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Distance and Similarity Measures Using Max-Min Operators of Neutrosophic Hypersoft Sets With Application in Site Selection for Solid Waste Management Systems

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ABSTRACT The idea of a neutrosophic hypersoft set (NHSS) was coined by Smarandache in 2018 as a generalization of the soft set. This structure is a hybrid of a neutrosophic set with a hypersoft set. It can be a valuable structure for dealing with multi-attributes, multi-objective problems with disjoint attributive values. Similarity measures (SM) play a vital role in measuring the similarity index that how much the things are similar. Different types of similarity measures were developed in literature with different fuzzy, intuitionistic, and neutrosophic theories. It is intended to merge the neutrosophic theory with the hypersoft set theory and propose different similarity measures with the help of new proposed distances with max-min operators. Also, we proved different theorems and properties of distance and similarity measures. Then as solid waste management is a global issue, and there are some Solid Waste Management Systems (SWMS) for environment protection, so an example will be given for the site selection for SWMS to check the validity of proposed techniques. To verify the validity and superiority of the suggested work, it is contrasted to several existing methodologies, which show that decision-making issues with more bifurcation attributes provide more accurate and precise outcomes and can only be solved using this technique. In the future, the presented methodologies could be used in case studies with several qualities that are further bifurcated and multiple decision-makers. This proposed work can also be extended to many existing hypersoft set hybrids, such as Fuzzy hypersoft sets (FHSs), Intuitionistic hypersoft sets (IHSs), bipolar hypersoft sets (Bi-HSs), m-polar HSs, and Pythagorean hypersoft sets (PHSs).

INDEX TERMS Neutrosophic hypersoft sets (NHSS), neutrosophic hypersoft matrices (NHSMs), distance measures (DM), similarity measures (SM), landfill, incinerator, composting, solid waste management system (SWMS), air quality index (AQI), multi-attributive decision making (MADM), truthness ($\mathbb{t}$), indeterminacy ($\mathbb{i}$) and falsity ($\mathbb{f}$).

I. INTRODUCTION

While addressing different real-life problems, we need to choose the best option from a list of many. MADM is a decision-making tool that helps us in such processes. The majority of everyday decisions are fraught with ambiguity, and they must be tailored to solve various issues in the real world. Uncertain data is one of the most challenging factors in tackling these difficulties. Many mathematical theories have been developed to overcome these difficulties including Fuzzy Sets (FS) [1], Intuitionistic Fuzzy Sets (IFSs) [2] Pythagorean Fuzzy Sets (PFSs) [3], and

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Generalized orthopair [4] etc. In these sets uncertainty depends upon different functions, including membership and non-membership functions. Then in 1998, Smarandache [5] proposed neutrosophic set (NS) theory as a generalization of the theories mentioned above. He considered truthness ($\mathbb{t}$), indeterminacy ($\mathbb{i}$),and falsity ($\mathbb{f}$) independently. Single valued neutrosophic Set (SVNS) is a robust formal framework proposed by Wang *et al.* [6] and applied to solve real-life problems. Molodsove [7] proposed a Soft set (SS) as a parametrized family of sets with aggregation operators. Maji *et al.* [8], [9] redefined the aggregation operators of SS, and developed a decision-making algorithm using choice values of objects. He applied the proposed algorithm in the house selection problem. Later on, in 2004, Roy and Maji [10] proposed the theoretic approach of Fuzzy soft sets (FSSs) and developed an algorithm for real-life problems.

Many researchers have hybridized the neutrosophic set theory with the soft set theory and extended it in different directions. Wang *et al.* [6], [11] proposed SVNS and Interval Valued neutrosophic sets (IVNS). Peng *et al.* [12] have coined the term simplified neutrosophic sets (SNSs) that have been proposed with the primary goal of dealing with challenges involving a collection of a particular number of attributes, Maji *et al.* [13] worked on NSS, Ye [14] proposed TOPSIS technique based on SVN Linguistic numbers, Tian *et al.* [15] proposed simplified Linguistic normalized weighted numbers in neutrosophic environment and applied it in the investment plans in metal companies, Wang and Li [16] worked on multi-valued neutrosophic soft sets (MVNS) and Broumi *et al.* [17] developed rough neutrosophic soft sets, Jun *et al.* [18] proposed neutrosophic cubic sets. Furthermore, a large number of aggregate operators have been proposed by various researchers based on different techniques. The operators that researchers proposed are Dombi t-norm and t-conorms by Dombi [19], Riaz *et al.* [20] proposed cubic m-polar aggregation operators with Dombi t-norms. Power average, exponential operational law, prioritized average, by Yager *et al.* [21], [22]. Riaz and Hashmi [23] presented m-polar neutrosophic soft mappings and applied them in multiple personality disorders associated with mental disorders. Hashmi *et al.* [24] proposed m-polar neutrosophic generalized and m-polar generalized Einstein operators and used them in the diagnostic process of COVID-19.

Furthermore, a variety of information metrics for the SVNS model have been presented over the years, including similarity measures, distance measures, entropy measures, inclusion measures, and correlation coefficients and graphs in the neutrosophic environment. Due to the contributions of Broumi and Smarandache [25], Cui and Ye [26], Ye [27]–[33], sahin *et al.* [34], Ye and Zhang [35], Ye and Fu. [36], Majumdar and Samanta [37], Peng and Smarandache [38], Mondal and Pramanik [39], Chai *et al.* [40], Liu [41], Jafar *et al.* [42], Garg and Nancy [43], Akram *et al.* [44], [45], some of the most important research publications on similarity and distance measures for SVNSs have been published. In a row of these contributions, Yang *et al.* [46] presented a robust clustering method, Yang and Lin [47] proposed inclusion and similarity measures between Type-2 fuzzy sets. Hung and Yang [48] given similarity measures of Type-2 fuzzy sets. Hussain and Yang [49] worked on distance measures in Pythagorean fuzzy sets. Ejegwa *et al.* worked a lot in uncertain environments. They proposed different techniques of decision making using Pythagorean fuzzy sets like correlation coefficients [50], Some new statistical viewpoints of correlation [51], Distance Measures [52], fuzzy algorithms using correlation measures [53], and proposed modified Zhang and Xu's distance measures [54] in PFS.

The idea of the hypersoft set (HSS) was coined by Smarandache [55] in 2018 as an extension of the soft set. It is useful for dealing with multi-attributes, multi-objective problems with disjoint attributive values. This structure has been extended in different uncertain environments. Jafar *et al.* [56] proposed Fuzzy hypersoft sets (FHS) and their aggregation operators. Saeed *et al.* [57], [58] proposed complex multi-fuzzy hypersoft sets to solve MCDM problems using entropy and similarity measures. They advocated the use of entropy and similarity of efficient complex fuzzy hypersoft sets in the assessment of SWMS. Jafar *et al.* [59] proposed the matrix theory of IFHSs, proposed the MADM algorithm, and applied it in staff selection. Complex neutrosophic hypersoft mappings were also suggested by Saeed *et al.* [60], [61] and used to diagnose infectious disorders and diagnose hepatitis. Saqlain *et al.* [62]–[65] extended HSSs to NHSSs and proposed the TOPSIS in the NHSS environment by using similarity measures. Considered single and multi-valued NHSSs, and tangent similarity measures for single-valued neutrosophic hypersoft sets (SVNHSS), further bring an extension of NHSS to m-Polar and Interval-valued m-Polar NHSS. Application in decision-making in NHSSs theory is given by Rahman *et al.* [66].

The similarity between two sets of objects is a crucial way of measuring how much is similar. SM for extension of FSs has been applied and proposed in different fields of daily life, including medical diagnoses, economics, agriculture, multi-criteria decision making (MCDM), and pattern recognition. Trigonometric similarities measures can be defined by using the trigonometric functions including cosine, tangent, and cotangent applied in various fields of our daily lives. Garg and Wang [67] proposed an algorithm for multi attributive problems. Similarity Measure on m-polar Interval-valued neutrosophic set by Saeed *et al.* [68]. In strategic decision-making, cosine similarity measures are given by Wei [69]. Cotangent similarity measures of single value neutrosophic soft set and its application in fault diagnosis in the steam turbine are given by Ye *et al.* [33], Khan *et al.* [70], and Ahsan *et al.* [71] CMFHS mapping and applied in HIV diagnosis with treatment. Jafar *et al.* [72] proposed Trigonometric Similarity measures and applied them in the renewable energy source selection.

TABLE 1. Similarity measures of NSs, SVNSs and SNNHSs.

Researchers	Similarity Measures	Title of the Article
Broumi and Samandrache [25]	SVNSs	"Several similarity measures of neutrosophic sets"
Cui and Ye [26]	SVNSs	"Improved symmetry measures of simplified neutrosophic sets and their decision making method based on a sine entropy weight model"
Ye [30]	SVNSs	" Clustering methods using distance-based similarity measures of single-valued neutrosophic sets "
Sahin et al. [34]	SVNSs	"new similarity measure based on falsify value between single valued neutrosophic sets based on the centroid points of transformedsingle valued neutrosophic numbers with applications to patternrecognition"
Ye and Zang [35]	SVNSS	"Single valued neutrosophic similarity measures for multiple attribute decision making
Ye and Fu. [36]	SVNSs	Multi-period medical diagnosis method using a single valued neutrosophic similarity measure based on tangent function"
Majumdar and Samanta [37]	NSs	"On similarity and entropy of neutrosophic sets.
Peng and Samandrache [38]	NSs	New multi-parametric similarity measure for neutrosophic set with big data industry evaluation"
Mondal and Pramanik [39]	SVNSs	"Neutrosophic tangent similarity measure and its application to multiple attribute decision making"
Chai et al.[40]	SVNSs	"New similaritymeasures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems"
Liu et al.[41]	SVNSs	"Some similarity measures of neutrosophic sets based on the Euclidean distance and their application in medical diagnosis"
Jafar et al.[42]	SVNSs	"Similarity measures of tangent, cotangent and cosines in neutrosophic environment and their application in selection of academic programs"
Garg and Nancy [43]	SVNSs	"Some new parametric distance measures on single-valued neutrosophic sets with applications to pattern"
Saqlain et al[63]	SVNHSS	"Distance and similarity measures for neutrosophic hypersoft set (NHSS) with construction of NHSS-TOPSIS and applications"
Jafar.et.al [72]	SVNHSS	"Trigonometric Similarity Measures for neutrosophic Hypersoft Sets with Application to Renewable Energy Source Selection"

The human population is increasing at an exponential rate along with drastic effects on the environment, which has put the earth's sustainability at stake. One of the most prominent consequences of the increasing human population is urbanization. Urbanization has disturbed the equilibrium of the natural environment. In every urban sector, a huge amount of solid waste is generated. Proper disposal and management of this solid waste is a challenging task nowadays. After using the 3R (Reduce, Reuse, And Recycle) strategy, some waste still needs to be properly disposed of. There are some management systems in practice like Incineration, composting, and landfill. Many factors need to be considered to install any of those management systems that include environmental, economic, social, political, and ecological factors. An organic portion of the solid waste can be used to make good quality compost for agricultural purposes. Incinerators can be installed to produce energy. The remaining waste is dumped into a landfill site. The most challenging task to implement any of the SWMS mentioned above is site selection. Many factors need to be considered, including land cost, groundwater level, social concerns, distance from urban areas, distance from main roads, the geology of the land, slope of the land, Air quality index (AQI), etc.

This study has chosen four distinct factors to analyze which type of site will be more suitable for which type of solid waste management system. These factors include AQI, distance from the population, economic values, and slope of the land. Air Quality Index (AQI) is one of the essential environmental factors; all the three proposed management systems will affect the AQI of the regions differently, so present AQI of any region will help in decision making. Experts have given the criteria that describe which AQI level will best suit the SWMS. Distance from the populated area is another one of the most important social concerns. All three SWMS have their proposed ideal and acceptable distances proposed by experts that must be followed. The slope of the land is also one of the most important geographical factors for any of these three SWMS, The ideal slope of the land is flat land, but to some extent, less than 20 percent is acceptable.

Economic factors include the cost of the land, building cost, operating cost, etc. Many researchers used different DM techniques in environmental issues and gave the solutions to such problems, like Luo *et al.* [73] used the best worst ANP decision-making technique in incineration plant site selection. Guiqin *et al.* [74] applied informational technology and the Analytical Hierarchy Process (AHP) in landfill site selection. Ming-Lui *et al.* [75] used the hybrid modified multi-attributive decision making (MADM) technique in composting for sustainable developments. Here in the following table, it is intended to show some research article which we will show the comparison of our proposed study with the existing similarity measures.

We employ Distance and similarity metrics to investigate the best SWMS site selection problem according to geography as a mathematical model. Under certain technical attribution variables, the results suggest the optimal geographical place for installing SWM systems.

The rest of the research is organized as follows. We review the basic notions of SSs, HSSs, and NHSSs in Section II. In Section III, we propose six distance measures for NHSSs. In section IV, we provide operators, theorems, and assertions concerning these distance similarity measurements. In Section V, given resources were used to determine the accuracy of the similarity measurements. In this section, we gave a real-life case study, and finally, the conclusion portion is in Section VI, and future investigations are also covered.

II. PRELIMINARIES

This section reviews the definitions of soft sets, hypersoft sets, and neutrosophic hypersoft sets.

Definition ([07]): the terminology of Soft set was given by French Philosopher D. Molodstove in 1999. 07 For dealing with vague scenarios and unpredictable decision-making problems, it is defined as *Let* $\mathfrak{M} = \{\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \ldots \mathfrak{N}_s\}$ be a finite set of alternatives, and $\mathfrak{P}$ be a set of attributes. Let $P(\mathfrak{M})$ denote the power set of $\mathfrak{M}$ and $A \subset \mathfrak{P}$. A pair (λ, A) is called a soft set over $\mathfrak{M}$, where the relation λ is given by

$$\lambda : A \to P(\mathfrak{M}) \tag{2.1}$$

Definition ([62]): Let P be a set of parameters and *Let* $\mathcal{Y} = \{y_1, y_2, y_3, \ldots y_s\}$ be a finite set. The power set of $\mathcal{Y}$ is denoted by $P(\mathcal{Y})$. Let $\upsilon^1, \upsilon^2, \upsilon^3 \ldots \upsilon^n$ for $n \geq 1$ be n well-defined features, whose corresponding feature values are the sets $\mathfrak{z}^1, \mathfrak{z}^2, \mathfrak{z}^3, \ldots \mathfrak{z}^n$ with $\mathfrak{z}^l \cap \mathfrak{z}^m = \emptyset$ for $l \neq m$, $l, m = 1, 2 \ldots n$, respectively, and let their relation be $\omega = \mathfrak{z}^1 \times \mathfrak{z}^2 \times \mathfrak{z}^3 \times \ldots \times \mathfrak{z}^n$. Then the pair $(\wp, W)$ is called an NHSS over $\mathcal{Y}$, where $\wp : \mathfrak{z}^1 \times \mathfrak{z}^2 \times \mathfrak{z}^3 \times \ldots \times \mathfrak{z}^n \to P(\mathbb{Y})$ and $\wp\left(\mathfrak{z}^1 \times \mathfrak{z}^2 \times \mathfrak{z}^3 \times \ldots \times \mathfrak{z}^t\right) = \wp(\omega)$, where $t \leq n$

$$= \{\langle y, \ \mathfrak{t}(\wp(\omega)), \mathfrak{i}(\wp(\omega)), \ \mathfrak{f}(\wp(\omega)), \ y \in \mathcal{Y}\rangle\}$$

where $\mathfrak{t}$, $\mathfrak{i}$, and $\mathfrak{f}$, are the belonging values of truthiness, indeterminacy and falsity respectively such that $\mathfrak{t}, \mathfrak{i}, \mathfrak{f} : \mathbb{Y} \to [0, 1]$ with

$$0 \leq \mathfrak{t}(\wp(\omega)) + \mathfrak{i}(\wp(\omega)) + \mathfrak{f}(\wp(\omega)) \leq 3. \tag{2.2}$$

Definition.

Let $\mathbb{P}$ and $\mathbb{Q}$ be the two NHSs then Addition and Multiplication, Subtraction and Division of $\mathbb{P}$ and $\mathbb{Q}$ are defined as follow

Addition:

$$\begin{aligned}\mathbb{P} \oplus \mathbb{Q} &= \{\langle \omega, \mathfrak{t}_{\mathbb{P}}(\wp(\omega)) + \mathfrak{t}_{\mathbb{Q}}(\wp(\omega)) \\ &\quad - \mathfrak{t}_{\mathbb{P}}(\wp(\omega))\mathfrak{t}_{\mathbb{Q}}(\wp(\omega)), \mathfrak{i}_{\mathbb{P}}(\wp(\omega))\mathfrak{i}_{\mathbb{Q}}(\wp(\omega)), \\ &\quad \times \mathfrak{f}_{\mathbb{P}}(\wp(\omega))\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\rangle\}\end{aligned} \tag{2.3}$$

Multiplication:

$$\begin{aligned}\mathbb{P} \otimes \mathbb{Q} &= \{\langle \omega, \mathfrak{t}_{\mathbb{P}}(\wp(\omega))\mathfrak{t}_{\mathbb{Q}}(\wp(\omega)), \\ &\quad \times \mathfrak{i}_{\mathbb{P}}(\wp(\omega)) + \mathfrak{i}_{\mathbb{Q}}(\wp(\omega)) - \mathfrak{i}_{\mathbb{P}}(\wp(\omega))\mathfrak{i}_{\mathbb{Q}}(\wp(\omega)), \\ &\quad \times \mathfrak{f}_{\mathbb{P}}(\wp(\omega)) + \mathfrak{f}_{\mathbb{Q}}(\wp(\omega)) - \mathfrak{f}_{\mathbb{P}}(\wp(\omega))\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\rangle\}\end{aligned} \tag{2.4}$$

Subtraction:

$$\mathbb{P} \ominus \mathbb{Q} = \left\{\left\langle \omega, \frac{\mathfrak{t}_{\mathbb{P}}(\wp(\omega)) - \mathfrak{t}_{\mathbb{Q}}(\wp(\omega))}{1 - \mathfrak{t}_{\mathbb{Q}}(\wp(\omega))} \times \frac{\mathfrak{i}_{\mathbb{P}}(\wp(\omega))}{\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))}, \frac{\mathfrak{f}_{\mathbb{P}}(\wp(\omega))}{\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))}\right\rangle\right\} \tag{2.5}$$

Which is valid under the conditions $\mathbb{P} \geq \mathbb{Q}$, $\mathfrak{t}_{\mathbb{Q}}(\wp(\omega)) \neq 1$, $\mathfrak{i}_{\mathbb{Q}}(\wp(\omega)) \neq 0$, $\mathfrak{f}_{\mathbb{Q}}(\wp(\omega)) \neq 0$

Division:

$$\mathbb{P} \oslash \mathbb{Q} = \left\{\left\langle \omega, \frac{\mathfrak{t}_{\mathbb{P}}(\wp(\omega))}{\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))} \frac{\mathfrak{i}_{\mathbb{P}}(\wp(\omega)) - \mathfrak{i}_{\mathbb{Q}}(\wp(\omega))}{1 - \mathfrak{i}_{\mathbb{Q}}(\wp(\omega))}, \times \frac{\mathfrak{f}_{\mathbb{P}}(\wp(\omega)) - \mathfrak{f}_{\mathbb{Q}}(\wp(\omega))}{1 - \mathfrak{f}_{\mathbb{Q}}(\wp(\omega))}\right\rangle\right\} \tag{2.6}$$

Which is valid under the conditions $\mathbb{P} \leq \mathbb{Q}$, $\mathfrak{t}_{\mathbb{Q}}(\wp(\omega)) \neq 0$, $\mathfrak{i}_{\mathbb{Q}}(\wp(\omega)) \neq 1$, $\mathfrak{f}_{\mathbb{Q}}(\wp(\omega)) \neq 1$

Definition

Let $\mathbb{P}$ and $\mathbb{Q}$ be the two NHSs then Compliment, Inclusion, Equality, Union and Intersection of $\mathbb{P}$ and $\mathbb{Q}$ are defined as follow

Complement:

$$\mathbb{P}^c = \{\langle \omega, \mathfrak{f}_{\mathbb{P}}(\wp(\omega)), 1 - \mathfrak{i}_{\mathbb{P}}(\wp(\omega)), \ \mathfrak{t}_{\mathbb{P}}(\wp(\omega))\rangle\} \tag{2.7}$$

The case in this definition is based on pendency neutrosophic theory, all truthiness, and indeterminacy, and falsity are dependent.

$$\text{And } \mathbb{P}^c = \{\langle 1 - \mathfrak{t}_{\mathbb{P}}(\wp(\omega)), 1 - \mathfrak{i}_{\mathbb{P}}(\wp(\omega)), 1 - \mathfrak{f}_{\mathbb{P}}(\wp(\omega))\rangle\} \tag{2.8}$$

The case in this definition is based on independency neutrosophic theory, all truthiness, and indeterminacy, and falsity are independent.

Inclusion:

$$\begin{aligned}&\mathbb{P} \subseteq \mathbb{Q} \text{ if and only if } \mathfrak{t}_{\mathbb{P}}(\wp(\omega)) \leq \mathfrak{t}_{\mathbb{Q}}(\wp(\omega)), \\ &\mathfrak{i}_{\mathbb{P}}(\wp(\omega)) \leq \mathfrak{i}_{\mathbb{Q}}(\wp(\omega))\, \mathfrak{f}_{\mathbb{P}}(\wp(\omega)) \geq \mathfrak{f}_{\mathbb{Q}}(\wp(\omega)) \\ &\text{for any } \wp(\omega)\end{aligned} \tag{2.9}$$

Equality:

$$\mathbb{P} = \mathbb{Q} \text{ if and only if } \mathbb{P} \subseteq \mathbb{Q} \text{ and } \mathbb{Q} \subseteq \mathbb{P} \tag{2.10}$$

Union:

$$\begin{aligned}\mathbb{P} \cup \mathbb{Q} = \{\langle \omega, \mathfrak{t}_{\mathbb{P}}(\wp(\omega)) \vee \mathfrak{t}_{\mathbb{Q}}(\wp(\omega)), \ \mathfrak{i}_{\mathbb{P}}(\wp(\omega)) \\ \times \wedge \mathfrak{i}_{\mathbb{Q}}(\wp(\omega)), \mathfrak{f}_{\mathbb{P}}(\wp(\omega)) \wedge \mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\end{aligned} \tag{2.11}$$

where remember that

Intersection:

$$\mathbb{P}\cap\mathbb{Q}=\{\langle\omega,\mathfrak{t}_{\mathbb{P}}(\wp(\omega))\wedge\mathfrak{t}_{\mathbb{Q}}(\wp(\omega)),\mathfrak{i}_{\mathbb{P}}(\wp(\omega))\times\vee\mathfrak{i}_{\mathbb{Q}}(\wp(\omega)),\mathfrak{f}_{\mathbb{P}}(\wp(\omega))\vee\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\tag{2.12}$$

Definition

Let $\mathbb{P}$ be SVNHS over the common universe $\mathcal{Y}$. Then $\mathbb{P}$ is said to be **Absolute** SVNHS if

$$\mathfrak{t}_{\mathbb{P}}(\wp(\omega))=1,\ \mathfrak{i}_{\mathbb{P}}(\wp(\omega))=0\text{ and }\mathfrak{f}_{\mathbb{P}}(\wp(\omega))=0\tag{2.13}$$

Let $\mathbb{P}$ be SVNHSS over the common universe $\mathcal{Y}$. Then $\mathbb{P}$ is said to be **Empty** SVNHS if

$$\mathfrak{t}_{\mathbb{P}}(\wp(\omega))=0,\ \mathfrak{i}_{\mathbb{P}}(\wp(\omega))=0\text{ and }\mathfrak{f}_{\mathbb{P}}(\wp(\omega))=1\tag{2.14}$$

III. DISTANCE MEASURES WITH THEORY AND APPLICATIONS

Based on the axiomatic concept of distance and similarity between SVNHSs, we offer numerous new formulas for SVNHS distance and similarity measures in this section.

A. DISTANCE MEASURES FOR SVNHSS

A real-valued function $D:\beta(\mathcal{Y})\times\beta(\mathcal{Y})\to[0,1]$ is called a distance measure and D satisfies the following axioms for $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}\subseteq\beta(\mathcal{Y})$

D$_1$): $0\leq D(\mathbb{P},\mathbb{Q})\leq 1$

D$_2$): $D(\mathbb{P},\mathbb{Q})=0$ If and only if $\mathbb{P}=\mathbb{Q}$

D$_3$): $D(\mathbb{P},\mathbb{Q})=D(\mathbb{Q},\mathbb{P})$

D$_4$): $\mathbb{P}\subseteq\mathbb{Q}\subseteq\mathbb{R}$ Then $D(\mathbb{P},\mathbb{R})\geq D(\mathbb{P},\mathbb{Q})$ and $D(\mathbb{P},\mathbb{R})\geq D(\mathbb{Q},\mathbb{R})$

Broumi and Florentin [25] proposed Hausdroff distance in NS, and we convert that distance in NHSs as follows:

$$\begin{aligned}d_H^*(\mathbb{P},\mathbb{Q})&=\frac{1}{n}\sum_{\tau=1}^{n}\max\{|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau|,\\&\times|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau|,\\&\times|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau|\}\end{aligned}\tag{3.1}$$

where the proofs of **D$_1$**)– **D$_3$**) are very simple according to the above definition

We will discuss **D$_4$**)

The proof of **D$_4$**) for the above definition between two NHSs

Since $\mathbb{P}\subseteq\mathbb{Q}\subseteq\mathbb{R}$ implies

$$\begin{aligned}\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau&\leq\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau\leq\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\\\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau&\geq\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau\geq\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\\\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau&\geq\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau\geq\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\end{aligned}$$

We have to prove that $D(\mathbb{P},\mathbb{R})\geq D(\mathbb{P},\mathbb{Q})$, so for this we will discuss three cases

Case-I

So, as

$$\begin{aligned}&|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|\\&\geq|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau|\\&\geq|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau|\\d_H(\mathbb{P},\mathbb{Q})&=|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|\end{aligned}$$

But we have for all τ from discourse set we have

$$\begin{aligned}|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau|&\leq|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau|\\&\leq|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|\end{aligned}\tag{a}$$

$$\begin{aligned}|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau|&\leq|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau|\\&\leq|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|\end{aligned}\tag{b}$$

$$\begin{aligned}|\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau|&\leq|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau|\\&\leq|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|\end{aligned}\tag{c}$$

$$\begin{aligned}|\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau|&\leq|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau|\\&\leq|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|\end{aligned}\tag{d}$$

On the other hand

$$|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau|\leq|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|$$

And

$$|\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|\leq|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|$$

Combining **(a)-(d)** we obtain

$$\begin{aligned}&\frac{1}{n}\sum_{\tau=1}^{n}\max\{|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau|\}\\&\leq\frac{1}{n}\sum_{\tau=1}^{n}\max\{|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau|\}\end{aligned}$$

And

$$\begin{aligned}&\frac{1}{n}\sum_{\tau=1}^{n}\max\{|\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau|\}\\&\leq\frac{1}{n}\sum_{\tau=1}^{n}\max\{|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau|,\\&\quad\times|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau|\}\end{aligned}$$

So we conclude that

$$d_H(\mathbb{P},\mathbb{Q})\leq d_H(\mathbb{P},\mathbb{R})\text{ and }d_H(\mathbb{Q},\mathbb{R})\leq d_H(\mathbb{P},\mathbb{R})$$

Case-II

So, as

$$\begin{aligned}&\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&d_H(\mathbb{P},\mathbb{R})=\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}$$

But we have for all τ from discourse set we have

$$\begin{aligned}\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{a}')$$

And

$$\begin{aligned}\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{b}')$$

Now

$$\begin{aligned}\left|\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{c}')$$

And finally

$$\begin{aligned}\left|\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{d}')$$

Combining $(\boldsymbol{a}')-(\boldsymbol{d}')$ we obtain

$$\begin{aligned}&\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau\right|\Big\}\\&\leq\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\Big\}\end{aligned}$$

And

$$\begin{aligned}&\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\Big\}\\&\leq\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\Big\}\end{aligned}$$

So we conclude that

$$d_H(\mathbb{P},\mathbb{Q})\leq d_H(\mathbb{P},\mathbb{R})\ \text{ and }\ d_H(\mathbb{Q},\mathbb{R})\leq d_H(\mathbb{P},\mathbb{R})$$

Case-III

So, as

$$\begin{aligned}&\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&d_H(\mathbb{P},\mathbb{R})=\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}$$

But we have for all τ from discourse set we have

$$\begin{aligned}\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{a}'')$$

And

$$\begin{aligned}\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{b}'')$$

Now

$$\begin{aligned}\left|\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{c}'')$$

And finally

$$\begin{aligned}\left|\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|&\leq\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\\&\leq\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|\end{aligned}\qquad(\boldsymbol{d}'')$$

Combining $(\boldsymbol{a}'')-(\boldsymbol{d}'')$ we obtain

$$\begin{aligned}&\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau\right|\Big\}\\&\leq\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\Big\}\end{aligned}$$

And

$$\begin{aligned}&\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\Big\}\\&\leq\frac{1}{\mathrm{n}}\sum_{\tau=1}^{n}\max\Big\{\left|\mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{t}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{i}_{\mathbb{R}}(\wp(\omega))_\tau\right|,\\&\qquad\times\left|\mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau-\mathfrak{f}_{\mathbb{R}}(\wp(\omega))_\tau\right|\Big\}\end{aligned}$$

So we conclude that

$$d_H(\mathbb{P},\mathbb{Q}) \leq d_H(\mathbb{P},\mathbb{R}) \text{ and } d_H(\mathbb{Q},\mathbb{R}) \leq d_H(\mathbb{P},\mathbb{R})$$

So finally, by combining all three cases, we can obtain $\mathbf{D_4}$) ■

Here in this section, we will define different distance measures in NHSs environment

Theorem: **Let** $\mathbb{P}$ and $\mathbb{Q}$ be two NHSs then $d^k(\mathbb{P},\mathbb{Q})$ for k = 1, 2, 3 . . . 6 is a distance between NHSs then we define all the distances like

1.

$$\boldsymbol{d^1}(\mathbb{P},\mathbb{Q}) = \frac{1}{3|\mathrm{y}|}\sum_{\tau}\Big(\left|\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right| + \left|\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right| + \left|\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right|\Big) \tag{3.2}$$

2.

$$\boldsymbol{d^2}(\mathbb{P},\mathbb{Q}) = \frac{1}{3|\mathrm{y}|}\sum_{\tau}\Big|\Big(\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau - \left(\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right) - \left(\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)\Big|\Big) \tag{3.3}$$

3.

$$\boldsymbol{d^3}(\mathbb{P},\mathbb{Q}) = \frac{1}{|\mathrm{y}|}\sum_{\tau}\Big(\left|\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right| \vee \left|\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right| \vee \left|\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right|\Big) \tag{3.4}$$

4. $\boldsymbol{d^4}(\mathbb{P},\mathbb{Q})$, as shown at the bottom of the next page.

5.

$$\boldsymbol{d^5}(\mathbb{P},\mathbb{Q}) = 1 - \alpha\frac{\sum_\tau\left(\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau \wedge \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)}{\sum_\tau\left(\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau \vee \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)} - \beta\frac{\sum_\tau\left(\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau \wedge \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)}{\sum_\tau\left(\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau \vee \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)} - \gamma\frac{\sum_\tau\left(\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau \wedge \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)}{\sum_\tau\left(\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau \vee \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)} \tag{3.6}$$

where $\alpha+\beta+\gamma=1$ and $\alpha,\ \beta,\ \gamma\ \epsilon[0,1]$

6.

$$\boldsymbol{d^6}(\mathbb{P},\mathbb{Q}) = 1 - \frac{\alpha}{|\mathrm{y}|}\sum_\tau\frac{\left(\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau \wedge \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)}{\left(\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau \vee \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)} - \frac{\beta}{|\mathrm{y}|}\sum_\tau\frac{\left(\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau \wedge \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)}{\left(\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau \vee \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)} - \frac{\gamma}{|\mathrm{y}|}\sum_\tau\frac{\left(\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau \wedge \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)}{\left(\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau \vee \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)} \tag{3.7}$$

where $\alpha+\beta+\gamma=1$ and $\alpha,\ \beta,\ \gamma\ \epsilon[0,1]$

In the light of the $\mathbf{D_1}$)$-$ $\mathbf{D_4}$) for $d^k(\mathbb{P},\mathbb{Q})$ for k=1, 2, 3…6, if $d^k(\mathbb{P},\mathbb{Q})$ satisfied the all for axioms of the distance, they are qualified for validity. So you can easily understand just like the previous definition we can easily prove all four properties.

Theorem: Let $\mathbb{P}$ and $\mathbb{Q}$ be two NHSs then $d^k(\mathbb{P},\mathbb{Q})$ for k = 1, 2, 3 . . . 6. Then $d^k(\mathbb{P},\mathbb{Q})$ holds the following.

i. $d^k(\mathbb{P},\mathbb{Q}^c) = d^k(\mathbb{P}^c,\mathbb{Q})$

ii. $d^k(\mathbb{P},\mathbb{Q}) = d^k(\mathbb{P}\cap\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})$

iii. $d^k(\mathbb{P},\ \mathbb{P}\cap\mathbb{Q}) = d^k(\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})$

iv. $d^k(\mathbb{P},\ \mathbb{P}\cup\mathbb{Q}) = d^k(\mathbb{Q},\ \mathbb{P}\cap\mathbb{Q})$

Proof:

i. $\boldsymbol{d^1}(\mathbb{P},\mathbb{Q}^c) = \boldsymbol{d^1}(\mathbb{P}^c,\mathbb{Q})$

Let

$$\mathbb{P} = \left\{\left\langle \mathfrak{t}_{\mathbb{P}}(\wp(\omega))_\tau, \mathfrak{i}_{\mathbb{P}}(\wp(\omega))_\tau, \mathfrak{f}_{\mathbb{P}}(\wp(\omega))_\tau, \right\rangle\right\}$$
$$\mathbb{Q} = \left\{\left\langle \mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau, \mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau, \mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau \right\rangle\right\}$$
$$\mathbb{Q}^c = \left\{\left\langle \mathfrak{f}_{\mathbb{Q}}(\wp(\omega))_\tau, 1-\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))_\tau, \mathfrak{t}_{\mathbb{Q}}(\wp(\omega))_\tau \right\rangle\right\}$$

Then by definition (Eq. **3.2**) we have

1.

$$\boldsymbol{d^1}(\mathbb{P},\mathbb{Q}) = \frac{1}{3|\mathrm{y}|}\sum_{\tau}\Big(\left|\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right| + \left|\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right| + \left|\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right|\Big)$$

Then

$$\boldsymbol{d^1}(\mathbb{P},\mathbb{Q}^c) = \frac{1}{3|\mathrm{y}|}\sum_{\tau}\Big(\left|\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right| + \left|\mathfrak{i}^2_{\mathbb{P}}(\wp(\omega))_\tau - \left(1-\mathfrak{i}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right)\right| + \left|\mathfrak{f}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{t}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right|\Big)$$

$$= \frac{1}{3|\mathrm{y}|}\sum_{\tau}\Big(\left|\mathfrak{t}^2_{\mathbb{P}}(\wp(\omega))_\tau - \mathfrak{f}^2_{\mathbb{Q}}(\wp(\omega))_\tau\right|$$

$$
\begin{aligned}
&+\left|\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}+\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}-1\right| \\
&+\left|\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\Big) \\
&=\frac{1}{3|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
&+\left|1-\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
&+\left|\mathfrak{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\Big) \\
&=\boldsymbol{d}^{\mathbf{1}}\left(\mathbb{P}^{\mathbf{c}},\mathbb{Q}\right)
\end{aligned}
$$

Note: The theorem can be proved with the refined complement on the same line. ■

ii.

$$
\begin{aligned}
&d^{1}(\mathbb{P},\mathbb{Q})=d^{1}(\mathbb{P}\cap\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})=d^{1}(\mathbb{P}\cap\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q}) \\
&=\frac{1}{3|\mathsf{y}|}\sum_{\tau}\Big(\Big|\left(\min\left(\mathfrak{t}_{\mathbb{P}}(\wp(\omega)),\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2} \\
&-\left(\max\left(\mathfrak{t}_{\mathbb{P}}(\wp(\omega)),\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\Big| \\
&+\Big|\left(\max\left(\mathfrak{i}_{\mathbb{P}}(\wp(\omega)),\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2} \\
&-\left(\min\left(\mathfrak{i}_{\mathbb{P}}(\wp(\omega)),\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\Big| \\
&+\Big|\left(\max\left(\mathfrak{f}_{\mathbb{P}}(\wp(\omega)),\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2} \\
&-\left(\min\left(\mathfrak{f}_{\mathbb{P}}(\wp(\omega)),\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\Big| \\
&=\frac{1}{3|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathfrak{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
&+\left|\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
&+\left|\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\Big) \\
&=d^{1}(\mathbb{P},\mathbb{Q})
\end{aligned}
$$

■

ii.

$$
\begin{aligned}
&d^{1}(\mathbb{P},\ \mathbb{P}\cap\mathbb{Q}) \\
&=d^{1}(\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})=\frac{1}{3|\mathsf{y}|}\sum_{\tau} \\
&\times\Big(\left|\mathfrak{t}_{\mathbb{P}}^{2}(\wp(\omega))-\left(\min\left(\mathfrak{t}_{\mathbb{P}}(\wp(\omega)),\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&+\left|\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))-\left(\max\left(\mathfrak{i}_{\mathbb{P}}(\wp(\omega)),\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&+\left|\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))-\left(\max\left(\mathfrak{f}_{\mathbb{P}}(\wp(\omega)),\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&=\frac{1}{3|\mathsf{y}|}\sum_{\tau} \\
&\times\Big(\left|\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))-\left(\max\left(\mathfrak{t}_{\mathbb{P}}(\wp(\omega)),\mathfrak{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&+\left|\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))-\left(\min\left(\mathfrak{i}_{\mathbb{P}}(\wp(\omega)),\mathfrak{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&+\left|\mathfrak{f}_{\mathbb{Q}}^{2}(\wp(\omega))-\left(\min\left(\mathfrak{f}_{\mathbb{P}}(\wp(\omega)),\mathfrak{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&\because\left|\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
&=\left|\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\right| \\
&=d^{1}(\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})
\end{aligned}
$$

■

In these proofs, iii and iv are similar in being verified in a similar pattern.

Here these four proofs with six definitions are 24 proofs and for sample we prove some results remaining you can verify easily.

IV. SIMILARITY MEASURES WITH THEORY AND APPLICATIONS

Let $\mathbb{P}$ and $\mathbb{Q}$ be two SVNHSSs and a mapping S such that $S\ :\ \beta(\mathcal{Y})\times\beta(\mathcal{Y})\rightarrow[0,1]$ is called a similarity measure between $\mathbb{P}$ and $\mathbb{Q}$ if S satisfies the following axioms for $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}\ \subseteq\beta(\mathcal{Y})$

S_1): $0\leq S(\mathbb{P},\mathbb{Q})\leq 1$

S_2): $S(\mathbb{P},\mathbb{Q})=0$ If and only if $\mathbb{P}=\mathbb{Q}$

S_3): $S(\mathbb{P},\mathbb{Q})=S(\mathbb{Q},\ \mathbb{P})$

S_4): $\mathbb{P}\subseteq\mathbb{Q}\subseteq\mathbb{R}$ Then

$S(\mathbb{P},\mathbb{R})\leq S(\mathbb{P},\mathbb{Q})$ and $S(\mathbb{P},\mathbb{R})\leq S(\mathbb{Q},\mathbb{R})$

Theorem: Let $\mathbb{P}$ and $\mathbb{Q}$ be two SVNHSSs then $S^{k}(\mathbb{P},\mathbb{Q})$ for $k\ =\ 1,2,3\ldots 6$ are the similarity measures in between SVNHSSs $\mathbb{P}$ and $\mathbb{Q}$.

1.

$$
\begin{aligned}
&\boldsymbol{S}^{\mathbf{1}}(\mathbb{P},\mathbb{Q}) \\
&=1-\frac{1}{3|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathfrak{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
&+\left|\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
&+\left|\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\Big)
\end{aligned}
\tag{4.1}
$$

2.

$$
\begin{aligned}
&\boldsymbol{S}^{\mathbf{2}}(\mathbb{P},\mathbb{Q}) \\
&=1-\frac{1}{3|\mathsf{y}|}\sum_{\tau}\Big|\Big(\mathfrak{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau} \\
&-\left(\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right) \\
&-\left(\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)\Big|\Big)
\end{aligned}
\tag{4.2}
$$

$$
\boldsymbol{d}^{\mathbf{4}}(\mathbb{P},\mathbb{Q})=\frac{\sum_{\tau}\left(\left|\mathfrak{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\vee\left|\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\vee\left|\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\right)}{\sum_{\tau}\left(\mathbf{1}+\left(\left|\mathfrak{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\vee\left|\mathfrak{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\vee\left|\mathfrak{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathfrak{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\right)\right)}
\tag{3.5}
$$

3.

$$S^{3}(\mathbb{P},\mathbb{Q}) = 1-\frac{1}{|\mathrm{y}|}\sum_{\tau}\left(\left|\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \vee \left|\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \vee \left|\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\right) \tag{4.3}$$

4. $S^{4}(\mathbb{P},\mathbb{Q})$, as shown at the bottom of the next page.

5.

$$S^{5}(\mathbb{P},\mathbb{Q}) = \alpha\frac{\sum_{\tau}\left(\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\wedge \mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)}{\sum_{\tau}\left(\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\vee \mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)} +\beta\frac{\sum_{\tau}\left(\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\wedge \mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)}{\sum_{\tau}\left(\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\vee \mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)} +\gamma\frac{\sum_{\tau}\left(\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\wedge \mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)}{\sum_{\tau}\left(\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\vee \mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)} \tag{4.5}$$

where $\alpha+\beta+\gamma=1$ and $\alpha,\ \beta,\ \gamma\,\epsilon[0,1]$

6.

$$S^{6}(\mathbb{P},\mathbb{Q}) = \frac{\alpha}{|\mathrm{y}|}\sum_{\tau}\frac{\left(\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\wedge \mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)}{\left(\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\vee \mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)} +\frac{\beta}{|\mathrm{y}|}\sum_{\tau}\frac{\left(\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\wedge \mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)}{\left(\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\vee \mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)} +\frac{\gamma}{|\mathrm{y}|}\sum_{\tau}\frac{\left(\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\wedge \mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)}{\left(\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\vee \mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right)} \tag{4.6}$$

where $\alpha+\beta+\gamma=1$ and $\alpha,\ \beta,\ \gamma\,\epsilon[0,1]$

To check the validity of the proposed similarity measure, we verify all four ($S_1 - S_4$) Axioms of the similarity measure.

S_1, S_3 and S_4 are straightforward so we only prove the conditions S_2 and S_4 in the interest of brevity we only present the proof of $S^k(\mathbb{P},\mathbb{Q})$ for $k = 1$, and the proofs for $k = 2, 3, 4\ldots 6$ can be generated in a similar way. So, for $k = 1$, $S^k(\mathbb{P},\mathbb{Q})$ is

$$S^{1}(\mathbb{P},\mathbb{Q}) = 1-\frac{1}{3|\mathrm{y}|}\sum_{\tau}\left(\left|\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\right)$$

$S_1)$: $\boldsymbol{S^1}(\mathbb{P},\mathbb{Q}) = \mathbf{1}$ if and only if $\mathbb{P}=\mathbb{Q}$

$$1-\frac{1}{3|\mathrm{y}|}\sum_{\tau}\left(\left|\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\right) = 1$$

$$(\Rightarrow)\sum_{\tau}\left(\left|\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\right) = 0$$

$(\Rightarrow)$ Which is only possible when

$$\left|\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| + \left|\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\Big) = 0$$

$$(\Rightarrow)\left|\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| = 0,\quad \left|\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| = 0,\quad \left|\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\Big) = 0$$

$$(\Rightarrow)\,\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau} = 0,\quad \mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau} = 0\quad \mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau} = 0$$

$$(\Rightarrow)\,\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau} = \mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau},\quad \mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau} = \mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\ \mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau} = \mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}$$

$$\mathrm{t}_{\mathbb{P}}(\wp(\omega))_{\tau} = \mathrm{t}_{\mathbb{Q}}(\wp(\omega))_{\tau},\ \mathrm{i}_{\mathbb{P}}(\wp(\omega))_{\tau} = \mathrm{i}_{\mathbb{Q}}(\wp(\omega))_{\tau}\quad \mathrm{f}_{\mathbb{P}}(\wp(\omega))_{\tau} = \mathrm{f}_{\mathbb{Q}}(\wp(\omega))_{\tau}$$

$(\Rightarrow)\ \mathbb{P}=\mathbb{Q}$ **Hence**

Conversely suppose that $\mathbb{P}=\mathbb{Q}$ and we have to prove that $\boldsymbol{S^1}(\mathbb{P},\mathbb{Q}) = \mathbf{1}$

So, as $\mathbb{P}=\mathbb{Q}$ that implies

$$(\Leftarrow)\,\mathrm{t}_{\mathbb{P}}(\wp(\omega))_{\tau} = \mathrm{t}_{\mathbb{Q}}(\wp(\omega))_{\tau},\ \mathrm{i}_{\mathbb{P}}(\wp(\omega))_{\tau} = \mathrm{i}_{\mathbb{Q}}(\wp(\omega))_{\tau}\quad \mathrm{f}_{\mathbb{P}}(\wp(\omega))_{\tau} = \mathrm{f}_{\mathbb{Q}}(\wp(\omega))_{\tau}$$

$$(\Leftarrow)\,\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau} = \mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau},\quad \mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau} = \mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\quad \mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau} = \mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}$$

$$(\Leftarrow)\,\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau} = 0,\quad \mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau} = 0\quad \mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau} = 0$$

$$(\Leftarrow)\left|\mathrm{t}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{t}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| = 0,\quad \left|\mathrm{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| = 0,\quad \left|\mathrm{f}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathrm{f}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right|\Big) = 0$$

$$(\Leftarrow)\ \frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\times\Big(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\Big)=0$$

$$(\Leftarrow)\ 1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\times\Big(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\Big)=1-0$$
$$=\boldsymbol{S^{1}}(\mathbb{P},\mathbb{Q})=\mathbf{1}$$

■

Theorem: Let $\mathbb{P}$ and $\mathbb{Q}$ be two SVNHSSs then $S^k(\mathbb{P},\mathbb{Q})$ for $k = 1, 2, 3\ldots 6$ are the similarity measures in between SVNHSSs $\mathbb{P}$ and $\mathbb{Q}$, then for $\alpha=\beta=\gamma=\frac{1}{3}$ we have

- **i.** $S^k(\mathbb{P},\mathbb{Q}^c)=S^k(\mathbb{P}^c,\mathbb{Q})$
- **ii.** $S^k(\mathbb{P},\mathbb{Q})=S^k(\mathbb{P}\cap\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})$
- **iii.** $S^k(\mathbb{P},\ \mathbb{P}\cap\mathbb{Q})=S^k(\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})$
- **iv.** $S^k(\mathbb{P},\ \mathbb{P}\cup\mathbb{Q})=S^k(\mathbb{Q},\ \mathbb{P}\cap\mathbb{Q})$

In brevity point of view we only prove the properties **i-iii** for $S^k(\mathbb{P},\mathbb{Q})$ when $k = 1$, it can be easily shown $S^k(\mathbb{P},\mathbb{Q})$ for $k = 2, 3\ldots 6$. The proof of property **IV** is similar to **iii** and is therefore omitted.

Proof:

i. $\boldsymbol{S^{1}}(\mathbb{P},\mathbb{Q}^{\mathbf{c}})=\boldsymbol{S^{1}}(\mathbb{P}^{\mathbf{c}},\mathbb{Q})$

Let

$$\mathbb{P}=\left\{\left\langle\mathbb{t}_{\mathbb{P}}(\wp(\omega))_{\tau},\mathbb{i}_{\mathbb{P}}(\wp(\omega))_{\tau},\mathbb{f}_{\mathbb{P}}(\wp(\omega))_{\tau},\right\rangle\right\}$$
$$\mathbb{Q}=\left\{\left\langle\mathbb{t}_{\mathbb{Q}}(\wp(\omega))_{\tau},\mathbb{i}_{\mathbb{Q}}(\wp(\omega))_{\tau},\mathbb{f}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right\rangle\right\}$$
$$\mathbb{Q}^{\mathbf{c}}=\left\{\left\langle\mathbb{f}_{\mathbb{Q}}(\wp(\omega))_{\tau},1-\mathbb{i}_{\mathbb{Q}}(\wp(\omega))_{\tau},\mathbb{t}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right\rangle\right\}$$

Then by definition **(4.1)** we have

$$\boldsymbol{S^{1}}(\mathbb{P},\mathbb{Q})=1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\Big)$$

Then

$$\boldsymbol{S^{1}}(\mathbb{P},\mathbb{Q}^{\mathbf{c}})=1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\left(1-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right)\right|+\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\Big)$$
$$=1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}+\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}-1\right|+\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\Big)$$
$$=1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|1-\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\Big)$$
$$=\boldsymbol{S^{1}}\left(\mathbb{P}^{\mathbf{c}},\mathbb{Q}\right)$$

Note: The theorem can be proved with the refined complement (Eq. **2.8**) on the same line. ■

ii.

$$S^{1}(\mathbb{P},\mathbb{Q})=S^{1}(\mathbb{P}\cap\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})=S^{1}(\mathbb{P}\cap\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})$$
$$=1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\Big(\Big|\left(\min\left(\mathbb{t}_{\mathbb{P}}(\wp(\omega)),\mathbb{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}-\left(\max\left(\mathbb{t}_{\mathbb{P}}(\wp(\omega)),\mathbb{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\Big|+\Big|\left(\max\left(\mathbb{i}_{\mathbb{P}}(\wp(\omega)),\mathbb{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}-\left(\min\left(\mathbb{i}_{\mathbb{P}}(\wp(\omega)),\mathbb{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\Big|+\Big|\left(\max\left(\mathbb{f}_{\mathbb{P}}(\wp(\omega)),\mathbb{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}-\left(\min\left(\mathbb{f}_{\mathbb{P}}(\wp(\omega)),\mathbb{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\Big|$$
$$=1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\Big(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|+\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\Big)$$
$$=S^{1}(\mathbb{P},\mathbb{Q})$$

■

iii.

$$S^{1}(\mathbb{P},\ \mathbb{P}\cap\mathbb{Q})=S^{1}(\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})=1-\frac{1}{3\,|\mathsf{y}|}\sum_{\tau}\times\Big(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))-\left(\min\left(\mathbb{t}_{\mathbb{P}}(\wp(\omega)),\mathbb{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right|$$

$$\boldsymbol{S^{4}}(\mathbb{P},\mathbb{Q})=\frac{\sum_{\tau}1-\left(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\vee\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\vee\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\right)}{\sum_{\tau}\left(\mathbf{1}+\left(\left|\mathbb{t}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{t}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\vee\left|\mathbb{i}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{i}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\vee\left|\mathbb{f}^{2}_{\mathbb{P}}(\wp(\omega))_{\tau}-\mathbb{f}^{2}_{\mathbb{Q}}(\wp(\omega))_{\tau}\right|\right)\right)}\quad(4.4)$$

$$
\begin{aligned}
&+\left|\mathbb{i}_{\mathbb{P}}^{2}(\wp(\omega))-\left(\max\left(\mathbb{i}_{\mathbb{P}}(\wp(\omega)),\mathbb{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&+\left|\mathbb{f}_{\mathbb{P}}^{2}(\wp(\omega))-\left(\max\left(\mathbb{f}_{\mathbb{P}}(\wp(\omega)),\mathbb{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
=&\frac{1}{3|\mathbb{y}|}\sum_{\tau} \\
&\times\left(\left|\mathbb{t}_{\mathbb{Q}}^{2}(\wp(\omega))-\left(\max\left(\mathbb{t}_{\mathbb{P}}(\wp(\omega)),\mathbb{t}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right|\right. \\
&+\left|\mathbb{i}_{\mathbb{Q}}^{2}(\wp(\omega))-\left(\min\left(\mathbb{i}_{\mathbb{P}}(\wp(\omega)),\mathbb{i}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right| \\
&+\left.\left|\mathbb{f}_{\mathbb{Q}}^{2}(\wp(\omega))-\left(\min\left(\mathbb{f}_{\mathbb{P}}(\wp(\omega)),\mathbb{f}_{\mathbb{Q}}(\wp(\omega))\right)\right)^{2}\right|\right) \\
\because&\left|\mathbb{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}-\mathbb{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}\right| \\
=&\left|\mathbb{i}_{\mathbb{Q}}^{2}(\wp(\omega))_{\tau}-\mathbb{i}_{\mathbb{P}}^{2}(\wp(\omega))_{\tau}\right| \\
=&S^{1}(\mathbb{Q},\ \mathbb{P}\cup\mathbb{Q})
\end{aligned}
$$

■

V. ALGORITHM AND ILLUSTRATIVE EXAMPLES

This portion intends to discuss an algorithm depending upon the proposed work, then use that algorithm in site selection of solid-waste management systems.

A. THE ALGORITHM BASED ON NHSS SIMILARITY MEASURES

Let $\mathfrak{Q}^1, \mathfrak{Q}^2, \mathfrak{Q}^3, \ldots \mathfrak{Q}^n$ represent different sites separate set of geographical areas. $\mathfrak{p}^1, \mathfrak{p}^2, \mathfrak{p}^3, \ldots \mathfrak{p}^n$ Based on a set of geographical region norms $S^1, S^2, S^3 \ldots\ S^n$ is the set of possibilities for SWM Systems in each geographical region. Using a decision-making technique, a decision-maker can evaluate $\mathfrak{Z}$ regions and $\mathfrak{s}$ SWM system type under $\mathfrak{p}$ norms. This analysis can be used to determine which SWM systems should be deployed in which geographic region. As a result, it will be able to select the greatest match between geographic locations and SWMS.

Now we'll go over how to put the proposed distance similarity metrics for neutrosophic hypersoft sets into practice.

Step 1: To begin, geographical regions should be assessed and SWMS that can be used in these areas. Following that, the regional norms and SWMS should be determined. A decision matrix in terms of neutrosophic hypersoft sets should be used to show the relationship between geographical regions and norms.

Step 2: The decision matrix in terms of neutrosophic hypersoft sets should be used to show the relationship between the norms and the alternatives, which is the type of SWMS (see, e.g., below Table 2).

Step 3: Using equations (Eq. **4.1**)-(Eq. **4.6**) to calculate for the association between norms and alternatives.

Step 4: The best choice is determined by selecting the greatest value, which indicates the best option for each geographical location. The bold font is used to emphasize this value (see, e.g., below Table.4-Table.9).

To better understand the concept, we've included a flow chart of the proposed algorithm.

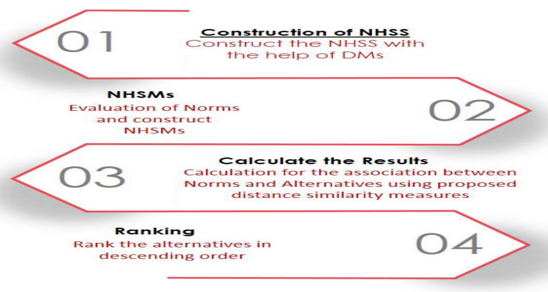

FIGURE 1. Flow Chart of the proposed algorithm.

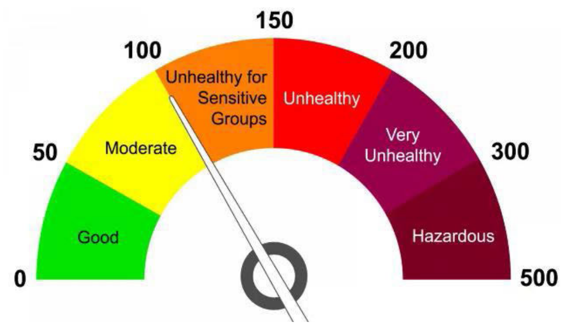

FIGURE 2. Air quality index (AQI).

VI. APPLICATIONS

A huge amount of solid-waste is generated in every urban sector. Proper disposal and management of this solid waste is a challenging task nowadays. There are many factors that need to be considered for proper disposal and management, including environmental, economic, ecological, social, and political factors. After using 3R (Reduce, Reuse, and Recycle) strategy, there is still some waste to be disposed of. Incineration is a way in which energy can be produced through solid waste. The organic portion of the solid waste can be used for composting to produce good fertilizers for crops. The final fate of all the remaining waste is landfilling. As mentioned earlier, for all three waste, the most challenging task is site selection. For the solution of this issue, in this study, we try to overcome the world problem by developing a mathematical technique. For solving this problem, we assume ten different geographical regions with linked with key factors that we discussed earlier

Let $\mathfrak{Z} = \left\{\mathfrak{Q}^1, \mathfrak{Q}^2, \mathfrak{Q}^3, \mathfrak{Q}^4, \mathfrak{Q}^5, \mathfrak{Q}^6, \mathfrak{Q}^7, \mathfrak{Q}^8, \mathfrak{Q}^9, \mathfrak{Q}^{10}\right\}$

Now we will take a set of the most useable WSM Systems

$\mathfrak{s} = \{\text{Landfill, Composting, Incinerator}\}$

Now we will suppose the most suitable criteria's that effects both regions and SWMSs

$$
\text{Let }\ \mathfrak{p} = \left\{\begin{array}{c}
\mathfrak{p}^1\ (\text{Air Quality Index } (\text{AQI})) \\
\mathfrak{p}^2\ (\text{Slope}) \\
\\
\mathfrak{p}^3\ (\text{Distance From Population}) \\
\mathfrak{p}^4\ (\text{Economical Factor})
\end{array}\right\}
$$

TABLE 2. Decision matrix between criteria's and geographical regions.

Geographical Regions	AQI (51-100)	Slope (2-8)%	Distance (1Km-2Km)	Economy Moderate
$\mathfrak{Q}^1$	(0.3,0.3,0.6)	(0.6,0.4,0.4)	(0.8,0.1,0.4)	(0.4,0.3,0.6)
$\mathfrak{Q}^3$	(0.6,0.3,0.2)	(0.4,0.4,0.4)	(0.9,0.3,0.1)	(0.3,0.4,0.5)
$\mathfrak{Q}^7$	(0.9,0.1,0.4)	(0.7,0.1,0.1)	(0.3,0.4,0.3)	(0.5,0.3,0.6)
$\mathfrak{Q}^9$	(0.5,0.4,0.4)	(0.6,0.3,0.1)	(0.7,0.3,0.1)	(1.0,0,0.1)

TABLE 3. Decision matrix between criteria's and geographical regions.

Power type	AQI (51-100)	Slope (2-8)%	Distance (1Km-2Km)	Economy Moderate
Landfill	(0.5,0.3,0.2)	(0.6,0.4,0.4)	(0.3,0.4,0.6)	(0.1,0.4,0.6)
Composting	(0.7,0.2,0.4)	(0.7,0.4,0.7)	(0.8,0.2,0.3)	(0.4,0.4,0.5)
Incinerator	(0.9,0.1,0.1)	(0.8,0.3,0.1)	(0.8,0.3,0.5)	(0.6,0.3,0.4)

TABLE 4. Distance similarity measures using $S^1(\mathbb{P},\mathbb{Q})$.

Similarity measures	*geographical regions*	Landfill	Composting	Incinerator
$s^1_{nj}(\mathbb{P},\mathbb{Q})$	$\mathfrak{Q}^1$	0.9466	**0.9536**	0.9260
	$\mathfrak{Q}^3$	0.9523	**0.9586**	0.9390
	$\mathfrak{Q}^7$	0.9410	0.9360	**0.9510**
	$\mathfrak{Q}^9$	**0.9090**	0.9046	0.8996

TABLE 5. Distance similarity measures using $S^2(\mathbb{P},\mathbb{Q})$.

Similarity measures	*geographical regions*	Landfill	Composting	Incinerator
$s^2_{nj}(\mathbb{P},\mathbb{Q})$	$\mathfrak{Q}^1$	0.9960	**1.000**	0.9940
	$\mathfrak{Q}^3$	0.9803	**1.000**	0.9856
	$\mathfrak{Q}^7$	0.9910	0.9910	**0.9950**
	$\mathfrak{Q}^9$	**0.9983**	0.9946	0.9716

$$\mathbb{P}:\left(\mathfrak{p}^1\times\mathfrak{p}^2\times\mathfrak{p}^3\times\mathfrak{p}^4\right)\rightarrow P(\mho)\text{ and}$$

$\mathbb{Q}:\left(\mathfrak{p}^1\times\mathfrak{p}^2\times\mathfrak{p}^3\times\mathfrak{p}^4\right)\rightarrow P(\mathcal{S})$ Now we evaluate $\left\{\mathfrak{Q}^1,\mathfrak{Q}^3,\mathfrak{Q}^7,\mathfrak{Q}^9\right\}$ And Landfill, composting, and incinerator. So

$$\mathfrak{p}^1=\left\{\begin{array}{c}0-50\ \text{(Good)},\ 51-100\text{(Moderate)}\\ 100-150\ \text{(Unhelthy for sensitive)},\\ 150-200\ \text{(unhealthy)},\ 200-300\ \text{(V.Unh)}\\ >300\ \text{(Hazardous)}\end{array}\right\}$$

$$\mathfrak{p}^2=\left\{\begin{array}{c}0\ (ideal),\ \ (2\%-09\%)\ good,\\ (10\%-20\%)\ moderate\end{array}\right\}$$

$$\mathfrak{p}^3=\{1Km,\ 1Km-2Km,2>km\},$$

$$\mathfrak{p}^4=\{\text{Very costly},\ \text{costly},\ \text{Moderately costly}\}$$

Now, we construct NHSs, with the association of Attributive values Geographical regions with and SWMS through the following mappings. So, according to our definition, we construct the association between geographical regions and SWMS and showing in the following table

VII. RESULT DISCUSSIONS AND COMPARISON WITH EXISTING TECHNIQUES

We compared proposed similarity measures methodologies to existing distance similarity measures in this research on the basis of attribution and sub-attributions for SVNS, Broumi and Samandrache [25], Cui and Ye [26], Ye [30], sahin *et al.* [34], Ye and Zang [35], Ye and Fu. [36], Majumdar and Samanta [37], Peng and Samandrache [38], Mondal and Pramanik [39], Chai *et al.* [40], Liu *et al.* [41], Jafar *et al.* [42], Garg and Nancy [43]. Here the comparison is proposed based on additional bifurcation qualities and their corresponding attributive values. The existing structure of SVNSS are not dealing sub-attributions, But the proposed structure can deal with sub-attributions. This comparison is not based on numerical values. Saqlain *et al.* [63] suggested distance similarity measures in NHSS, and finally, even we compared our results with Jafar *et al.* [72] who given trigonometric similarity measures in NHSS's environment considered sub-attributions, but our proposed structure is uniquely presented using Max-Min operators. In this study, the NHSS proposes six different distance similarity measures that deal with multi-attributive values and multi-objective decision-making problems. NHSS's

TABLE 6. Distance similarity measures using S^3 (ℙ, ℚ).

Similarity measures	*geographical regions*	Landfill	Composting	Incinerator
$s^3_{nj}(\mathbb{P},\mathbb{Q})$	$\mathfrak{Q}^1$	0.8980	**0.9090**	0.8710
	$\mathfrak{Q}^3$	0.8980	**0.9306**	0.8780
	$\mathfrak{Q}^7$	0.8780	0.8540	**0.8950**
	$\mathfrak{Q}^9$	**0.8340**	0.8290	0.8280

TABLE 7. Distance similarity measures using S^4 (ℙ, ℚ).

Similarity measures	*geographical regions*	Landfill	Composting	Incinerator
$s^4_{nj}(\mathbb{P},\mathbb{Q})$	$\mathfrak{Q}^1$	0.5936	**0.6293**	0.5122
	$\mathfrak{Q}^3$	0.5968	**0.7241**	0.5325
	$\mathfrak{Q}^7$	0.5325	0.4652	**0.5841**
	$\mathfrak{Q}^9$	**0.4134**	0.4010	0.3986

TABLE 8. Distance similarity measures using S^5 (ℙ, ℚ).

Similarity measures	*geographical regions*	Landfill	Composting	Incinerator
$s^5_{nj}(\mathbb{P},\mathbb{Q})$	$\mathfrak{Q}^1$	0.4729	**0.6510**	0.4593
	$\mathfrak{Q}^3$	0.4806	**0.6705**	0.5002
	$\mathfrak{Q}^7$	0.4506	0.4874	**0.5851**
	$\mathfrak{Q}^9$	**0.3856**	0.3826	0.3850

TABLE 9. Distance similarity measures using S^6 (ℙ, ℚ).

Similarity measures	*geographical regions*	Landfill	Composting	Incinerator
$s^6_{nj}(\mathbb{P},\mathbb{Q})$	$\mathfrak{Q}^1$	0.1973	**0.2607**	0.1862
	$\mathfrak{Q}^3$	0.2019	**0.2654**	0.1919
	$\mathfrak{Q}^7$	0.1909	0.2199	**0.2468**
	$\mathfrak{Q}^9$	**0.1843**	0.1788	0.1840

have a split construction that provides greater accuracy and efficiency. We also made a comparison table in Table.10 to demonstrate the originality of the suggested strategies and how they differ from current parallels. The numerical results are presented in Table 4-Table.9 shows the results that for the SWMS where we install these systems, the result shows that $\mathfrak{Q}^1$ should be selected for composting according to all of the proposed distance similarity measures for NHSS are consistent with each other. For composting, all of our processes have the same $\mathfrak{Q}^3$ Weight. Furthermore, the results presented in our theory suggest that Incinerator has the highest value against $\mathfrak{Q}^7$, indicating that $\mathfrak{Q}^7$ should be chosen for Incinerator for environmentally friendly solid waste treatment. Similarly, landfills are the most significant method for managing solid waste, but because they take more space and cash to install, according to our recommended techniques, landfills have the highest value versus, $\mathfrak{Q}^9$ hence $\mathfrak{Q}^9$ should be chosen for the landfill. As a result, according to all of the proposed distance similarity measures, the proportion of solid waste treatment systems in the regions is calculated and shown in the above tables. We discovered that this method of determining the best system selection for solid waste treatment systems is a highly beneficial selection tool.

VIII. CONCLUSION

For dealing with material that is partial, indeterminate, uncertain, or imprecise, neutrosophic hypersoft Sets (NHSS's) can be a strong mathematical paradigm. NHSS is more effective at dealing with uncertain and ambiguous information than fuzzy sets and intuitionistic fuzzy sets in general. In the case of NHSS, however, no one has examined distance similarity measures using Max-Min operators. Six different newly developed distances and applied in similarity measures to the NHSS environment also developed an algorithm to solve MCDM using the provided similarity metrics. Some properties, Theorems, Axioms as well as many results relating to distance and similarity measures are proved and applied to the site selection of solid waste management systems (SWMS) using the proposed six distance and similarity measures. As a result, we proposed a mathematical model as a solution to

TABLE 10. A comparison of the proposed study with the existing studies on the basis of multi-Attributions.

Researcher	Similarity Measure	Truthness	Indeterminacy	Falsity	Similarity-Based on distance Measures	Weighted Distance Measures	Max-Min Distance Measures
Broumi and Samandrache [25]	SVNSs	Yes	Yes	Yes	Yes	No	No
Cui and Ye [26]	SVNSs	Yes	Yes	Yes	Yes	No	No
Ye [30]	SVNSs	Yes	Yes	Yes	Yes	No	No
Sahin et al. [34]	SVNSs	Yes	Yes	Yes	Yes	No	No
Ye and Zang [35]	SVNSS	Yes	Yes	Yes	Yes	No	No
Ye and Fu. [36]	SVNSs	Yes	Yes	Yes	Yes	No	No
Majumdar and Samanta [37]	NSs	Yes	Yes	Yes	Yes	No	No
Peng and Samandrache [38]	NSs	Yes	Yes	Yes	Yes	No	No
Mondal and Pramanik [39]	SVNSs	Yes	Yes	Yes	Yes	No	No
Chai et al.[40]	SVNSs	Yes	Yes	Yes	Yes	No	No
Liu et al.[41]	SVNSs	Yes	Yes	Yes	Yes	No	No
Jafar et al.[42]	SVNSs	Yes	Yes	Yes	Yes	No	No
Garg and Nancy [43]	SVNSs	Yes	Yes	Yes	No	No	No
Saqlain et al. [63]	SVNHSS	Yes	Yes	Yes	Yes	No	No
Jafar et al. [72]	SVNHSS	Yes	Yes	Yes	Yes	No	No
Proposed	SVNHSS	Yes	Yes	Yes	Yes	Yes	Yes

a global problem. Finally, compare the proposed theory to the existing theories in Table 10. Supplier selection, manufacturing frameworks, and several other management frameworks could all benefit from the NHSS-Similarity metrics. In the future, the presented methodologies could be used in case studies with several qualities that are further bifurcated, as well as multiple decision-makers. This proposed work can also be extended to many existing hybrids of the hypersoft set, such as FHS's, IHSs, Bi-polar HSs, m-polar HSs, Pythagorean HSs (with its hybrids, fuzzy, intuitionistic, and neutrosophic), and many others.

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