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Improved Delay-Dependent Stability Analysis of Fixed-Point State-Space Digital Filters With Time-Varying Delay and Generalized Overflow Arithmetic

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ABSTRACT This paper is concerned with the stability analysis of fixed-point state-space digital filters with generalized overflow arithmetic and a time-varying delay. This paper aims to derive a delay and nonlinear function bound dependent asymptotical stability criterion with less conservatism. Firstly, a new Lyapunov functional with several augmented terms, including extra free matrices and overflow nonlinear function, is constructed such that it has a relaxed positive condition. Then, for bounding the summation term arising in the forward difference of Lyapunov functional, a new lemma is developed to introduce the terms for linking the delayed states and the overflow nonlinear function, the Wirtinger-based summation inequality and several zero-value terms are applied to add more cross terms. As a result, a stability criterion with less conservatism is established and its conservatism. Finally, several numerical examples are given to illustrate the advantages of the proposed method.

INDEX TERMS Asymptotic stability, digital filter, generalized overflow arithmetic, time-varying delay.

I. INTRODUCTION

As an effective device that produces the desired discrete-time output signal from the original input signal, digital filter becomes a necessary element of everyday electronics like radios, cell phones, and stereo receivers. Due to its large-scale applications in many areas such as radar, image processing, telecommunications, signal processing, the analysis of properties and performances of the digital filters has attracted considerable attention in the past few decades (see [1]–[3] and references therein).

During the practical implementation of a digital filter via hardware using the fixed-point arithmetic, the complex operations within the hardware require increasing wordlength to deal with the signals. On the other side, because of the limitation of register length, the quantization and overflow correction mechanisms are commonly required to reduce the wordlength [2]. Therefore, nonlinearities, including magnitude truncation, roundoff, or value truncation due to quantization and saturation, zeroing, two's complement, or triangular for overflow, are unavoidable [4], [5]. Those nonlinearities in turn lead to undesirable behaviors, for example, performance degradation, oscillations, and limit cycles [6]. The stability problem of digital filters with different nonlinearity has been considered. Under the consideration that the influence of quantization and that of overflow can be studied separately if the total number of quantization steps (or internal wordlength) is large sufficiently [7], a stability criterion of digital filters with saturation arithmetic was presented in [8] and was improved in [9]. Linear matrix inequality (LMI) based stability criteria for direct form digital filters utilizing single saturation nonlinear were developed in [7] and [10].

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In [11], the stability analysis of fixed-point state-space digital filters with generalized saturation nonlinear was discussed. Due to the fact that the hardware implementation of saturation arithmetic is more expensive than that of two's complement arithmetic, the digital filters using two's complement arithmetic were investigated, and the stability criteria for such type of filters were also proposed [12]–[14]. Stability criteria for direct-form digital filters utilizing two's complement nonlinearity were proposed in [15] and [16]. By taking into account the possibility of influence of both overflow and quantization, stability criteria for digital filters with different combinations of overflow and quantization nonlinearities were established [4], [17], [18]. Furthermore, in order to analyze the possible effect of external disturbances, different performances of digital filters were successively investigated, for example, the H_∞, l_2-l_∞, and l_∞ performances [19]–[22], the input-to-state stability (ISS) and the input/output-to-state stability (IOSS) analysis [5], [6], [23]–[26], the dissipativity analysis [27], [28], local stability analysis [29], [30], and so on.

Besides the nonlinearities and external disturbances mentioned above, time delay is frequently encountered in many systems [31]–[46] and also exists in digital filters. For example, a causal digital filter with a fixed order and cutoff frequency will delay different frequency signals [47]. For the discrete-time systems with quantization/overflow nonlinearities and time delays, a delay-independent stability criterion was proposed in [48]. For digital filters with delay and overflow nonlinearity, the exponential stability analysis methods [1] and the robust stability criterion [2] has been proposed, respectively. However, the delays concerned in [1] and [2] are all constant. In [49], a delay-dependent criterion was developed by using free-weighting matrix approach for the asymptotic stability of a class of uncertain discrete-time state-delayed systems with the combination of quantization and overflow nonlinearities. For the digital filters with generalized overflow nonlinearity, a stability condition depends not only on the delay bounds but also on the bounds of nonlinear function was reported in [47] with the help of Jensen-based summation inequality. In [50], the extended dissipativity analysis for digital filters with a time-varying delay and Markovian jumping parameters was investigated, and a criterion was given by putting forward a general form of nonlinearity functions and employing the reciprocally convex combination approach. While the criteria reported in [47], [49], [50] are all based on a simple Lyapunov functional, which is simple but conservative. In [3], the ISS problem of digital filters in the presence of both external disturbance and time-varying delay was discussed and stability criteria were derived by using simple Lyapunov functionals and Jensen-like summation inequality.

Based on the above discussions, there still remains room for further investigation on the analysis of digital filters with the overflow nonlinearity and the time-varying delay. From the research on digital filters point of view, there are only a few works on digital filters considering the time-varying delay [3], [47], [49], [50], and the techniques used therein are all conservative in comparison to the ones developed for the time-delay systems. Recently, many more effective methods have been developed for dealing with time-varying delays, such as augmented Lyapunov functionals [51]–[53], new inequalities [42], [54]–[56], extended reciprocally convex matrix inequalities [57]–[59], etc. From the viewpoint of techniques dealing with the discrete-time delay systems, every term in the Lyapunov functionals is usually required to positive in order to guarantee the positive-definiteness of the functionals. Such strict requirement leads to the conservatism. The conservatism of the methods may lead to inaccurate results. Therefore, how to reduce the conservatism motivates the current research.

In this paper, the stability analysis problem of digital filters with generalized overflow nonlinearity and a time-varying delay is further investigated. The main contribution of the paper is that a new delay and nonlinearity bounds dependent stability criterion with less conservatism is developed, and the proposed criterion can provide more accurate delay stable region (namely, the allowably maximal delay region such that the stability of the digital filter with any delay belonging to such region is guaranteed). The advantage of the proposed stability criterion is illustrated based on several numerical examples. The main techniques, different from the previous publications, are summarized as follows.

- The first aspect is on the construction of the Lyapunov functional. Several augmented terms, especially the one with the information of overflow nonlinearity, are introduced into the Lyapunov functional and the condition of positive-definiteness of functional is relaxed by requiring the sum of all terms, instead of each term, be positive. Those treatments can provide extra freedom for the feasibility of the obtained criterion.
- The second aspect relies on the estimation of the forward difference of the functional. A new lemma is developed to introduce new cross terms for constructing the link between the delay states and the overflow nonlinear function. Moreover, several methods (such as the Wirtinger-based summation inequality, the extended reciprocally convex matrix inequality, and zero-value equations), which are not used in the literature on delayed digital filters, are applied to estimate the forward difference of the functional as accurate as possible.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ respectively denote the set of all the n-dimensional vectors and that of all the $m \times n-$dimensional real matrices; $\|\cdot\|$ denotes the Euclidean norm; the superscripts T and -1 stand for the transpose and the inverse of a matrix, respectively; $diag\{\cdots\}$ denotes a block-diagonal matrix; $P > 0\ (\geq 0)$ means that P is a positive-definite (semi-positive-definite) symmetric matrix; I and 0 represent the identity matrix and the zero-matrix with appropriate dimensions, respectively; the symmetric term in a symmetric matrix is denoted by $*$; and $Sym\{X\} = X + X^T$.

II. PROBLEM FORMULATION AND PRELIMINARY

Consider the following digital filter with a time-varying delay:

$$\begin{cases} x(k+1) = f(y(k)), \\ y(k) = Ax(k) + A_d x(k-\tau(k)), \\ x(k) = \phi(k), \quad k \in \{-\tau_2, \ldots, 0\}, \end{cases} \tag{1}$$

where $x(k) = [x_1(k), x_2(k), \cdots, x_n(k)]^T \in \mathbb{R}^n$ is the state vector; $\phi(k) = [\phi_1(k), \phi_2(k), \cdots, \phi_n(k)]^T \in \mathbb{R}^n$ is the initial condition with $|\phi_i(k)| \leq 1$, $y(k) = [y_1(k), y_2(k), \cdots, y_n(k)]^T \in \mathbb{R}^n$ is the filter output vector; $\tau(k)$ is the time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2, \tag{2}$$

with τ_1 and τ_2 being constant; A and A_d are the known interconnection weight matrices; the nonlinearity function $f(\cdot)$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined as follows [47]

$$\begin{cases} -1 \leq l_i \leq f_i(y_i(k)) \leq l_{1i} \leq 1, & y_i(k) > 1, \\ f_i(y_i(k)) = y_i(k), & -1 \leq y_i(k) \leq 1, \\ -1 \leq -l_{2i} \leq f_i(y_i(k)) \leq -l_i \leq 1, & y_i(k) < -1, \end{cases} \tag{3}$$

with $i = 1, 2, \ldots, n$, l_i, l_{1i} and l_{2i} being known real scalars.

Remark 1: As mentioned in [47], the nonlinear relationships shown in (3) include various overflow arithmetics by fixing the values of l_i, l_{1i}, and l_{2i}. For example, (3) gives saturation nonlinearity for $l_i = l_{1i} = l_{2i} = 1$; (3) indicates zeroing nonlinearity for $l_i = l_{1i} = l_{2i} = 0$; and (3) shows two's complement nonlinearity for $l_i = -1, l_{1i} = l_{2i} = 1$. That is, the stability criterion developed in this paper can be used to check the stability of digital filters with the above three types of overflow nonlinearities.

In order to analyze the influence of the time-varying delay on the stability of digital filter (1), this paper aims to develop a less conservative delay-dependent stability criterion.

The following lemmas to be used for handling time delays are given.

Lemma 1: (Wirtinger-based inequality [54]). For a given positive definite matrix R, integers $b \geq a$, any sequence of discrete-time variable $x : \mathcal{Z}[a, b] \rightarrow \mathbb{R}^n$, the following inequality holds:

$$(b-a)\sum_{i=a}^{b-1} \triangle x^T(i)R\triangle x(i) \geq \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}, \tag{4}$$

where

$$\triangle x(k) = x(k+1) - x(k),$$
$$\chi_1 = x(b) - x(a),$$
$$\chi_2 = x(b) + x(a) - \frac{2}{b-a+1}\sum_{i=a}^{b} x(i).$$

Lemma 2: (Jensen-based inequality [60]). For a given positive definite matrix R, integers $b \geq a$, any sequence of discrete-time variable $x : \mathcal{Z}[a, b] \rightarrow \mathbb{R}^n$, the following inequality holds:

$$\sum_{i=a}^{b-1} x^T(i)Rx(i) \geq \frac{1}{b-a}\left(\sum_{i=a}^{b-1} x(i)\right)^T R \left(\sum_{i=a}^{b-1} x(i)\right). \tag{5}$$

Lemma 3: (Extended reciprocally convex matrix inequality [57], [58]). For a real scalar $0 < \alpha < 1$, positive-definite symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$, and any matrix $N \in \mathbb{R}^{n\times n}$, the following inequality holds:

$$\begin{bmatrix} \frac{1}{\alpha}X & 0 \\ 0 & \frac{1}{1-\alpha}Y \end{bmatrix} \geq \begin{bmatrix} X + (1-\alpha)T_1 & N \\ * & Y + \alpha T_2 \end{bmatrix}, \tag{6}$$

where $T_1 = X - NY^{-1}N^T$ and $T_2 = Y - N^T X^{-1} N$.

The following lemmas related to the overflow nonlinearity are given.

Lemma 4: [47] Let $\hat{l}_i = min\{l_i, 0\}$. Then the following inequality holds for nonlinear functions $f_i(\cdot)$ satisfying condition (3):

$$\Big[y_i(k) - f_i(y_i(k))\Big]\Big[f_i(y_i(k)) - \hat{l}_i y_i(k)\Big] \geq 0. \tag{7}$$

Lemma 5: [14] For given digital filter (1) satisfying condition (3), if there exist matrix $S = diag(s_1, s_2, \ldots, s_n) > 0$ and any matrices $M = [m_{ij}]_{n\times n}$, $N = [n_{ij}]_{n\times n}$ satisfying

$$s_i \geq \sum_{j=1}^{n} |m_{ji}| + \sum_{j=1}^{n} |n_{ji}|, \quad i = 1, 2, \ldots, n, \tag{8}$$

then the following inequality holds:

$$\Big[y^T(k)S + x^T(k)M + f^T(y(k))N\Big]\Big[y(k) - f(y(k))\Big] \geq 0. \tag{9}$$

Lemma 6: For given digital filter (1) satisfying condition (3), if there exist matrices $S_1 = diag(s_{11}, s_{12}, \ldots, s_{1n}) > 0$, and $S_2 = diag(s_{21}, s_{22}, \ldots, s_{2n}) > 0$, and any matrices $M_1 = [m_{1ij}]_{n\times n}$ and $M_2 = [m_{2ij}]_{n\times n}$ satisfying

$$s_{1i} \geq \sum_{j=1}^{n} |m_{1ji}|, \quad i = 1, 2, \ldots, n, \tag{10}$$

$$s_{2i} \geq \sum_{j=1}^{n} |m_{2ji}|, \quad i = 1, 2, \ldots, n, \tag{11}$$

then the following inequalities hold:

$$\Big[y^T(k)S_1 + x^T(k-\tau(k))M_1\Big]\Big[y(k) - f(y(k))\Big] \geq 0, \tag{12}$$

$$\Big[y^T(k)S_2 + x^T(k-1)M_2\Big]\Big[y(k) - f(y(k))\Big] \geq 0. \tag{13}$$

Proof: It is easy to find that (12) holds if $|y_i(k)| \leq 1$ (i.e., $y_i(k) = f_i(y_i(k))$ based on (3)). For the case of $|y_i(k)| > 1$, the left-hand side of (12) can be rewritten as

$$\sum_{i=1}^{n}\Big[y_i(k)s_{1i} + \sum_{j=1}^{n} x_j(k-\tau(k))m_{1ji}\Big]\Big[y_i(k) - f_i(y_i(k))\Big]$$

$$= \sum_{i=1}^{n} y_i^2(k)\Big[s_{1i} + \sum_{j=1}^{n} \frac{x_j(k-\tau(k))}{y_i(k)} m_{1ji}\Big]\Big[1 - \frac{f_i(y_i(k))}{y_i(k)}\Big]. \tag{14}$$

Then, it follows from $|y_i(k)| > 1$, $|f_j(y_j(k))| \leq 1$, $|x_j(k-\tau(k))| \leq 1$ (obtained from (1)), and (10) that

$$1 - \frac{f_i(y_i(k))}{y_i(k)} > 0, \tag{15}$$

and

$$\begin{aligned} s_{1i} &+ \sum_{j=1}^{n} \frac{x_j(k-\tau(k))}{y_i(k)} m_{1ji} \\ &\geq s_{1i} - \sum_{j=1}^{n} \left|\frac{x_j(k-\tau(k))}{y_i(k)}\right| |m_{1ji}| \\ &\geq s_{1i} - \sum_{j=1}^{n} |m_{1ji}| \\ &\geq 0. \end{aligned} \tag{16}$$

Combining (14), (15), and (16) leads that (12) holds for the case of $|y_i(k)| > 1$. Thus, (12) holds for all $y_i(k)$.

Similar, the holding of (13) can be proved if (11) holds. ■

Remark 2: Compared with (9) used in [3], [47], [50], in which only delay-free states, $y(k)$ and $x(k)$, are linked with the nonlinear function $f(y(k))$, (12) and (13) in Lemma 6 introduce many additional cross terms related to the delayed states, $x(k-\tau(k))$ and $x(k-1)$, and overflow nonlinear function, $f(y(k))$, which constructs the link between delayed states and overflow nonlinear function. In fact, the holding of (7), (9), (12), and (13) is based on the special feature of nonlinear function caused by overflow correction mechanism. The usage of those information is an important difference in comparison to the linear discrete-time delayed systems [42], [54]–[56] or traditional Lur'e nonlinear discrete-time delayed systems [61], [62], and it is also the one of important treatments for reducing the conservatism.

III. MAIN RESULTS

In this section, a new delay-dependent stability criterion is derived by constructing an augmented Lypunov functional and using several new techniques to estimate the forward difference of the functional.

Before giving the main results, the following notations are defined to simplify the expression of the proof of stability criterion.

$$\upsilon_1(k) = \sum_{i=k-\tau_1}^{k-1} x(i),$$

$$\upsilon_2(k) = \sum_{i=k-\tau(k)}^{k-\tau_1-1} x(i),$$

$$\upsilon_3(k) = \sum_{i=k-\tau_2}^{k-\tau(k)-1} x(i),$$

$$\upsilon_4(k) = \sum_{i=k-\tau(k)}^{k-\tau_1} \frac{x(i)}{\tau(k)-\tau_1+1},$$

$$\upsilon_5(k) = \sum_{i=k-\tau_2}^{k-\tau(k)} \frac{x(i)}{\tau_2-\tau(k)+1},$$

$$\begin{aligned} \xi(k) = \Big[&f^T(y(k)), \\ &x^T(k), x^T(k-\tau_1), x^T(k-\tau(k)), x^T(k-\tau_2), \\ &\upsilon_1^T(k), \upsilon_2^T(k), \upsilon_3^T(k), \upsilon_4^T(k), \upsilon_5^T(k), x^T(k-1)\Big]^T, \end{aligned}$$

$$v_6(k) = \sum_{i=k-\tau_1}^{k-1} x(i),$$

$$v_7(k) = \sum_{i=k-\tau_2}^{k-\tau_1-1} x(i),$$

$$\begin{aligned} \bar{\xi}(k) = \Big[&x^T(k), v_6^T(k), v_7^T(k), \\ &x^T(k-\tau_1), x^T(k-\tau_2), x^T(k-1)\Big]^T, \end{aligned}$$

The stability criterion developed is shown as follows.

Theorem 1: For given scalars l_i, τ_1 and τ_2, digital filter (1) with time-varying satisfying (2) is asymptotically stable if there exist symmetric matrices P_1, P_2, Z, Q_1, Q_2, R_1, R_2, T_1, T_2, positive definite diagonal matrices S, S_1, S_2, D, and any matrices X, U_1, U_2, M, M_1, M_2, N, such that conditions (8), (10), and (11), and the following LMIs are feasible:

$$R_i > 0, \quad i = 1, 2, \tag{17}$$

$$\hat{\Phi}_1 = \begin{bmatrix} 0 & 0 \\ 0 & R_2 \end{bmatrix} + Z > 0, \tag{18}$$

$$\hat{\Phi}_2 = \tau_{12}^2 Z + \begin{bmatrix} Q_1 & 0 \\ 0 & \tau_{12}^2 R_2 + \tau_1 R_1 \end{bmatrix} > 0, \tag{19}$$

$$\hat{\Phi}_3 = \tau_{12} Z + \begin{bmatrix} Q_2 & 0 \\ 0 & \tau_{12} R_2 \end{bmatrix} > 0, \tag{20}$$

$$\begin{aligned} \hat{\Phi}_4 = &\begin{bmatrix} \bar{e}_2 \\ \bar{e}_1-\bar{e}_4 \end{bmatrix}^T \frac{\hat{\Phi}_2}{\tau_1} \begin{bmatrix} \bar{e}_2 \\ \bar{e}_1-\bar{e}_4 \end{bmatrix} + \begin{bmatrix} \bar{e}_3 \\ \bar{e}_4-\bar{e}_5 \end{bmatrix}^T \frac{\hat{\Phi}_3}{\tau_{12}} \begin{bmatrix} \bar{e}_3 \\ \bar{e}_4-\bar{e}_5 \end{bmatrix} \\ &+ \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix}^T P_1 \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix} + \begin{bmatrix} \bar{e}_6 \\ \bar{e}_1 \end{bmatrix}^T P_2 \begin{bmatrix} \bar{e}_6 \\ \bar{e}_1 \end{bmatrix} \\ &+ \tau_1(\tau_1-1)[\bar{e}_1-\bar{e}_{11}]^T R_1 [\bar{e}_1-\bar{e}_{11}] \\ &> 0, \end{aligned} \tag{21}$$

$$\begin{bmatrix} \Psi|_{\tau(k)=\tau_1} & E_2^T X \\ * & -\Xi_1 \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \Psi|_{\tau(k)=\tau_2} & E_3^T X^T \\ * & -\Xi_2 \end{bmatrix} < 0, \tag{23}$$

$$Z + \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} > 0, \tag{24}$$

$$Z + \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} > 0, \tag{25}$$

where

$$\Psi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 - \Phi_7 - \Phi_8,$$

$$\Phi_1 = \Pi_1^T P_1 \Pi_1 - \Pi_2^T P_1 \Pi_2 + \Pi_3^T P_2 \Pi_3 - \Pi_4^T P_2 \Pi_4,$$
$$\Pi_1 = \begin{bmatrix} e_1 \\ e_6 + e_2 - e_3 \\ e_7 + e_8 + e_3 - e_5 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} e_2 \\ e_6 \\ e_7 + e_8 \end{bmatrix},$$
$$\Pi_3 = \begin{bmatrix} e_2 \\ e_1 \end{bmatrix}, \quad \Pi_4 = \begin{bmatrix} e_{11} \\ e_2 \end{bmatrix},$$
$$\Phi_2 = e_2^T Q_1 e_2 - e_3^T (Q_1 - Q_2) e_3 - e_5^T Q_2 e_5,$$
$$\Phi_3 = \Pi_5^T (\tau_1^2 R_1 + \tau_{12}^2 R_2) \Pi_5 + \tau_{12}^2 \begin{bmatrix} e_2 \\ \Pi_5 \end{bmatrix}^T Z \begin{bmatrix} e_2 \\ \Pi_5 \end{bmatrix},$$
$$\Pi_5 = e_1 - e_2,$$
$$\Phi_4 = \tau_{12}\Big[e_3^T T_1 e_3 - e_4^T (T_1 - T_2) e_4 - e_5^T T_2 e_5\Big],$$
$$\Phi_5 = Sym\Big\{[\Pi_6 - e_1]^T D [e_1 - L\Pi_6]\Big\},$$
$$+Sym\Big\{[\Pi_6^T S + e_2^T M + e_1^T N][\Pi_6 - e_1]\Big\},$$
$$+Sym\Big\{[\Pi_6^T S_1 + e_4^T M_1][\Pi_6 - e_1]\Big\},$$
$$+Sym\Big\{[\Pi_6^T S_2 + e_{11}^T M_2][\Pi_6 - e_1]\Big\},$$
$$[2pt]\Pi_6 = A e_2 + A_d e_4,$$
$$\Phi_6 = Sym\Big\{e_g^T U_1[(\tau(k) - \tau_1 + 1)e_9 - e_7 - e_3]\Big\}$$
$$+Sym\Big\{e_g^T U_2[(\tau_2 - \tau(k) + 1)e_{10} - e_8 - e_4]\Big\},$$
$$\Phi_7 = E_1^T \begin{bmatrix} R_1 & 0 \\ 0 & 3R_1 \end{bmatrix} E_1,$$
$$E_1 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - \frac{2}{\tau_1+1}(e_2 + e_6) \end{bmatrix},$$
$$\Phi_8 = \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \frac{2\tau_2 - \tau(k) - \tau_1}{\tau_{12}} \Xi_1 & X \\ * & \frac{\tau_2 + \tau(k) - 2\tau_1}{\tau_{12}} \Xi_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} e_7 \\ e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix},$$
$$E_3 = \begin{bmatrix} e_8 \\ e_4 - e_5 \\ e_4 + e_5 - 2e_{10} \end{bmatrix},$$
$$\Xi_1 = \left[\begin{array}{c|c} Z + \begin{bmatrix} 0 & T_1 \\ T_1 & R_2 + T_1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0 \;\; 0] & 3R_2 \end{array}\right],$$
$$\Xi_2 = \left[\begin{array}{c|c} Z + \begin{bmatrix} 0 & T_2 \\ T_2 & R_2 + T_2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0 \;\; 0] & 3R_2 \end{array}\right],$$
$$e_i = \Big[0_{n\times(i-1)n}, \; I_n, \; 0_{n\times(11-i)n}\Big] \quad (i = 1, 2, \cdots, 11),$$
$$e_g = \Big[e_3^T, \; e_4^T, \; e_7^T, \; e_8^T, \; e_9^T, \; e_{10}^T\Big]^T,$$
$$\bar{e}_i = [0_{n\times(i-1)n}, I_n, 0_{n\times(6-i)n}], \quad i = 1, 2, \cdots, 6,$$
$$L = diag\{\hat{l}_1, \hat{l}_2, \cdots, \hat{l}_n\},$$
$$\hat{l}_i = min\{l_i, 0\}.$$

Proof: Firstly, choose the following functional candidate:

$$V(k) = \sum_{i=1}^{4} V_i(k), \tag{26}$$

where

$$V_1(k) = \eta_1^T(k) P_1 \eta_1(k) + \eta_2^T(k) P_2 \eta_2(k),$$
$$V_2(k) = \sum_{i=k-\tau_1}^{k-1} x^T(i) Q_1 x(i) + \sum_{i=k-\tau_2}^{k-\tau_1-1} x^T(i) Q_2 x(i),$$
$$V_3(k) = \tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_1 \triangle x(j)$$
$$+\tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_2 \triangle x(j),$$
$$V_4(k) = \tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \eta_3^T(j) Z \eta_3(j),$$

with $P_1, P_2, Z, Q_1, Q_2, R_1,$ and R_2 being symmetric matrices, $R_i > 0, \; i = 1, 2,$ and

$$\tau_{12} = \tau_2 - \tau_1$$
$$\eta_1(k) = \left[x^T(k), \; \sum_{i=k-\tau_1}^{k-1} x^T(i), \; \sum_{i=k-\tau_2}^{k-\tau_1-1} x^T(i)\right]^T,$$
$$\eta_2(k) = \Big[x^T(k-1), \; f^T(y(k-1))\Big]^T,$$
$$\eta_3(i) = \Big[x^T(i), \; \triangle x^T(i)\Big]^T.$$

The second step is to find the conditions to ensure the positive-definiteness of functional (26). Based on $R_1 > 0$ in (17), the following holds

$$\tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_1 \triangle x(j)$$
$$= \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j)$$
$$+\tau_1 \sum_{i=-\tau_1+1}^{-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_1 \triangle x(j)$$
$$> \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j)$$
$$+\tau_1 \sum_{i=-\tau_1+1}^{-1} \sum_{j=k-1}^{k-1} \triangle x^T(j) R_1 \triangle x(j)$$
$$= \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j)$$
$$+\tau_1 \sum_{i=-\tau_1+1}^{-1} \triangle x^T(k-1) R_1 \triangle x(k-1)$$

$$
\begin{aligned}
&= \tau_1(\tau_1-1)[x(k)-x(k-1)]^T R_1[x(k)-x(k-1)]\\
&\quad +\tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j)R_1\triangle x(j). \qquad (27)
\end{aligned}
$$

Based on $\hat{\Phi}_1 > 0$ in (18), the following is correct

$$
\begin{aligned}
&\tau_{12}\sum_{i=-\tau_2}^{-\tau_1-1}\sum_{j=k+i}^{k-1}\triangle x^T(j)R_2\triangle x(j)+V_4(k)\\
&= \tau_{12}\sum_{i=-\tau_2}^{-\tau_1-1}\sum_{j=k+i}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&= \tau_{12}\sum_{j=k-\tau_2}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\tau_{12}\sum_{i=-\tau_2+1}^{-\tau_1-1}\sum_{j=k+i}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&> \tau_{12}\sum_{j=k-\tau_2}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\tau_{12}\sum_{i=-\tau_2+1}^{-\tau_1-1}\sum_{j=k-\tau_1}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&= \tau_{12}\sum_{j=k-\tau_1}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\tau_{12}\sum_{j=k-\tau_2}^{k-\tau_1-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\tau_{12}(\tau_{12}-1)\sum_{j=k-\tau_1}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&= \tau_{12}^2\sum_{j=k-\tau_1}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\tau_{12}\sum_{j=k-\tau_2}^{k-\tau_1-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}. \qquad (28)
\end{aligned}
$$

Combining (26), (27), and (28) yields

$$
\begin{aligned}
&V_2(k)+V_3(k)+V_4(k)\\
&> \sum_{i=k-\tau_1}^{k-1}x^T(i)Q_1x(i)+\sum_{i=k-\tau_2}^{k-\tau_1-1}x^T(i)Q_2x(i)\\
&\quad +\tau_1\sum_{j=k-\tau_1}^{k-1}\triangle x^T(j)R_1\triangle x(j)\\
&\quad +\tau_1(\tau_1-1)[x(k)-x(k-1)]^T R_1[x(k)-x(k-1)]\\
&\quad +\tau_{12}^2\sum_{j=k-\tau_1}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\tau_{12}\sum_{j=k-\tau_2}^{k-\tau_1-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_1\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&= \tau_1(\tau_1-1)[x(k)-x(k-1)]^T R_1[x(k)-x(k-1)]\\
&\quad +\sum_{j=k-\tau_1}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_2\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\sum_{j=k-\tau_2}^{k-\tau_1-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_3\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}, \qquad (29)
\end{aligned}
$$

where $\hat{\Phi}_2$ and $\hat{\Phi}_3$ are defined in (19) and (20), respectively. Furthermore, using $\hat{\Phi}_2 > 0$, $\hat{\Phi}_3 > 0$ and applying (5) to estimate the summation terms in (29) yield

$$
\begin{aligned}
&\sum_{j=k-\tau_1}^{k-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_2\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&\quad +\sum_{j=k-\tau_2}^{k-\tau_1-1}\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}^T\hat{\Phi}_3\begin{bmatrix}x(j)\\ \triangle x(j)\end{bmatrix}\\
&> \begin{bmatrix}\sum_{i=k-\tau_1}^{k-1}x(i)\\ x(k)-x(t-\tau_1)\end{bmatrix}^T\frac{\hat{\Phi}_2}{\tau_1}\begin{bmatrix}\sum_{i=k-\tau_1}^{k-1}x(i)\\ x(k)-x(t-\tau_1)\end{bmatrix}\\
&\quad +\begin{bmatrix}\sum_{i=k-\tau_2}^{k-\tau_1-1}x(i)\\ x(t-\tau_1)-x(t-\tau_2)\end{bmatrix}^T\frac{\hat{\Phi}_3}{\tau_{12}}\begin{bmatrix}\sum_{i=k-\tau_2}^{k-\tau_1-1}x(i)\\ x(t-\tau_1)-x(t-\tau_2)\end{bmatrix}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&V_1(k)+V_2(k)+V_3(k)+V_4(k)\\
&> \begin{bmatrix}x(k)\\ \sum_{i=k-\tau_1}^{k-1}x(i)\\ \sum_{i=k-\tau_2}^{k-\tau_1-1}x(i)\end{bmatrix}^T P_1\begin{bmatrix}x(k)\\ \sum_{i=k-\tau_1}^{k-1}x(i)\\ \sum_{i=k-\tau_2}^{k-\tau_1-1}x(i)\end{bmatrix}\\
&\quad +\begin{bmatrix}x(k-1)\\ f(y(k-1))\end{bmatrix}^T P_2\begin{bmatrix}x(k-1)\\ f(y(k-1))\end{bmatrix}\\
&\quad +\tau_1(\tau_1-1)[x(k)-x(k-1)]^T R_1[x(k)-x(k-1)]\\
&\quad +\begin{bmatrix}\sum_{i=k-\tau_1}^{k-1}x(i)\\ x(k)-x(t-\tau_1)\end{bmatrix}^T\frac{\hat{\Phi}_2}{\tau_1}\begin{bmatrix}\sum_{i=k-\tau_1}^{k-1}x(i)\\ x(k)-x(t-\tau_1)\end{bmatrix}\\
&\quad +\begin{bmatrix}\sum_{i=k-\tau_2}^{k-\tau_1-1}x(i)\\ x(t-\tau_1)-x(t-\tau_2)\end{bmatrix}^T\frac{\hat{\Phi}_3}{\tau_{12}}\begin{bmatrix}\sum_{i=k-\tau_2}^{k-\tau_1-1}x(i)\\ x(t-\tau_1)-x(t-\tau_2)\end{bmatrix}\\
&= \begin{bmatrix}\bar{e}_1\bar{\xi}(k)\\ \bar{e}_2\bar{\xi}(k)\\ \bar{e}_3\bar{\xi}(k)\end{bmatrix}^T P_1\begin{bmatrix}\bar{e}_1\bar{\xi}(k)\\ \bar{e}_2\bar{\xi}(k)\\ \bar{e}_3\bar{\xi}(k)\end{bmatrix}+\begin{bmatrix}\bar{e}_6\bar{\xi}(k)\\ \bar{e}_1\bar{\xi}(k)\end{bmatrix}^T P_2\begin{bmatrix}\bar{e}_6\bar{\xi}(k)\\ \bar{e}_1\bar{\xi}(k)\end{bmatrix}\\
&\quad +\tau_1(\tau_1-1)[\bar{e}_1-\bar{e}_{11}]^T R_1[\bar{e}_1-\bar{e}_{11}]\\
&\quad +\begin{bmatrix}\bar{e}_2\bar{\xi}(k)\\ \bar{e}_1\bar{\xi}(k)-\bar{e}_4\bar{\xi}(k)\end{bmatrix}^T\frac{\hat{\Phi}_2}{\tau_1}\begin{bmatrix}\bar{e}_2\bar{\xi}(k)\\ \bar{e}_1\bar{\xi}(k)-\bar{e}_4\bar{\xi}(k)\end{bmatrix}\\
&\quad +\begin{bmatrix}\bar{e}_3\bar{\xi}(k)\\ \bar{e}_4\bar{\xi}(k)-\bar{e}_5\bar{\xi}(k)\end{bmatrix}^T\frac{\hat{\Phi}_3}{\tau_{12}}\begin{bmatrix}\bar{e}_3\bar{\xi}(k)\\ \bar{e}_4\bar{\xi}(k)-\bar{e}_5\bar{\xi}(k)\end{bmatrix}\\
&= \bar{\xi}^T(k)\hat{\Phi}_4\bar{\xi}(k), \qquad (30)
\end{aligned}
$$

where $\hat{\Phi}_4$ is defined in (21).

It follows from (21) and (30) that

$$V(k) > \bar{\xi}^T(k)\hat{\Phi}_4\bar{\xi}(k) \geq \epsilon||x(k)||^2, \tag{31}$$

where ϵ is a sufficient small positive scalar.

The third step is to find the conditions to ensure the negative-definiteness of forward difference of functional (26). Defining the forward difference of Lyapunov functional as $\Delta V(k) = V(k+1) - V(k)$ and calculating it along the trajectories of digital filter (1) yield

$$\Delta V(k) = \sum_{i=1}^{4} \Delta V_i(k), \tag{32}$$

where $\Delta V_1(k)$ is given as

$$\begin{aligned}
&\Delta V_1(k) \\
&= V_1(k+1) - V_1(k) \\
&= \eta_1^T(k+1)P_1\eta_1(k+1) - \eta_1^T(k)P_1\eta_1(k) \\
&\quad +\eta_2^T(k+1)P_2\eta_2(k+1) - \eta_2^T(k)P_2\eta_2(k) \\
&= \begin{bmatrix} x(k+1) \\ \sum_{i=k-h_1+1}^{k} x(i) \\ \sum_{i=k-h_2+1}^{k-h_1} x(i) \end{bmatrix}^T P_1 \begin{bmatrix} x(k+1) \\ \sum_{i=k-h_1+1}^{k} x(i) \\ \sum_{i=k-h_2+1}^{k-h_1} x(i) \end{bmatrix} \\
&\quad - \begin{bmatrix} x(k) \\ \sum_{i=k-h_1}^{k-1} x(i) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix}^T P_1 \begin{bmatrix} x(k) \\ \sum_{i=k-h_1}^{k-1} x(i) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix} \\
&\quad + \begin{bmatrix} x(k) \\ f(y(k)) \end{bmatrix}^T P_2 \begin{bmatrix} x(k) \\ f(y(k)) \end{bmatrix} - \begin{bmatrix} x(k-1) \\ f(y(k-1)) \end{bmatrix}^T P_2 \begin{bmatrix} x(k-1) \\ f(y(k-1)) \end{bmatrix} \\
&= \begin{bmatrix} e_1\xi(k) \\ (e_6+e_2-e_3)\xi(k) \\ (e_7+e_8+e_3-e_5)\xi(k) \end{bmatrix}^T P_1 \begin{bmatrix} e_1\xi(k) \\ (e_6+e_2-e_3)\xi(k) \\ (e_7+e_8+e_3-e_5)\xi(k) \end{bmatrix} \\
&\quad - \begin{bmatrix} e_2\xi(k) \\ e_6\xi(k) \\ (e_7+e_8)\xi(k) \end{bmatrix}^T P_1 \begin{bmatrix} e_2\xi(k) \\ e_6\xi(k) \\ (e_7+e_8)\xi(k) \end{bmatrix} \\
&\quad + \begin{bmatrix} e_2\xi(k) \\ e_1\xi(k) \end{bmatrix}^T P_2 \begin{bmatrix} e_2\xi(k) \\ e_1\xi(k) \end{bmatrix} - \begin{bmatrix} e_{11}\xi(k) \\ e_2\xi(k) \end{bmatrix}^T P_2 \begin{bmatrix} e_{11}\xi(k) \\ e_2\xi(k) \end{bmatrix} \\
&= \xi^T(k)(\Pi_1^T P_1 \Pi_1 - \Pi_2^T P_1 \Pi_2 + \Pi_3^T P_2 \Pi_3 - \Pi_4^T P_2 \Pi_4)\xi(k) \\
&= \xi^T(k)\Phi_1\xi(k),
\end{aligned} \tag{33}$$

with Π_1, Π_2, and Φ_1 being defined in Theorem 1.

$\Delta V_2(k)$ is given as

$$\begin{aligned}
&\Delta V_2(k) \\
&= V_2(k+1) - V_2(k) \\
&= \sum_{i=k-\tau_1+1}^{k} x^T(i)Q_1x(i) - \sum_{i=k-\tau_1}^{k-1} x^T(i)Q_1x(i) \\
&\quad + \sum_{i=k-\tau_2+1}^{k-\tau_1} x^T(i)Q_2x(i) - \sum_{i=k-\tau_2}^{k-\tau_1-1} x^T(i)Q_2x(i) \\
&= x^T(k)Q_1x(k) - x^T(k-\tau_1)Q_1x(k-\tau_1) \\
&\quad +x^T(k-\tau_1)Q_2x(k-\tau_1) - x^T(k-\tau_2)Q_2x(k-\tau_2) \\
&= \xi^T(k)(e_2^TQ_1e_2 - e_3^T(Q_1-Q_2)e_3 - e_5^TQ_2e_5)\xi(k) \\
&= \xi^T(k)\Phi_2\xi(k),
\end{aligned} \tag{34}$$

with Φ_2 being defined in Theorem 1.

$\Delta V_3(k)$ is given as

$$\begin{aligned}
\Delta V_3(k) &= V_3(k+1) - V_3(k) \\
&= \tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i+1}^{k} \Delta x^T(j)R_1\Delta x(j) \\
&\quad -\tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i}^{k-1} \Delta x^T(j)R_1\Delta x(j) \\
&\quad +\tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i+1}^{k} \Delta x^T(j)R_2\Delta x(j) \\
&\quad -\tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \Delta x^T(j)R_2\Delta x(j) \\
&= \tau_1^2\Delta x^T(k)R_1\Delta x(k) + \tau_{12}^2\Delta x^T(k)R_2\Delta x(k) \\
&\quad -J_1 - J_2 - J_3,
\end{aligned} \tag{35}$$

with

$$\begin{aligned}
J_1 &= \tau_1 \sum_{j=k-\tau_1}^{k-1} \Delta x^T(j)R_1\Delta x(j), \\
J_2 &= \tau_{12} \sum_{j=k-\tau(t)}^{k-\tau_1-1} \Delta x^T(j)R_2\Delta x(j), \\
J_3 &= \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(t)-1} \Delta x^T(j)R_2\Delta x(j).
\end{aligned}$$

$\Delta V_4(k)$ is given as

$$\begin{aligned}
\Delta V_4(k) &= V_4(k+1) - V_4(k) \\
&= \tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i+1}^{k} \eta_3^T(j)Z\eta_3(j) \\
&\quad -\tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \eta_3^T(j)Z\eta_3(j) \\
&= \tau_{12}^2\eta_3^T(k)Z\eta_3(k) - J_4 - J_5,
\end{aligned} \tag{36}$$

with

$$\begin{aligned}
J_4 &= \tau_{12} \sum_{j=k-\tau(t)}^{k-\tau_1-1} \eta_3^T(j)Z\eta_3(j), \\
J_5 &= \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(t)-1} \eta_3^T(j)Z\eta_3(j).
\end{aligned}$$

For symmetric matrix T_1 and T_2, the following two zero-value equations are satisfied:

$$\begin{aligned}\mathcal{Z}_1 &= \tau_{12}x(k-\tau_1)^T T_1 x(k-\tau_1) \\ &\quad -\tau_{12}x(k-\tau(k))^T T_1 x(k-\tau(k)) \\ &\quad -\tau_{12}\sum_{j=k-\tau(k)}^{k-\tau_1-1}\eta_3^T(i)\begin{bmatrix}0 & T_1\\ T_1 & T_1\end{bmatrix}\eta_3(i) \\ &= 0, \end{aligned}\tag{37}$$

$$\begin{aligned}\mathcal{Z}_2 &= \tau_{12}x(k-\tau(k))^T T_2 x(k-\tau(k)) \\ &\quad -\tau_{12}x(k-\tau_2)^T T_2 x(k-\tau_2) \\ &\quad -\tau_{12}\sum_{j=k-\tau_2}^{k-\tau(k)-1}\eta_3^T(i)\begin{bmatrix}0 & T_2\\ T_2 & T_2\end{bmatrix}\eta_3(i) \\ &= 0, \end{aligned}\tag{38}$$

which implies

$$\begin{aligned}&\mathcal{Z}_1+\mathcal{Z}_2 \\ &= \tau_{12}\xi^T(k)\Big[e_3^T T_1 e_3 - e_4^T(T_1-T_2)e_4 - e_5^T T_2 e_5\Big]\xi(k) \\ &\quad -J_6-J_7,\end{aligned}\tag{39}$$

where

$$J_6 = \tau_{12}\sum_{j=k-\tau(k)}^{k-\tau_1-1}\eta_3^T(i)\begin{bmatrix}0 & T_1\\ T_1 & T_1\end{bmatrix}\eta_3(i),$$

$$J_7 = \tau_{12}\sum_{j=k-\tau_2}^{k-\tau(k)-1}\eta_3^T(i)\begin{bmatrix}0 & T_2\\ T_2 & T_2\end{bmatrix}\eta_3(i).$$

Based on Lemmas 4-6, the following inequalities hold, if (8), (10), and (11) hold, for any positive diagonal matrices S, S_1, S_2 and D and any matrices N, M, M_1, and M_2:

$$\begin{aligned}\mathcal{Z}_3 &= 2\Big[y(k)-f(y(k))\Big]^T D\Big[f(y(k))-Ly(k)\Big] \\ &\geq 0,\end{aligned}\tag{40}$$

$$\begin{aligned}\mathcal{Z}_4 &= 2\Big[y^T(k)S + x^T(k)M + f^T(y(k))N\Big]\Big[y(k)-f(y(k))\Big] \\ &\geq 0,\end{aligned}\tag{41}$$

$$\begin{aligned}\mathcal{Z}_5 &= 2\Big[y^T(k)S_1 + x^T(k-\tau(k))M_1\Big]\Big[y(k)-f(y(k))\Big] \\ &\geq 0,\end{aligned}\tag{42}$$

$$\mathcal{Z}_6 = 2\Big[y^T(k)S_2 + x^T(k-1)M_2\Big]\Big[y(k)-f(y(k))\Big] \geq 0. \tag{43}$$

Moreover, based on the definition of $\xi(k)$, it can be found that several vectors therein satisfy the following conditions:

$$\upsilon_2(k) = (\tau(k)-\tau_1+1)\upsilon_4(k) - x(k-\tau_1),$$
$$\upsilon_3(k) = (\tau_2-\tau(k)+1)\upsilon_5(k) - x(k-\tau(k)),$$

which can lead to the following zero-value terms

$$\mathcal{Z}_7 = 2\xi^T(k)e_g^T U_1\Big[(\tau(k)-\tau_1+1)e_9 - e_7 - e_3\Big]\xi(k) = 0, \tag{44}$$

$$\mathcal{Z}_8 = 2\xi^T(k)e_g^T U_2\Big[(\tau_2-\tau(k)+1)e_{10} - e_8 - e_4\Big]\xi(k) = 0, \tag{45}$$

with U_1 and U_2 being any matrices.

Combining (32)-(45) yields

$$\begin{aligned}\Delta V(k) &= \sum_{i=1}^{4}\Delta V_i(k) \\ &\leq \sum_{i=1}^{4}\Delta V_i(k) + \sum_{i=1}^{8}\mathcal{Z}_i \\ &= \xi^T(k)\left(\sum_{i=1}^{6}\Phi_i\right)\xi(k) - \sum_{i=1}^{7}J_i.\end{aligned}\tag{46}$$

Using $R_1 > 0$ and applying (4) of Lemma 1 to estimate J_1 yield

$$\begin{aligned}J_1 &= \tau_1\sum_{j=k-\tau_1}^{k-1}\Delta x^T(j)R_1\Delta x(j) \\ &\geq \begin{bmatrix}\kappa_1(k)\\ \kappa_2(k)\end{bmatrix}^T\begin{bmatrix}R_1 & 0\\ 0 & 3R_1\end{bmatrix}\begin{bmatrix}\kappa_1(k)\\ \kappa_2(k)\end{bmatrix} \\ &= \xi^T(k)\begin{bmatrix}e_2-e_3\\ e_2+e_3-\frac{2(e_2+e_6)}{\tau_1+1}\end{bmatrix}^T\begin{bmatrix}R_1 & 0\\ 0 & 3R_1\end{bmatrix} \\ &\quad\times\begin{bmatrix}e_2-e_3\\ e_2+e_3-\frac{2(e_2+e_6)}{\tau_1+1}\end{bmatrix}\xi(k) \\ &= \xi^T(k)\Phi_7\xi(k),\end{aligned}\tag{47}$$

where

$$\kappa_1(k) = x(k) - x(k-\tau_1),$$
$$\kappa_2(k) = x(k) + x(k-\tau_1) - \frac{2}{\tau_1+1}(\upsilon_1(k)+x(k)).$$

Using $R_2 > 0$ and applying (4) of Lemma 1 to estimate J_2 yield

$$\begin{aligned}J_2 &= \tau_{12}\sum_{j=k-\tau(k)}^{k-\tau_1-1}\Delta x^T(i)R_2\Delta x(i) \\ &\geq \frac{\tau_{12}}{\tau(k)-\tau_1}\begin{bmatrix}\kappa_3(k)\\ \kappa_4(k)\end{bmatrix}^T\begin{bmatrix}R_2 & 0\\ 0 & 3R_2\end{bmatrix}\begin{bmatrix}\kappa_3(k)\\ \kappa_4(k)\end{bmatrix} \\ &= \frac{\tau_{12}}{\tau(k)-\tau_1}\xi^T(k)\begin{bmatrix}e_3-e_4\\ e_3+e_4-2e_9\end{bmatrix}^T\begin{bmatrix}R_2 & 0\\ 0 & 3R_2\end{bmatrix} \\ &\quad\times\begin{bmatrix}e_3-e_4\\ e_3+e_4-2e_9\end{bmatrix}\xi(k) \\ &= \xi^T(k)\frac{\tau_{12}\Phi_{81}}{\tau(k)-\tau_1}\xi(k),\end{aligned}\tag{48}$$

where

$$\kappa_3(k) = x(k-\tau_1) - x(k-\tau(k)),$$
$$\kappa_4(k) = x(k-\tau_1) + x(k-\tau(k)) - 2\upsilon_4(k),$$

$$\Phi_{81} = \begin{bmatrix}e_3-e_4\\ e_3+e_4-2e_9\end{bmatrix}^T\begin{bmatrix}R_2 & 0\\ 0 & 3R_2\end{bmatrix}\begin{bmatrix}e_3-e_4\\ e_3+e_4-2e_9\end{bmatrix}.$$

Using $R_2 > 0$ and applying (4) of Lemma 1 to estimate J_3 yield

$$J_3 = \tau_{12}\sum_{j=k-\tau_2}^{k-\tau(k)-1}\Delta x^T(i)R_2\Delta x(i)$$

$$
\begin{aligned}
&\geq \frac{\tau_{12}}{\tau_2-\tau(k)}\begin{bmatrix}\kappa_5(k)\\ \kappa_6(k)\end{bmatrix}^T\begin{bmatrix}R_2 & 0\\ 0 & 3R_2\end{bmatrix}\begin{bmatrix}\kappa_5(k)\\ \kappa_6(k)\end{bmatrix}\\
&= \frac{\tau_{12}}{\tau_2-\tau(k)}\xi^T(k)\begin{bmatrix}e_4-e_5\\ e_4+e_5-2e_{10}\end{bmatrix}^T\begin{bmatrix}R_2 & 0\\ 0 & 3R_2\end{bmatrix}\\
&\quad\times\begin{bmatrix}e_4-e_5\\ e_4+e_5-2e_{10}\end{bmatrix}\xi(k)\\
&= \xi^T(k)\frac{\tau_{12}\Phi_{82}}{\tau_2-\tau(k)}\xi(k), \qquad (49)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_5(k) &= x(k-\tau(k)) - x(k-\tau_2),\\
\kappa_6(k) &= x(k-\tau(k)) + x(k-\tau_2) - 2\upsilon_5(k),\\
\Phi_{82} &= \begin{bmatrix}e_4-e_5\\ e_4+e_5-2e_{10}\end{bmatrix}^T\begin{bmatrix}R_2 & 0\\ 0 & 3R_2\end{bmatrix}\begin{bmatrix}e_4-e_5\\ e_4+e_5-2e_{10}\end{bmatrix}.
\end{aligned}
$$

Using (24) and applying (5) Lemma 2 to estimate $J_4 + J_6$ yield

$$
\begin{aligned}
J_4 + J_6 &= \tau_{12}\sum_{j=k-\tau(k)}^{k-\tau_1-1}\eta_3^T(i)\left(Z+\begin{bmatrix}0 & T_1\\ T_1 & T_1\end{bmatrix}\right)\eta_3(i)\\
&\geq \frac{\tau_{12}}{\tau(k)-\tau_1}\begin{bmatrix}\upsilon_2(k)\\ \kappa_3(k)\end{bmatrix}^T\left(Z+\begin{bmatrix}0 & T_1\\ T_1 & T_1\end{bmatrix}\right)\begin{bmatrix}\upsilon_2(k)\\ \kappa_3(k)\end{bmatrix}\\
&= \frac{\tau_{12}}{\tau(k)-\tau_1}\xi^T(k)\begin{bmatrix}e_7\\ e_3-e_4\end{bmatrix}^T\left(Z+\begin{bmatrix}0 & T_1\\ T_1 & T_1\end{bmatrix}\right)\\
&\quad\times\begin{bmatrix}e_7\\ e_3-e_4\end{bmatrix}\xi(k)\\
&= \xi^T(k)\frac{\tau_{12}\Phi_{83}}{\tau(k)-\tau_1}\xi(k), \qquad (50)
\end{aligned}
$$

where

$$
\Phi_{83} = \begin{bmatrix}e_7\\ e_3-e_4\end{bmatrix}^T\left(Z+\begin{bmatrix}0 & T_1\\ T_1 & T_1\end{bmatrix}\right)\begin{bmatrix}e_7\\ e_3-e_4\end{bmatrix}.
$$

Using (25) and applying (5) Lemma 2 to estimate $J_5 + J_7$ yield

$$
\begin{aligned}
&J_5 + J_7\\
&= \tau_{12}\sum_{j=k-\tau_2}^{k-\tau(k)-1}\eta_3^T(i)\left(Z+\begin{bmatrix}0 & T_2\\ T_2 & T_2\end{bmatrix}\right)\eta_3(i)\\
&\geq \frac{\tau_{12}}{\tau_2-\tau(k)}\begin{bmatrix}\upsilon_3(k)\\ \kappa_4(k)\end{bmatrix}^T\left(Z+\begin{bmatrix}0 & T_2\\ T_2 & T_2\end{bmatrix}\right)\begin{bmatrix}\upsilon_3(k)\\ \kappa_4(k)\end{bmatrix}\\
&= \frac{\tau_{12}}{\tau_2-\tau(k)}\xi^T(k)\begin{bmatrix}e_8\\ e_4-e_5\end{bmatrix}^T\left(Z+\begin{bmatrix}0 & T_2\\ T_2 & T_2\end{bmatrix}\right)\\
&\quad\times\begin{bmatrix}e_8\\ e_4-e_5\end{bmatrix}\xi(k)\\
&= \xi^T(k)\frac{\tau_{12}\Phi_{84}}{\tau_2-\tau(k)}\xi(k), \qquad (51)
\end{aligned}
$$

where

$$
\Phi_{84} = \begin{bmatrix}e_8\\ e_4-e_5\end{bmatrix}^T\left(Z+\begin{bmatrix}0 & T_2\\ T_2 & T_2\end{bmatrix}\right)\begin{bmatrix}e_8\\ e_4-e_5\end{bmatrix}.
$$

It follows from (48)-(51) that

$$
\begin{aligned}
\sum_{i=2}^{7}J_i &\geq \xi^T(k)\left[\frac{\tau_{12}(\Phi_{81}+\Phi_{83})}{\tau(k)-\tau_1}+\frac{\tau_{12}(\Phi_{82}+\Phi_{84})}{\tau_2-\tau(k)}\right]\xi(k)\\
&= \xi^T(k)\left[\frac{\tau_{12}\Phi_{813}}{\tau(k)-\tau_1}+\frac{\tau_{12}\Phi_{824}}{\tau_2-\tau(k)}\right]\xi(k)\\
&= \xi^T(k)\left[\frac{\tau_{12}E_2^T\Xi_1E_2}{\tau(k)-\tau_1}+\frac{\tau_{12}E_3^T\Xi_2E_3}{\tau_2-\tau(k)}\right]\xi(k), \qquad (52)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_{813} &= \begin{bmatrix}e_7\\ e_3-e_4\\ e_3+e_4-2e_9\end{bmatrix}^T\left[\begin{array}{c|c}Z+\begin{bmatrix}0 & T_1\\ T_1 & R_2+T_1\end{bmatrix} & \begin{bmatrix}0\\ 0\end{bmatrix}\\ \hline \begin{bmatrix}0 & 0\end{bmatrix} & 3R_2\end{array}\right]\\
&\quad\times\begin{bmatrix}e_7\\ e_3-e_4\\ e_3+e_4-2e_9\end{bmatrix}\\
&= E_2^T\Xi_1E_2,\\
\Phi_{824} &= \begin{bmatrix}e_8\\ e_4-e_5\\ e_4+e_5-2e_{10}\end{bmatrix}^T\left[\begin{array}{c|c}Z+\begin{bmatrix}0 & T_2\\ T_2 & R_2+T_2\end{bmatrix} & \begin{bmatrix}0\\ 0\end{bmatrix}\\ \hline \begin{bmatrix}0 & 0\end{bmatrix} & 3R_2\end{array}\right]\\
&\quad\times\begin{bmatrix}e_8\\ e_4-e_5\\ e_4+e_5-2e_{10}\end{bmatrix}\\
&= E_3^T\Xi_2E_3.
\end{aligned}
$$

For any matrix X, it follows from (6) of Lemma 3 that

$$
\begin{aligned}
&\frac{\tau_{12}E_2^T\Xi_1E_2}{\tau(k)-\tau_1}+\frac{\tau_{12}E_3^T\Xi_2E_3}{\tau_2-\tau(k)}\\
&\geq \begin{bmatrix}E_2\\ E_3\end{bmatrix}^T\begin{bmatrix}\frac{2\tau_2-\tau(k)-\tau_1}{\tau_{12}}\Xi_1 & X\\ * & \frac{\tau_2+\tau(k)-2\tau_1}{\tau_{12}}\Xi_2\end{bmatrix}\begin{bmatrix}E_2\\ E_3\end{bmatrix}\\
&\quad-\frac{\tau_2-\tau(k)}{\tau_{12}}E_2^TX\Xi_1^{-1}X^TE_2\\
&\quad-\frac{\tau(k)-\tau_1}{\tau_{12}}E_3^TX^T\Xi_2^{-1}XE_3\\
&= \Phi_8-\bar{\Phi}_8, \qquad (53)
\end{aligned}
$$

where

$$
\bar{\Phi}_8 = \frac{\tau_2-\tau(k)}{\tau_{12}}E_2^TX\Xi_1^{-1}X^TE_2+\frac{\tau(k)-\tau_1}{\tau_{12}}E_3^TX^T\Xi_2^{-1}XE_3.
$$

Based on (46), (47), (52), and (53), it can be obtained that

$$
\begin{aligned}
\Delta V(k) &\leq \xi^T(k)\left(\sum_{i=1}^{6}\Phi_i-\Phi_7-\Phi_8+\bar{\Phi}_8\right)\xi(k)\\
&= \xi^T(k)\left(\Psi+\bar{\Phi}_8\right)\xi(k). \qquad (54)
\end{aligned}
$$

It can be checked that $\Psi + \bar{\Phi}_8$ is convex with respect to $\tau(t)$, which from the convex combination technique shows the following holds

$$
\begin{aligned}
&\Psi+\bar{\Phi}_8 < 0, \quad \forall\tau(t)\in\{\tau_1,\tau_2\} \qquad (55)\\
\Rightarrow\ &\Psi+\bar{\Phi}_8 < 0, \quad \forall\tau(t)\in[\tau_1,\tau_2]. \qquad (56)
\end{aligned}
$$

Based on the Schur complement, (55) is equivalent to LMIs (22) and (23). Thus, if LMIs (22) and (23) holds, then (56) holds, which combined with (54) leads to

$$\Delta V(k) \leq -\varepsilon ||x(k)||^2, \tag{57}$$

for a sufficient small $\varepsilon > 0$.

Therefore, under conditions (8), (10), and (11), and (17)-(25), digital filter (1) is stable. This completes the proof. ■

Remark 3: compared with the simple functionals used for discussing the digital filters in the previous literature [3], [47], [49], [50], extra free matrices are introduced by the augmented terms, $V_1(k)$ and $V_4(k)$, of the proposed augmented functional (26), especially, $f(y(k-1))$ included in $V_1(k)$ is used to construct functional at the first time, which can provide extra freedom during checking the feasibility of criterion. Thus, Theorem 1 in this paper has less conservatism than the ones in the previous literature.

Remark 4: compared with the criteria of digital filters developed in [3], [47], [49], [50], Theorem 1 in this paper has less conservatism due to the improvements during the estimation of $\Delta V(k)$. In order to accurately estimate the summation term arising in $\Delta V(k)$, i.e., $J_i, i = 1, 2, \cdots, 7$, Lemma 6 and several other techniques, which are proved to be helpful to reduce the conservatism during the investigation of discrete-time delay systems, are applied to reduce the estimation error, summarized as follows:

(1) Due to the adding of $\mathcal{Z}_5$ and $\mathcal{Z}_6$ respectively defined in (42) and (43) into the $\Delta V(k)$, several cross terms are introduced to give the relationship between the delayed states, $x(k-\tau(k))$ and $x(k-1)$, and the nonlinear function, $f(y(k))$, which are not used in the literature [3], [47], [49], [50] and provide extra freedom for finding the solutions of conditions in Theorem 1 so as to reduce the conservatism.

(2) The summation term defined by J_1, J_2, and J_3 are bounded by using the Wirtinger-based summation inequality, while the similar terms are estimated based on a more conservative inequality, i.e., the Jensen-based summation inequality, in the previous literature [3], [47], [49], [50].

(3) Two zero-value equations $\mathcal{Z}_7$ and $\mathcal{Z}_8$, respectively defined in (44) and (45), are developed and introduced into the $\Delta V(k)$, which adds many cross-terms into Theorem 1. The presence of free matrices U_1 and U_2 in Theorem 1 can increase the feasibility of the conditions of Theorem 1 so as to reduce the conservatism.

Remark 5: Remarks 3 and 4 show the improvements of Theorem 1 compared with the ones for digital filters reported in literature. In fact, compared with the techniques developed for the analysis of linear discrete-time system with a time-varying delay, novel treatments are also used to develop Theorem 1. More specifically, compared with the functionals used in the related works, in which each term of functionals is usually required to be positive, the functional (26) constructed in this paper is relaxed by considering all terms together and requiring the sum of all terms be positive. That is to say, the positive-definite condition of functional (26) is relaxed, (i.e., $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, and $Z > 0$ are removed), which helps to reduce the conservatism. Moreover, $x(k-1)$ included in $V_1(k)$ has not been used for the investigation of linear discrete-time system with a time-varying delay.

In order to easily show the advantage of the proposed Theorem 1, the following corollary is developed by requiring each term of LKF is positive and setting $P_2 = 0, S_1 = 0, S_2 = 0, M_1 = 0, M_2 = 0$.

Corollary 1: For given scalars l_i, τ_1 and τ_2, digital filter (1) with time-varying satisfying (2) is asymptotically stable if there exist positive definite symmetric matrices $P_1, Z, Q_1, Q_2, R_1, R_2$, symmetric matrices T_1, T_2, positive diagonal matrices S, D, and any matrices X, U_1, U_2, M, N, such that condition (8) and LMIs (22)-(25) are feasible.

In order to further verify the advantage of the zero-value equations (37) and (38) for Theorem 1, the following corollary is developed by setting $T_1 = 0, T_2 = 0$.

Corollary 2: For given scalars l_i, τ_1 and τ_2, digital filter (1) with time-varying satisfying (2) is asymptotically stable if there exist symmetric matrices $P_1, P_2, Z, Q_1, Q_2, R_1, R_2$, positive definite diagonal matrices S, S_1, S_2, D, and any matrices $X, U_1, U_2, M, M_1, M_2, N$, such that conditions (8), (10), (11), and LMIs (17)-(24) are feasible.

Remark 6: Although Theorem 1 has less conservatism, compared with the ones in [3], [47], [50], with the help of the techniques summarized in Remarks 3-5, it is still a sufficient criterion and provides conservative results. Many new techniques developed for linear time-delay systems, such as novel Lyapunov functionals [39], [45], [46], improved bounding inequalities [52], and relaxed quadratic function transformation technologies [71], can be used to further reduce the conservatism. In addition, the method proposed in this paper can be extended to deal with other types of systems with time delays and/or noise [50], [63], the practical systems affected by communication delays and/or saturation constraint such as power systems [64]–[67], teleoperation systems [68], microgrids [69], vehicle swarm systems [70], and the systems with relaxed nonlinear conditions than the one given in (3).

Remark 7: The introduction of slack matrices reduces the conservatism of Theorem 1, compared with the ones in [3], [47], [50], at the cost of the computation complexity. The number of decision variables of Theorem 1 is $44.5n^2 + 10.5n$, those of criteria in [47], [50], and [3] are $5n^2 + 5n$, $6n^2 + 6n$, and $9n^2 + 5n$, respectively.

IV. NUMERICAL EXAMPLES

In this section, two numerical examples are given to demonstrate the effectiveness and advantages of the proposed method.

Example 1: Consider digital filter (1) with the following parameters

$$l_1 = -1, \quad l_2 = 0, \tag{58}$$

$$l_{11} = l_{12} = l_{21} = l_{22} = 1, \tag{59}$$

$$A = \begin{bmatrix} 0.3 & -0.4 \\ 0.5 & 0.7 \end{bmatrix}, \tag{60}$$

$$A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \tag{61}$$

For different given lower bounds of $\tau(k)$, τ_1, the allowably maximal values of τ_2 calculated by Theorem 1, Corollary 1, Corollary 2 and the ones provided by the criteria of [3], [47], [50] are listed in Table 1. It is clearly seen that Theorem 1 provides less conservative results (i.e., bigger allowably maximal values) than the ones reported in [3], [47], [50], which shows that the improvements, summarized in Remarks 3 and 4, indeed reduce the conservatism. By comparing the results of Theorem 1 and Corollary 2, it is shown that the zero-value equations (37) and (38) used in Theorem 1 can greatly reduce the conservatism. Moreover, the results provided by Theorem 1 are less conservative than those provided by Corollary 1, which means that the relaxed positive-definite condition in Theorem 1 and the cross terms introduced by Lemma 6 indeed reduce the conservatism as analyzed in Remarks 4 and 5. Therefore, the advantages of the proposed method on the conservatism-reduction is verified. Note that

TABLE 1. Allowably maximal value of τ_2, τ_{max}, for various τ_1 (Example 1).

	τ_1			
Criteria	1	3	5	10
Theorem 1 [47]	8	10	12	17
Corollary 1 [50]	11	13	15	20
Theorem 2 [3]	11	13	15	20
Corollary 1	15	17	19	24
Corollary 2	13	15	17	22
Theorem 1	43	45	47	52

When the lower bound of time-varying delay is 1, i.e., $\tau(k) \geq \tau_1 = 1$, it is calculated from Theorem 1 that the allowably maximal value of τ_2, τ_{max}, is 43 and the related feasible solutions of the conditions of Theorem 1 are given as $P_1, P_2, Q_1, Q_2, R_1, R_2$, and Z, at the top of the next page. That is, digital filter (1) with parameters given in (58)-(61) and time-varying delay satisfying $1 \leq \tau(k) \leq 43$ is asymptotically stable since one can find the Lyapunov functional (26) with matrices $P_1, P_2, Q_1, Q_2, R_1, R_2, Z$ being given above. Simulation test is carried out to verify this conclusion. Let the initial condition $x(k) = [-0.4,\ 0.1]^T$, $k \in [-43, 0]$, and the delay is random value within [1, 43]. The responses of two state variables of digital filter and the corresponding time delay are shown in Fig. 1. It is clearly observed that digital filter is stable. That is, the simulation result is consistent with the calculated result from the proposed method. Therefore, the effectiveness of the proposed criterion is verified.

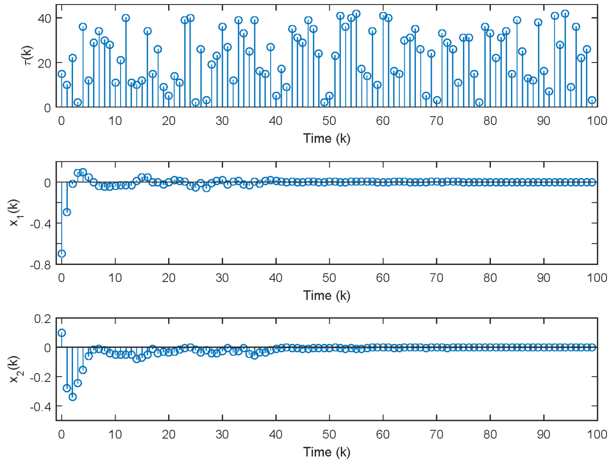

FIGURE 1. The time delay and state trajectories (Example 1).

TABLE 2. Allowably maximal value of τ_2, τ_{max}, for various τ_1 (Example 2).

	τ_1				
Criteria	5	10	15	20	25
Theorem 1 [47]	33	38	43	48	53
Corollary 1 [50]	34	39	43	48	53
Theorem 2 [3]	34	39	43	48	53
Corollary 1	47	52	57	62	67
Corollary 2	14	19	24	29	34
Theorem 1	49	54	59	64	69

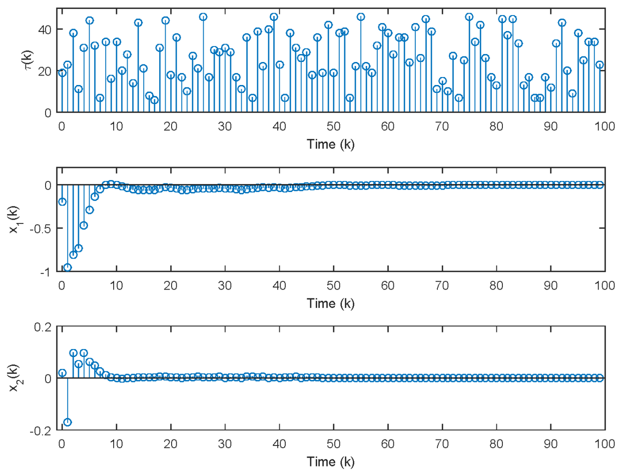

FIGURE 2. The time delay and state trajectories (Example 2).

Example 2: Consider digital filter (1) with the following parameters

$$l_1 = -1, \quad l_2 = 0,\ l_3 = 1, \tag{62}$$

$$l_{11} = l_{21} = 0, \quad l_{12} = l_{22} = l_{13} = l_{23} = 1, \tag{63}$$

$$A = \begin{bmatrix} 0.8 & -1.75 & -2.5 \\ -0.1 & -0.5 & -0.6 \\ 0.1 & 0.1 & 0.5 \end{bmatrix}, \tag{64}$$

$$P_1 = \begin{bmatrix} 1.8501 & 0.7162 & 0.0050 & 0.0085 & -0.0020 & 0.0017 \\ 0.7162 & 1.1126 & -0.0043 & 0.0031 & -0.0011 & -0.0013 \\ 0.0050 & -0.0043 & 0.0006 & 0.0003 & -0.0007 & 0.0006 \\ 0.0085 & 0.0031 & 0.0003 & 0.0050 & -0.0007 & -0.0003 \\ -0.0020 & -0.0011 & -0.0007 & -0.0007 & -0.0002 & 0.0001 \\ 0.0017 & -0.0013 & 0.0006 & -0.0003 & 0.0001 & 0.0003 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.0082 & -0.0036 & -0.0104 & -0.0022 \\ -0.0036 & 0.0089 & 0.0109 & -0.0024 \\ -0.0104 & 0.0109 & 0.9051 & 0.1608 \\ -0.0022 & -0.0024 & 0.1608 & 0.6397 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.0702 & 0.0129 \\ 0.0129 & 0.0411 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.0333 & 0.0053 \\ 0.0053 & 0.0177 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0.0116 & -0.0006 \\ -0.0006 & 0.0074 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.3433 & -0.0088 \\ -0.0088 & 0.1474 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.0013 & 0.0007 & 0.0010 & -0.0004 \\ 0.0007 & 0.0010 & 0.0013 & 0.0001 \\ 0.0010 & 0.0013 & 0.0023 & 0.0001 \\ -0.0004 & 0.0001 & 0.0001 & 0.0006 \end{bmatrix}$$

$$A_d = \begin{bmatrix} 0.01 & 0.01 & -0.01 \\ 0 & 0.01 & 0 \\ -0.01 & 0.01 & 0.01 \end{bmatrix}. \quad (65)$$

For different given lower bounds of $\tau(k)$, τ_1, the allowably maximal values of τ_2 calculated by Theorem 1, Corollary 1, Corollary 2, and the ones provided by the criteria of [3], [47], [50] are listed in Table 2. It is clearly seen that Theorem 1 provides less conservative results (i.e., bigger allowably maximal values) than the others, which again shows the advantages of the proposed Theorem 1. And the results of Corollary 2 are all lower than the others, it shows that the zero-value equations (37) and (38) has a big improvement on reducing the conservatism for some numerical examples.

From Table 2, digital filter (1) with parameters given in (62)-(65) and time-varying delay satisfying $5 \leq \tau(k) \leq 49$ is asymptotically stable. Simulation test is carried out to verify this observation. Let the initial condition $x(k) = [-0.4, 0.02, 0.3]^T$, $k \in [-49, 0]$, and the delay is random value within $[5, 49]$. The responses of two state variables of digital filter, together with the time delay, are shown in Fig. 2. It is clearly observed that digital filter is stable. That is, the simulation result is consistent with the calculated result from the proposed method. Therefore, the effectiveness of the proposed criterion is verified.

V. CONCLUSION

This paper has investigated the stability analysis of fixed-point state-space digital filters with generalized overflow arithmetic and a time-varying delay. A new stability criterion has been developed to assess the influence of the time delay on the stability of digital filter. The criterion has less conservatism in comparison to the ones reported in the previous literature due to two aspects of improvements. A new Lyapunov functional with several augmented terms and relaxed positive-definite condition has been constructed, and free matrices therein provides extra freedom of checking the conditions of stability criterion. And, Lemma 6, together with several new techniques (such as Wirtinger-based inequality, zero-value equations, the extended reciprocally convex matrix inequality), has been used to estimate the summation terms arising in the forward difference of functional, which leads to a smaller estimate gap than the methods used in the literature. Finally, two numerical examples have been given to show the effectiveness and advantages of the proposed stability criterion. How to extend the proposed method to other problems of digital filters and other practical systems is our future work.

REFERENCES

[1] P. Kokil, C. G. Parthipan, S. Jogi, and H. Kar, "Criterion for realizing state-delayed digital filters subjected to external interference employing saturation arithmetic," *Cluster Comput.*, vol. 22, no. S6, pp. 15187–15194, Nov. 2019, doi: 10.1007/s10586-018-2530-3.

[2] N. Agarwal and H. Kar, "Improved criterion for robust stability of discrete-time state-delayed systems with quantization/overflow nonlinearities," *Circuits, Syst., Signal Process.*, vol. 38, no. 11, pp. 4959–4980, Nov. 2019.

[3] W. Xia, W. X. Zheng, and S. Xu, "Realizability condition for digital filters with time delay using generalized overflow arithmetic," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 66, no. 1, pp. 141–145, Jan. 2019.

[4] L.-J. Leclerc and P. H. Bauer, "New criteria for asymptotic stability of one- and multidimensional state-space digital filters in fixed-point arithmetic," *IEEE Trans. Signal Process.*, vol. 42, no. 1, pp. 46–53, Jan. 1994.

[5] M. K. Kumar, P. Kokil, and H. Kar, "A new realizability condition for fixed-point state-space interfered digital filters using any combination of overflow and quantization nonlinearities," *Circuits, Syst., Signal Process.*, vol. 36, no. 8, pp. 3289–3302, Aug. 2017.

[6] P. Kokil, S. Jogi, and C. K. Ahn, "Stability analysis of digital filters subjected to interference using generalized overflow nonlinearities," *Signal Process.*, vol. 148, pp. 1–8, Jul. 2018.

[7] V. Singh, "LMI approach to stability of direct form digital filters utilizing single saturation overflow nonlinearity," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 54, no. 10, pp. 878–881, Oct. 2007.

[8] D. Liu and A. N. Michel, "Asymptotic stability of discrete-time systems with saturation nonlinearities with applications to digital filters," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 39, no. 10, pp. 798–807, Oct. 1992.

[9] V. Singh, "Modified form of Liu-Michel's criterion for global asymptotic stability of fixed-point state-space digital filters using saturation arithmetic," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, no. 12, pp. 1423–1425, Dec. 2006.

[10] V. Singh, "Stability analysis of a class of digital filters utilizing single saturation nonlinearity," *Automatica*, vol. 44, no. 1, pp. 282–285, Jan. 2008.

[11] T. Li and W. X. Zheng, "New stability criterion for fixed-point state-space digital filters with generalized overflow arithmetic," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 59, no. 7, pp. 443–447, Jul. 2012.

[12] V. Singh, "Modified criterion for global asymptotic stability of fixed-point state-space digital filters using two's complement arithmetic," *Automatica*, vol. 46, no. 2, pp. 475–478, Feb. 2010.

[13] T. Shen and Z. Yuan, "Stability of fixed-point state-space digital filters using two's complement arithmetic: Further insight," *Automatica*, vol. 46, no. 12, pp. 2109–2111, Dec. 2010.

[14] T. Shen, Z. Yuan, and X. Wang, "An improved stability criterion for fixed-point state-space digital filters using two's complement arithmetic," *Automatica*, vol. 48, no. 1, pp. 243–245, Jan. 2012.

[15] T. Shen and Z. Yuan, "Stability criterion for a class of fixed-point digital filters using two's complement arithmetic," *Appl. Math. Comput.*, vol. 219, no. 9, pp. 4880–4883, Jan. 2013.

[16] V. Singh, "A novel LMI-based criterion for the stability of direct-form digital filters utilizing a single two's complement nonlinearity," *Nonlinear Anal.: Real World Appl.*, vol. 14, no. 1, pp. 684–689, Feb. 2013.

[17] N. Agarwal and H. Kar, "An improved criterion for the global asymptotic stability of fixed-point state-space digital filters with combinations of quantization and overflow," *Digit. Signal Process.*, vol. 28, pp. 136–143, May 2014.

[18] M. Rehan, M. B. Mobeen, M. Tufail, and C. K. Ahn, "A novel method for guaranteed overflow oscillation elimination in digital filters subject to quantization," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 65, no. 9, pp. 1279–1283, Sep. 2018.

[19] T. Li, N. Sun, Q. Lin, and J. Li, "Improved criterion for the elimination of overflow oscillations in digital filters with external disturbance," *Adv. Difference Equ.*, vol. 2012, no. 1, pp. 1–6, Nov. 2012.

[20] C. K. Ahn, "L_2-L_∞ stability criterion for fixed-point state-space digital filters with saturation nonlinearity," *Int. J. Electron.*, vol. 100, no. 9, pp. 1309–1316, 2013.

[21] P. Rani, P. Kokil, and H. Kar, "L_2-L_∞ suppression of limit cycles in interfered digital filters with generalized overflow nonlinearities," *Circuits, Syst., Signal Process.*, vol. 36, no. 7, pp. 2727–2741, Jul. 2017.

[22] P. Kokil, S. X. Arockiaraj, S. Jogi, and H. Kar, "New realizability criterion for digital filters with external disturbance and saturation arithmetic," *AEU–Int. J. Electron. Commun.*, vol. 85, pp. 179–182, Feb. 2018.

[23] C. K. Ahn, "Two new criteria for the realization of interfered digital filters utilizing saturation overflow nonlinearity," *Signal Process.*, vol. 95, pp. 171–176, Feb. 2014.

[24] P. Kokil and S. S. Shinde, "Asymptotic stability of fixed-point state-space digital filters with saturation arithmetic and external disturbance: An IOSS approach," *Circuits, Syst., Signal Process.*, vol. 34, no. 12, pp. 3965–3977, Dec. 2015.

[25] C. K. Ahn, "Some new results on the stability of direct-form digital filters with finite wordlength nonlinearities," *Signal Process.*, vol. 108, pp. 549–557, Mar. 2015.

[26] M. K. Kumar and H. Kar, "ISS criterion for the realization of fixed-point state-space digital filters with saturation arithmetic and external interference," *Circuits, Syst., Signal Process.*, vol. 37, no. 12, pp. 5664–5679, Dec. 2018.

[27] C. K. Ahn and P. Shi, "Generalized dissipativity analysis of digital filters with finite-wordlength arithmetic," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 63, no. 4, pp. 386–390, Apr. 2016.

[28] C. K. Ahn and P. Shi, "Strict dissipativity and asymptotic stability of digital filters in direct form with saturation nonlinearity," *Nonlinear Dyn.*, vol. 85, no. 1, pp. 453–461, Jul. 2016.

[29] I. Arif, M. Rehan, and M. Tufail, "Toward local stability analysis of externally interfered digital filters under overflow nonlinearity," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 64, no. 5, pp. 595–599, May 2017.

[30] A. Shams, M. Rehan, M. Tufail, C. K. Ahn, and W. Ahmed, "Local stability analysis and H_∞ performance for Lipschitz digital filters with saturation nonlinearity and external interferences," *Signal Process.*, vol. 153, pp. 101–108, Dec. 2018.

[31] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*. Boston, MA, USA: Birkhäuser, 2014.

[32] Y. He, J. Zeng, M. Wu, and C.-K. Zhang, "Robust stabilization and controllers design for stochastic genetic regulatory networks with time-varying delays and structured uncertainties," *Math. Biosci.*, vol. 236, no. 1, pp. 53–63, Mar. 2012.

[33] C. Yang, L. Huang, and F. Li, "Exponential synchronization control of discontinuous nonautonomous networks and autonomous coupled networks," *Complexity*, vol. 2018, Oct. 2018, Art. no. 6164786.

[34] L. Duan, X. Fang, and C. Huang, "Global exponential convergence in a delayed almost periodic Nicholson's blowflies model with discontinuous harvesting," *Math. Methods Appl. Sci.*, vol. 41, no. 5, pp. 1954–1965, Mar. 2018.

[35] L. Duan, L. H. Z. Y. Huang Guo, and X. W. Fang, "Periodic attractor for reaction-diffusion high-order hopfield neural networks with time-varying delays," *Comput. Math. Appl.*, vol. 73, no. 2, pp. 233–245, 2017.

[36] C. Jin, K. Gu, S.-I. Niculescu, and I. Boussaada, "Stability analysis of systems with delay-dependent coefficients: An overview," *IEEE Access*, vol. 6, pp. 27392–27407, 2018.

[37] C. Huang, B. Liu, X. Tian, L. Yang, and X. Zhang, "Global convergence on asymptotically almost periodic SICNNs with nonlinear decay functions," *Neural Process. Lett.*, vol. 49, no. 2, pp. 625–641, Apr. 2019.

[38] C. Huang, H. Zhang, and L. Huang, "Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term," *Commun. Pure Appl. Anal.*, vol. 18, no. 6, pp. 3337–3349, 2019.

[39] Z. Lian, Y. He, C. K. Zhang, P. Shi, and M. Wu, "Robust H_∞ control for T–S fuzzy systems with state and input time-varying delays via delay-product-type functional method," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 10, pp. 1917–1930, Oct. 2019.

[40] C. Huang and H. Zhang, "Periodicity of non-autonomous inertial neural networks involving proportional delays and non-reduced order method," *Int. J. Biomath.*, vol. 12, no. 2, 2019, Art. no. 1950016.

[41] C. Huang, Z. Yang, T. Yi, and X. Zou, "On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities," *J. Differ. Equ.*, vol. 256, no. 7, pp. 2101–2114, Apr. 2014.

[42] C. Gong, G. Zhu, and L. Wu, "New weighted integral inequalities and its application to exponential stability analysis of time-delay systems," *IEEE Access*, vol. 4, pp. 6231–6237, 2016.

[43] J. Cao, L. Guerrini, and Z. Cheng, "Stability and Hopf bifurcation of controlled complex networks model with two delays," *Appl. Math. Comput.*, vol. 343, pp. 21–29, Feb. 2019.

[44] X. Yang and J. Cao, "Hybrid adaptive and impulsive synchronization of uncertain complex networks with delays and general uncertain perturbations," *Appl. Math. Comput.*, vol. 227, pp. 480–493, Jan. 2014.

[45] F. Long, C. K. Zhang, L. Jiang, Y. He, and M. Wu, "Stability analysis of systems with time-varying delay via improved Lyapunov–Krasovskii functionals," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 4, pp. 2457–2466, Apr. 2021, doi: 10.1109/TSMC.2019.2914367.

[46] F. Long, C.-K. Zhang, Y. He, L. Jiang, Q.-G. Wang, and M. Wu, "Stability analysis of Lur'e systems with additive delay components via a relaxed matrix inequality," *Appl. Math. Comput.*, vol. 328, pp. 224–242, Jul. 2018.

[47] T. Li, Q. Zhao, J. Lam, and Z. Feng, "Multi-bound-dependent stability criterion for digital filters with overflow arithmetics and time delay," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 61, no. 1, pp. 31–35, Jan. 2014.

[48] V. Krishna Rao Kandanvli and H. Kar, "An LMI condition for robust stability of discrete-time state-delayed systems using quantization/overflow nonlinearities," *Signal Process.*, vol. 89, no. 11, pp. 2092–2102, Nov. 2009.
[49] V. K. R. Kandanvli and H. Kar, "Delay-dependent LMI condition for global asymptotic stability of discrete-time uncertain state-delayed systems using quantization/overflow nonlinearities," *Int. J. Robust Nonlinear Control*, vol. 21, no. 14, pp. 1611–1622, Sep. 2011.
[50] W. Xia, W. X. Zheng, and S. Xu, "Extended dissipativity analysis of digital filters with time delay and Markovian jumping parameters," *Signal Process.*, vol. 152, pp. 247–254, Nov. 2018.
[51] C.-K. Zhang, Y. He, L. Jiang, M. Wu, and H.-B. Zeng, "Delay-variation-dependent stability of delayed discrete-time systems," *IEEE Trans. Autom. Control*, vol. 61, no. 9, pp. 2663–2669, Sep. 2016.
[52] C.-K. Zhang, Y. He, L. Jiang, M. Wu, and H.-B. Zeng, "Summation inequalities to bounded real lemmas of discrete-time systems with time-varying delay," *IEEE Trans. Autom. Control*, vol. 62, no. 5, pp. 2582–2588, May 2017.
[53] M.-J. Park, S.-H. Lee, O.-M. Kwon, and J.-H. Ryu, "Augmented Lyapunov–Krasovskii functional approach to stability of discrete systems with time-varying delays," *IEEE Access*, vol. 5, pp. 24389–24400, 2017.
[54] A. Seuret, F. Gouaisbaut, and E. Fridman, "Stability of discrete-time systems with time-varying delays via a novel summation inequality," *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 2740–2745, Oct. 2015.
[55] C.-K. Zhang, Y. He, L. Jiang, and M. Wu, "An improved summation inequality to discrete-time systems with time-varying delay," *Automatica*, vol. 74, pp. 10–15, Dec. 2016.
[56] J. Chen, S. Xu, X. Jia, and B. Zhang, "Novel summation inequalities and their applications to stability analysis for systems with time-varying delay," *IEEE Trans. Autom. Control*, vol. 62, no. 5, pp. 2470–2475, May 2017.
[57] C.-K. Zhang, Y. He, J. Liang, M. Wu, and Q.-G. Wang, "An extended reciprocally convex matrix inequality for stability analysis of systems with time-varying delay," *Automatica*, vol. 85, pp. 481–485, Nov. 2017.
[58] C.-K. Zhang, Y. He, L. Jiang, Q.-G. Wang, and M. Wu, "Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality," *IEEE Trans. Cybern.*, vol. 47, no. 10, pp. 3040–3049, Oct. 2017.
[59] Y. Xue, H. Li, and X. Yang, "An improved reciprocally convex inequality and application to stability analysis of time-delay systems based on delay partition approach," *IEEE Access*, vol. 6, pp. 40245–40252, 2018.
[60] X. Jiang, Q.-L. Han, and X. Yu, "Stability criteria for linear discrete-time systems with interval-like time-varying delay," in *Proc. Amer. Control Conf.*, Jun. 2005, pp. 2817–2822.
[61] S. Ramasamy, G. Nagamani, and T. Radhika, "Further results on dissipativity criterion for Markovian jump discrete-time neural networks with two delay components via discrete Wirtinger inequality approach," *Neural Process. Lett.*, vol. 45, no. 3, pp. 939–965, Jun. 2017.
[62] S. Ramasamy, G. Nagamani, and Q. X. Zhu, "Robust dissipativity and passivity analysis for discrete-time stochastic T–S fuzzy Cohen–Grossberg Markovian jump neural networks with mixed time delays," *Nonlinear Dyn.*, vol. 85, no. 4, pp. 2777–2799, Sep. 2016.
[63] Q. Zhu, "Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control," *IEEE Trans. Autom. Control*, vol. 64, no. 9, pp. 3764–3771, Sep. 2019.
[64] L. Jin, C.-K. Zhang, Y. He, L. Jiang, and M. Wu, "Delay-dependent stability analysis of multi-area load frequency control with enhanced accuracy and computation efficiency," *IEEE Trans. Power Syst.*, vol. 34, no. 5, pp. 3687–3696, Sep. 2019.
[65] W. Yao, L. Jiang, J. Wen, Q. H. Wu, and S. Cheng, "Wide-area damping controller of FACTS devices for inter-area oscillations considering communication time delays," *IEEE Trans. Power Syst.*, vol. 29, no. 1, pp. 318–329, Jan. 2014.
[66] C. K. Zhang, L. Jiang, Q. H. Wu, Y. He, and M. Wu, "Further results on delay-dependent stability of multi-area load frequency control," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4465–4474, Nov. 2013.
[67] C. K. Zhang, L. Jiang, Q. H. Wu, Y. He, and M. Wu, "Delay-dependent robust load frequency control for time delay power systems," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 2192–2201, Aug. 2013.
[68] Y. Li, Y. Yin, and D. Zhang, "Adaptive task-space synchronization control of bilateral teleoperation systems with uncertain parameters and communication delays," *IEEE Access*, vol. 6, pp. 5740–5748, 2018.
[69] C. Dou, D. Yue, Z. Zhang, and J. M. Guerrero, "Hierarchical delay-dependent distributed coordinated control for DC ring-bus microgrids," *IEEE Access*, vol. 5, pp. 10130–10140, 2017.
[70] A. Zhang, D. Zhou, M. Yang, and P. Yang, "Finite-time formation control for unmanned aerial vehicle swarm system with time-delay and input saturation," *IEEE Access*, vol. 7, pp. 5853–5864, 2019.
[71] C.-K. Zhang, F. Long, Y. He, W. Yao, L. Jiang, and M. Wu, "A relaxed quadratic function negative-determination lemma and its application to time-delay systems," *Automatica*, vol. 113, Mar. 2020, Art. no. 108764.

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