IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received December 15, 2021, accepted January 12, 2022, date of publication January 18, 2022, date of current version January 27, 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3143899

Locally Repairable Codes Based on Permutation Cubes and Latin Squares

EHSAN YAVARI[1], MORTEZA ESMAEILI[1,2], AND JOSEP RIFÀ[3], (Life Senior Member, IEEE)
[1]Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran
[2]Department of Electrical and Computer Engineering, University of Victoria, Victoria, BC V8P 5C2, Canada
[3]Department of Information and Communications Engineering, Universitat Autònoma de Barcelona, 08193 Barcelona, Spain

Corresponding author: Morteza Esmaeili (mesmaeili@iut.ac.ir)

The work of Josep Rifà was supported in part by Spanish Grant PID2019-104664GB-I00 (AEI/10.13039/501100011033).

ABSTRACT The importance of Locally Repairable Codes (LRCs) lies on their applications in distributed storage systems (DSSs). There are two approaches to repair multiple failed nodes: the parallel approach, in which a set of failed nodes are simultaneously repaired; and the sequential method, wherein the failed nodes are repaired successively by making use of known nodes, including those already repaired. LRCs in the joint sequential-parallel mode were investigated with the aim of reducing the repair time in the sequential mode, and in this study, we continue the investigation by providing LRCs with higher repair tolerance. We propose a construction of generator matrices for binary LRCs based on back-circulant Latin squares and t-dimensional permutation cubes. The codes based on back-circulant Latin squares have locality $r = 5$ and availability $t \geq 4$ in the parallel mode and, in the case where $t + 1$ is neither even nor a multiple of 3, they possess a short block length compared to their counterparts [Tamo and Barg (2014) and Wang and Zhang (2015)]. We present LRCs based on t-dimensional permutation m-cubes with block length m^t, locality $r = 2m - 3$, and availability $t \geq 3$ in parallel mode. These codes can repair any set of failed nodes of size up to $2^t - 1$ in the sequential mode in at most $t - 1$ steps. It is shown that these codes are *overall local* with repair tolerance $2t$ in the parallel mode. Finally, we introduce the LRC-AL class of codes, a new class of LRCs which satisfies the property that for any pair of nodes i, j, there is a repair set for the failed node i that contains the live node j.

INDEX TERMS Distributed storage, Latin square, locally repairable codes, multiple erasures, parallel repair, permutation cubes, repair time, sequential repair.

I. INTRODUCTION

Distributed storage systems (DSSs) are used to store large amounts of data and improve reliability by distributing data over multiple storage nodes. *Locally repairable codes* (LRCs) are important because of their applications in DSSs. Let $\mathcal{C}$ denote an $[n, k, d]$ linear code with block length n, dimension k, and minimum distance d. The i-th coordinate of $\mathcal{C}$, where $1 \leq i \leq n$, is said to have locality r if the value at this coordinate can be repaired by accessing at most r ($r \ll k$) other coordinates and performing a linear computation with them. Furthermore, if the i-th coordinate, $1 \leq i \leq n$, can be repaired by accessing r_i other coordinates (referred to as a *repair set*), then, $\mathcal{C}$ is called an LRC with locality $r = \max\{r_i | 1 \leq i \leq n\}$ and average locality $\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$. Codes with *locality* r were proposed independently in [1]–[3]. The average locality was investigated in [4] and [5] and a lower bound on $\bar{r}$ was given, which is useful for single erasure (node failure).

Example 1: The binary code $\mathcal{C}$ with the following generator matrix G

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

is an $[8, 4, 3]$ linear code with locality $r = 2$.

Each coordinate of $\mathcal{C}$ can be repaired by $r = 2$ other coordinates. For instance, the last coordinate of a codeword $\mathbf{c} \in \mathcal{C}$ can be recovered by adding the second and fourth coordinates.

The associate editor coordinating the review of this manuscript and approving it for publication was Xueqin Jiang.

In a DSS, multiple failed nodes may occur simultaneously, and the DSS must be able to repair the failed nodes as much as possible. In general, in the case of multiple failed nodes, the repair efficiency depends on the applied repair method, which is repairing either simultaneously or one by one. These methods are referred to as parallel and sequential approaches, respectively, in [6]. The parallel approach has been widely studied in [7]–[25].

Codes with locality r and availability t are a subclass of LRCs in the parallel mode (PM). A code $\mathcal{C}$ is called an LRC with locality r and availability t if for every coordinate, there exist t pairwise disjoint repair sets of size at most r, which enables $\mathcal{C}$ to simultaneously repair any t failures [9], [10], [18]–[21]. Another subclass of LRCs in the PM, studied in [26] and [8], is known as the *overall local code*. For any positive integer n, we define $[n]$ as the set $\{1, 2, \cdots, n\}$. A given $[n, k]$-code $\mathcal{C}$ is called an overall local with repair tolerance t, if for any $E \subset [n]$ of size up to t and each $i \in E$, the i-th coordinate of $\mathcal{C}$ has a repair set included in $[n] \setminus E$.

For the sequential approach, there are several papers that distinguish functional repair and exact repair [6], [26]–[31]. A class of codes in the sequential mode (SM) was given in [27], they are (n, k, r, u)-ELRC (exact locally repairable code) where u is the erasure tolerance (repair tolerance). The construction of codes by the direct product method was considered in [28]. Some structures of LRCs with locality r and erasure tolerance u in the SM, constructed based on graphs of the girth at least $(u + 1)$, are given in [30]. In addition, a construction of LRCs with locality r is given in [29] by using $u - 3$ orthogonal Latin squares of order r. These codes have length $n = r^2 + r(u - 1) + 1$ and erasure tolerance u in the SM. It is worth mentioning that a family of binary non-MDS (Maximum Distance Separable) erasure codes has been constructed in [32] based on incident matrices coming from Latin squares by a method quite different from our approach in this paper.

From a practical perspective, the number of steps needed for repairing, referred to as the *repair time*, is important in the SM. If a DSS uses an LRC in the SM with erasure tolerance u then in general the repair time for repairing u failed nodes is u. Hence, reducing the repair time in the SM is important. In [33], LRCs were investigated under the *joint sequential-parallel* mode (SPM) with the aim of reducing the repair time in the SM. If a DSS uses an LRC (with parameters (n, k, r, t, u)) in the joint SPM with erasure tolerance u (in the SM) and t (in the PM), then the repair time for u failed nodes is at most $u - t + 1$ ($u \geq t$).

Let $\mathcal{C}$ be a code with locality r and availability t. This property is equivalent to saying that for each coordinate i of $\mathcal{C}$, there exist t distinct codewords in $\mathcal{C}^{\perp}$, each of which has a support of size at most $r + 1$ with the property that any two of these t supports have only the i-th coordinate in common.

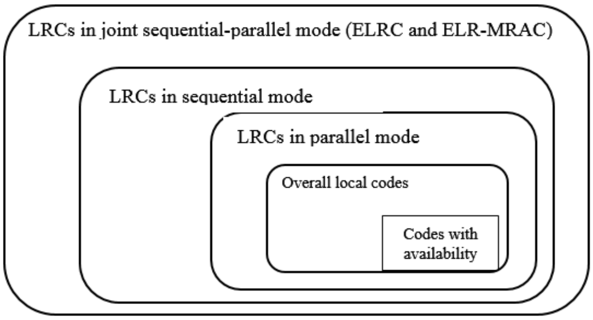
This realization led us to construct an LRC by focusing on its dual code $\mathcal{C}^{\perp}$ and, hence, considering the structure of the parity-check matrices of the LRC [33]. In [33], we realized that parity-check matrices possessing the aforementioned property can be constructed by using orthogonal Latin squares and variational permutation cubes (a generalization of orthogonal Latin squares), because the existence of a set of t mutually orthogonal Latin squares guarantees the existence of the required t distinct codewords. We have to mention that, in this approach, there is a restriction on the applied Latin squares and an arbitrary set of Latin squares will not necessarily produce a desired parity-check matrix. In addition, in combinatorial designs, there are some limitations on constructing orthogonal Latin squares\variational cubes, and there are not enough mutually orthogonal Latin squares\variational cubes of arbitrary order. Due to this limitation of this approach, we are trying to find a better method for constructing LRCs for the joint SPM.

In this research line, we observe that the binary matrices corresponding to Latin squares (not necessarily orthogonal) and permutation cubes can be considered as *generator matrices* of the overall local codes, because each element in a Latin square\permutation cube appears only once in each row and column. This leads to the present research paper, which focuses on the structure of generator matrices of LRCs, which are based on the structure of Latin squares and permutation cubes.

Therefore, from a theoretical point of view, the novelty of this work with respect to [33] lies in using generator matrices instead of parity-check matrices, based on the reasons described above. From an output perspective, the novelty of this study, compared with [33], is that owing to the elimination of the limiting orthogonality constraint, we have been able to construct codes that have a higher repair tolerance. More precisely, the codes constructed by using t'-dimensional permutation cubes have repair tolerance $t = 2t'$ (in the PM) and $u = 2^{t'} - 1$ in the SM, whereas the codes constructed in [33], using t'-dimensional variational permutation cubes, have erasure tolerance $u = 3$ (in the SM) and $t = 2$ (in the PM). Further, for the codes presented in this paper, finding a repair set for a failed node is simpler than that in the codes presented in [33]; this is due to the fact that for the codes given in this paper finding a repair set is done by using just the cell labels of a cube, while in the structure given in [33], to find a repair set we need both the cell labels and the values of the cells of all permutation cubes involved in the structure.

As previously mentioned, the general novelty of the joint sequential-parallel approach is that the repair time of the SM can be reduced by using the PM along with the SM. Specifically, the code constructed by making use of t-dimensional permutation cubes can repair $2^t - 1$ failed nodes in at most $t - 1$ steps, whereas the repair time spent in other constructions in the SM is not low; for instance, the repair time of some $u = 2^t - 1$ failed nodes is 2^{t-1} for the code constructed in [30]. Further, choosing a repair set (for a failed node) that includes a specified desired node, as discussed in this paper, might be useful in practice.

In comparison with other previous constructions [21], [27], [29], [30], the new constructions of LRCs given in this paper



Properties of LRCs based on permutation cubes and Latin squares			
Binary structure	Short block length	High repair tolerance	Low repair time

FIGURE 1. Relationship between different subclasses of LRCs, and Properties of LRCs constructed by using permutation cubes and Latin squares.

achieve higher repair tolerance and reduced repair time. All the codes constructed here are binary, and hence, are easily implemented. In this study, back-circulant Latin squares are used to construct LRCs with availability t, which possess a shorter block length compared to the previous constructions (such as codes presented in [21] and direct product code in [18]) with availability t. In addition, it is very simple to construct a back-circulant Latin square, so the corresponding code can be easily constructed. Furthermore, it is easy to find a repair set for a failed node in this construction. Codes constructed based on permutation cubes can repair failed nodes in the PM and have a high repair tolerance in the SM, unlike the existing ones that can either repair failed nodes in the sequential or PM. For example, the codes presented in [30] constructed based on graphs of girth $\geq (u+1)$ can repair u failed nodes only in the SM and have a large block length, versus codes constructed based on permutation cubes (see Table 4). In addition, the codes based on permutation cubes have high flexibility in choosing a repair set for a failed node; that is, there are different options in choosing a repair set for a failed node. Indeed, by considering different rows of a cube, different repair sets can be obtained. Furthermore, we show that these new codes can be introduced as (r, t, x)-LRCs (a new generalization of LRCs investigated in [34]).

Figure 1 illustrates the chain relationship that holds between five subclasses of LRCs. The figure also provides properties of LRCs based on permutation cubes and Latin squares.

Some bounds on the parameters of LRCs are given in [14]–[16], [18], [26], [30]. An upper bound on the rate (lower bound on the block length) of an LRC with locality r and erasure tolerance u in the SM was given in [30] as

$$\frac{k}{n} \leq \frac{r^{s+1}}{r^{s+1} + 2\sum_{i=1}^{s} r^i + (u-2s)}, \tag{1}$$

where $s = \lceil (u-1)/2 \rceil$. In Theorem 9, we obtain a lower bound on the block length under the joint SPM by using the same technical approach as that in [30] to derive (1). For a sufficiently large n, say, $n > 50$, our bound is almost equal to this bound in (1), and the difference is negligible. A lower bound on the block length of LRCs in the SM is given in [26]. This bound for a code that can repair $u = 3$ failed nodes in the SM is given by

$$n \geq k + \left\lceil \frac{2k + \lceil \frac{k}{r} \rceil}{r} \right\rceil. \tag{2}$$

Some examples of LRCs constructed by Latin squares are presented in Table 3, whose block lengths are closer to this bound up to a few units.

Finally, it is worth to briefly give a comparison between LRCs and MDS codes. An $[n, k, d]$ MDS code $\mathcal{C}$ has maximum transmission rate among all linear codes of length n and minimum distance d. From this point of view, although the codes constructed in this paper are non-MDS, but, in using a given $[n, k, d]$ MDS code $\mathcal{C}$ for a DSS, the code needs to transfer $r = k$ symbols to repair a single failed node, and this leads to high disc I/O. In order to overcome this I/O disc problem, the parameter r needs to be considerably smaller than k, and LRCs are constructed to fulfill this condition.

The paper is organized as follows: in Section II, we provide some necessary basic concepts about LRCs, as well as the definition of availability for LRCs in the PM, and (E, r)-repairable set for LRCs in the sequential approach. In addition, some basic concepts about Latin squares, back-circulant Latin squares, and m-dimensional permutation cubes are presented. In Section III, first, a bound on the block length under the joint sequential-parallel approach is derived, and second, we assign a binary matrix to a given permutation cube, and provide an important relationship between permutation cubes and their assigned binary matrices. In Section IV, we present some constructions of LRCs summarized in Theorems 13, 14 and 17 and their corollaries; in Theorem 13, we present LRCs with locality $r = 5$, by making use of back-circulant Latin squares; in Theorem 14, we apply Latin squares to construct LRCs with repair tolerance $t = 3$, and then generalize these codes for repair tolerance $t \geq 3$ in Theorem 17, where we make use of t-dimensional permutation cubes to construct LRCs that can repair t failed nodes in the PM and $2^t - 1$ failed nodes in the SM. Specific codes are given in Corollaries 15 and 18 that achieve the bounds given in Theorems 14 and 17, respectively. In Theorem 22, we show that these codes can repair $2t$ failed nodes in the PM. We also prove that the codes constructed in Theorem 17 belong to the class of (r, t, x)-LRC introduced in [34]. In Section V, we introduce the LRC-AL class of codes, a new class of LRC for which for any pair of nodes i, j there is a repair set for a failed node i containing the live node j (a node that has not failed), and present a binary construction for this class of codes. This new class of codes might be useful for engineering design because it may be necessary to have a certain node in a repair set. In Section VI we compare our constructions with the previous well-known constructions in the literature. Finally, in Section VII, we give the most important conclusions.

II. PRELIMINARY

Some background definitions and notations for LRCs, Latin squares, and m-dimensional permutation cubes are provided in this section.

A. LOCALLY REPAIRABLE CODES IN THE PARALLEL AND SEQUENTIAL MODE

Recall that the set $\{1, 2, \cdots, n\}$ is denoted by $[n]$. Let $\mathcal{C}$ be a q-ary linear $[n, k, d]$-code of length n, dimension k, and minimum distance d over the field $\mathbb{F}_q$, where q is a prime power (if the minimum distance is not relevant, we simply refer to $\mathcal{C}$ as an $[n, k]$-code). Let $G = (g_1, \ldots, g_n)$ and $H = (h_1, \ldots, h_n)$ be a generator and a parity-check matrix for $\mathcal{C}$, respectively, where g_i and h_i denote the i-th column of G and H, respectively. Then, $GH^T = 0$ and H is a generator matrix for the dual code $\mathcal{C}^{\perp}$. For any $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{F}_q^n$, the support of $\mathbf{c}$ is defined by $supp(\mathbf{c}) = \{i \in [n] : c_i \neq 0\}$ and $\mathrm{wt}(\mathbf{c}) = |supp(\mathbf{c})|$ is the Hamming weight of $\mathbf{c}$. The minimum distance of $\mathcal{C}$, denoted by $d(\mathcal{C})$, is the minimum weight of the nonzero codewords in $\mathcal{C}$.

Codes with locality r and availability t, in which each coordinate has t pairwise disjoint repair sets with locality r have been introduced in [19] and [10] as a class of LRCs.

Definition 1: Let $\mathcal{C}$ be a code of length n with generator matrix $G = (g_1, \ldots, g_n)$ and $i \in [n]$. A set $R(i) \subseteq [n] \setminus \{i\}$ of the maximum size r is called an $(r, \mathcal{C})$-repair set *for the coordinate i if $g_i = \sum_{l \in R(i)} a_{i_l} g_l$, where $a_{i_l} \in F_q$. This code is called a locally repairable code (LRC) with locality r and availability t if each coordinate of $\mathcal{C}$ has t pairwise disjoint $(r, \mathcal{C})$-repair sets.*

Example 2: Let $\mathcal{C}$ be the code in Example 1. Each coordinate of $\mathcal{C}$ has $t = 2$ pairwise disjoint $(3, \mathcal{C})$-repair sets; hence, it is an LRC with $r = 3$ and $t = 2$. For instance, for $i = 5$ we have $R_1(5) = \{1, 2\}$, $R_2(5) = \{6, 7, 8\}$, and the $(r, \mathcal{C})$-repair sets for $i = 8$ are $R_1(8) = \{2, 4\}$, $R_2(8) = \{5, 6, 7\}$.

Definition 2: Let $E = \{i_1, i_2 \cdots, i_u\}$ be an ordered set of indexes belonging to $[n]$ and $E' = [n] \setminus E$. A linear $[n, k]$-code $\mathcal{C}$ is said to be (E, r)-repairable if there exists a collection of sets $\{R(i_1), R(i_2), \cdots, R(i_u)\}$ such that, for each $j \in [u]$, we have $R(i_j) \subset E' \cup \{i_1, \cdots, i_{j-1}\}$ and $|R(i_j)| \leq r$, where $R(i_j)$ is an $(r, \mathcal{C})$-repair set for i_j. In other words, a coordinate i_j can be computed once its preceding coordinates $i_1, \cdots$, and i_{j-1} are computed.

By this definition, if an $[n, k]$-code $\mathcal{C}$ is (E, r)-repairable, then we can repair a set E of the failed nodes in the SM.

Example 3: Let $\mathcal{C}$ be the $[12, 4, 6]$ linear code with generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The code $\mathcal{C}$ is $(E, 2)$-repairable for $E = \{4, 5, 6, 7, 8\} \subset [12]$,. Indeed, taking the elements in E in the following order $5, 4, 6, 7, 8$, we have the collection $\{R(5) = \{1, 2\}, R(4) = \{12, 5\}, R(6) = \{2, 3\}, R(7) = \{3, 4\}, R(8) = \{4, 1\}\}$. Note that, in this example, as $5 \in R(4)$, we need to compute node 5 before computing node 4. In addition, it is easy to check that $\mathcal{C}$ is an LRC with locality $r = 2$ and availability $t = 4$.

Now, we combine Definitions 1 and 2 and introduce a class of LRCs with the property of repairing failed nodes in both the parallel and the SMs.

Definition 3: An $[n, k]$-code $\mathcal{C}$ with locality r and availability t is called (n, k, r, t, u)-exact locally repairable code *(ELRC), if it is (E, r)-repairable for all $E \subseteq [n]$ of size up to u $(t \leq u)$.*

Example 4: Consider $\mathcal{C}$ given in Example 3. It is easy to verify that any 5 failed nodes of $\mathcal{C}$ can be repaired in the SM. For example, if nodes $4, 5, 6, 7, 8$ fail, then they can be repaired in the SM, as explained in Example 3. Thus $\mathcal{C}$ is a $(12, 4, 2, 4, 5)$-ELRC with minimum distance $d = 6$.

The next definition introduces the concept of *overall local codes in the joint SPM*, denoted by ELR-MRAC (exact locally repairable with multiple repair alternatives code).

Definition 4 [33]: An $[n, k]$ linear code $\mathcal{C}$ is called an (n, k, r, t, u)-exact locally repairable with multiple repair alternatives code (ELR-MRAC) *if it is (E, r)-repairable for each $E \subseteq [n]$ of size up to u, and for any $S \subset [n]$ of size t and any $i \in S$, the i-th coordinate has a repair set with locality r contained in $[n] \setminus S$.*

We should note that ELRCs are a subclass of ELR-MRACs. Example 5 illustrates the difference between ELRC and ELR-MRAC.

Example 5: Let $r = 3$ and $\mathcal{C}$ be the $[8, 4, 4]$ linear code with generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Each failed node can be repaired using $r = 3$ live nodes, but there are no 2 distinct $(r = 3, \mathcal{C})$-repair sets for the first node; thus, the availability of this code is 1. Furthermore, if S is a set of $t = 3$ failed nodes, then each node in S has a repair set contained in $[8] \setminus S$. For instance, for $S = \{1, 2, 5\}$, we have repair sets $R(1) = \{3, 6, 8\}$, $R(2) = \{3, 6, 7\}$ and $R(5) = \{3, 4, 6\}$ contained in $[8] \setminus S$, but these repair sets are not distinct. Furthermore, it is obvious that each set of three failed nodes can be repaired in the SM. Therefore, $\mathcal{C}$ is an $(8, 4, 3, 1, 3)$-ELRC and $(8, 4, 3, 3, 3)$-ELR-MRAC.

Based on Definitions 3 and 4, if an (n, k, r, t, u)-ELRC or (n, k, r, t, u)-ELR-MRAC is used in a DSS with n nodes, then the DSS is able not only to repair any t-set of failed nodes simultaneously, but also to repair any u-set of failed nodes in the SM $(u \geq t)$.

B. LATIN SQUARES AND PERMUTATION CUBES

Definition 5: An $n \times n$ array L of n objects, say $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$, is called a Latin square if each object occurs once in each row and column of L. Each position in L is called a cell, and hence L has n^2 cells.

Definition 6: For each n, the Latin square $\mathcal{L} = (\ell_{(i,j)})$, where $\ell_{(i,j)} = i + j \pmod n$ for $0 \leq i, j \leq n-1$, is called back-circulant Latin square of order n.

Definition 7: An m-dimensional permutation cube *of order n is an m-dimensional $n \times n \times \cdots \times n$ array, where each column (i.e., each sequence of objects parallel to an edge of the cube) is a permutation of the elements of $\mathbb{Z}_n$. Accordingly, a 2-dimensional permutation cube is a Latin square of order n.*

Definition 8: Let $\mathcal{L}$ be an m-dimensional permutation cube of order n. Then, an $(m-1)$-dimensional plane *of $\mathcal{L}$ is an $(m-1)$-dimensional subcube of $\mathcal{L}$. In particular, a 1-dimensional plane is called a* line, *and for convenience, a 2-dimensional plane is called a plane. Two planes that do not intersect are said to be parallel. It is obvious that any cube of dimension m has n parallel $(m-1)$-dimensional planes. Two* lines *are said to be* perpendicular *if they have exactly one common cell. In general, two $(m-1)$-dimensional planes are said to be* perpendicular *if they have exactly one common $(m-2)$-dimensional plane.*

For brevity, henceforth we use 'cube' instead of 'permutation cube'.

III. BLOCK LENGTH AND GENERATOR MATRIX OF AN LRC

This section is divided into two parts. First, a bound on the block length in the joint SPM is obtained by using the same approach as in [30]. In the second subsection, we assign a binary matrix to a given cube and provide a result on the relationship between the permutation cubes and their assigned binary matrices.

A. A LOWER BOUND ON THE BLOCK LENGTH UNDER THE JOINT SEQUENTIAL-PARALLEL APPROACH

We obtain a lower bound on the block length in the joint SPM by using the technical proof in [30, Appendix A]. This bound, derived for ELR-MRACs, also holds for ELRCs because ELR-MRAC is a generalization of ELRC.

Theorem 9: Let $\mathcal{C}$ denote an (n, k, r, t, u)-ELR-MRAC over $\mathbb{F}_2$ and let $r \geq 3, t \geq 4$. Then

$$n \geq \frac{k}{A} + \frac{(t-3)(1-A)}{3A}, \tag{3}$$

wherein

$$A = \frac{r^{s+1}}{r^{s+1} + 2\sum_{i=1}^{s} r^i + (u - 2s)}.$$

In other words, the rate of $\mathcal{C}$ is upper bounded by

$$\frac{k}{n} \leq A - \frac{(1-A)(t-3)}{3n}. \tag{4}$$

Proof: Considering the proof presented in [30], we provide a proof for odd values of u; the process for even values of u is similar. Let $\mathcal{B}$ be the linear span of the set $\{\mathbf{c}' \in \mathcal{C}^{\perp} : \mathrm{wt}(\mathbf{c}') \leq r+1\}$. Let m be the dimension of $\mathcal{B}$ and assume that $\mathbf{c}'_1, \mathbf{c}'_2, \cdots, \mathbf{c}'_m$ is a basis of $\mathcal{B}$ such that $\mathrm{wt}(\mathbf{c}'_i) \leq r+1$, for each $1 \leq i \leq m$. Let H be the $m \times n$ matrix whose i-th row is $\mathbf{c}'_i$. Assuming that there is a column in H of weight at least t, inequality (55) in [30] can be rewritten as follows:

$$m(r+1) \geq a_0 + 2(\sum_{i=1}^{s+1} a_i) + 3(n - \sum_{i=0}^{s+1} a_i - 1) + t \quad \text{with} \quad s = \frac{(u-1)}{2}$$

or equivalently,

$$m(r+1) \geq 3n - 2a_0 - \sum_{i=1}^{s+1} a_i + (t-3).$$

Therefore, inequality (66) of [30] is changed to the following:

$$m\frac{3(r+1)}{2}\left(1 + \frac{1}{r^s + \frac{(r^s-1)(r+1)}{(r-1)}}\right) \geq 3n + (t-3). \tag{5}$$

Using the same algebraic manipulations that has been applied in [30] we derive inequality (3) from (5).

Hence, we only need to show that H contains a column of weight at least t. Suppose that $\mathbf{c}'_1 \in \mathcal{C}^{\perp}$ is a vector of weight at most $r+1$ such that $1 \in supp(\mathbf{c}'_1)$. Let $E_2 = \{1, i_2\}$ where $1 \neq i_2 \in supp(\mathbf{c}'_1)$. As $t \geq 4$ and $|E_2| = 2 \leq t$, there is $\mathbf{c}'_2 \in \mathcal{C}^{\perp}$ of weight at most $r+1$ such that $E_2 \cap supp(\mathbf{c}'_2) = \{1\}$. Now, set $E_3 = \{1, i_2, i_3\}$ where $1 \neq i_3 \in supp(\mathbf{c}'_2)$. As $|E_3| \leq t$, there is $\mathbf{c}'_3 \in \mathcal{C}^{\perp}$ of weight at most $r+1$ such that $E_3 \cap supp(\mathbf{c}'_3) = \{1\}$. Repeating this process again, we obtain $E_t = \{1, i_2, \cdots, i_t\}$ and t vectors $\mathbf{c}'_1, \mathbf{c}'_2, \cdots, \mathbf{c}'_t$ of weight at most $r+1$ such that $1 \neq i_j \in supp(\mathbf{c}'_{j-1})$ and $E_j \cap supp(\mathbf{c}'_j) = \{1\}$ for $2 \leq j \leq t$.

Now, we show that the set $\{\mathbf{c}'_i : 1 \leq i \leq t\}$ is linearly independent. Suppose that $\sum_{i=1}^{t} \alpha_i \mathbf{c}'_i = 0$ and $\alpha_i \in \mathbb{F}_2$. We have $\alpha_1 = 0$ because $i_2 \in supp(\mathbf{c}'_1)$ and $i_2 \notin supp(\mathbf{c}'_j)$ for $2 \leq j \leq t$. In addition, $\alpha_2 = 0$ because $\alpha_1 = 0$, $i_3 \in supp(\mathbf{c}'_2)$, and $i_3 \notin supp(\mathbf{c}'_j)$ for $3 \leq j \leq t$. By repeatedly using the same argument, we conclude that $\alpha_i = 0$ for $1 \leq i \leq t$, and thus $\{\mathbf{c}'_i : 1 \leq i \leq t\}$ is a part of a basis of $\mathcal{B}$. Hence, the first column of H has a weight of at least t, and the proof is complete. □

B. THE RELATIONSHIP BETWEEN GENERATOR MATRIX AND CUBE

Given a Latin square $\mathcal{L}_{n \times n} = (\ell_{ij})$ over $\{0, 1, 2, \cdots, n-1\}$, assume that the cells of $\mathcal{L}$ are labeled from 1 to n^2, as demonstrated in Table 1; hence the label of ℓ_{ij} is $in+j+1$ for $0 \leq i, j \leq n-1$. An $n \times n^2$ binary matrix $A_{\mathcal{L}} = (a_{st})$ is assigned to $\mathcal{L}$, where $a_{st} = 1$ if and only if the value of the t-th cell of L is s, for $0 \leq s \leq n-1$ and $1 \leq t \leq n^2$.

TABLE 1. Labeling the cells of a Latin square.

1	2	$\cdots$	$n-1$	n
$n+1$	$n+2$	$\cdots$	$2n-1$	$2n$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$(n-1)n+1$	$(n-1)n+2$	$\cdots$	n^2-1	n^2

Example 6: For the Latin square $\mathcal{L}$ given below, the corresponding matrix $A_{\mathcal{L}}$ is as follows. For instance, for $s=1$, the second row of $A_{\mathcal{L}}$, the values of the 4th, 6th, 9th, and 15th cells of the Latin square are 1. Hence, in the second row of $A_{\mathcal{L}}$, only the 4th, 6th, 9th, and 15th entries are 1 and the rest are zero.

$$\mathcal{L}=\begin{array}{|cccc|}\hline 2 & 3 & 0 & 1\\ 0 & 1 & 2 & 3\\ 1 & 2 & 3 & 0\\ 3 & 0 & 1 & 2\\ \hline\end{array},$$

$$A_{\mathcal{L}}=\begin{pmatrix} 0\,0\,1\,0 & 1\,0\,0\,0 & 0\,0\,0\,1 & 0\,1\,0\,0\\ 0\,0\,0\,1 & 0\,1\,0\,0 & 1\,0\,0\,0 & 0\,0\,1\,0\\ 1\,0\,0\,0 & 0\,0\,1\,0 & 0\,1\,0\,0 & 0\,0\,0\,1\\ 0\,1\,0\,0 & 0\,0\,0\,1 & 0\,0\,1\,0 & 1\,0\,0\,0 \end{pmatrix}$$

□

Clearly, that there is exactly one 1 in each column of the matrix $A_{\mathcal{L}}$ and exactly n in each row of $A_{\mathcal{L}}$.

Similarly, the cells of an m-dimensional cube of size n are labeled from 1 to n^m (as illustrated by Figure 2 in cell labeling of the given 3-dimensional cube of size 5). Note that the cell $(i_1, i_2, \cdots, i_m)$ has label $\sum_{j=1}^{m} i_j n^{j-1}+1$, where $0 \le i_j \le n-1$.

Given an m-dimensional cube $\mathcal{L}$ over $\{0, 1, \cdots, n-1\}$, a binary $n \times n^m$ matrix $A_{\mathcal{L}}=(a_{st})$ is assigned to $\mathcal{L}$, in which $a_{st}=1$ if and only if the t-th cell (the cell with label t) of $\mathcal{L}$ contains the symbol s for $0 \le s \le n-1$ and $1 \le t \le n^m$.

Proposition 10: Let $\{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_w\}$ be a set of w permutation cubes of dimension m (or Latin squares in the case of dimension 2) of size $n \ge 3$. Assume that $A_{\mathcal{L}_i}$ is the matrix corresponding to $\mathcal{L}_i$ as mentioned above. Let

$$G=\begin{pmatrix} A_{\mathcal{L}_1}\\ A_{\mathcal{L}_2}\\ \vdots\\ A_{\mathcal{L}_w} \end{pmatrix}$$

and $E=\{i_1, i_2, \cdots, i_r\} \subset [n^m]$. Let $\mathcal{L}_j(E)$ be the set of values of the cells in $\mathcal{L}_j$ whose labels are in E. If the number of appearances of any element of the set $\mathcal{L}_j(E)$ is even, for $1 \le j \le w$, then the set $\{g_{i_1}, g_{i_2}, \cdots, g_{i_r}\}$ of columns of G is linearly dependent (over $\mathbb{F}_2$).

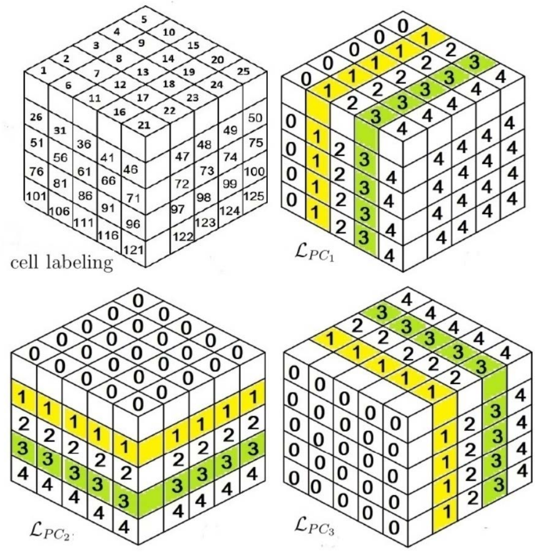


FIGURE 2. Cell labeling and 3-dimensional particular cubes of order 5, denoted by $\mathcal{L}_{PC_1}$, $\mathcal{L}_{PC_2}$, $\mathcal{L}_{PC_3}$.

Proof: Consider the submatrix $G'=(g_{i_1}\ g_{i_2}\ \cdots\ g_{i_r})$ of G whose columns are those indexed by E.

In each row of G', there is an even number of 1, because the number of appearances of each element of set $\mathcal{L}_j(E)$ is even, for $1 \le j \le w$. Thus, $\sum_{s=1}^{r} g_{i_s}=0$. □

Therefore, for each m-dimensional cube (or Latin square) $\mathcal{L}$ of order n, $\mathcal{L}(E)$ is the set of values of the cells of $\mathcal{L}$ whose labels are in $E \subset [n^m]$.

Example 7: Let $\mathcal{L}_1$, $\mathcal{L}_2$ be the following two Latin squares of size 3. The matrix G defined in Theorem 10 is as follows.

$$\mathcal{L}_1=\begin{array}{|ccc|}\hline 0 & 1 & 2\\ 1 & 2 & 0\\ 2 & 0 & 1\\ \hline\end{array},\quad \mathcal{L}_2=\begin{array}{|ccc|}\hline 0 & 2 & 1\\ 2 & 1 & 0\\ 1 & 0 & 2\\ \hline\end{array}$$

$$G=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\ \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $E=\{1, 3, 5, 8\}$. Then $\mathcal{L}_1(E)=\{0, 2, 2, 0\}$ and $\mathcal{L}_2(E)=\{0, 1, 1, 0\}$. The number of appearances of any element in each set $\mathcal{L}_j(E)$ is 2; hence, the columns $\{g_1, g_3, g_5, g_8\}$ of G are linearly dependent.

Corollary 11: Consider the matrix introduced in Theorem 10 as a generator matrix for $\mathcal{C}$, an LRC of length n^m. If $E=\{i_1, i_2, \cdots, i_{r+1}\} \subset [n^m]$ is a subset of the cell labels, such that the number of appearances of any element in $\mathcal{L}_j(E)$

is even, for $1 \leq j \leq w$, then any coordinate $i_{i_0} \in E$ of $\mathcal{C}$ can be repaired by using r other coordinates $\{i_1, i_2, \cdots, i_{r+1}\} \backslash \{i_{i_0}\}$.

In other words, suppose that the value of the i-th cell of $\mathcal{L}_j$ is x_j, for $1 \leq j \leq w$. Now, if there exists a subset $R(i) = \{i_1, i_2, \cdots, i_r\} \subset [n^m]$ of cell labels, such that $i \notin R(i)$ and the number of appearances of $x_j \in \mathcal{L}_j(R(i))$ is odd and the number of appearances of any other element in $\mathcal{L}_j(R(i))$, different from x_j, is even, for $1 \leq j \leq w$, then the coordinate i of $\mathcal{C}$ can be repaired using the coordinates $\{i_1, i_2, \cdots, i_r\}$.

Remark 1: One of the most important features of the structure presented in Corollary 11 is that it easily provides many ways for selecting a repair set for a failed node, which can be achieved by only using the structure of cubes (or Latin squares) regardless of the code structure. □

The Kronecker product of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{r\times s}$, is defined as $A \otimes B = (a_{ij}B)_{mr\times ns}$. Consider matrices $J_n = I_n \otimes 1^n$ and $R_n = 1^n \otimes I_n$, where I_n is the $n \times n$ identity matrix and 1^n denotes the all-one vector of length n:

$$J_n = I_n \otimes 1^n = \begin{pmatrix} \overbrace{11\cdots1}^{n} & 00\cdots0 & \cdots & 00\cdots0 \\ 00\cdots0 & \overbrace{11\cdots1}^{n} & \cdots & 00\cdots0 \\ \vdots & \vdots & \ddots & \vdots \\ 00\cdots0 & 00\cdots0 & 00\cdots0 & \overbrace{11\cdots1}^{n} \end{pmatrix}$$

$$R_n = 1^n \otimes I_n = \begin{pmatrix} I_n & I_n & \cdots & I_n \end{pmatrix}. \tag{6}$$

J_n and R_n correspond, respectively, to the following squares $\mathcal{L}_{n\times n}$ and $\mathcal{L}'_{n\times n}$ which are not Latin squares:

$$\mathcal{L}_{n\times n} = \begin{array}{|cccc|} \hline 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & \cdots & n-1 \\ \hline \end{array},$$

$$\mathcal{L}'_{n\times n} = \begin{array}{|ccccc|} \hline 0 & 1 & \cdots & n-2 & n-1 \\ 0 & 1 & \cdots & n-2 & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & n-2 & n-1 \\ \hline \end{array}.$$

The definition of matrices R_n and J_n given by (6) are extended to m-dimensional cubes.

Definition 12 (Particular Cubes and Their Corresponding Matrices): As an extension of squares $\mathcal{L}_{n\times n}$ and $\mathcal{L}'_{n\times n}$, given above, define $\mathcal{L}_{PC_i}$ as an m-dimensional cube of order n for $1 \leq i \leq m$, such that when partitioned into n parallel $m-1$-dimensional planes, all cells on each plane have the same value.

It is obvious that these cubes are not permutation cubes, but they can still be used to construct LRCs.

Example 8: Let $m = 3$ and $n = 5$. In this case, we have three particular cubes $\mathcal{L}_{PC_i}$, as shown in Figure 2. The corresponding matrices are as follows:

$$\begin{aligned} A_{\mathcal{L}_{PC_1}} &= 1^5 \otimes J_5 = 1^5 \otimes I_5 \otimes 1^5, \\ A_{\mathcal{L}_{PC_2}} &= J_5 \otimes 1^5 = I_5 \otimes 1^5 \otimes 1^5, \\ A_{\mathcal{L}_{PC_3}} &= 1^5 \otimes R_5 = 1^5 \otimes 1^5 \otimes I_5. \end{aligned}$$

Remark 2: In Corollary 11, if $A_{\mathcal{L}_{PC_i}}$ (or R_n, J_n for dimension 2) is added to the generator matrix G, then the code-rate of $\mathcal{C}$ can increase. In this case, if we want to select a repair set of size at most r to repair a coordinate $i \in [n^m]$ (with the terms stated in Corollary 11), we need a set $R(i) = \{i_1, i_2, \cdots, i_r\} \subset [n^m]$ of cell labels, such that in addition to the conditions stated in Corollary 11, there must be an even number of cell labels for each $(m-1)$-dimensional plane of the cube (or, of each row and each column in a Latin square) in $R(i) \cup \{i\}$.

IV. CODE CONSTRUCTIONS

In this section, by using Theorem 10 and Corollary 11, we provide four theorems describing some new constructions of LRCs.

In Theorem 13, we present LRCs with locality $r = 5$ by using back-circulant Latin squares. The lengths of these codes are much shorter than those of their counterparts given in [18] and [21]. In Theorem 14, we apply Latin squares to construct LRCs with repair tolerance $t = 3$, and then generalize these codes for repair tolerance $t \geq 3$ in Theorem 17, which is one of the most important results of this section. Theorem 17 gives us a family of LRCs with locality $r = 2n_0 - 3$ and availability $t \geq 3$ in the PM and repair tolerance $u = 2^t - 1$ in the SM based on t-dimensional cubes of size n_0. Then, in Theorem 20, we show that these codes are able to repair any set of failed nodes of size up to $u = 2^t - 1$ in at most $t - 1$ steps.

Each coordinate of the code $\mathcal{C}$ generated in Theorem 17 has $(n_0 - 1)(t - 1)t$ repair sets, any two of which intersect at most $x = n_0 - 2$ coordinates (proved in Corollary 21). At the end of this section, in Theorem 22, we prove that the code $\mathcal{C}$ generated in Theorem 17 has a good repair tolerance in the PM.

Theorem 13: Let $r = 5$ and $t \geq 4$, such that $n_0 = t+1$ and $2 \nmid n_0$, $3 \nmid n_0$. Let $\mathcal{L}$ be a back-circulant Latin square of order n_0. Then, the code $\mathcal{C}$ with the following generator matrix G is an $(n_0^2, k = 3n_0 - 2, r = 5, t = n_0 - 1, u = n_0 - 1)$-ELRC with minimum distance $d = n_0 = \sqrt{n}$, where R_{n_0}, J_{n_0} are defined in (6).

$$G = \begin{pmatrix} R_{n_0} \\ J_{n_0} \\ A_{\mathcal{L}} \end{pmatrix}$$

Proof: Suppose that the f-th coordinate of $\mathcal{C}$ fails. The f-th coordinate corresponds to the (i, j)-th cell (the cell in row i and column j) of the Latin square, where $f = in_0 + j + 1$ (see Figure 1). We want to find t pairwise disjoint repair sets of order 5. We use the ordered pair (x, y) instead of the corresponding label $z = xn_0 + y + 1$.

Let $0 \leq k \leq t = n_0 - 1$ and assume that $k \neq j$. The k-th $(5, \mathcal{C})$-repair set for this cell is $R_k(f)$ and note that all computations are made modulo n_0:

$$R_k(f) = \{(i, k), (i - 2k + 2j, 2k - j), (i - k + j, 2k - j), (i - k + j, j), (i - 2k + 2j, k)\}.$$

To prove that $R_k(f)$ is a repair set for the f-th coordinate, it is sufficient to show that $\mathfrak{B} = \mathcal{L}\big(R_k(f)\big) \bigcup \{\mathcal{L}(i, j)\}$ is a set satisfying the conditions of Corollary 11. According to Definition 6 of the back-circulant Latin squares, we have

$$\begin{aligned} \mathcal{L}(i, j) &= i + j, \\ \mathcal{L}(i, k) &= i + k, \\ \mathcal{L}(i - 2k + 2j, 2k - j) &= i + j, \\ \mathcal{L}(i - k + j, 2k - j) &= i + k, \\ \mathcal{L}(i - k + j, j) &= i - k + 2j, \\ \mathcal{L}(i - 2k + 2j, k) &= i - k + 2j. \end{aligned}$$

So, $\mathcal{L}(i, j) = \mathcal{L}(i - 2k + 2j, 2k - j) = i + j$; $\mathcal{L}(i, k) = \mathcal{L}(i - k + j, 2k - j) = i + k$; $\mathcal{L}(i - k + j, j) = \mathcal{L}(i - 2k + 2j, k) = i - k + 2j$. Moreover, $i + j \neq i + k$, $i + j \neq i - k + 2j$, and $i + k \neq i - k + 2j$; otherwise, in each case, we obtain $k = j$, which is false. Thus, the number of appearances of any element in the set $\mathfrak{B}$ is even. In addition, because R_{n_0} and J_{n_0} are submatrices of G, according to Remark 2, it is necessary that $R_k(f) \bigcup \{f\}$ have an even number of cell labels from each row and each column of $\mathcal{L}$. We analyse it. The cell labels in $R_k(f) \bigcup \{f\}$ are in columns $j, 2k - j, k$ and in rows $i, i - 2k + 2j, i - k + j$ in such a way that there are exactly two elements in each column as well as two elements in each row.

Finally, we prove that these t repair sets are pairwise disjoint, that is, $R_k(f) \bigcap R_{k'}(f) = \emptyset$ for any two different numbers $0 \leq k, k' \leq t$ and distinct from j. Comparing the elements in $R_k(f)$ and $R_{k'}(f)$, it is obvious that there are no common elements in almost all cases. The cases that are not so obvious are as follows:

- If $(i, k) = (i - 2k' + 2j, 2k' - j)$ then $2k' = 2j$ and since $2 \nmid n_0$ then $k' = j$, which is false. Hence, $(i, k) \neq (i - 2k' + 2j, 2k' - j)$.
- If $(i - 2k + 2j, 2k - j) = (i - k' + j, 2k' - j)$ then $2k = 2k'$, which is not possible since $2 \nmid n_0$. Hence, $(i - 2k + 2j, 2k - j) \neq (i - k' + j, 2k' - j)$.
- If $(i-k+j, 2k-j) = (i-2k'+2j, k')$ then $-k = -2k'+j$ and $2k-j = k'$, so $3k = 3k'$ and since $3 \nmid n_0$ then $k = k'$, which is false. Hence $(i-k+j, 2k-j) \neq (i-2k'+2j, k')$

Therefore, any $t = n_0 - 1$ of failed nodes can be repaired in the PM; hence, any $u = t = n_0 - 1$ of failed nodes can be repaired in the SM.

Now we show that the dimension of $\mathcal{C}$ is $3n_0 - 2$. The sum of the rows in R_{n_0} is $1^{n_0} = (1, 1, \cdots, 1)$, which coincides with the sum of the rows of J_{n_0} and the sum of the rows of $A_{\mathcal{L}}$. Hence, $rank(G) \leq 3n_0 - 2$.

On the other hand, let $S = \{1, 2, \cdots, 2n_0, 2n_0 + 1, 3n_0 + 1, 4n_0 + 1, \cdots, (n_0 - 1)n_0 + 1\} \subset [n_0^2]$. The set S consists of the cell labels of the first two rows and the first column of the Latin square. The values of these cells are as shown below.

0	1	$\cdots$	$n_0 - 2$	$n_0 - 1$
1	2	$\cdots$	$n_0 - 1$	0
2				
$\vdots$				
$n_0 - 1$				

We claim that the set $\{g_s \mid s \in S\}$ of columns in G is linearly independent over $\mathbb{F}_2$; otherwise, we have $\sum_{s \in S} a_s g_s = 0$ for some a_is in $\mathbb{F}_2$, where at least one of the a_is is nonzero. Set $E = \{i \in S \mid a_i = 1\}$. Then

$$\sum_{i \in E} a_i g_i = \sum_{i \in E} g_i = 0,$$

and hence, each row of the following matrix G' contains an even number of 1:

$$G' = \left(g_{i_1}\ g_{i_2}\ \cdots\ g_{i_{|E|}}\right), \quad i_j \in E.$$

Then, according to the matrix J_{n_0}, the set E must contain an even number of cell labels in each row of the Latin square; and hence, E does not contain any cell labels outside the first two rows of the Latin square (because $E \subset S$ and S contain one cell label from the r-th row with $r > 2$).

Based on the structure of matrix R_{n_0}, the set E must contain an even number of cell labels from each column of the Latin square. Based on the properties of matrix $A_{\mathcal{L}}$ the number of appearances of any element of the set $\mathcal{L}(E)$ must be even. Therefore, according to the structure of the back-circulant Latin square, E must contain all cell labels of the first and second rows of the Latin square; and hence, $E = \{1, 2, \cdots, 2n_0\}$. However, n_0 is an odd number, and hence E contains an odd number of cell labels from the first row of the Latin square, which is a contradiction. Therefore, the set of columns $\{g_s \mid s \in S\}$ in G is linearly independent, and $rank(G) = 3n_0 - 2$.

Finally, because this code can repair $n_0 - 1$ failed nodes, $d \geq n_0$. In addition, note that the weight of the first row in G is n_0, so $d = n_0$. □

The minimum distance of the constructed code $\mathcal{C}$ is $d = n_0 = t + 1$, and the length of $\mathcal{C}$ is much shorter than that of its counterparts constructed in the PM (see comparisons provided in Table 2).

In this code, assume that i, j are two arbitrary coordinates and that we have a repair set $\{i_x | 1 \leq x \leq 5\}$ for the i-th coordinate. If we arrive at the j-th label from the i-th label by a movement pattern on the back-circulant Latin square, and then, using the same movement pattern, we arrive at the j_x-th label from the i_x-th label (for $1 \leq x \leq 5$), then $\{j_x | 1 \leq x \leq 5\}$ is a repair set for the j-th coordinate. For instance, see Figures 3 and 4 in Example 9.

Algorithm 1 describes how Theorem 13 is working.

Algorithm 1 Construction of ELRC Based on Back Circulant Latin Square (Theorem 13)

Input: n_0 is neither even nor a multiple of 3.

1: Construct the back-circulant Latin square $\mathcal{L} = (\ell_{(i,j)})$ of order n_0, where $\ell_{(i,j)} = i + j \pmod{n_0}$ for $0 \le i, j \le n_0 - 1$
2: Calculate the binary matrix $A_{\mathcal{L}}$ associate to the Latin square $\mathcal{L}$
3: Construct the generator matrix

$$G = \begin{pmatrix} R_{n_0} \\ J_{n_0} \\ A_{\mathcal{L}} \end{pmatrix}$$

Output: $(n_0^2, 3n_0 - 2, r = 5, t = n_0 - 1, u = n_0 - 1)$-ELRC with generator matrix G .

Example 9: Let $\mathcal{L}$ be a back-circulant Latin square of order $n_0 = 5$; then, the matrix G from Theorem 13 is the equation can be derived, as shown at the bottom of the page.

The code $\mathcal{C}$ with generator matrix G is a $(25, k = 13, r = 5, t = 4, u = 4)$-ELRC. Suppose the first coordinate of $\mathcal{C}$ fails; this coordinate corresponds to the label 1 of the Latin square $\mathcal{L}$ (see Figure 3 and Table 1). According to Theorem 13, there are four disjoin repair sets for this node (see Figure 3); for instance, $R_3(1) = \{4, 11, 12, 22, 24\}$.

Now, suppose that the second coordinate of $\mathcal{C}$ also fails. This coordinate corresponds to the label 2 of the Latin square $\mathcal{L}$. In the Latin square, we move from the first label to the second label by cyclic shifting one unit to the right. Then, we obtain a repair set for the second node by moving all labels of any repair set for the first node one unit to the right.

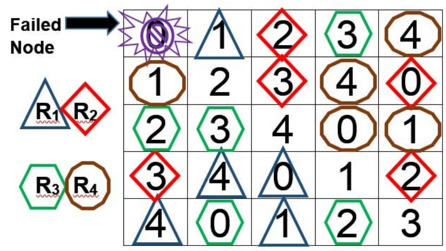


FIGURE 3. Repair sets R_1, R_2, R_3, and R_4 for the first node.

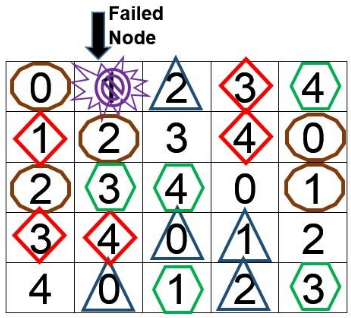


FIGURE 4. Repair sets R_1, R_2, R_3, and R_4 for the second node obtained from the repair sets of the first node.

$$G = \begin{pmatrix}
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
\\
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
\\
1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0
\end{pmatrix}.$$

Theorem 14: Let $\{\mathcal{L}_i \mid 1 \leq i \leq w\}$ be a set of w Latin squares of size n_0, and assume that $A_{\mathcal{L}_i}$ is the binary matrix corresponding to $\mathcal{L}_i$. The code $\mathcal{C}$ with the following generator matrix G is an $(n = (n_0)^2, k, r = 2n_0-3, t = 3, u = 3)$-ELR-MRAC for $n_0 \geq 4$, and an $(n = 9, k, r = 3, t = 2, u = 2)$-ELR-MRAC for $n_0 = 3$, where k depends on the structure of the Latin squares $\mathcal{L}_i$ and $k \leq (n_0 - 1)w + 1$. The minimum distance d of $\mathcal{C}$ satisfies $4 \leq d \leq n_0 = \sqrt{n}$ for $n_0 \geq 4$, and $d = 3$ if $n_0 = 3$.

$$G = \begin{pmatrix} A_{\mathcal{L}_1} \\ A_{\mathcal{L}_2} \\ \vdots \\ A_{\mathcal{L}_w} \end{pmatrix}$$

Proof: First suppose $n_0 = 4$. Assume that $E \subset [n]$ is a set of at most $t = 3$ failed nodes and f is an arbitrary coordinate of E. The f-th coordinate corresponds to the (a, b)-th cell (the cell in row a and column b) of a Latin square, where $f = an_0 + b + 1$. Note that the joint set of values of a row and a column of the Latin square $\mathcal{L}_x$, except the value of its intersection cell, has $2n_0-2$ elements for each $1 \leq x \leq w$, and the number of appearances of any element in this set is 2. Then, according to Corollary 11, for each coordinate $f \in E$, it is sufficient to find a row a and a column b, such that the joint set of cells of a, b, except its common cell, contains f and contains no other coordinates of $E \setminus \{f\}$.

- Suppose that the a-th row of a Latin square contains no coordinate of $E \setminus \{f\}$.
  In this case, as $n_0 \geq 4$, there are at least three columns other than b. Because $E \setminus \{f\}$ has at most two members, then there is a column b that contains no coordinates of $E \setminus \{f\}$. Hence, row a and column b have the required conditions.
- Suppose that the b-th column of a Latin square contains no coordinate of $E \setminus \{f\}$.
  The previous argument applies to this case.
- Suppose that the a-th row of a Latin square contains a coordinate f' of $E \setminus \{f\}$ and the b-th column contains a coordinate f'' of $E \setminus \{f\}$.
  In this case, assume that $f' = an_0 + d + 1$. Hence, row a and column d have the required conditions.

As $u \geq t$ then 3 is a lower bound for u.

Note that the dimension of $\mathcal{C}$ depends on the structure of the w Latin squares $\mathcal{L}_i$. However, we can easily find an upper bound for k: the rows in the matrix G are not independent if $w > 1$, because by adding the vectors in any $A_{\mathcal{L}_i}$, we always obtain the all-one vector, and hence, $k \leq w(n_0 - 1) + 1$. We see that there exists a row of weight n_0 in G, and hence, the minimum distance d is at most n_0. Because $u \geq 3$, we have $d \geq 4$; thus, $4 \leq d \leq \sqrt{n}$.

A similar argument shows that for $n_0 = 3$, the code $\mathcal{C}$ is an $(n = 9, k, r = 3, t = 2, u = 2)$-ELR-MRAC with minimum distance $d = 3$. □

In the following corollary, $w = n_0-1$ Latin squares of size $n_0 \geq 4$ are given, and the associated code from Theorem 14 achieves the dimension bound with equality, so $k = w(n_0 - 1) + 1$. Hence, the rate of this code is $\frac{n-2\sqrt{n}+2}{n}$.

Corollary 15: Let $n_0 \geq 4$. For each $1 \leq x \leq n_0 - 1$, the Latin square $\mathcal{L}_x = (\ell^x_{(i,j)})$ of size n_0 is defined as follows:
$\ell^x_{(0,0)} = n_0 - 1$; $\ell^x_{(i,0)} = i + x - 2 \pmod{n_0 - 1}$ for $1 \leq i \leq n_0 - 1$; $\ell^x_{(i,j)} = \ell^x_{(i,j-1)} + 1 \pmod{n_0}$ for $0 \leq i \leq n_0 - 1$ and $1 \leq j \leq n_0 - 1$.

From Theorem 14, the code associated with these $w = n_0 - 1$ Latin squares is an ELR-MRAC with dimension $k = (n_0 - 1)^2 + 1$ and minimum distance $d = 4$.

Proof: Assume that $A_{\mathcal{L}_i}$ is the binary matrix corresponding to $\mathcal{L}_i$, and R_{i_j} is the j-th row of $A_{\mathcal{L}_i}$. As $supp(R_{1_1} + R_{2_2}) = \{2, 3, n_0^2 - n_0 + 3, n_0^2 - n_0 + 2\}$ and $d \geq 4$, we have $d = 4$.

Now, we show that $k = (n_0 - 1)^2 + 1$ by proving that the elements in the set $S = \{R_{i_j} \mid 1 \leq i, j \leq n_0 - 1\} \cup \{R_{1_{n_0}}\}$ are linearly independent. Suppose $\sum_{i=1}^{n_0-1} \sum_{j=1}^{n_0-1} a_{i_j} R_{i_j} + a_{1_{n_0}} R_{1_{n_0}} = 0$, wherein $a_{i_j} \in \mathbb{F}_2$.

Because the value of the first cell of each Latin square is $n_0 - 1$, the first coordinate of each vector in S is zero, except in $R_{1_{n_0}}$, and hence,

$$a_{1_{n_0}} = 0. \tag{7}$$

We show that $a_{i_j} = 0$, for all $1 \leq i, j \leq n_0 - 1$. Without loss of generality, we assume that $a_{1_1} \neq 0$. According to the construction of the Latin squares, and looking at the content of $supp(R_{1_1})$, there are at least 3 indices i, j, k (i.e., three Latin squares labeled by i, j, and k) such that none of them is 1 and $a_{i_1} a_{j_j} a_{k_{(k+1)}} \neq 0$. Assume that $i = \alpha, j = \beta, k = \gamma$ and hence $a_{\alpha_1} a_{\beta_{\beta'}} a_{\gamma_{\gamma'}} \neq 0$, wherein $\beta' = \beta$ and $\gamma' = \gamma + 1$.

Again, according to the construction of the Latin squares and looking at the content of $supp(R_{\alpha_1} + R_{\beta_{\beta'}} + R_{\gamma_{\gamma'}})$, there are at least three indices i, j, and k such that $i \neq \beta$ and $j \neq \alpha \neq k$ and $a_{i_{\beta'}} a_{j_{j'}} a_{k_{k'}} \neq 0$ for some j' and k'.

By repeating this argument, we conclude that $a_{1_{n_0}}$ and all a_{i_j}, for $1 \leq i, j \leq n_0 - 1$, are nonzero. This contradicts (7). □

Example 10: Let $n_0 = 4$ and that $\{\mathcal{L}_i \mid 1 \leq i \leq 3\}$ be a set of $w = 3$ Latin squares of size 4, as follows:

$$\mathcal{L}_1 = \begin{array}{|cccc|} \hline 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ \hline \end{array},$$

$$\mathcal{L}_2 = \begin{array}{|cccc|} \hline 3 & 0 & 1 & 2 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ \hline \end{array},$$

$$\mathcal{L}_3 = \begin{array}{|cccc|} \hline 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ \hline \end{array}$$

$$G=\begin{pmatrix}
0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1\\
0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\
1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0\\
\\
0&1&0&0&0&0&0&1&0&0&1&0&1&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&1&0&1&0&0&1&0&0&0&0&0&1&0\\
1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1\\
\\
0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&1&0&1&0&0&1&0&0&0\\
0&0&0&1&1&0&0&0&0&0&1&0&0&1&0&0\\
1&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0
\end{pmatrix}.$$

Then, the code $\mathcal{C}$ with generator matrix G is a $(16, 10, r = 5, t = 3, u = 3)$-ELR-MRAC, with minimum distance $d = 4$. For instance, if the first, second, and fifth nodes fail, then these nodes can be repaired simultaneously with repair sets $\{6, 7, 8, 9, 13\}$, $\{6, 10, 13, 15, 16\}$, and $\{4, 6, 7, 12, 16\}$, respectively.

We now give an example in which the locality parameter is $r = 97$ and the code rate is 0.961. Let $n_0 = 50$ and suppose that $\{\mathcal{L}_i \mid 1 \le i \le 49\}$ is a set of $w = 49$ Latin squares of size 50 constructed from Corollary 15. The code associated to these Latin squares is an ELR-MRAC with code rate $\frac{k}{n} = \frac{2402}{2500} = 0.961$.

As a generalization, in the following theorem, by using cubes of dimension t, we can construct some LRCs with availability t. The code-rate can be increased by increasing the number of cubes.

Before proceeding to the theorem, we introduce a useful lemma. In this lemma, points (marked cells) represent the failed nodes.

Lemma 16: Let t and x be two positive integers with $t \ge 2$ and $x \le 2^t - 1$. If we have x points (marked cells) in a t-dimensional n-cube, then there exists a $(t-1)$-dimensional subcube containing at most $2^{t-1} - 1$ points.

Proof: If $x = 1$, then there is nothing to prove, so suppose that $x \ge 2$. Consider all $(t-1)$-dimensional subcubes perpendicular to one specific direction and assume that all x points (marked cells) are not in the same $(t-1)$-dimensional subcube, so it is obvious that the statement is fulfilled. Otherwise, we can repeat the same argument by taking all the different t directions. Hence, the statement is proved; otherwise, all x points are in the intersection of t perpendicular $(t-1)$-dimensional subcubes. This intersection is reduced to one point, and thus, it should be $x = 1$, which is impossible because we assume that $x \ge 2$. □

Theorem 17: Let $t \ge 3$ and $r = 2n_0 - 3$, for some integer $n_0 \ge 3$. Assume that $\{\mathcal{L}_i \mid 1 \le i \le w\}$ is a set of w t-dimensional permutation cubes of order n_0, and $A_{\mathcal{L}_i}$ is the matrix corresponding to $\mathcal{L}_i$. Then, code $\mathcal{C}$ with the generator matrix

$$G=\begin{pmatrix} A_{\mathcal{L}_1}\\ A_{\mathcal{L}_2}\\ \vdots\\ A_{\mathcal{L}_w}\end{pmatrix}$$

is an $((n_0)^t, k, r = 2n_0 - 3, t, u = 2^t - 1)$-ELRC, with minimum distance $2^t \le d \le n_0^{t-1}$. The dimension k depends on the structure of the permutation cubes $\mathcal{L}_i$ and $k \le w(n_0 - 1) + 1$.

Proof: Suppose that the f-th coordinate of $\mathcal{C}$ fails. The f-th coordinate corresponds to the $(i_1, i_2, \cdots, i_t)$-th cell of a permutation cube, where $f = \sum_{j=1}^{t} i_j n_0^{j-1} + 1$. We want to find t pairwise disjoint $(r, \mathcal{C})$-repair sets for this coordinate. We use the ordered t-tuples $(x_1, x_2, \cdots, x_t)$ instead of the corresponding label $z = \sum_{j=1}^{t} x_j n_0^{j-1} + 1$.

The first $(r, \mathcal{C})$-repair set is R_1 (all calculations are made modulo n_0):

$$R_1 = \big\{(i_1 + x, i_2, \cdots, i_t), (i_1 - 1, i_2 + y, i_3, \cdots, i_t) \\ \mid 1 \le x \le n_0 - 2, \quad 1 \le y \le n_0 - 1\big\}.$$

The size of the repair set R_1 is $|R_1| = 2n_0 - 3$ and the set $A_1 = R_1 \bigcup \big\{(i_1, i_2, \cdots, i_t)\big\}$ contains exactly two perpendicular lines of a cube, except their intersection (for instance, see Figure 5). Then, the number of appearances of any element in the collection $\mathcal{L}_x(A_1)$ is two, for each $1 \le x \le w$. Hence, according to Corollary 11, R_1 is an $(r, \mathcal{C})$-repair set for the f-th coordinate. Analogously, the other $(r, \mathcal{C})$-repair sets are as follows:

$$R_2 = \big\{(i_1, i_2 + x, \cdots, i_t), (i_1, i_2 - 1, i_3 + y, \cdots, i_t) \\ \mid 1 \le x \le n_0 - 2, \quad 1 \le y \le n_0 - 1\big\};$$
$$\vdots$$
$$R_t = \big\{(i_1, i_2, \cdots, i_t + x), (i_1 + y, i_2, i_3, \cdots, i_t - 1) \\ \mid 1 \le x \le n_0 - 2, \quad 1 \le y \le n_0 - 1\big\}.$$

It is obvious that $R_i \bigcap R_j = \emptyset$ for each $i \ne j$. Therefore, code $\mathcal{C}$ is an LRC with locality r and availability t in the PM.

For example, for $t = 3, n_0 = 5, r = 7$, the repair sets R_1, R_2, R_3 are shown in Figure 5. Each R_i is a subset of the labels of two perpendicular lines without their intersection and without the failed node.

In the SM, suppose that we have at most $2^t - 1$ failed nodes. From Lemma 16, we can find a $(t-1)$-dimensional subcube with at most $2^{t-1} - 1$ failed nodes; by repeatedly using Lemma 16, we end up with a 2-dimensional subcube (plane) with at most three failed nodes, denoted by $\mathcal{L}'$. If $n_0 \ge 4$, then these failed nodes can be repaired using Theorem 14.

Now, suppose that $n_0 = 3$. If there exists a line $\mathscr{X}$ (row or column) in $\mathcal{L}'$ such that $\mathscr{X} \cap E = \emptyset$, then there exists a line $\mathscr{Y}$ in $\mathcal{L}'$ perpendicular to $\mathscr{X}$ satisfying $\mathscr{Y} \cap E = \{f\}$;

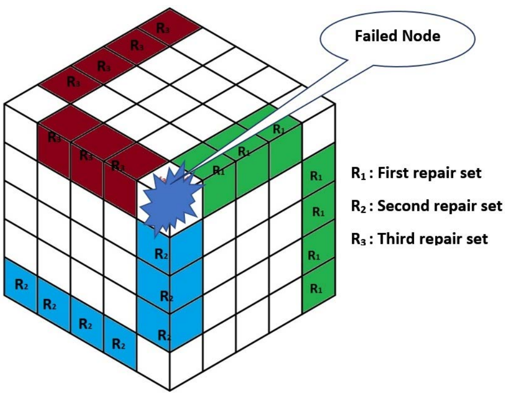


FIGURE 5. Repair sets R_1, R_2, and R_3 for a 3-dimensional permutation cube of order 5.

and hence, the failed node f can be repaired based on a proof similar to that given for Theorem 14. Otherwise, if each line in $\mathcal{L}'$ contains a failed node of E, then from among the $n_0 = 3$ parallel planes perpendicular to $\mathcal{L}'$, there is at least one plane that contains one or two failed nodes, and these nodes can be repaired by Theorem 14. Finally, using the same argument, we can repair the remaining failed nodes.

The dimension of the constructed code depends on the structure of the cubes. However, by the same argument used in the proof of Theorem 14, we have $k \le w(n_0 - 1) + 1$. For the minimum distance of $\mathcal{C}$, we see that the weight of one row in G is n_0^{t-1}, and thus $d \le n_0^{t-1}$. Finally, because $u = 2^t - 1$, then $d \ge 2^t$. Therefore, $2^t \le d \le n_0^{t-1}$. □

In the following construction, we introduce $w = (n_0 - 1)^2$ permutation cubes of size $n_0 \ge 4$ such that the code from Theorem 17 achieves the dimension bound with equality, and thus $k = w(n_0 - 1) + 1$. Hence, the code rate is $\frac{(n_0-1)^3+1}{n_0^3}$. Note that this construction can be generalized to the case of t-dimensional cubes with $3 < t$, and the obtained code has dimension $k = (n_0 - 1)^t + 1$ and minimum distance $d = 2^t$.

Corollary 18: Let $n_0 \ge 4$ and $\mathcal{L}_x = (\ell^x_{(i,j)})$, $1 \le x \le n_0 - 1$, be the Latin squares introduced in Corollary 15. For each pair $1 \le x, y \le n_0 - 1$, a permutation cube $\mathcal{L}_{x_y} = (\ell^{x_y}_{(i,j,k)})$ of size n_0 is defined as follows, wherein $0 \le i, j, k \le n_0 - 1$:

$$\begin{cases} \ell^{x_y}_{(i,j,0)} = \ell^{x}_{(i,j)}; \\ \ell^{x_y}_{(i,j,k)} = \ell^{x_y}_{(i+k+y-1,j,1)} \quad \text{for } 1 \le k \le n_0 - y; \\ \ell^{x_y}_{(i,j,k)} = \ell^{x_y}_{(i+k+y,j,1)} \quad \text{for } n_0 - y + 1 \le k \le n_0 - 1. \end{cases}$$

From Theorem 17, the code assigned to these $w = (n_0 - 1)^2$ permutation cubes is an ELRC with dimension $k = (n_0 - 1)^3 + 1$ and minimum distance $d = 8$.

Proof: The proof is similar to the one given in Corollary 15. □

Algorithm 2 describes how Theorem 17 is working.

The result in Theorem 17 can be improved for the case of $n_0 = 3$ when the availability is greater than that given in the general case. In Theorem 22, we show that the codes presented in Theorem 17 can repair $2t$ failed nodes in the PM for $n_0 \ge 4$, although the availability is not $2t$.

Algorithm 2 Construction of ELRC Based on t-Dimensional Permutation Cubes (Theorem 17)

Input: Given $n_0 \ge 4$ and $t \ge 3$.
1: Construct a set of w permutation cubes of dimension t and size n_0
2: Calculate the binary matrix $A_{\mathcal{L}}$ associate to these permutation cubes
3: Construct generator matrix G
Output: $((n_0)^t, k, r = 2n_0 - 3, t, u = 2^t - 1)$-ELRC.

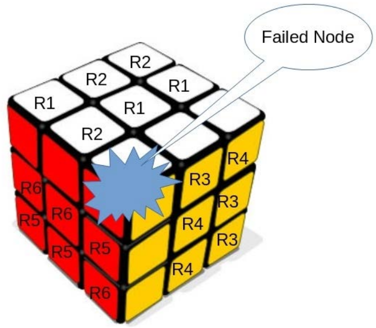


FIGURE 6. 3-dimensional permutation cube of order $m = 3$. There are $2t = 6$ disjoint repair sets R_1, R_2, R_3, R_4, R_5, and R_6 for the specified failed node.

Corollary 19: For $n_0 = 3$, the codes obtained in Theorem 17 are $(3^t, k, r = 3, 2t, u = 2^t - 1)$-ELRC. These codes have an availability of $2t$ instead of t.

Proof: With the same notation as in Theorem 17, there are t pairwise disjoint repair sets for the f-th coordinate of the code $\mathcal{C}$, and another t pairwise disjoint repair sets are as follows:

$$\begin{aligned} R_{t+1} &= \{(i_1 + 2, i_2, \cdots, i_t), (i_1 + 1, i_2 + 1, i_3, \cdots, i_t), \\ &\quad (i_1 + 1, i_2 + 2, i_3, \cdots, i_t)\}; \\ R_{t+2} &= \{(i_1, i_2+2, i_3, \cdots, i_t), (i_1, i_2+1, i_3 + 1, i_4, \cdots, i_t), \\ &\quad (i_1, i_2 + 1, i_3 + 2, i_4, \cdots, i_t)\}; \\ &\vdots \\ R_{2t} &= \{(i_1, i_2, \cdots, i_{t-1}, i_t+2), (i_1 + 1, i_2, \cdots, i_{t-1}, i_t+1), \\ &\quad (i_1 + 2, i_2, \cdots, i_{t-1}, i_t + 1)\}. \end{aligned}$$

It is obvious that all these $2t$ repair sets are pairwise disjoint.

For instance, if $t = 3$, then $R_1, R_2, R_3, R_4, R_5, R_6$ are $2t = 6$ pairwise disjoint $(r, \mathcal{C})$-repair sets (see Theorem 6). □

Example 11: Let $n_0 = t = 3$. We use four cubes to construct the generator matrix G the equation can be derived, as shown at the bottom of next page.

The code $\mathcal{C}$ with generator matrix G is an $(n = 27, k = 9, r = 3, 2t = 6, u = 7)$-ELRC with minimum distance $d = 8$ (calculated using magma [35]) and code rate $\frac{1}{3}$. The code with the same locality and availability constructed in [21] has block length $n = \binom{9}{6} = 84$, code rate $\frac{1}{3}$, and minimum distance $d = 7$. Thus, our code achieves a better minimum distance and shorter block length.

In Example 11, we constructed an LRC with the same rate as in [14], but with shorter block length and a high minimum distance. To do this, we used $w = 4$ cubes. It may be possible to increase the code rate by increasing w.

The repair time is an important issue in the SM. In [33], we showed that the repair time for $2^t - 1$ failed nodes in the direct product of t copies of $[r+1, r]$ single-parity-check codes is at most t rounds. In the next corollary, we show that the repair time for repairing $2^t - 1$ failed nodes using the codes obtained in Theorem 17 is at most $t - 1$. In addition, it is shown in Theorem 22 that these codes can repair $2t$ failed nodes simultaneously in one step.

Corollary 20: The codes obtained in Theorem 17 can repair $2^t - 1$ failed nodes in at most $t - 1$ steps.

Proof: We prove the corollary by induction on t. For $t = 3$, take all planes of a cube (they can intersect), such that each of them contains at most three failed nodes. If $n_0 \geq 4$, then the failed nodes in each of these planes can be repaired in one step, based on Theorem 14. Suppose that $x > 0$ failed nodes are not repaired at this step (for $x = 0$, the repair time is 1). All x failed nodes are in the same plane, so a proof similar to that given for Lemma 16, shows that $x = 1$. This failed node can then be repaired in the second step. Hence, the repair time is at most $t - 1 = 2$. In addition, if $n_0 = 3$, then the repair time is $t - 1 = 2$, as 6 failed nodes can be repaired simultaneously based on Corollary 19.

Now, suppose $t \geq 4$ and that the claim has been proven for $t - 1$. Take all $(t-1)$-dimensional subcubes of a cube (they can intersect), such that each of them contains at most $2^{t-1} - 1$ failed nodes. By applying the induction hypothesis, the failed nodes in each of these $(t-1)$-dimensional subcubes can be repaired in at most $t - 2$ steps. Suppose that $x > 0$ failed nodes are not repaired at these $t - 2$ steps (for $x = 0$, the repair time is $t - 2$). A proof similar to that given for Theorem 16 shows that $x = 1$. This failed node can then be repaired in step $t - 1$. Hence, the repair time is at most $t - 1$. $\square$

In [34], a new generalization of LRC ((r, t, x)-LRC) was introduced. A linear $[n, k]$-code $\mathcal{C}$ is said to be (r, t, x)-LRC, if for each coordinate $i \in [n]$ there are t $(r, \mathcal{C})$-repair sets with the property that each pair of them can intersect in at most x coordinates. In the next corollary, we want to show that the code $\mathcal{C}$ obtained in Theorem 17 is contained in the class of (r, t, x)-LRCs.

Corollary 21: The code $\mathcal{C}$ obtained in Theorem 17 is $(2n_0 - 3, (n_0 - 1)t, n_0 - 3)$-LRC. Furthermore, $\mathcal{C}$ is also a $(2n_0 - 3, (n_0 - 1)(t - 1)t, n_0 - 2)$-LRC.

Proof: Let f be an arbitrary coordinate (a failed node) of the code $\mathcal{C}$. The cell f is at the intersection of t lines of a t-dimensional cube, and let $\mathscr{X}$ be one of these lines. If there is a line $\mathscr{Y}$ perpendicular to $\mathscr{X}$ and $\mathscr{X} \cap \mathscr{Y} = \{f'\} \neq \{f\}$, then with the same argument used in the proof of Theorem 17, $R = (\mathscr{X} \cup \mathscr{Y}) \setminus \{f, f'\}$ is an $(r, \mathcal{C})$-repair set for the f-th coordinate.

Note that in the first part of the corollary, in a plane that contains $\mathscr{X}$, there are $n_0 - 1$ lines $\{\mathscr{Y}(a) \mid f \neq a \in \mathscr{X}\}$ satisfying $\mathscr{X} \cap \mathscr{Y}(a) = \{a\}$. Hence, we have $n_0 - 1$ $(r, \mathcal{C})$-repair sets $R_a = (\mathscr{X} \cup \mathscr{Y}(a)) \setminus \{f, a\}, a \in \mathscr{X} \setminus \{f\}$ for the f-th coordinate. For each pair $a, a' \in \mathscr{X} \setminus \{f\}$, we have $R_a \cap R_{a'} = \mathscr{X} \setminus \{f, a, a'\}$. As f is on t lines, there exist $t(n_0 - 1)$ $(r, \mathcal{C})$-repair sets for the f-th coordinate, such that the size of the intersection of any two repair sets is at most $x = n_0 - 3$ coordinates. Therefore, $\mathcal{C}$ is a $(2n_0 - 3, (n_0 - 1)t, n_0 - 3)$-LRC.

$$G = \begin{pmatrix}
1&0&0&0&0&1&0&1&0&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&1&0&0\\
0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&1&0&0&1&0&0&0&0&1&0&1&0\\
0&0&1&0&1&0&1&0&0&1&0&0&0&0&1&0&1&0&0&1&0&1&0&0&0&0&1\\
\\
1&0&0&0&1&0&0&0&1&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0\\
0&1&0&0&0&1&1&0&0&1&0&0&0&1&0&0&0&1&0&0&1&1&0&0&0&1&0\\
0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1&0&0&0&1\\
\\
1&0&0&0&1&0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&1&0\\
0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&1&0&1&0&0&0&1&0&0&0&1\\
0&0&1&1&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&1&1&0&0\\
\\
1&0&0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&1\\
0&1&0&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0\\
0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0
\end{pmatrix}.$$

Regarding the second part of the corollary, as $\mathscr{X}$ is on the $t-1$ planes of a cube, for each cell $a \in \mathscr{X}$ and $a \neq f$, there are $(t-1)$ lines $\mathscr{Y}_1(a), \mathscr{Y}_2(a), \cdots, \mathscr{Y}_{t-1}(a)$, such that $\mathscr{X} \cap \mathscr{Y}_i(a) = \{a\}, 1 \leq i \leq t-1$. Then, there exists $(n_0 - 1)(t-1)$ $(r, \mathcal{C})$-repair sets $R_{i_a} = (\mathscr{X} \cup \mathscr{Y}_i(a)) \setminus \{f, a\}$, which satisfy $R_{i_a} \cap R_{j_a} = \mathscr{X} \setminus \{f, a\}$ and $R_{i_a} \cap R_{j_{a'}} = \mathscr{X} \setminus \{f, a, a'\}$ for $1 \leq i, j \leq t-1$, with $a, a' \in \mathscr{X}$; thus, the number of intersections of two repair sets is at most $n_0 - 2$. Finally, as f is on t lines, there are $t(n_0-1)(t-1)$ $(r, \mathcal{C})$-repair sets for the f-th coordinate, such that each pair intersects at most $x = n_0 - 2$ coordinates. Therefore, $\mathcal{C}$ is a $(2n_0 - 3, (n_0 - 1)(t-1)t, n_0 - 2)$-LRC. $\square$

Example 12: The code $\mathcal{C}$ given in Example 11 is an $(r = 3, t = 12, x = 1)$-LRC with a code rate of $1/3$.

To conclude this section, we show that the code $\mathcal{C}$ in Theorem 17 is an ELR-MRAC with erasure tolerance $2t$ in the PM. This result shows that these codes in the joint SPM have high erasure tolerance and short block length compared to previously published works (e.g., [21], [28]).

Theorem 22: For $n_0 \geq 4$, the codes obtained in Theorem 17, are $((n_0)^t, k, r = 2n_0 - 3, 2t, u = 2^t - 1)$-ELR-MRAC.

Proof: Suppose that $E \subset [n]$ is a set of at most $2t$ failed nodes, and f is an arbitrary coordinate in E. The f-th coordinate corresponds to the f-th cell of a t-dimensional permutation cube. By the same argument used in the proof of Theorem 17, it is sufficient to show that there are two perpendicular lines $\mathscr{X}, \mathscr{Y}$ of a cube, such that the set of cells of $\mathscr{X}, \mathscr{Y}$, except its intersection cell, contains f and does not contain any coordinates of $E \setminus \{f\}$.

Note that for an arbitrary line $\mathscr{X}$ of a cube and a cell g on $\mathscr{X}$, there are $t-1$ lines with intersection cell g that are perpendicular to $\mathscr{X}$. In addition, there are t lines $\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_t$ of a cube, such that $\mathscr{X}_i$ contains the cell f and $\mathscr{X}_i, \mathscr{X}_j$ are perpendicular if $i \neq j$.

There are two cases:

- Suppose that there is a line $\mathscr{X}_i$ such that $\mathscr{X}_i \cap E = \{f\}$. As $n_0 \geq 4$, there are at least $3(t-1)$ lines (separate from the set $B = \{\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_t\}$) perpendicular to $\mathscr{X}_i$. Because $E \setminus \{f\}$ has at most $2t-1$ cells and $3t - 3 > 2t - 1$, there is a line $\mathscr{Y}$ perpendicular to $\mathscr{X} = \mathscr{X}_i$ such that $\mathscr{Y} \cap E = \emptyset$.
- Suppose that for each line $\mathscr{X}_i$ in $B = \{\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_t\}$, we have $|\mathscr{X}_i \cap E| \geq 2$.
  In this case, suppose there is a set of b lines $B' = \{\mathscr{X}_{i_1}, \cdots, \mathscr{X}_{i_b}\} \subseteq B$, such that $\mathscr{X}_{i_j} \cap E = \{f, a_j\}$ for each $\mathscr{X}_{i_j} \in B'$, and $|\mathscr{X}_i \cap E| \geq 3$ for each $\mathscr{X}_i \in B \setminus B'$. There is a set of b lines $B'' = \{\mathscr{Y}_1, \cdots, \mathscr{Y}_b\}$, such that $\mathscr{X}_{i_j} \cap \mathscr{Y}_j = \{a_j\}$ (perpendicular lines with intersection cell a_j) and $\mathscr{Y}_j \cap \mathscr{Y}_{j'} = \emptyset$ (no pairs of them are on the same plane), for any $j, j' \in [b]$ and $j \neq j'$.
  As $|\mathscr{X}_i \cap E| \geq 3$ for each $\mathscr{X}_i \in B \setminus B'$, at least $2(t-b)+1$ cells of E are on the lines in $B \setminus B'$. In addition, b cells of $E \setminus \{f\}$ are on the lines of B'; thus, as $|E| - (2(t-b)+1+b) = b-1$ and $|B''| = b$, there is a line $\mathscr{Y} = \mathscr{Y}'_{j_0} \in B''$ that is perpendicular to $\mathscr{X} = \mathscr{X}_{i_{j_0}} \in B'$. Hence, $\mathscr{X}$ and $\mathscr{Y}$ have the desired conditions.

The other parameters of this code are the same as those given in Theorem 17. $\square$

The direct product of t copies of the $[r+1, r]$ single parity-check code is an $((r+1)^t, r^t, r, t, u = 2^t - 1)$-ELRC, and the code based on Theorem 22 is an $((\frac{r+3}{2})^t, k, r, 2t, u = 2^t - 1)$-ELR-MRAC for any odd r. However, if we use the permutation cubes introduced in Corollary 18, or their extension to t-dimensional cubes with $t > 3$, we obtain an $((\frac{r+3}{2})^t, (\frac{r+1}{2})^t + 1, r, 2t, u = 2^t - 1)$-ELR-MRAC. Thus, for every given parameter t and odd r, the repair tolerance of our code in the PM is twice that of the aforementioned direct product code, and the block length is shorter than that.

V. THE LRC-AL, A NEW CLASS OF LRC

In this section, we introduce a new class of LRC, a *locally repairable code with an arbitrary live node* (denoted as LRC-AL). The codes in this class have the property that for a failed node i and each arbitrary node j, there is a repair set for node i containing live node j. We also provide a binary construction for these codes in Theorem 24.

Definition 23: Suppose $\mathcal{C}$ is an $[n, k]$-linear code, and r is an integer number. If for any two coordinates $i, j \in [n]$, there is at least λ $(r, \mathcal{C})$-repair sets $R_a(i)$ $(1 \leq a \leq \lambda)$ for the i-th coordinate such that $j \in R_a(i)$, then we say that $\mathcal{C}$ is an (n, r, λ)-locally repairable code with an arbitrary live node (LRC-AL).

Note that the λ repair sets in an (n, r, λ)-LRC-AL may have intersections between them.

By this definition, if a DSS with n nodes uses an (n, r, λ)-LRC-AL as the storage code, any desired live node can be used in the process of repairing a failed node.

Theorem 24: Let $\{\mathcal{L}_i \mid 1 \leq i \leq w\}$ be a set of w Latin squares of size $n_0 \geq 4$, and let $A_{\mathcal{L}_i}$ be the binary matrix corresponding to $\mathcal{L}_i$. Then, code $\mathcal{C}$ with the generator matrix

$$G = \begin{pmatrix} A_{\mathcal{L}_1} \\ A_{\mathcal{L}_2} \\ \vdots \\ A_{\mathcal{L}_w} \end{pmatrix}$$

is an $(n = (n_0)^2, r = 2n_0 - 3, \lambda = 2)$-LRC-AL.

Proof: Assume that the i-th coordinate of $\mathcal{C}$ fails, and that $j \neq i$ is an arbitrary node (coordinate) of code $\mathcal{C}$. The i-th coordinate corresponds to the (a, b)-th cell (the cell in row a and column b) of a Latin square, where $i = an_0 + b + 1$, and the j-th coordinate corresponds to the (c, d)-th cell, where $j = cn_0 + d + 1$. We use the ordered pair (x, y) instead of the corresponding label $z = xn_0 + y + 1$.

Two $(r, \mathcal{C})$-repair sets for the i-th coordinate are $R_1(i)$ and $R_2(i)$, each of which contains the j-th coordinate:

- ► If $a \neq c$, $b \neq d$, then let
  $R_1(i) = \{(a, y), (x, d) \mid 0 \leq x, y \leq n_0 - 1; y \notin \{b, d\}; x \neq a\}$,
  $R_2(i) = \{(x, b), (c, y) \mid 0 \leq x, y \leq n_0 - 1; x \notin \{a, c\}; y \neq b\}$.

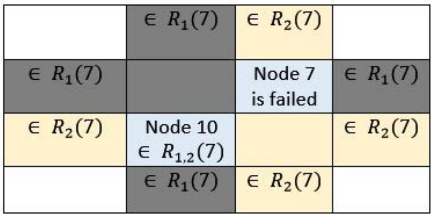


FIGURE 7. Two repair sets for the failed node 7 which contain the live node 10.

- If $a = c$, then let $0 \leq v_1, v_2 \leq n_0 - 1$, where $v_1, v_2 \notin \{b, d\}$ and
$R_1(i) = \{(a, y), (x, v_1) \mid 0 \leq x, y \leq n_0 - 1; y \notin \{b, v_1\}; x \neq a\}$,
$R_2(i) = \{(a, y), (x, v_2) \mid 0 \leq x, y \leq n_0 - 1; y \notin \{b, v_2\}; x \neq a\}$.
- If $b = d$, then let $0 \leq v_1, v_2 \leq n_0 - 1$, where $v_1, v_2 \notin \{a, c\}$ and
$R_1(i) = \{(x, b), (v_1, y) \mid 0 \leq x, y \leq n_0 - 1; x \notin \{a, v_1\}; y \neq b\}$,
$R_2(i) = \{(x, b), (v_2, y) \mid 0 \leq x, y \leq n_0 - 1; x \notin \{a, v_2\}; y \neq b\}$.

Note that all calculations are done modulo n_0. In all items, $\mathcal{L}_x(R_1(i)) \cup \mathcal{L}_x(\{i\})$ is a collection containing the values of a row and a column of the Latin square $\mathcal{L}_x$, except for the common value of them, for each $1 \leq x \leq w$. The number of appearances of any element of this collection is two; and hence, according to Corollary 11, $R_1(i)$ is an $(r, \mathcal{C})$-repair set for the i-th coordinate. With the same argument, $R_2(i)$ is an $(r, \mathcal{C})$-repair set for the i-th coordinate. The sizes of $R_1(i)$ and $R_2(i)$ are $2n_0 - 3$ and each repair set contains the j-th coordinate. □

Example 13: Consider the same parameters as in Example 10. Hence, $n_0 = 4$ and $\{\mathcal{L}_i \mid 1 \leq i \leq 4\}$ is a set of four Latin squares of size 4. The code $\mathcal{C}$ with generator matrix G presented in Example 10 is a $(16, r = 5, \lambda = 2)$-LRC-AL. For instance, assume that the 7-th node (coordinate) of code $\mathcal{C}$ fails, and consider node 10 as a live node; then, $R_1(7) = \{5, 8, 2, 10, 14\}$, $R_2(7) = \{3, 15, 9, 10, 12\}$ are two repair sets, containing live node 10, for node 7. These two repair sets are shown in Figure 7.

VI. COMPARISON WITH OTHER CONSTRUCTIONS

In the literature, there are LRCs constructed either in the PM or in the SM. In this paper, we presented LRCs that could repair in the joint SPM. Now, we provide some examples and compare them with other constructions.

Table 2 provides a comparison of LRCs of short block length constructed following Theorem 13 with their counterparts constructed in the PM.

Based on Theorem 14 and applying w Latin squares of order n_0, we constructed some $(n, k, r, t = 3, u = 3)$-ELR-MRACs, which are listed in Table 3. The block lengths of these codes are compared with the bounds given by (2) and the codes presented in [21]. The results confirm that our codes have a code rate nearly equal to the codes given in [21] (with the same erasure tolerance $t = 3$ in the PM), but our codes have shorter block lengths. There may also be other Latin squares (except for these w in our constructions) that can increase the code rate if they are added to the generator matrix.

TABLE 2. Codes constructed using Theorem 13 with short block length versus other constructions in the PM for $r = 5$.

Back circulant Latin square of size n_0		Code constructed by Theorem 13		Code presented in [21]		Direct product code in [18], [27]	
n_0	t	n	Code rate	$n = \binom{r+t}{t}$	Rate $= \frac{r}{r+t}$	$n = (r+1)^t$	Rate
5	4	25	0.52	126	0.55	1296	0.47
7	6	49	0.39	462	0.45	46656	.033
11	10	121	0.26	3003	0.33	$> 6 \times 10^7$	0.16
13	12	169	0.22	6188	0.29	$> 2 \times 10^9$	0.11

TABLE 3. Making use of w Latin squares of size n_0 in Theorem 14 to construct LRCs with short block length.

	w Latin square of size n_0		Code constructed by Theorem 14, with erasure tolerance $t = 3$				Construction in [21], with availability $t = 3$	
r	n_0	w	n	k	Code rate	Bound (2)	n	Rate $= \frac{r}{r+3}$
5	4	4	16	10	0.625	15	56	0.625
7	5	4	25	17	0.68	23	120	0.7
9	6	6	36	27	0.75	34	220	0.75
11	7	7	49	37	0.755	45	364	0.786
23	13	12	169	145	0.858	158	2600	0.885
47	25	24	625	577	0.923	601	19600	0.94
97	50	49	2500	2402	0.961	2452	161700	0.97

A method of constructing codes with locality r, erasure tolerance u in the SM, and block length $n = r^2 + r(u-1) + 1$, by using $u - 3$ orthogonal Latin squares of order r, has been addressed in [29]. However, this approach does not cover arbitrary values for the parameters r or u. For instance, for $r = 3$, there are at most two orthogonal Latin squares of order 3. In addition, the corresponding codes cannot repair two failed nodes $E = \{r^2 + 1, n\} \subset [n]$ simultaneously, and hence, the erasure tolerance of the codes in the PM is $t = 1$. Some optimal LRCs with locality r and erasure tolerance u (in the SM), constructed using graphs of girth at least $u + 1$, are given in [30]. Let $\mathcal{C}$ be the code introduced in [30] associated with a graph G_∞. The code $\mathcal{C}$ has erasure tolerance $t \leq 2$ in the PM, because the set $E = \{e_0, e_1, e_2\}$ consisting of three failed nodes cannot be repaired simultaneously, where $e_0 \in E_0, e_1 \in E_1, e_2 \in E_2$ are three edges of G_∞, and $e_0e_1e_2$ is a path of this graph. Some examples of these codes are presented in Table 4. The lower bound on the block length of the codes (in Table 4) is calculated using the Moor Bound (described in [30, Theorem 9]). Considering this, it is worth noting that our codes, in addition to the erasure tolerance $2^t - 1$ in the SM, can repair $2t$ failed nodes in the PM, according to Theorem 22. The direct product of t copies of the $[r+1, r]$ single-parity-check code, with block length $(r+1)^t$, can repair $2^t - 1$ failed nodes in the SM, and t failed nodes in the PM. A comparison of these codes with our codes is given in Table 4.

Furthermore, LRCs in the joint SPM were presented in [33], based on t-dimensional variational permutation cubes

TABLE 4. **A set of codes with parameters *r* and *u*, constructed by different methods, and a comparison between them from the erasure tolerance *t* in the PM and block length *n* perspectives.**

given		Codes based on Corollary 19 and Theorem 22, using n_0-cubes			Codes given in [29], using $u-3$ orthogonal Latin squares of order r		Codes given in [30] based on graphs of girth $\geqslant u+1$		Direct product codes in [18], [27]	
r	u	n_0	t	n	t	n	$t \leqslant$	$n \geqslant$	t	n
3	7	3	6	27	does not exist		2	160	3	64
5	7	4	6	64	1	56	2	936	3	216
7	15	5	8	625	does not exist		2	7686400	4	4096
13	15	8	8	4096	1	352	2	9.5×10^8	4	38416
13	7	8	6	512	1	248	2	3×10^4	3	2744

that are capable of repairing 2 failed nodes in the PM and 3 failed nodes in the SM. In this study, we utilized t-dimensional permutation cubes (not necessarily variational permutation cubes) to construct LRCs that can repair $2t$ failed nodes in the PM and 2^t-1 failed nodes in the SM. In addition, these codes can repair 2^t-1 failed nodes in at most $t-1$ steps.

VII. CONCLUSION

In this paper, we present an explicit construction of generator matrices of (n, k, r, t, u)-ELRCs and (n, k, r, t, u)-ELR-MRACs by using Latin squares, permutation cubes, and back-circulant Latin squares. The constructed codes based on t-dimensional permutation cubes are LRCs with availability $t \geq 3$, and can repair every set of failed nodes of size 2^t-1 in the SM in at most $t-1$ steps. Further, we showed that these codes are overall local with repair tolerance $2t$ in the PM. Finally, we have shown that these codes are also a type of (r, t, x)-LRC, in which the repair sets can intersect in at most x coordinates.

One of the most important features of the structures presented here is that they provide many options for selecting a repair set for a failed node. For instance, for the LRC-AL class of codes given in Section V, for each failed node i and for each live node j, we can choose a repair set for node i containing live node j.

For all constructed codes, without using either the generator matrix or the parity check matrix of our code, we can find a repair set for each failed node by using only the corresponding cubes (or Latin squares) of the code. Therefore, finding a repair set for a failed node is much simpler in our construction than in previous constructions.

Obtaining a theoretical bound on the minimum distance of the joint sequential-parallel LRCs may be considered as a future research direction.

REFERENCES

[1] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, "On the locality of codeword symbols," *IEEE Trans. Inf. Theory*, vol. 58, no. 11, pp. 6925–6934, Aug. 2012.

[2] F. Oggier and A. Datta, "Self-repairing homomorphic codes for distributed storage systems," in *Proc. IEEE INFOCOM*, Shanghai, China, Apr. 2011, pp. 1215–1223.

[3] D. S. Papailiopoulos, J. Luo, A. G. Dimakis, C. Huang, and J. Li, "Simple regenerating codes: Network coding for cloud storage," in *Proc. IEEE INFOCOM*, Orlando, FL, USA Mar. 2012, pp. 2801–2805.

[4] M. Shahabinejad, M. Khabbazian, and M. Ardakani, "On the average locality of locally repairable codes," *IEEE Trans. Commun.*, vol. 66, no. 7, pp. 2773–2783, Jul. 2018.

[5] M. Shahabinejad, M. Ardakani, and M. Khabbazian, "An erasure code with reduced average locality for distributed storage systems," in *Proc. Int. Conf. Comput., Netw. Commun. (ICNC)*, Jan. 2017, pp. 427–431.

[6] N. Prakash, V. Lalitha, and P. V. Kumar, "Codes with locality for two erasures," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Honolulu, HI, USA, Jun. 2014, pp. 1962–1966.

[7] N. Prakash, G. M. Kamath, V. Lalitha, and P. V. Kumar, "Optimal linear codes with a local-error-correction property," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Cambridge, MA, USA, Jul. 2012, pp. 2776–2780.

[8] L. Pamies-Juarez, H. D. L. Hollmann, and F. Oggier, "Locally repairable codes with multiple repair alternatives," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Istanbul, Turkey, Jul. 2013, pp. 892–896.

[9] Y. S. Su, "On the construction of local parities for (r, t) -availability in distributed storage," *IEEE Trans. Commun.*, vol. 65, no. 6, pp. 2332–2344, Jun. 2017.

[10] A. S. Rawat, D. S. Papailiopoulos, A. G. Dimakis, and S. Vishwanath, "Locality and availability in distributed storage," *IEEE Trans. Inf. Theory*, vol. 62, no. 8, pp. 4481–4493, Aug. 2016.

[11] A. S. Rawat, A. Mazumdar, and S. Vishwanath, "On cooperative local repair in distributed storage," in *Proc. 48th Annu. Conf. Inf. Sci. Syst. (CISS)*, Princeton, NJ, USA, Mar. 2014, pp. 1–5.

[12] P. Huang, E. Yaakobi, H. Uchikawa, and P. H. Siegel, "Linear locally repairable codes with availability," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2015, pp. 1871–1875.

[13] W. Song, S. Day, C. Yuen, and T. Li, "Optimal locally repairable linear codes," *IEEE J. Sel. Areas Commun.*, vol. 32, no. 5, pp. 6925–6934, May 2014.

[14] A. Agarwal, A. Barg, S. Hu, A. Mazumdar, and I. Tamo, "Combinatorial alphabet-dependent bounds for locally recoverable codes," *IEEE Trans. Inf. Theory*, vol. 64, no. 5, pp. 3481–3492, May 2018.

[15] I. Tamo, A. Barg, and A. Frolov, "Bounds on the parameters of locally recoverable codes," *IEEE Trans. Inf. Theory*, vol. 62, no. 6, pp. 3070–3083, Jun. 2016.

[16] A. Wang, Z. Zhang, and D. Lin, "Bounds for binary linear locally repairable codes via a sphere-packing approach," *IEEE Trans. Inf. Theory*, vol. 65, no. 7, pp. 4167–4179, Jul. 2019.

[17] A. Barg, I. Tamo, and S. Vlăduţ, "Locally recoverable codes on algebraic curves," *IEEE Trans. Inf. Theory*, vol. 63, no. 8, pp. 4928–4939, Aug. 2017.

[18] I. Tamo and A. Barg, "Bounds on locally recoverable codes with multiple recovering sets," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Honolulu, HI, USA, Jun. 2014, pp. 691–695.

[19] A. Wang and Z. Zhang, "Repair locality with multiple erasure tolerance," *IEEE Trans. Inf. Theory*, vol. 60, no. 11, pp. 6979–6987, Nov. 2014.

[20] Y.-S. Su, "On constructions of a class of binary locally repairable codes with multiple repair groups," *IEEE Access*, vol. 5, pp. 3524–3528, 2017.

[21] A. Wang, Z. Zhang, and M. Liu, "Achieving arbitrary locality and availability in binary codes," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2015, pp. 1866–1870.

[22] J. Hao and S.-T. Xia, "Constructions of optimal binary locally repairable codes with multiple repair groups," *IEEE Commun. Lett.*, vol. 20, no. 6, pp. 1060–1063, Jun. 2016.

[23] A. Wang and Z. Zhang, "Repair locality from a combinatorial perspective," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Honolulu, HI, USA, Jun. 2014, pp. 1972–1976.

[24] A. Wang, Z. Zhang, and D. Lin, "Two classes of (r, t)-locally repairable codes," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Barcelona, Spain, Jul. 2016, pp. 445–449.

[25] I. Tamo and A. Barg, "A family of optimal locally recoverable codes," *IEEE Trans. Inf. Theory*, vol. 60, no. 8, pp. 4661–4676, May 2014.

[26] W. Song and C. Yuen, "Locally repairable codes with functional repair and multiple erasure tolerance," 2015, *arXiv:1507.02796*.

[27] W. Song and C. Yuen, "Binary locally repairable codes—Sequential repair for multiple erasures," in *Proc. IEEE Global Commun. Conf. (GLOBECOM)*, Washington, DC, USA, Dec. 2016, pp. 1–6.

[28] W. Song, K. Cai, C. Yuen, K. Cai, and G. Han, "On sequential locally repairable codes," *IEEE Trans. Inf. Theory*, vol. 64, no. 5, pp. 3513–3527, May 2018.

[29] S. B. Balaji, K. P. Prasanth, and P. V. Kumar, "Binary codes with locality for multiple erasures having short block length," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Barcelona, Spain, Jul. 2016, pp. 655–659.

[30] S. B. Balaji, G. R. Kini, and P. V. Kumar, "A tight rate bound and matching construction for locally recoverable codes with sequential recovery from any number of multiple erasures," *IEEE Trans. Inf. Theory*, vol. 66, no. 2, pp. 1023–1052, Feb. 2020.
[31] B. S. Babu, G. R. Kini, and P. V. Kumar, "A rate-optimal construction of codes with sequential recovery with low block length," in *Proc. 24th Nat. Conf. Commun. (NCC)*, Hyderabad, India, Feb. 2018, pp. 1–6.
[32] D. Gligoroski and K. Kralevska, "Families of optimal binary non-MDS erasure codes," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Honolulu, HI, USA, Jun. 2014, pp. 3150–3154.
[33] E. Yavari and M. Esmaeili, "Locally repairable codes: Joint sequential–parallel repair for multiple node failures," *IEEE Trans. Inf. Theory*, vol. 66, no. 1, pp. 222–232, Jan. 2020.
[34] S. Kruglik, M. Dudina, V. Potapova, and A. Frolov, "On one generalization of LRC codes with availability," in *Proc. IEEE Inf. Theory Workshop (ITW)*, Nov. 2017, pp. 26–30.
[35] W. Bosma, J. Cannon, and C. Playoust, "The Magma algebra system I: The user language," *J. Symbolic Comput.*, vol. 24, no. 3–4, pp. 235–265, 1997. [Online]. Available: http://dx.doi.org/10.1006/jsco.1996.0125.

EHSAN YAVARI received the B.Sc. degree in mathematics from Shahid Rajaee Teacher Training University, Tehran, Iran, in 2011, the M.Sc. degree in mathematics from the Sharif University of Technology, Tehran, in 2013, and the Ph.D. degree in mathematics (coding theory) from the Isfahan University of Technology, Isfahan, Iran, in 2020. He is currently with the Ministry of Education of Iran. His research interests include coding theory and its applications to distributed data storage, applications of discrete mathematics to problems in computer science, machine learning, blockchain, and combinatorial designs.

MORTEZA ESMAEILI received the M.S. degree in mathematics from Kharazmi University (formerly the Teacher Training University of Tehran), Tehran, Iran, in 1988, and the Ph.D. degree in mathematics (coding theory) from Carleton University, Ottawa, Canada, in 1996. For the following two years, he was a Postdoctoral Fellow with the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Canada. Since September 1998, he has been with the Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran, where he is currently a Professor. He joined the Department of Electrical and Computer Engineering, University of Victoria, Victoria, BC, Canada, in July 2009, as an Adjunct Professor. His current research interests include coding and information theory, cryptography, machine learning, and the IoT.

JOSEP RIFÀ (Life Senior Member, IEEE) was born in Manlleu, Catalonia, Spain, in July 1951. He received the degree in sciences (mathematical section) from the University of Barcelona, in 1973, and the Ph.D. degree in sciences (computer sciences section) from the Autonomous University of Barcelona (UAB), in 1987. Since 1974, he has been an Assistant Professor with the Mathematics Department, Barcelona University. In 1987, he joined UAB, where he has been a Full Professor, since 1992, and is currently a Professor Emeritus. He was the former Head of the Information and Communications Engineering Department, UAB, and the former Vice-Chairperson of the Spanish Chapter of Information Theory of IEEE. He has worked in several projects of Spanish CICYT and other organizations on subjects related to digital communications, error correcting codes, and encryption of digital information. His research interests include information theory, coding theory, and cryptography.