$

and

$$\eta_1 = \sup_{[0,\omega]} |x_i^A(t)|, \eta_2 = \max_A \left\{|f_j^A(0)|\right\}, \eta_3 = \max_A \left\{|g_j^A(0)|\right\}.$$

*Theorem 3.1: Assume that assumptions $(H_1)$-$(H_3)$ hold. Then system (1.1) has at least an $\frac{\omega}{2}$-anti-periodic solution.*

*Proof:* Let $u_i(t) = x_i(t) - r_i(t)x_i(t - \tau_i(t)) \in \mathcal{A}$, then $x_i(t) = u_i(t) + r_i(t)x_i(t - \tau_i(t))$ and system (1.1) can be described as following differential equations

$$u_i'(t) = -c_i(t)u_i(t) - c_i(t)r_i(t)x_i(t - \tau_i(t)) + \sum_{j=1}^n a_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^n b_{ij}(t) \times g_j\big(x_j(t - \gamma_{ij}(t))\big) + I_i(t), \tag{3.1}$$

where $i = 1, 2, \cdots, n$.

It is well known that an $\frac{\omega}{2}$-anti-periodic solution of system (3.1) is equivalent to find an $\frac{\omega}{2}$-anti-periodic solution of the integral equation

$$u_i(t) = \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\Big[-c_i(s)r_i(s) x_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s)f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big) + I_p(s)\Big]ds,$$

that is,

$$x_i(t) = r_i(t)x_i(t-\tau_i(t)) + \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \times\Big[-c_i(s)r_i(s)x_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s) \times f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big) + I_i(s)\Big]ds, \tag{3.2}$$

where $i = 1, 2, \cdots, n$.

Let $E \subset \mathbb{X}$ is a closed subset, $E = \Big\{x : x(t) \in C(\mathbb{R}, \mathcal{A}^n), x(t+\frac{\omega}{2}) = -x(t), \|x\|_{\mathbb{X}} \le \xi\Big\}$, we define one mapping $T$ as follows

$$(Tx)(t) = \big((Tx)_1(t), (Tx)_2(t), \cdots, (Tx)_n(t)\big)^T,$$

where $(Tx)_i(t) \in \mathcal{A}$ and

$$(Tx)_i(t) = r_i(t)x_i(t-\tau_i(t)) + \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \times\Big[-c_i(s)r_i(s)x_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s) \times f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big) + I_i(s)\Big]ds, \quad i = 1, 2, \cdots, n. \tag{3.3}$$

For any $x \in E$ and $t \ge 0$, by $(H_1)$, from (3.3) we have that

$$\begin{aligned}
(Tx)_i(t+\frac{\omega}{2}) &= r_i\Big(t+\frac{\omega}{2}\Big)x_i\Big(t+\frac{\omega}{2}-\tau_i\Big(t+\frac{\omega}{2}\Big)\Big) + \int_{t+\frac{\omega}{2}}^{t+\frac{\omega}{2}+\frac{\omega}{2}} -\frac{e^{\int_{t+\frac{\omega}{2}}^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \\
&\times\Big[-c_i(s)r_i(s)x_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s)f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s) \times g_j\big(x_j(s-\gamma_{ij}(s))\big) + I_i(s)\Big]ds \\
&= -r_i(t)x_i(t-\tau_i(t)) + \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^{\nu} c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \\
&\times\Big[-c_i\Big(\nu+\frac{\omega}{2}\Big)r_i\Big(\nu+\frac{\omega}{2}\Big)x_i\Big(\nu+\frac{\omega}{2} - \tau_i\Big(\nu+\frac{\omega}{2}\Big)\Big) + \sum_{j=1}^n a_{ij}\Big(\nu+\frac{\omega}{2}\Big)f_j\Big(x_j\Big(\nu+\frac{\omega}{2}\Big)\Big) \\
&+ \sum_{j=1}^n b_{ij}(\nu+\frac{\omega}{2})j_q\Big(x_j\Big(\nu+\frac{\omega}{2}-\gamma_{ij}\Big(\nu+\frac{\omega}{2}\Big)\Big)\Big) + I_i\Big(\nu+\frac{\omega}{2}\Big)\Big]d\nu \\
&= -r_i(t)x_i(t-\tau_i(t)) + \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^{\nu} c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \times\Big[c_i(\nu)r_i(\nu)x_i(\nu-\tau_i(\nu)) - \sum_{j=1}^n a_{ij}(\nu)f_j\big(x_j(\nu)\big) \\
&- \sum_{j=1}^n b_{ij}(\nu)g_j\big(x_j(\nu-\gamma_{pq}(\nu))\big) - I_i(\nu)\Big]d\nu = -(Tx)_i(t),
\end{aligned}$$

which shows that $(Tx)(t)$ is $\frac{\omega}{2}$-anti-periodic.

Next, we show that $\|Tx\|_{\mathbb{X}} \le \xi$. For any $x \in E$, $i = 1, 2, \cdots, n$, we have

$$\begin{aligned}
\|(Tx)(t)\|_{\mathbb{X}} &= \|(Tx)_i(t)\|_{\mathcal{A}^n} = \max_{1\le i\le n}\Big\{\|(Tx)_i(t)\|_{\mathcal{A}}\Big\} \\
&= \max_{1\le i\le n}\Big\{\Big\|r_i(t)x_i(t-\tau_i(t)) + \int_t^{t+\frac{\omega}{2}} -e^{\int_t^s c_i(\mu)d\mu} \times \frac{1}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\Big[-c_i(s)r_i(s)x_i(s-\tau_i(s)) \\
&+\sum_{j=1}^n a_{ij}(s)f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big) + I_i(s)\Big]ds\Big\|_{\mathcal{A}}\Big\} \\
&\le \max_{1\le i\le n}\Big\{r_i^+\|x_i(t-\tau_i(t))\|_{\mathcal{A}} + \int_t^{t+\frac{\omega}{2}} e^{\int_t^s c_i(\mu)d\mu} \times \frac{1}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\Big[c_i^+r_i^+\|x_i(s-\tau_i(s))\|_{\mathcal{A}} \\
&+\sum_{j=1}^n \|a_{ij}(s)\|_{\mathcal{A}}\Big(\|f_j(x_j(s)) - f_j(0)\|_{\mathcal{A}} + \|f_j(0)\|_{\mathcal{A}}\Big) \\
&+\sum_{j=1}^n \|b_{ij}(s)\|_{\mathcal{A}}\Big(\|g_j\big(x_j(s-\gamma_{ij}(s))\big) - g_j(0)\|_{\mathcal{A}} + \|g_j(0)\|_{\mathcal{A}}\Big) + \|I_i(s)\|_{\mathcal{A}}\Big]ds\Big\} \\
&\le \max_{1\le i\le n}\Big\{r_i^+\|x_i(t-\tau_p(t))\|_{\mathcal{A}} + \int_t^{t+\frac{\omega}{2}} e^{\int_t^s c_i(\mu)d\mu}
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{1}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[ c_i^+ r_i^+ \|x_i(s-\tau_i(s))\|_{\mathcal{A}} \\
&+\sum_{j=1}^n a_{ij}^+ L_f \|x_j(s)\|_{\mathcal{A}} + \sum_{j=1}^n b_{ij}^+ L_g \|x_j(s-\gamma_{ij}(s))\|_{\mathcal{A}} \\
&+\sum_{j=1}^n a_{ij}^+ \|f_j(0)\|_{\mathcal{A}} + \sum_{j=1}^n b_{ij}^+ \|g_j(0)\|_{\mathcal{A}} + \|I_i(s)\|_{\mathcal{A}} \Big] ds \Big\} \\
&\le \frac{1}{c_i^-}\Big[ c_i^- r_i^+ + c_i^+ r_i^+ + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij} L_g \Big] \eta_1 \\
&\quad + \sum_{j=1}^n \frac{a_{ij}^+ \eta_2}{c_i^-} + \sum_{j=1}^n \frac{b_{ij}^+ \eta_3}{c_i^-} + \frac{I_i}{c_i^-} \le \xi.
\end{aligned}
$$

Hence, we have $(Tx)(t) \in E$.

Finally, we show $T$ is a contraction mapping. For any $x$, $x^* \in E$, $i = 1, 2, \cdots, n$, we have

$$
\begin{aligned}
&\|(Tx)(t) - (Tx^*)(t)\|_{\mathcal{A}^n} \\
&= \max_{1\le i\le n} \Big\{ \|(Tx)_i(t) - (Tx^*)_i(t)\|_{\mathcal{A}} \Big\} \\
&= \max_{1\le i\le n} \Big\{ \Big\| r_i(t)\big(x_i(t-\tau_i(t)) - x_i^*(t-\tau_i(t))\big) \\
&\quad + \int_t^{t+\frac{\omega}{2}} - \frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[ -c_i(s) r_i(s) \\
&\quad \times \Big( x_i(s-\tau_i(s)) - x_i^*(s-\tau_i(s)) \Big) + \sum_{j=1}^n a_{ij}(s) \\
&\quad \times \Big( f_j\big(x_j(s)\big) - f_j\big(x_j^*(s)\big) \Big) + \sum_{j=1}^n b_{ij}(s) \\
&\quad \times \Big( g_j\big(x_j(s-\gamma_{ij}(s))\big) - g_j\big(x_j^*(s-\gamma_{ij}(s))\big) \Big) \Big] ds \Big\|_{\mathcal{A}} \Big\} \\
&\le \max_{1\le i\le n} \Big\{ r_i^+ \big\| x_i(t-\tau_i(t)) - x_i^*(t-\tau_i(t)) \big\|_{\mathcal{A}} \\
&\quad + \int_t^{t+\frac{\omega}{2}} \frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[ c_i^+ r_i^+ \big\| x_i(s-\tau_i(s)) \\
&\quad - x_i^*(s-\tau_i(s)) \big\|_{\mathcal{A}} + \sum_{j=1}^n a_{ij}^+ L_f \big\| x_j(s) - x_j^*(s) \big\|_{\mathcal{A}} \\
&\quad + \sum_{j=1}^n b_{ij}^+ L_g \big\| x_j(s-\gamma_{ij}(s)) - x_j^*(s-\gamma_{ij}(s)) \big\|_{\mathcal{A}} \Big] ds \Big\} \\
&\le \frac{1}{c_i^-} \Big[ c_i^- r_i^+ + c_i^+ r_i^+ + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij} L_g \Big] \\
&\quad \times \|x - x^*\|_{\mathbb{X}} \le \delta \|x - x^*\|_{\mathbb{X}},
\end{aligned}
$$

that is,

$$
\|(Tx)(t) - (Tx^*)(t)\|_{\mathbb{X}} =\le \delta \|x - x^*\|_{\mathbb{X}}.
$$

Thus, $T$ is a contraction mapping.

Therefore, by **Lemma 2.1**, system (1.1) has at least an $\frac{\omega}{2}$-anti-periodic solution. The proof is completed. □

Next, in order to investigate drive-response synchronization, we will consider neural network system (1.1) as the master system, and the slave system is given by

$$
\begin{aligned}
&[y_i(t) - r_i(t) y_i(t-\tau_i(t))]' \\
&\quad = -c_i(t) y_i(t) + \sum_{j=1}^n a_{ij}(t) f_j\big(y_j(t)\big) \\
&\qquad + \sum_{j=1}^n b_{ij}(t) g_j\big(y_j(t-\gamma_{ij}(t))\big) + I_i(t) + \varepsilon_i(t), \quad (3.4)
\end{aligned}
$$

where $i = 1, 2, \cdots, n$, $y_i(t) : \mathbb{R} \to \mathcal{A}$ denotes the state of the response system, $\varepsilon_i(t) \in \mathcal{A}$ is a state-feedback controller, other notations are the same as those in system (1.1).

The initial value of system (3.4) is the following

$$
y_i(s) = \psi_i(s), \quad s \in [-\theta, 0],
$$

where $\psi_i \in C\Big([-\theta, 0], \mathcal{A}\Big)$, $i = 1, 2, \cdots, n$.

In order to realize synchronization between (1.1) and (3.4), the controller $\varepsilon_i$ is designed as

$$
\varepsilon_i(t) = -\sigma_i(t) z_i(t) + \sum_{j=1}^n \mu_{ij}(t) h_j(z_j(t-\alpha_{ij}(t))), \quad (3.5)
$$

where $i = 1, 2, \cdots, n$, $\sigma_i, \alpha_{ij} : \mathbb{R} \longrightarrow \mathbb{R}_+$, $\mu_{ij}, h_j \in \mathcal{A}$.

We are now in a position to discuss the problem of systems (1.1) and (3.4). Let $z_i = y_i - x_i$, $i = 1, 2, \ldots, n$, $Z_i(t) = z_i(t) - r_i(t) z_i(t-\tau_i(t))$, then the error system is given by

$$
\begin{aligned}
Z_i'(t) &= -c_i(t) z_i(t) + \sum_{j=1}^n a_{ij}(t) \Big( f_j\big(y_j(t)\big) \\
&\quad - f_j\big(x_j(t)\big) \Big) + \sum_{j=1}^n b_{ij}(t) \Big( g_j\big(y_j(t-\gamma_{ij}(t))\big) \\
&\quad - g_j\big(x_j(t-\gamma_{ij}(t))\big) \Big) - \sigma_i(t) z_i(t) + \sum_{j=1}^n \mu_{ij}(t) \\
&\quad \times h_j(z_j(t-\alpha_{ij}(t))). \qquad (3.6)
\end{aligned}
$$

System (3.6) is supplemented with initial values given by

$$
z_i(s) = \psi_i(s) - \varphi_i(s), \quad s \in [-\theta, 0]. \qquad (3.7)
$$

*Definition 3.3: The response system (3.4) and the drive system (1.1) are said to be globally exponentially synchronized, if there exist constants $\lambda > 0$ and $M > 0$ such that*

$$
\|y(t) - x(t)\|_{\mathbb{X}} \le M \|\psi - \varphi\|_{\mathbb{X}} e^{-\lambda t}, \forall t > 0,
$$

*where*

$$
\|y(t) - x(t)\|_{\mathbb{X}} = \max_{1\le i\le n} \Big\{ \|y_i(t) - x_i(t)\|_{\mathcal{A}} \Big\},
$$

*and*

$$
\|\psi - \varphi\|_{\mathbb{X}} = \max_{1\le i\le n} \Big\{ \|\psi_i - \varphi_i\|_{\mathcal{A}} \Big\}.
$$

*Theorem 3.2: Assume that $(H_1)$-$(H_3)$ hold. If the following conditions are satisfied:*

$(H_4)$ *For $i,j = 1,2,\cdots,n$, $\sigma_i(t), \alpha_{ij} \in C(\mathbb{R},\mathbb{R}_+)$, $\mu_{ij}(t), h_j(\cdot) \in \mathcal{A}$, there exists positive constant $\omega$ such that*

$$\sigma_i(t+\frac{\omega}{2}) = \sigma_i(t), \alpha_{ij}(t+\frac{\omega}{2}) = \alpha_{ij}(t),$$
$$\mu_{ij}(t+\frac{\omega}{2})h_j(u) = -\mu_{ij}(t)h_j(-u);$$

$(H_5)$ *For $j = 1,2,\cdots,n$, $h_j(0) = 0$, there exists a positive constant $L_h$ such that*

$$\|h_j(u) - h_j(v)\|_{\mathcal{A}} \le L_h\|u-v\|_{\mathcal{A}};$$

$(H_6)$ *There exists a positive constant $\lambda$ such that*

$$\nu := 1 - r_i^+ e^{\lambda\tau} > 0$$

*and*

$$0 < \frac{1}{c_i^- + \sigma_i^- - \lambda}\Big[(c_i^+ + \sigma_i^+)r_i^+ e^{\lambda\tau} + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij}^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n \mu_{ij}^+ L_h e^{\lambda\alpha}\Big]\frac{1}{\nu} < 1,$$

*where*

$$\sigma_i^- = \inf_{[0,\frac{\omega}{2}]} \sigma_i(t), \sigma_i^+ = \sup_{[0,\frac{\omega}{2}]} \sigma_i(t),$$
$$\mu_{ij}^+ = \max_{1\le i,j\le n} \|\mu_{ij}(t)\|_{\mathcal{A}}, \alpha = \max_{1\le i,j\le n}\Big\{\sup_{t\in[0,\frac{\omega}{2}]} \alpha_{ij}(t)\Big\}.$$

*Then the drive system* (1.1) *and the response system* (3.4) *are globally exponentially synchronized.*

*Proof:* Let $Z_i(t) = z_i(t) - r_i(t)z_i(t-\tau_i(t))$, we have that

$$\begin{aligned} e^{\lambda t}\|z_i(t)\|_{\mathcal{A}} &= e^{\lambda t}\|z_i(t) - r_i(t)z_i(t-\tau_i(t)) + r_i(t) \times z_i(t-\tau_i(t))\|_{\mathcal{A}} \\ &\le e^{\lambda t}\|z_i(t) - r_i(t)z_i(t-\tau_i(t))\|_{\mathcal{A}} + r_i^+ \times e^{\lambda\tau}e^{\lambda t}\|z_i(t)\|_{\mathcal{A}}. \end{aligned}$$

Hence, we have

$$e^{\lambda t}\|z_i(t)\|_{\mathcal{A}} \le \frac{e^{\lambda t}\|Z_i(t)\|_{\mathcal{A}}}{1 - r_i^+ e^{\lambda\tau}}.$$

By $(H_6)$, let

$$M := (c_i^- + \sigma_i^-)\Big[(c_i^+ + \sigma_i^+)r_i^+ e^{\lambda\tau} + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij}^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n \mu_{ij}^+ L_h e^{\lambda\alpha}\Big]^{-1}\nu > 1,$$

then

$$\begin{aligned} \frac{1}{M} &= \frac{1}{c_i^- + \sigma_i^-}\Big[(c_i^+ + \sigma_i^+)r_i^+ e^{\lambda\tau} + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij}^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n \mu_{ij}^+ L_h e^{\lambda\alpha}\Big]\frac{1}{\nu} \\ &\le \frac{1}{c_i^- + \sigma_i^- - \lambda}\Big[(c_i^+ + \sigma_i^+)r_i^+ e^{\lambda\tau} + \sum_{j=1}^n a_{ij}^+ L_f + \sum_{j=1}^n b_{ij}^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n \mu_{ij}^+ L_h e^{\lambda\alpha}\Big]\frac{1}{\nu}. \end{aligned}$$

From (3.6), For $i = 1,2\cdots,n$, we can have that

$$\begin{aligned} Z_i(t) &= Z_i(0)e^{-\int_0^t (c_i(\xi)+\sigma_i(\xi))d\xi} + \int_0^t e^{-\int_s^t (c_i(\xi)+\sigma_i(\xi))d\xi}\Big[-\big(c_i(s) + \sigma_i(s)\big)r_i(s)z_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s) \\ &\quad \times \Big(f_j\big(y_j(s)\big) - f_j\big(x_j(s)\big)\Big) + \sum_{j=1}^n b_{ij}(s) \times \Big(g_j\big(y_j(s-\gamma_{ij}(s))\big) - g_j\big(x_j(s-\gamma_{ij}(s))\big)\Big) \\ &\quad + \sum_{j=1}^n \mu_{ij}(s)h_j(z_j(s-\alpha_{ij}(s)))\Big]ds. \end{aligned}$$

When $t \in [-\theta, 0]$, it is easy to see that there exist two constants $\epsilon > 0$ and $M > 1$ such

$$\|Z_i(0)\|_{\mathcal{A}} < \|\phi\|_{\mathbb{X}} + \epsilon$$

and

$$\|Z(t)\|_{\mathbb{X}} = \max_{1\le i\le n}\Big\{\|Z_i(t)\|_{\mathcal{A}}\Big\} < M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t},$$

that is,

$$\|z(t)\|_{\mathbb{X}} < \frac{M}{1 - r_i^+ e^{\lambda\tau}}(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t},$$

where $\|\phi\|_{\mathbb{X}} = \|\psi - \varphi\|_{\mathbb{X}}$. We claim that

$$\|Z(t)\|_{\mathbb{X}} < M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t}, \quad t \in [0, +\infty). \tag{3.8}$$

If it is not true, then there must be some $\hat{t} > 0$ such that

$$\|Z(\hat{t})\|_{\mathbb{X}} = \max_{1\le i\le n}\{\|Z_i(\hat{t})\|_{\mathcal{A}}\} = M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda\hat{t}} \tag{3.9}$$

and

$$\|Z(t)\|_{\mathbb{X}} < M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t}, \quad t \in [-\theta, \hat{t}).$$

Hence, we have

$$\begin{aligned} \|Z_i(\hat{t})\|_{\mathcal{A}} = \Big\| & Z_i(0)e^{-\int_0^{\hat{t}} (c_i(\xi)+\sigma_i(\xi))d\xi} + \int_0^{\hat{t}} e^{-\int_s^{\hat{t}} (c_i(\xi)+\sigma_i(\xi))d\xi}\Big[-\big(c_i(s) + \sigma_i(s)\big)r_i(s) \\ & \times z_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s)\Big(f_j\big(y_j(s)\big) - f_j\big(x_j(s)\big)\Big) \\ & + \sum_{j=1}^n b_{ij}(s)\Big(g_j\big(y_j(s-\gamma_{ij}(s))\big) - g_j\big(x_j(s-\gamma_{ij}(s))\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\mu_{ij}(s)h_j(z_j(s-\alpha_{ij}(s)))\Big]ds\Big\|_{\mathcal{A}} \\
&\le \|Z_i(0)\|_{\mathcal{A}}e^{-\int_0^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi}+\int_0^{\hat{t}}e^{-\int_s^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi} \\
&\quad\times\Big[(c_i^{+}+\sigma_i^{+})r_i^{+}\|z_i(s-\tau_i(s))\|_{\mathcal{A}}+\sum_{j=1}^{n}\|a_{ij}(s)\|_{\mathcal{A}} \\
&\quad\times\Big\|\Big(f_j\big(y_j(s)\big)-f_j\big(x_j(s)\big)\Big)\Big\|_{\mathcal{A}}+\sum_{j=1}^{n}\|b_{ij}(s)\|_{\mathcal{A}} \\
&\quad\times\Big\|\Big(g_j\big(y_j(s-\gamma_{ij}(s))\big)-g_j\big(x_j(s-\gamma_{ij}(s))\big)\Big)\Big\|_{\mathcal{A}} \\
&\quad+\sum_{j=1}^{n}\|\mu_{ij}(s)\|_{\mathcal{A}}\|h_j(z_j(s-\alpha_{ij}(s)))\|_{\mathcal{A}}\Big]ds \\
&\le (\|\phi\|_{\mathbb{X}}+\epsilon)e^{-(c_i^{-}+\sigma_i^{-})\hat{t}}+M(\|\phi\|_{\mathbb{X}}+\epsilon) \\
&\quad\int_0^{\hat{t}}e^{-\int_s^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi}\Big[(c_i^{+}+\sigma_i^{+})r_i^{+}e^{\lambda\tau} \\
&\quad+\sum_{j=1}^{n}a_{ij}^{+}L_f+\sum_{j=1}^{n}b_{ij}^{+}L_g e^{\lambda\gamma}+\sum_{j=1}^{n}\mu_{ij}^{+}L_h e^{\lambda\alpha}\Big]\times\frac{e^{-\lambda s}}{\nu}ds \\
&\le M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}\Big\{\frac{e^{(\lambda-c_i^{-}-\sigma_i^{-})\hat{t}}}{M}+\frac{1}{c_i^{-}+\sigma_i^{-}-\lambda} \\
&\quad\Big[(c_i^{+}+\sigma_i^{+})r_i^{+}e^{\lambda\tau}+\sum_{j=1}^{n}a_{ij}^{+}L_f+\sum_{j=1}^{n}b_{ij}^{+}L_g e^{\lambda\gamma} \\
&\quad+\sum_{j=1}^{n}\mu_{ij}^{+}L_h e^{\lambda\alpha}\Big]\frac{1-e^{(\lambda-c_i^{-}-\sigma_i^{-})\hat{t}}}{\nu}\Big\} \\
&\le M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}\Big\{e^{(\lambda-c_i^{-}-\sigma_i^{-})\hat{t}}\Big(\frac{1}{M} \\
&\quad-\frac{1}{c_i^{-}+\sigma_i^{-}-\lambda}\Big[(c_i^{+}+\sigma_i^{+})r_i^{+}e^{\lambda\tau}+\sum_{j=1}^{n}a_{ij}^{+}L_f \\
&\quad+\sum_{j=1}^{n}b_{ij}^{+}L_g e^{\lambda\gamma}+\sum_{j=1}^{n}\mu_{ij}^{+}L_h e^{\lambda\alpha}\Big]\frac{1}{\nu}\Big) \\
&\quad+\frac{1}{c_i^{-}+\sigma_i^{-}-\lambda}\Big[(c_i^{+}+\sigma_i^{+})r_i^{+}e^{\lambda\tau}+\sum_{j=1}^{n}a_{ij}^{+}L_f \\
&\quad+\sum_{j=1}^{n}b_{ij}^{+}L_g e^{\lambda\gamma}+\sum_{j=1}^{n}\mu_{ij}^{+}L_h e^{\lambda\alpha}\Big]\frac{1}{\nu}\Big\} \\
&\le M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}\Big\{\frac{1}{c_i^{-}+\sigma_i^{-}-\lambda}\Big[(c_i^{+}+\sigma_i^{+}) \\
&\quad\times r_i^{+}e^{\lambda\tau}+\sum_{j=1}^{n}a_{ij}^{+}L_f+\sum_{j=1}^{n}b_{ij}^{+}L_g e^{\lambda\gamma}+\sum_{j=1}^{n}\mu_{ij}^{+} \\
&\quad\times L_h e^{\lambda\alpha}\Big]\frac{1}{\nu}\Big\}<M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}.
\end{aligned}
$$

Hence,

$$\|Z(\hat{t})\|_{\mathbb{X}}<M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}},$$

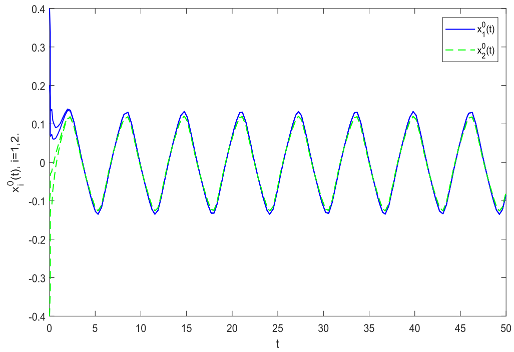


**FIGURE 2. Curves of $x_i^0(t)=(x_1^0(t),x_2^0(t))^T$ of system (4.1) with the initial values $(x_1^0(0),x_2^0(0))^T=(0.2,-0.1)^T,(0.4,-0.4)^T$.**

which contradicts the equality (3.9), and so (3.8) holds. Letting $\epsilon\to 0^{+}$, then

$$\|Z(t)\|_{\mathbb{X}}\le M\|\phi\|_{\mathbb{X}}e^{-\lambda t},$$

that is,

$$\|z(t)\|_{\mathbb{X}}\le\frac{M}{1-r_i^{+}e^{\lambda\tau}}\|\phi\|_{\mathbb{X}}e^{-\lambda t},$$

where

$$\|\phi\|_{\mathbb{X}}=\|\psi-\varphi\|_{\mathbb{X}}.$$

Therefore, the drive system (1.1) and the response system (3.4) are globally exponentially synchronized. The proof is complete. □

## IV. ILLUSTRATIVE EXAMPLE

In this section, we give one example to show the feasibility and effectiveness of main results.

*Example 4.1: Consider the following delayed Clifford-valued neutral-type recurrent neural networks with two neurons as the drive system:*

$$
\begin{aligned}
&[x_i(t)-r_i(t)x_i(t-\tau_i(t))]' \\
&\quad=-c_i(t)x_i(t)+\sum_{j=1}^{2}a_{ij}(t)f_j\big(x_j(t)\big) \\
&\qquad+\sum_{j=1}^{2}b_{ij}(t)g_j\big(x_j(t-\gamma_{ij}(t))\big)+I_i(t),
\end{aligned}\tag{4.1}
$$

*The corresponding response system is given by*

$$
\begin{aligned}
&[y_i(t)-r_i(t)y_i(t-\tau_i(t))]' \\
&\quad=-c_i(t)y_i(t)+\sum_{j=1}^{2}a_{ij}(t)f_j\big(y_j(t)\big) \\
&\qquad+\sum_{j=1}^{2}b_{ij}(t)g_j\big(y_j(t-\gamma_{ij}(t))\big) \\
&\qquad+I_i(t)+\varepsilon_i(t),
\end{aligned}\tag{4.2}
$$

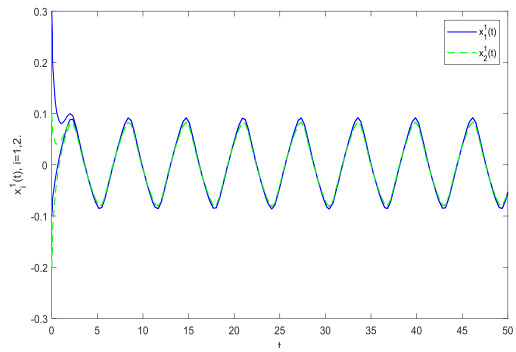

**FIGURE 3.** Curves of $x_i^1(t) = (x_1^1(t), x_2^1(t))^T$ of system (4.1) with the initial values $(x_1^1(0), x_2^1(0))^T = (-0.1, 0.1)^T, (0.3, -0.2)^T$.

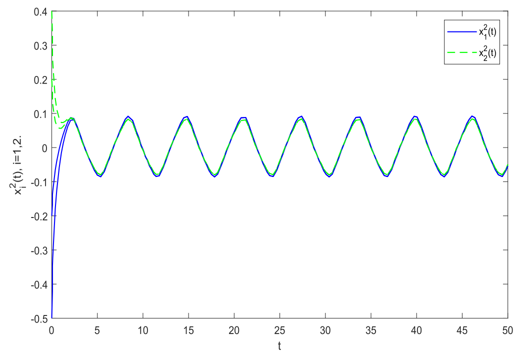

**FIGURE 4.** Curves of $x_i^2(t) = (x_1^2(t), x_2^2(t))^T$ of system (4.1) with the initial values $(x_1^2(0), x_2^2(0))^T = (-0.2, 0.2)^T, (-0.5, 0.4)^T$.

*and the controller is as follows*:

$$\varepsilon_i(t) = -\sigma_i(t)z_i(t) + \sum_{j=1}^{2} \mu_{ij}(t)h_j(z_j(t-\alpha_{ij}(t))), \quad (4.3)$$

*where* $m = 2$, $i = 1, 2$, $c_1(t) = 2.2 + 0.1\sin 2t$, $c_2(t) = 2.5 + 0.3\sin 2t$, $r_i(t) = \frac{1}{5} + \frac{1}{5}\sin 2t$, $\tau_i(t) = \frac{1}{15} + \frac{1}{15}\sin 2t$, $\gamma_{ij} = \frac{1}{10} + \frac{1}{20}\sin 2t$, $\sigma_i = 1.2 + 0.3\sin 2t$, $\alpha_{ij} = \frac{1}{15} + \frac{1}{30}\sin 2t$ *and*

$$\begin{aligned}
a_{11} &= 0.1e_0\sin t + 0.2e_1\sin t + 0.1e_2\sin t,\\
a_{12} &= 0.2e_0\sin t + 0.1e_1\sin t + 0.2e_{12}\sin t,\\
a_{21} &= 0.2e_0\sin t + 0.2e_2\sin t + 0.1e_{12}\sin t,\\
a_{22} &= 0.1e_1\sin t + 0.1e_2\sin t + 0.3e_{12}\sin t,\\
b_{11} &= 0.2e_0\sin t + 0.1e_1\sin t + 0.3e_{12}\sin t,\\
b_{12} &= 0.1e_0\sin t + 0.1e_1\sin t + 0.2e_2\sin t,\\
b_{21} &= 0.3e_1\sin t + 0.1e_2\sin t + 0.1e_{12}\sin t,\\
b_{22} &= 0.2e_0\sin t + 0.1e_2\sin t + 0.1e_{12}\sin t,\\
\mu_{11} &= 0.1e_0\sin t + 0.1e_2\sin t + 0.2e_{12}\sin t,\\
\mu_{12} &= 0.1e_1\sin t + 0.1e_2\sin t + 0.3e_{12}\sin t,\\
\mu_{21} &= 0.2e_0\sin t + 0.1e_1\sin t + 0.2e_2\sin t,\\
\mu_{22} &= 0.2e_0\sin t + 0.2e_1\sin t + 0.1e_2\sin t,\\
I_i &= 0.3e_0\sin t + 0.2e_1\sin t + 0.2e_2\sin t + 0.1e_{12}\sin t,\\
f_j &= \frac{1}{10}\big(\sin x_j^0 e_0 + \sin x_j^1 e_1 + \sin x_j^2 e_2 + \sin x_j^{12} e_{12}\big),
\end{aligned}$$

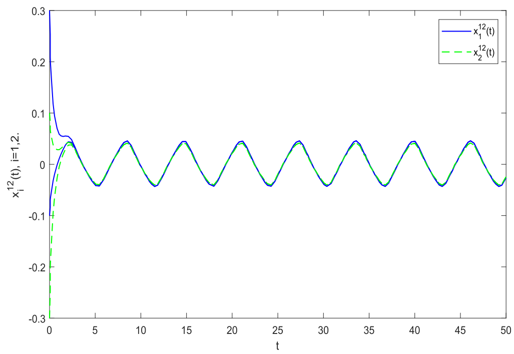

**FIGURE 5.** Curves of $x_i^{12}(t) = (x_1^{12}(t), x_2^{12}(t))^T$ of system (4.1) with the initial values $(x_1^{12}(0), x_2^{12}(0))^T = (0.3, -0.3)^T, (-0.1, 0.1)^T$.

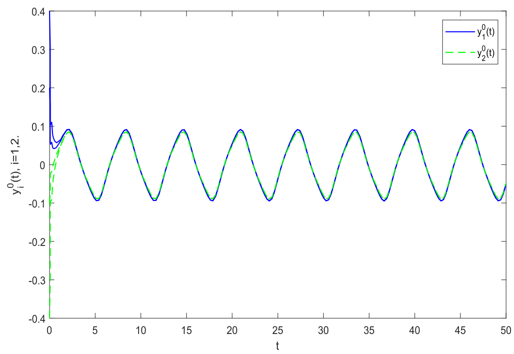

**FIGURE 6.** Curves of $y_i^0(t) = (y_1^0(t), y_2^0(t))^T$ of system (4.2) with the initial values $(y_1^0(0), y_2^0(0))^T = (0.2, -0.1)^T, (0.4, -0.4)^T$.

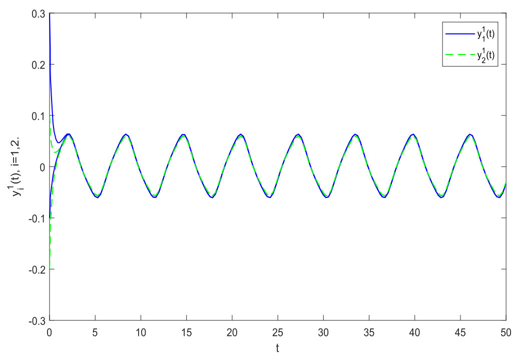

**FIGURE 7.** Curves of $y_i^1(t) = (y_1^1(t), y_2^1(t))^T$ of system (4.2) with the initial values $(y_1^1(0), y_2^1(0))^T = (-0.1, 0.1)^T, (0.3, -0.2)^T$.

$$\begin{aligned}
g_j &= \frac{1}{15}\big(\sin x_j^0 e_0 + \sin x_j^1 e_1 + \sin x_j^2 e_2 + \sin x_j^{12} e_{12}\big),\\
h_j &= \frac{1}{20}\big(\sin z_j^0 e_0 + \sin z_j^1 e_1 + \sin z_j^2 e_2 + \sin z_j^{12} e_{12}\big).
\end{aligned}$$

*Let* $\lambda = 0.3$, *and by calculating, we have*

$$\begin{aligned}
c_i^- &= 2.1,\ \ c_i^+ = 2.8,\ \ r_i^+ = \frac{2}{5},\ \ a_{ij}^+ = 0.3,\\
b_{ij}^+ &= 0.3,\ \ \mu_{ij}^+ = 0.3,\ \ \sigma_i^- = 0.9,\ \ \sigma_i^+ = 1.5,\\
L_f &= \frac{1}{10},\ \ L_g = \frac{1}{15},\ \ L_h = \frac{1}{20},\ \ \omega = 2\pi,\\
&\quad 1 - r_i^+ e^{\lambda\tau} \approx 0.5837 > 0,\\
\delta &:= \frac{1}{c_i^-}\Big[c_i^- r_i^+ + c_i^+ r_i^+ + \sum_{j=1}^{n} a_{ij}^+ L_f + \sum_{j=1}^{n} b_{ij}^+ L_g\Big]
\end{aligned}$$

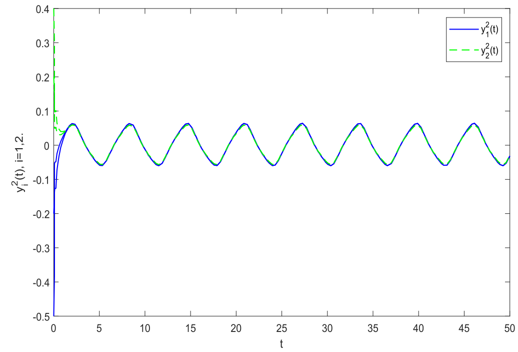


**FIGURE 8.** Curves of $y_i^2(t) = (y_1^2(t), y_2^2(t))^T$ of system (4.2) with the initial values $(y_1^2(0), y_2^2(0))^T = (-0.2, 0.2)^T, (-0.5, 0.4)^T$.

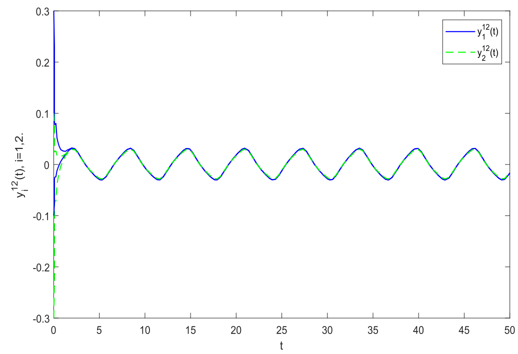


**FIGURE 9.** Curves of $y_i^{12}(t) = (y_1^{12}(t), y_2^{12}(t))^T$ of system (4.2) with the initial values $(y_1^{12}(0), y_2^{12}(0))^T = (0.3, -0.3)^T, (-0.1, 0.1)^T$.

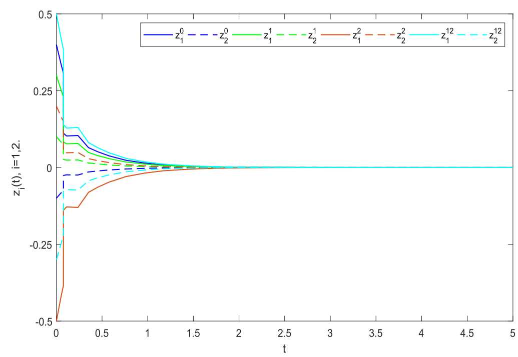


**FIGURE 10.** Synchronization errors $z_i(t) = y_i(t) - x_i(t)$.

$$\approx 0.9810 < 1,$$

*and*

$$0 < \frac{1}{c_i^- + \sigma_i^- - \lambda}\Big[(c_i^+ + \sigma_i^+)r_i^+ e^{\lambda\tau} + \sum_{j=1}^{n} a_{ij}^+ L_f + \sum_{j=1}^{n} b_{ij}^+ L_g e^{\lambda\gamma} + \sum_{j=1}^{n} \mu_{ij}^+ L_h e^{\lambda\alpha}\Big]\frac{1}{\nu} \approx 0.7690 < 1$$

*It is not difficult to verify that all conditions ($H_1$)-($H_6$) are satisfied. Therefore, by Theorem 3.1 and Theorem 3.2, we have that system (4.1) has a unique $\pi$-anti-periodic solution, and the system (4.1) and (4.2) are globally exponentially synchronized.*

## V. CONCLUSION

This paper deals with a class of delayed Clifford-valued neutral-type recurrent neural networks with $D$ operator. In order to overcome the complexity of the calculation, we obtain several sufficient condition for the existence of anti-periodic solutions for Clifford-valued neutral-type recurrent neural networks with $D$ operator by using non-decomposition method and the Banach fixed point theorem. By using the proof by contradiction and inequality techniques, we obtain the global exponential synchronization of anti-periodic solutions for Clifford-valued neutral-type recurrent neural networks with $D$ operator, one example is given. Our method can be extended to discuss the existence and synchronization (or stability) of anti-periodic (or almost periodic) solutions for other types Clifford-valued neural networks.

## DATA AVAILABILITY

No data were used to support this study.

## CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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