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Output Feedback Stabilization for a Class of Uncertain High-Order Nonlinear Systems

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ABSTRACT We investigate output feedback stabilization for a class of high-order nonlinear systems whose output function and nonlinear terms are unknown. First, a smooth state feedback control law is designed by adding a power integrator technique. Next, we design a high-order observer to estimate the unmeasurable state, and allocate gains of the observer one by one in an iterative way. Finally, a dynamic output compensator is achieved such that the closed-loop system converges to the equilibrium point quick. Two examples are provided to demonstrate the effectiveness of the proposed method.

INDEX TERMS Output feedback stabilization, high-order nonlinear systems, observer design, dynamic output compensator.

I. INTRODUCTION

It is challenge to investigate output feedback stabilization for nonlinear systems since it involves in observer designs. In order to avoid finite time escape, it is necessary to impose some restrictive conditions on the nonlinear terms when investigating output feedback control for nonlinear uncertain systems in [1]. Global output feedback stabilization for Lipschitz nonlinear systems is presented in [2].

When the nonlinear terms are not precisely known, a feedback domination method is presented, and it is shown that global exponential stabilization can be achieved under the linear growth condition with a priori knowledge of the growth rate in [3]. A homogeneous domination approach is established to handle higher-order nonlinearities, which the linear growth condition is relaxed in [4]. More recent works about global output feedback stabilization for nonlinear systems with uncertain growth and higher-order growth conditions can be found in [5]–[7].

Global output feedback stabilization for a class of homogeneous systems is presented in [8], [9]. For a class of high-order switched nonlinear systems, output-feedback control is also appeared in [10], [11]. For nonlinear systems with unknown output functions, global output feedback stabilization is explored in [12]–[14]. Adaptive output feedback tracking control for uncertain switched nonlinear systems with time delays is investigated in [15], [16]. For nonlinear stochastic switching systems, two control schemes are presented in [17], [18]. By adding a power integrator technique, global stabilization of high-order lower-triangular systems is shown in [19]. Robust regulation is designed for a chain of power integrators perturbed in [20]. When the output function of the high-order nonlinear systems [8]–[11] is unknown, how to design global output feedback controller becomes much more challenge.

This paper investigates output feedback stabilization for a class of high-order nonlinear systems with unknown output function and nonlinear terms. First, a state feedback law is designed by employing backstepping method and adding a power integrator technique. Then, by an iterative manner, a new observer design is presented, which gains of the observer can be appropriately enlarged to make the error arbitrarily small. Finally, a dynamic output compensator is achieved such that the closed-loop system is globally asymptotically stable quick.

The main contributions of this paper are as follows:

(1) a unified method is achieved to deal with high-order nonlinear systems with unknown output function and nonlinear terms;

(2) compared with [8]–[11], we deal with a class of uncertain nonlinear systems with weaker constraints on output and nonlinear terms. A new dynamic high-order observer design method is proposed. Gains of the observer are assigned one-by-one by an iterative manner, and can be appropriately enlarged to make the error arbitrarily small;

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(3) the advantage of dynamic output compensator is that the closed-loop system converges to the equilibrium point quick.

II. SYSTEM DESCRIPTION AND SOME LEMMAS

Consider the nonlinear system

$$\dot{x}_1 = x_2^p + \phi_1(t, x, u) \quad (1)$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n^p + \phi_{n-1}(t, x, u) \quad (2)$$
$$\dot{x}_n = u + \phi_n(t, x, u) \quad (3)$$
$$y = h(x_1) \quad (4)$$

where $x = (x_1, \cdots, x_n)^T \in R^n$, $u \in R$ and $y \in R$ are the system state, input and output, respectively, and $p \geq 1$ is an odd integer, and $h(\cdot) : R \to R$ with $h(0) = 0$ is unknown continuously differentiable, and the nonlinear terms $\phi_i : R^+ \times R^n \times R \to R, i = 1, \cdots, n$ are continuous.

The nonlinear function h is assumed satisfying Assumption 1 as follows:

Assumption 1: There exist two known positive constants $\underline{\theta}$ and $\overline{\theta}$ such that

$$\underline{\theta} \leq \frac{\partial h(s)}{\partial s} \leq \overline{\theta}, \quad \forall s \in R. \quad (5)$$

Remark 1: Owing to use sensors in practice, the relationship between the sensor output and state x_1 of the system is always nonlinear, uncertain and time-varying. As shown in [14], the sensor output y is an uncertain nonlinear function of the real displacement x_1 of the working region. However, the derivative of the nonlinear function $h(x_1)$ actually is bounded, which implies that (5) is a natural assumption. The simplest function satisfying (5) is a linear output $h(x_1) = \theta x_1$ with an unknown constant θ if the upper-bound and lower-bounder of θ are known. In addition, some nonlinear output functions are bounded, such as $h(x_1) = 2x_1 + \sin(x_1)$, satisfies Assumption 1 as well.

Further, the nonlinear terms $\phi_i : R^+ \times R^n \times R \to R, i = 1, \cdots, n$ satisfy Assumption 2 as follows:

Assumption 2: There exists a constant $C \geq 0$ such that

$$|\phi_i(t, x, u) - \phi_i(t, \hat{x}, u)| \leq C(|x_1 - \hat{x}_1| + \cdots + |x_i - \hat{x}_i|)\left(\sum_{j=1}^{i}(x_j^{p-1} + \hat{x}_j^{p-1})\right), \quad (6)$$

for $i = 1, \cdots, n$.

Remark 2: Assumption 2 can be viewed as a high-order version of Lipschitz-like condition. In fact, (10) is degenerated to global Lipschitz condition when $p = 1$,

$$|\phi_i(t, x, u) - \phi_i(t, \hat{x}, u)| \leq C(|x_1 - \hat{x}_1| + \cdots + |x_i - \hat{x}_i|).$$

The following lemmas from [8], [9] are needed in this paper.

Lemma 1: Let n and m be positive real numbers. If a, b and $\gamma > 0$ are continuous scalar-valued functions, for any real constant $c > 0$, it holds

$$\gamma|a|^n|b|^m \leq c|a|^{m+n} + \frac{m}{m+n}\left(\frac{n}{c(m+n)}\right)^{\frac{n}{m}}\gamma^{\frac{m+n}{m}}|b|^{m+n}. \quad (7)$$

Lemma 2: Given a positive real number $p \geq 1$, for any $x, y \in R$ and $c > 0$, it holds

$$|x^p - y^p| \leq c|x - y||(x - y)^{p-1} + y^{p-1}|. \quad (8)$$

Lemma 3: Given a positive real number p, for any $x_1, \cdots, x_n \in R$, it holds

$$(|x_1| + \cdots + |x_n|)^p \leq max(n^{p-1}, 1)(|x_1|^p + \cdots + |x_n|^p). \quad (9)$$

Lemma 4: For $x, y \in R, p \geq 1$, it holds

$$(x + y)^p \leq 2^{p-1}(|x|^p + |y|^p). \quad (10)$$

Lemma 5: Let x and y be any real numbers and $p > 0$ be an odd integer, it holds

$$-(x - y)(x^p - y^p) \leq -\frac{1}{2^{p-1}}(x - y)^{p+1}. \quad (11)$$

III. OUTPUT FEEDBACK STABILIZATION DESIGN AND STABILITY ANALYSIS

Under Assumptions 1–2, we will design a output feedback controller for the uncertain system (1)–(4).

Theorem 1: If Assumptions 1–2 hold, then the dynamic output compensator

$$\dot{\hat{x}} = \eta(\hat{x}, y), \quad \hat{x} \in R^n$$
$$u = u(\hat{x}, y) \quad (12)$$

globally stabilizes system (1)–(4).

Proof: The proof consists of three parts.

Part 1: Consider state feedback control law design if the state is measurable.

Step 1: Consider the Lyapunov candidate function

$$V_1(y) = \frac{1}{2}y^2. \quad (13)$$

Under Assumption 1, we obtain

$$|y|^p/\bar{\theta}^p \leq |x_1|^p \leq |y|^p/\underline{\theta}^p, \quad (14)$$

the proof is given in the appendix. Under Assumption 2, it can be deduced that system (1)–(4) satisfies the growth conditions

$$|\phi_i(t, x, u)| \leq \lambda_i(|x_1|^p + \cdots + |x_i|^p) \quad (15)$$

where $\lambda_i = 2iC \geq 0, i = 1, \cdots, n$.

With (14) and (15), the time derivative of V_1 along the trajectories of (1)–(4) for $n = 1$ is

$$\begin{aligned}\dot{V}_1(y) &= y\frac{\partial y}{\partial x_1}(x_2^p + \phi_1(t, x, u)) \\ &\leq y\frac{\partial y}{\partial x_1}x_2^p + \lambda_1|y||\frac{\partial y}{\partial x_1}||x_1|^p \\ &\leq y\frac{\partial y}{\partial x_1}x_2^p + \lambda_1\bar{\theta}\frac{1}{\underline{\theta}^p}y^{p+1}. \end{aligned} \quad (16)$$

Design a smooth state feedback control law

$$x_2 = x_2^*(y) = -\beta_1 y \tag{17}$$

where $\beta_1 = (\frac{n}{\underline{\theta}} + \lambda_1 \bar{\theta} \frac{1}{\underline{\theta}^p})^{\frac{1}{p}}$ is a non-negative constant, we get

$$\dot{V}_1(y) \leq -ny^{p+1}. \tag{18}$$

Step 2: Consider the Lyapunov candidate function

$$V_2 = V_2(y, \xi_2) = V_1(y) + \frac{\xi_2^2}{2}. \tag{19}$$

Let

$$\xi_1 = y, \quad \xi_2 = x_2 - x_2^*(\xi_1). \tag{20}$$

Under the coordinates transform (20), system (1)–(4) for $n = 2$ is transferred to

$$\dot{\xi}_1 = \Upsilon_1(\xi_1, \xi_2) \tag{21}$$
$$\dot{\xi}_2 = x_3^p + \Psi_2(\xi_1, \xi_2) \tag{22}$$

where

$$\Upsilon_1(\xi_1, \xi_2) = \frac{\partial \xi_1}{\partial x_1}\left(\phi_1(t, x, u) + (\xi_2 + x_2^*)^p\right),$$
$$\Psi_2(\xi_1, \xi_2) = \phi_2(t, x, u) - \frac{\partial x_2^*}{\partial \xi_1}\Upsilon_1(\xi_1, \xi_2).$$

Using Lemma 4, it is easy to show

$$\begin{aligned}|\Upsilon_1(\xi_1, \xi_2)| &\leq |\frac{\partial \xi_1}{\partial x_1}|\left(\lambda_1|x_1|^p + 2^{p-1}(|\xi_2|^p + |\beta_1\xi_1|^p)\right)\\ &\leq \bar{\theta}\left((\lambda_1\frac{1}{\underline{\theta}^p} + 2^{p-1}\beta_1^p)|\xi_1|^p + 2^{p-1}|\xi_2|^p\right)\\ &\leq \bar{\lambda}_1(|\xi_1|^p + |\xi_2|^p)\end{aligned} \tag{23}$$

where $\bar{\lambda}_1 = \max\{\bar{\theta}(\lambda_1\frac{1}{\underline{\theta}^p} + 2^{p-1}\beta_1^p), \bar{\theta}2^{p-1}\}$, and

$$\begin{aligned}&|\Psi_2(\xi_1, \xi_2)|\\ &\leq \lambda_2(|x_1|^p + |x_2|^p) + \beta_1\bar{\lambda}_1(|\xi_1|^p + |\xi_2|^p)\\ &\leq \lambda_2\left(\frac{1}{\underline{\theta}^p}|\xi_1|^p + 2^{p-1}(|\xi_2|^p + |\beta_1\xi_1|^p)\right) + \beta_1\bar{\lambda}_1\sum_{i=1}^{2}|\xi_i|^p\\ &= \left(\lambda_2(\frac{1}{\underline{\theta}^p} + 2^{p-1}\beta_1^p) + \beta_1\bar{\lambda}_1\right)|\xi_1|^p\\ &\quad + (2^{p-1}\lambda_2 + \beta_1\bar{\lambda}_1)|\xi_2|^p\\ &\leq \tilde{\lambda}_2(|\xi_1|^p + |\xi_2|^p)\end{aligned} \tag{24}$$

where $\tilde{\lambda}_2 = \max\{\lambda_2(\frac{1}{\underline{\theta}^p} + 2^{p-1}\beta_1^p) + \beta_1\bar{\lambda}_1, 2^{p-1}\lambda_2 + \beta_1\bar{\lambda}_1\}$ is a non-negative constant.

With the help of Lemma 1, the time derivative of $V_2(\cdot)$ along the trajectories of (21)–(22) is

$$\dot{V}_2(\xi_1, \xi_2) \leq -(n-1)\xi_1^{p+1} + \xi_2 x_3^p + (\tilde{\lambda}_2 + \tilde{\lambda}_2^{p+1})\xi_2^{p+1}. \tag{25}$$

Clearly, a smooth state feedback controller is chosen as follows

$$x_3 = x_3^*(\xi_2) = -\beta_2\xi_2 \tag{26}$$

where $\beta_2 = [n - 1 + (\tilde{\lambda}_2 + \tilde{\lambda}_2^{p+1})]^{\frac{1}{p}} > 0$ is a constant.

Substituting (26) into (25), we have

$$\dot{V}_2(\xi_1, \xi_2) \leq -(n-1)(\xi_1^{p+1} + \xi_2^{p+1}). \tag{27}$$

Step $k+1$: Suppose at step k, there exists a global change of coordinates

$$x_1^* = 0, \quad \xi_1 = y - x_1^*, \tag{28}$$
$$x_2^* = -\beta_1\xi_1, \quad \xi_2 = x_2 - x_2^*, \tag{29}$$
$$\vdots$$
$$x_k^* = -\beta_{k-1}\xi_{k-1}, \quad \xi_k = x_k - x_k^* \tag{30}$$

with constants $\beta_1 > 0, \cdots, \beta_{k-1} > 0$, transferring (1)–(4) for $n = k$ into a system of the form

$$\dot{\xi}_1 = \Upsilon_1(\xi_1, \xi_2) \tag{31}$$
$$\vdots$$
$$\dot{\xi}_{k-1} = \Upsilon_{k-1}(\xi_1, \cdots, \xi_k) \tag{32}$$
$$\dot{\xi}_k = x_{k+1}^p + \Psi_k(\xi_1, \cdots, \xi_k) \tag{33}$$

with

$$|\Upsilon_i(\xi_1, \cdots, \xi_{i+1})| \leq \bar{\lambda}_i(|\xi_1|^p + \cdots + |\xi_{i+1}|^p),$$
$$i = 1, \cdots, k-1.$$
$$|\Psi_k(\xi_1, \cdots, \xi_k)| \leq \tilde{\lambda}_k(|\xi_1|^p + \cdots + |\xi_k|^p),$$

where $\bar{\lambda}_1, \cdots, \bar{\lambda}_{k-1}, \tilde{\lambda}_k$ are non-negative constants.

And the Lyapunov candidate function is

$$V_k = V_k(\xi_1, \cdots, \xi_k) = V_{k-1} + \frac{\xi_k^2}{2} \tag{34}$$

and a smooth state feedback control law is

$$x_{k+1}^*(\xi_k) = -\beta_k\xi_k \tag{35}$$

where $\beta_k = \left(n - k + 1 + \left(\tilde{\lambda}_k + (k-1)\tilde{\lambda}_k^{p+1}\right)\right)^{\frac{1}{p}}$ is a non-negative constant, such that

$$\dot{V}_k \leq -(n-k+1)\sum_{i=1}^{k}\xi_i^{p+1}. \tag{36}$$

We claim that (36) also holds at step $k+1$. To prove the claim, we denote

$$\xi_{k+1} = x_{k+1} - x_{k+1}^*(\xi_k). \tag{37}$$

This, together with (31)–(33), yields the augmented system

$$\dot{\xi}_1 = \Upsilon_1(\xi_1, \xi_2) \tag{38}$$
$$\vdots$$
$$\dot{\xi}_k = \Upsilon_k(\xi_1, \cdots, \xi_{k+1}) \tag{39}$$
$$\dot{\xi}_{k+1} = x_{k+2}^p + \Psi_{k+1}(\xi_1, \cdots, \xi_{k+1}) \tag{40}$$

where

$$\begin{aligned}\Upsilon_k(\xi_1, \cdots, \xi_{k+1}) &= (\xi_{k+1} + x_{k+1}^*)^p\\ &\quad + \phi_k(t, x, u) - \frac{\partial x_k^*}{\partial \xi_{k-1}}\dot{\xi}_{k-1},\end{aligned} \tag{41}$$

$$\Psi_{k+1}(\xi_1,\cdots,\xi_{k+1}) = \phi_{k+1}(t,x,u) - \frac{\partial x^*_{k+1}}{\partial \xi_k}\dot{\xi}_k. \tag{42}$$

Using Lemma 4, by (14), (15), (28)–(30), we can deduce

$$\begin{aligned}
&|\Upsilon_k(\xi_1,\cdots,\xi_{k+1})| \\
&\le |(\xi_{k+1}+x^*_{k+1})^p| + \lambda_k \sum_{i=1}^{k}|x_i|^p + \beta_{k-1}|\dot{\xi}_{k-1}| \\
&\le 2^{p-1}(|\xi_{k+1}|^p + |\beta_k\xi_k|^p) + \lambda_k(\frac{1}{\theta^p}|\xi_1|^p \\
&\quad + 2^{p-1}(|\xi_2|^p + |\beta_1\xi_1|^p) + \cdots + 2^{p-1}(|\xi_k|^p \\
&\quad + |\beta_{k-1}\xi_{k-1}|^p)) + \beta_{k-1}\bar{\lambda}_{k-1}\sum_{i=1}^{k}|\xi_i|^p \\
&\le \bar{\lambda}_k(|\xi_1|^p + \cdots + |\xi_{k+1}|^p)
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
\bar{\lambda}_k = \max\{&\lambda_k(\frac{1}{\theta^p} + 2^{p-1}\beta_1^p) + \beta_{k-1}\bar{\lambda}_{k-1}, \\
&2^{p-1}\lambda_k(1+\beta_2^p) + \beta_{k-1}\bar{\lambda}_{k-1},\cdots, \\
&2^{p-1}\lambda_k(1+\beta_{k-1}^p) + \beta_{k-1}\bar{\lambda}_{k-1}, \\
&2^{p-1}(\lambda_k+\beta_k^p) + \beta_{k-1}\bar{\lambda}_{k-1}, 2^{p-1}\}.
\end{aligned}$$

Using (15), (35), (39) and (42), we have

$$\begin{aligned}
&|\Psi_{k+1}(\xi_1,\cdots,\xi_{k+1})| \\
&\le \lambda_{k+1}(|x_1|^p + \cdots + |x_{k+1}|^p) + \beta_k\bar{\lambda}_k\sum_{i=1}^{k+1}|\xi_i|^p \\
&\le \lambda_{k+1}(\frac{1}{\theta^p}|\xi_1|^p + 2^{p-1}(|\xi_2|^p + |\beta_1\xi_1|^p) + \cdots \\
&\quad + 2^{p-1}(|\xi_{k+1}|^p + |\beta_k\xi_k|^p)) + \beta_k\bar{\lambda}_k\sum_{i=1}^{k+1}|\xi_i|^p \\
&\le \tilde{\lambda}_{k+1}(|\xi_1|^p + \cdots + |\xi_{k+1}|^p)
\end{aligned} \tag{44}$$

with

$$\begin{aligned}
\bar{\lambda}_{k+1} = \max\{&\lambda_{k+1}(\frac{1}{\theta^p} + 2^{p-1}\beta_1^p) + \beta_k\bar{\lambda}_k, \\
&2^{p-1}\lambda_{k+1}(1+\beta_2^p) + \beta_k\bar{\lambda}_k,\cdots, \\
&2^{p-1}\lambda_{k+1}(1+\beta_k^p) + \beta_k\bar{\lambda}_k, \\
&2^{p-1}\lambda_{k+1} + \beta_k\bar{\lambda}_k\}.
\end{aligned}$$

Now, construct the Lyapunov candidate function as

$$V_{k+1} = V_{k+1}(\xi_1,\cdots,\xi_{k+1}) = V_k + \frac{\xi_{k+1}^2}{2}, \tag{45}$$

it holds

$$\begin{aligned}
\dot{V}_{k+1} &\le -(n-k+1)\sum_{i=1}^{k}\xi_i^{p+1} + \xi_{k+1}\dot{\xi}_{k+1} \\
&\le -(n-k+1)\sum_{i=1}^{k}\xi_i^{p+1} \\
&\quad + \xi_{k+1}(x_{k+2}^p + \Psi_{k+1}(\xi_1,\cdots,\xi_{k+1})).
\end{aligned} \tag{46}$$

Using an almost identical argument as proceeded in step 2, the following inequality can be deduced from (44), (46), and using Lemma 1

$$\begin{aligned}
\dot{V}_{k+1} &\le -(n-k+1)\sum_{i=1}^{k}\xi_i^{p+1} + \xi_{k+1}x_{k+2}^p \\
&\quad + \sum_{i=1}^{k}\xi_i^{p+1} + (\tilde{\lambda}_{k+1} + k\tilde{\lambda}_{k+1}^{p+1})\xi_{k+1}^{p+1}.
\end{aligned} \tag{47}$$

Clearly, choose a smooth state feedback control law

$$x_{k+2} = x^*_{k+2}(\xi_{k+1}) = -\beta_{k+1}\xi_{k+1} \tag{48}$$

where $\beta_{k+1} = \left(n-k+(\tilde{\lambda}_{k+1}+k\tilde{\lambda}_{k+1}^{p+1})\right)^{\frac{1}{p}}$ is a non-negative constant, yields

$$\dot{V}_{k+1}(\xi_1,\cdots,\xi_k) \le -(n-k)\sum_{i=1}^{k+1}\xi_i^{p+1}. \tag{49}$$

This completes the inductive step. The inductive argument shows that (36) is true for $k = 2,\cdots,n-1$.

Step n: Construct the Lyapunov candidate function

$$V_n = V_n(\xi_1,\cdots\xi_n) = V_{n-1} + \frac{1}{2}\xi_n^2. \tag{50}$$

Together with (38)–(40) and (48), choose a smooth state feedback control law $x_n^*(\xi_{n-1}) = -\beta_{n-1}\xi_{n-1}$ and denote

$$\xi_n = x_n - x_n^*(\xi_{n-1}). \tag{51}$$

Similarly, it is not difficult to obtain

$$\dot{\xi}_1 = \Upsilon_1(\xi_1,\xi_2) \tag{52}$$

$$\vdots$$

$$\dot{\xi}_{n-1} = \Upsilon_{n-1}(\xi_1,\cdots,\xi_n) \tag{53}$$

$$\dot{\xi}_n = u + \Psi_n(\xi_1,\cdots,\xi_n) \tag{54}$$

where

$$\begin{aligned}
\Upsilon_{n-1}(\xi_1,\cdots,\xi_n) &= (\xi_n + x_n^*)^p \\
&\quad + \phi_{n-1}(t,x,u) - \frac{\partial x^*_{n-1}}{\partial \xi_{n-2}}\dot{\xi}_{n-2} \\
\Psi_n(\xi_1,\cdots,\xi_n) &= \phi_n(t,x,u) - \frac{\partial x^*_n}{\partial \xi_{n-1}}\dot{\xi}_{n-1}
\end{aligned}$$

and have the properties

$$|\Upsilon_{n-1}(\xi_1,\cdots,\xi_n)| \le \bar{\lambda}_{n-1}(|\xi_1|^p + \cdots + |\xi_n|^p), \tag{55}$$

$$|\Psi_n(\xi_1,\cdots,\xi_n)| \le \tilde{\lambda}_n(|\xi_1|^p + \cdots + |\xi_n|^p), \tag{56}$$

where

$$\begin{aligned}
\bar{\lambda}_{n-1} = \max\{&\lambda_{n-1}(\frac{1}{\theta^p} + 2^{p-1}\beta_1^p) + \beta_{n-2}\bar{\lambda}_{n-2}, \\
&2^{p-1}\lambda_{n-1}(1+\beta_2^p) + \beta_{n-2}\bar{\lambda}_{n-2},\cdots, \\
&2^{p-1}\lambda_{n-1}(1+\beta_{n-2}^p) + \beta_{n-2}\bar{\lambda}_{n-2}, \\
&2^{p-1}(\lambda_{n-1}+\beta_{n-1}^p) + \beta_{n-2}\bar{\lambda}_{n-2}, 2^{p-1}\},
\end{aligned}$$

and

$$\tilde{\lambda}_n = \max\{\lambda_n(\frac{1}{\theta^p} + 2^{p-1}\beta_1^p) + \beta_{n-1}\bar{\lambda}_{n-1}, \\ 2^{p-1}\lambda_n(1+\beta_2^p) + \beta_{n-1}\bar{\lambda}_{n-1}, \cdots, \\ 2^{p-1}\lambda_n(1+\beta_{n-1}^p) + \beta_{n-1}\bar{\lambda}_{n-1}, \\ 2^{p-1}\lambda_n + \beta_{n-1}\bar{\lambda}_{n-1}\},$$

are non-negative constants. Time derivative of V_n is

$$\begin{aligned}\dot{V}_n &\le -2\sum_{i=1}^{n-1}\xi_i^{p+1} + \xi_n\dot{\xi}_n \\ &\le -2\sum_{i=1}^{n-1}\xi_i^{p+1} + \xi_n(u + \Psi_n(\xi_1, \cdots, \xi_n)) \\ &\le -2\sum_{i=1}^{n-1}\xi_i^{p+1} + \xi_n u + |\xi_n|\tilde{\lambda}_n\sum_{i=1}^{n}|\xi_i|^p \\ &\le -\sum_{i=1}^{n-1}\xi_i^{p+1} + \xi_n u + (\tilde{\lambda}_n + (n-1)\tilde{\lambda}_n^{p+1})\xi_n^{p+1}. \end{aligned} \quad (57)$$

Obviously, we have

$$\begin{aligned} u &= u(\xi_n) \\ &= -(\beta_n\xi_n)^p \\ &= -(\beta_n x_n + \beta_{n-1}(x_{n-1} + \cdots + \beta_2(x_2 + \beta_1 y)\cdots)))^p \end{aligned} \quad (58)$$

where $\beta_n = \left(1 + \left(\tilde{\lambda}_n + (n-1)\tilde{\lambda}_n^{p+1}\right)\right)^{\frac{1}{p}} > 0$ is a constant, such that

$$\dot{V}_n \le -\sum_{i=1}^{n}\xi_i^{p+1}. \quad (59)$$

So the original system (1)–(4) is globally stabilized by the control law (58).

Part 2: Nonlinear observer design

For the nonlinear system (1)–(4), if the state x is not measurable, the smooth state feedback controller (58) is not implementable directly. It is need to design an observer to estimate the state x. A nonlinear observer is designed as follows

$$\dot{\hat{x}}_1 = \hat{x}_2^p + \phi_1(t, \hat{x}, u) - L_1\hat{x}_1^p \quad (60)$$

$$\vdots$$

$$\dot{\hat{x}}_{n-1} = \hat{x}_n^p + \phi_{n-1}(t, \hat{x}, u) - L_{n-1}\cdots L_1\hat{x}_1^p \quad (61)$$

$$\dot{\hat{x}}_n = u + \phi_n(t, \hat{x}, u) - L_n\cdots L_1\hat{x}_1^p \quad (62)$$

where $L_1, \cdots, L_n \ge 0$ are the gain constants to be determined later.

Let $e_i = x_i - \hat{x}_i$, $i = 1, \cdots, n$ be the estimate errors. Then, the error dynamics is expressed as

$$\dot{e}_1 = (x_2^p - \hat{x}_2^p) + \Phi_1(\cdot) + L_1\hat{x}_1^p \quad (63)$$

$$\vdots$$

$$\dot{e}_{n-1} = (x_n^p - \hat{x}_n^p) + \Phi_{n-1}(\cdot) + L_{n-1}\cdots L_1\hat{x}_1^p \quad (64)$$

$$\dot{e}_n = \Phi_n(\cdot) + L_n\cdots L_1\hat{x}_1^p \quad (65)$$

where $\Phi_i(t, x, \hat{x}, u) = \phi_i(t, x, u) - \phi_i(t, \hat{x}, u)$, $i = 1, \cdots, n$.

For the error dynamics (63)–(65), choose a Lyapunov candidate function as follows

$$\begin{aligned} &U(e_1, \cdots, e_n) \\ &\quad = \frac{1}{2}(e_1^2 + (e_2 - L_2e_1)^2 + \cdots + (e_n - L_ne_{n-1})^2) \end{aligned} \quad (66)$$

which is positive definite and proper. Then, time derivative of $U(e_1, \cdots, e_n)$ along the trajectories of (63)–(65) is

$$\begin{aligned} &\dot{U}(e_1, \cdots, e_n) \\ &= e_1\dot{e}_1 + (e_2 - L_2e_1)(\dot{e}_2 - L_2\dot{e}_1) + \cdots \\ &\quad + (e_n - L_ne_{n-1})(\dot{e}_n - L_n\dot{e}_{n-1}) \\ &= e_1\left((1+L_2^2)\dot{e}_1 - L_2\dot{e}_2\right) + \sum_{i=2}^{n-1} e_i\left((1+L_{i+1}^2)e_i\right. \\ &\quad \left. - L_i\dot{e}_{i-1} - L_{i+1}e_{i+1}\right) + e_n\left(\dot{e}_n - L_n\dot{e}_{n-1}\right) \\ &\le -\sum_{i=1}^{n} L_ie_i(x_i^p - \hat{x}_i^p) + \sum_{i=1}^{n-1}(1+L_{i+1}^2)e_i \\ &\quad \times (x_{i+1}^p - \hat{x}_{i+1}^p) + \sum_{i=1}^{n-2} L_{i+1}|e_i(x_{i+2}^p - \hat{x}_{i+2}^p)| \\ &\quad + \sum_{i=2}^{n} L_i|e_i\Phi_{i-1}| + \sum_{i=1}^{n-1}(1+L_{i+1}^2)e_i\Phi_i \\ &\quad + \sum_{i=1}^{n-1} L_{i+1}|e_i\Phi_{i+1}| + e_n\Phi_n + L_1e_1x_1^p. \end{aligned} \quad (67)$$

First, by Lemma 5, we have

$$-\sum_{i=1}^{n} L_ie_i(x_i^p - \hat{x}_i^p) \le -\sum_{i=1}^{n}\frac{L_i}{2^{p-1}}e_i^{p+1}. \quad (68)$$

In what follows, we estimate the second term on the right hand side of (67), using Lemmas 1, 4 and (28)–(30), we have

$$\begin{aligned} &\sum_{i=1}^{n-1}(1+L_{i+1}^2)|e_i(x_{i+1}^p - \hat{x}_{i+1}^p)| \\ &\le \sum_{i=1}^{n-1}(1+L_{i+1}^2)|e_i|\left(|x_{i+1}|^p + 2^{p-1}(|x_{i+1}|^p + |e_{i+1}|^p)\right) \\ &\le \sum_{i=1}^{n-1}(1+L_{i+1}^2)2^{2p-1}|e_i|\left(|\xi_{i+1}|^p + |\beta_i\xi_i|^p + |e_{i+1}|^p\right) \\ &\le \frac{\tau_1}{12}(\sum_{i=2}^{n}\xi_i^{p+1} + \sum_{i=1}^{n-1}\xi_i^{p+1}) + \sum_{i=1}^{n-1}\alpha_i(L_{i+1})e_i^{p+1} + e_n^{p+1} \\ &\le \frac{\tau_1}{6}\sum_{i=1}^{n}\xi_i^{p+1} + \sum_{i=1}^{n-1}\alpha_i(L_{i+1})e_i^{p+1} + e_n^{p+1} \end{aligned} \quad (69)$$

where

$$0 < \tau_1 \le 1,$$

$$\alpha_1(L_2)=\left(\left(\frac{1}{p}(\frac{p-1}{p})^{p-1}2^{p(p-2)}(1+\beta_1^{p(p-1)})+1\right)^{p+1}+2(\frac{12}{\tau_1})^p\right)\frac{1}{p+1}(\frac{p}{p+1})^p\left((1+L_2^2)c\right)^{p+1},$$

$$\alpha_i(L_{i+1})=\left(\left(\frac{1}{p}(\frac{p-1}{p})^{p-1}2^{p(p-2)}(1+\beta_i^{p(p-1)})+1\right)^{p+1}+2(\frac{12}{\tau_1})^p\right)\frac{1}{p+1}(\frac{p}{p+1})^p\left((1+L_{i+1}^2)c\right)^{p+1}+1,\quad i=2,\cdots,n-1.$$

Similarly, we can prove

$$\begin{aligned}&\sum_{i=1}^{n-2}L_{i+1}|e_i(x_{i+2}^p-\hat{x}_{i+2}^p)|+L_1e_1x_1^p\\&\leq\sum_{i=1}^{n-2}L_{i+1}2^{2p-1}|e_i|(|\xi_{i+2}|^p+|\beta_{i+1}\xi_{i+1}|^p+|e_{i+2}|^p)\\&\quad+\frac{\tau_1}{6}\xi_1^{p+1}+\frac{1}{p+1}\left(\frac{6p}{\tau_1(p+1)}\right)^p(\frac{L_1}{\underline{\theta}^p})^{p+1}e_1^{p+1}\\&\leq\frac{\tau_1}{6}\sum_{i=1}^{n}\xi_i^{p+1}+\delta_1(L_1,L_2)e_1^{p+1}+\sum_{i=2}^{n-2}\delta_i(L_{i+1})e_i^{p+1}\\&\quad+e_{n-1}^{p+1}+e_n^{p+1}\end{aligned}\tag{70}$$

where

$$\delta_1(L_1,L_2)=\left(\left(\frac{1}{p}(\frac{p-1}{p})^{p-1}2^{p(p-2)}(1+\beta_2^{p(p-1)})+1\right)^{p+1}+2(\frac{12}{\tau_1})^p\right)\frac{1}{p+1}(\frac{p}{p+1})^p(cL_2)^{p+1}+\frac{1}{p+1}\left(\frac{6p}{\tau_1(p+1)}\right)^p(\frac{L_1}{\underline{\theta}^p})^{p+1},$$

$$\delta_i(L_{i+1})=\left(\left(\frac{1}{p}(\frac{p-1}{p})^{p-1}2^{p(p-2)}(1+\beta_{i+1}^{p(p-1)})+1\right)^{p+1}+2(\frac{12}{\tau_1})^p\right)\frac{1}{p+1}(\frac{p}{p+1})^p(cL_{i+1})^{p+1}+1,\quad i=2,\cdots,n-2.$$

Under Assumption 2, using Lemma 4, we have

$$\begin{aligned}&|\Phi_i(t,x,\hat{x},u)|\\&=|\phi_i(t,x,u)-\phi_i(t,\hat{x},u)|\\&\leq C(|e_1|+\cdots+|e_i|)\sum_{j=1}^{i}(x_j^{p-1}+(x_j-e_j)^{p-1})\\&\leq 2^{p-1}C(|e_1|+\cdots+|e_i|)\sum_{j=1}^{i}(x_j^{p-1}+e_j^{p-1})\\&\leq 2^{p-1}C(|e_1|+\cdots+|e_i|)(\frac{1}{\underline{\theta}^{p-1}}\xi_1^{p-1}+\sum_{j=2}^{i}(|\xi_j|\\&\quad+|\beta_{j-1}\xi_{j-1}|)^{p-1}+\sum_{j=1}^{i}e_j^{p-1})\\&\leq 2^{2p-3}C(|e_1|+\cdots+|e_i|)((\frac{1}{\underline{\theta}^{p-1}}+\beta_1^{p-1})\xi_1^{p-1}\\&\quad+\sum_{j=2}^{i}(1+\beta_j^{p-1})\xi_j^{p-1}+\sum_{j=1}^{i}e_j^{p-1})\\&\leq 2^{2p-3}C(|e_1|+\cdots+|e_i|)\sum_{j=1}^{i}(\bar{\beta}_i\xi_j^{p-1}+e_j^{p-1})\end{aligned}\tag{71}$$

where $\bar{\beta}_i=\max(\frac{1}{\underline{\theta}^{p-1}}+\beta_1^{p-1},1+\beta_2^{p-1},\cdots,1+\beta_i^{p-1})$, $i=1,\cdots,n$ are non-negative constants.

Using Lemma 1, we estimate every term of right-hand side of (71). First, it is

$$\begin{aligned}&2^{2p-3}C(|e_1|+\cdots+|e_i|)\sum_{j=1}^{i}\bar{\beta}_i\xi_j^{p-1}\\&\leq 2^{2p-3}C(|e_1|\sum_{j=1}^{i}\bar{\beta}_i\xi_j^{p-1}+\cdots+|e_i|\sum_{j=1}^{i}\bar{\beta}_i\xi_j^{p-1})\\&\leq\frac{1}{i}\sum_{j=1}^{i}|\xi_j|^p+i\frac{1}{p}(\frac{i(p-1)}{p})^{p-1}(2^{2p-3}\bar{\beta}_iC)^p|e_1|^p+\cdots\\&\quad+\frac{1}{i}\sum_{j=1}^{i}|\xi_j|^p+i\frac{1}{p}(\frac{i(p-1)}{p})^{p-1}(2^{2p-3}\bar{\beta}_iC)^p|e_i|^p\\&\leq\sum_{j=1}^{i}|\xi_j|^p+\sum_{j=1}^{i}\frac{1}{p}(\frac{p-1}{p})^{p-1}(2^{2p-3}i\bar{\beta}_iC)^p|e_j|^p.\end{aligned}\tag{72}$$

Similarly, we can prove that

$$\begin{aligned}&2^{2p-3}C(|e_1|+\cdots+|e_i|)\sum_{j=1}^{i}e_j^{p-1}\\&\leq 2^{2p-3}C\sum_{j=1}^{i}(1+\frac{1}{p}(\frac{p-1}{p})^{p-1}i^p)|e_j|^p.\end{aligned}\tag{73}$$

Substituting (72) and (73) into (71), we have

$$|\Phi_i(t,x,\hat{x},u)|\leq\sum_{j=1}^{i}|\xi_j|^p+\sum_{j=1}^{i}\kappa_i|e_j|^p\tag{74}$$

where $\kappa_i=\frac{1}{p}(\frac{p-1}{p})^{p-1}(2^{2p-3}i\bar{\beta}_iC)^p+2^{2p-3}C(1+\frac{1}{p}(\frac{p-1}{p})^{p-1}i^p)$, $i=1,\cdots,n$, are non-negative constants. Using (74), it holds

$$\begin{aligned}&\sum_{i=2}^{n}L_i|e_i\Phi_{i-1}(t,x,\hat{x},u)|\\&\leq\sum_{i=2}^{n}L_i|e_i|\left(\sum_{j=1}^{i-1}|\xi_j|^p+\sum_{j=1}^{i-1}\kappa_{i-1}|e_j|^p\right).\end{aligned}\tag{75}$$

With the help of Lemma 1, it can be deduced that

$$\begin{aligned}
&\sum_{i=2}^{n} L_i|e_i|\left(\sum_{j=1}^{i-1}|\xi_j|^p\right)\\
&\leq \sum_{i=2}^{n}\left(\frac{\tau_1}{6(n-1)}\sum_{j=1}^{i-1}\xi_j^{p+1}+\frac{i-1}{p+1}(\frac{6(n-1)p}{\tau_1(p+1)})^p L_i^{p+1}e_i^{p+1}\right)\\
&\leq \sum_{i=1}^{n-1}\frac{\tau_1}{6}\xi_i^{p+1}+\sum_{i=2}^{n}\frac{1}{p+1}(\frac{6p}{\tau_1(p+1)})^p\\
&\quad\times((n-1)L_i)^{p+1}e_i^{p+1},
\end{aligned}\tag{76}$$

and

$$\begin{aligned}
&\sum_{i=2}^{n} L_i|e_i|(\sum_{j=1}^{i-1}\kappa_{i-1}|e_j|^p)\\
&\leq \sum_{i=1}^{n-1}e_i^{p+1}+\sum_{i=2}^{n}\frac{1}{p+1}(\frac{p}{p+1})^p((n-1)\kappa_{i-1}L_i)^{p+1}e_i^{p+1}.
\end{aligned}\tag{77}$$

From (75)–(77), it holds

$$\sum_{i=2}^{n}L_i|e_i\Phi_{i-1}(t,x,\hat{x},u)|\leq\sum_{i=1}^{n}\frac{\tau_1}{6}\xi_i^{p+1}+\sum_{i=1}^{n}\zeta_i(L_i)e_i^{p+1}\tag{78}$$

where

$$\begin{aligned}
\zeta_1(L_1)&=1,\\
\zeta_i(L_i)&=1+\frac{((n-1)L_i)^{p+1}}{p+1}(\frac{p}{p+1})^p((\frac{6}{\tau_1})^p+\kappa_{i-1}^{p+1})\\
&\quad i=2,\cdots,n-1,\\
\zeta_n(L_n)&=\frac{1}{p+1}(\frac{p}{p+1})^p((n-1)L_n)^{p+1}((\frac{6}{\tau_1})^p+\kappa_{n-1}^{p+1}).
\end{aligned}$$

In a similar manner, we can prove

$$\begin{aligned}
&\sum_{i=1}^{n-1}(1+L_{i+1}^2)|e_i\Phi_i(t,x,\hat{x},u)|\\
&\qquad\leq\sum_{i=1}^{n}\frac{\tau_1}{6}\xi_i^{p+1}+\sum_{i=1}^{n-1}\omega_i(L_{i+1})e_i^{p+1},
\end{aligned}\tag{79}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n-1}L_{i+1}|e_i\Phi_{i+1}(t,x,\hat{x},u)|\\
&\qquad\leq\sum_{i=1}^{n}\frac{\tau_1}{6}\xi_i^{p+1}+\sum_{i=1}^{n-1}\eta_i(L_{i+1})e_i^{p+1}+e_n^{p+1},
\end{aligned}\tag{80}$$

and

$$\begin{aligned}
&|e_n\Phi_n(t,x,\hat{x},u)|\\
&\leq\frac{n}{p+1}(\frac{p}{p+1})^p((\frac{6}{\tau_1})^p+\kappa_n^{p+1})e_n^{p+1}\\
&\quad+\sum_{i=1}^{n}e_i^{p+1}+\sum_{i=1}^{n}\frac{\tau_1}{6}\xi_i^{p+1}
\end{aligned}\tag{81}$$

where

$$\begin{aligned}
\omega_i(L_{i+1})&=[(n-1)(1+L_{i+1}^2)]^{p+1}((\frac{6}{\tau_1})^p+\kappa_i^{p+1})\\
&\quad\times\frac{1}{p+1}(\frac{p}{p+1})^p+1,\\
\eta_i(L_{i+1})&=1+\frac{(nL_{i+1})^{p+1}}{p+1}(\frac{p}{p+1})^p((\frac{6}{\tau_1})^p+\kappa_{i+1}^{p+1}),\\
&\quad i=1,\cdots,n-1.
\end{aligned}$$

Substituting (68)–(81) into (67), it is not difficult to draw that

$$\begin{aligned}
&\dot{U}(e_1,\cdots,e_n)\\
&\leq\tau_1\sum_{i=1}^{n}\xi_i^{p+1}-\sum_{i=1}^{n-1}\left(\frac{L_i}{2^{p-1}}-\Psi_i(L_i,L_{i+1})\right)e_i^{p+1}\\
&\quad-\left(\frac{L_n}{2^{p-1}}-\Psi_n(L_n)\right)e_n^{p+1}
\end{aligned}\tag{82}$$

where

$$\begin{aligned}
\Psi_1(L_1,L_2)&=1+\alpha_1(L_2)+\delta_1(L_1,L_2)+\zeta_1(L_1)\\
&\quad+\omega_1(L_2)+\eta_1(L_2),\\
\Psi_i(L_i,L_{i+1})&=1+\alpha_i(L_{i+1})+\delta_i(L_{i+1})+\zeta_i(L_i)\\
&\quad+\omega_i(L_{i+1})+\eta_i(L_{i+1}),\\
&\quad i=2,\cdots,n-2,\\
\Psi_{n-1}(L_{n-1},L_n)&=2+\alpha_{n-1}(L_n)+\zeta_{n-1}(L_{n-1})\\
&\quad+\omega_{n-1}(L_n)+\eta_{n-1}(L_n),\\
\Psi_n(L_n)&=4+\zeta_n(L_n)+\frac{n}{p+1}(\frac{p}{p+1})^p\\
&\quad\times((\frac{6}{\tau_1})^p+\kappa_n^{p+1}),
\end{aligned}$$

are positive real constants.

Part 3: Output feedback control law design

Using the certainty equivalence principle, we replace the unmeasurable state $x=(x_1,\cdots,x_n)^T$ in the controller (58) by its estimate state $\hat{x}=(\hat{x}_1,\cdots,\hat{x}_n)^T$ generated with the observer (60)–(62). A output feedback control law is designed as follows

$$u=-\left(\sum_{i=2}^{n}\beta_i\cdots\beta_n\hat{x}_i+\beta_1\cdots\beta_n y\right)^p\tag{83}$$

where

$$\begin{aligned}
\beta_1&=(\frac{n}{\underline{\theta}}+\lambda_1\bar{\theta}\frac{1}{\underline{\theta}^p})^{\frac{1}{p}},\\
\beta_i&=\left(n-i+1+\left(\tilde{\lambda}_i+(i-1)\tilde{\lambda}_i^{p+1}\right)\right)^{\frac{1}{p}},\\
\tilde{\lambda}_i&=\max\left\{\lambda_i(\frac{1}{\underline{\theta}^p}+2^{p-1}\beta_1^p)+\beta_{i-1}\bar{\lambda}_{i-1},2^{p-1}\lambda_i(1+\right.\\
&\quad\beta_2^p)+\beta_{i-1}\bar{\lambda}_{i-1},\cdots,2^{p-1}\lambda_i(1+\beta_{i-1}^p)\\
&\quad\left.+\beta_{i-1}\bar{\lambda}_{i-1},2^{p-1}\lambda_i+\beta_{i-1}\bar{\lambda}_{i-1}\right\},\\
&\quad i=2,\cdots,n,
\end{aligned}$$

and

$$\bar{\lambda}_1 = \max\{\bar{\theta}(\lambda_1 \frac{1}{\theta^p} + 2^{p-1}\beta_1^p), \bar{\theta}2^{p-1}\},$$
$$\bar{\lambda}_k = \max\{\lambda_k(\frac{1}{\theta^p} + 2^{p-1}\beta_1^p) + \beta_{k-1}\bar{\lambda}_{k-1}, 2^{p-1}\lambda_k(1 + \beta_2^p) + \beta_{k-1}\bar{\lambda}_{k-1}, \cdots, 2^{p-1}\lambda_k(1 + \beta_{k-1}^p) + \beta_{k-1}\bar{\lambda}_{k-1}, 2^{p-1}(\lambda_k + \beta_k^p) + \beta_{k-1}\bar{\lambda}_{k-1}, 2^{p-1}\},$$
$$k = 2, \cdots, n-1.$$

Let

$$V = V(\xi_1, \cdots, \xi_n, e_1, \cdots, e_n) = V_n(\xi_1, \cdots, \xi_n) + U(e_1, \cdots, e_n). \tag{84}$$

Using (59) and (82), we have

$$\dot{V}(\xi_1, \cdots, \xi_n, e_1, \cdots, e_n) = \dot{V}_n(\xi_1, \cdots, \xi_n) + \dot{U}(e_1, \cdots, e_n) \leq -(1-\tau_1)\sum_{i=1}^{n} \xi_i^{p+1} - \left(\frac{L_n}{2^{p-1}} - \Psi_n(L_n)\right) e_n^{p+1} - \sum_{i=1}^{n-1}\left(\frac{L_i}{2^{p-1}} - \Psi_i(L_i, L_{i+1})\right) e_i^{p+1} \tag{85}$$

it is clear to choose

$$L_i \geq 2^{p-1}(\tau_2 + \Psi_i(L_i, L_{i+1})), \quad i = 1, \cdots, n-1$$
$$L_n \geq 2^{p-1}(\tau_2 + \Psi_i(L_n))$$

where $\tau_2 > 0$ is a constant, it yields

$$\dot{V} \leq -(1-\tau_1)\sum_{i=1}^{n} \xi_i^{p+1} - \tau_2 \sum_{i=1}^{n} e_i^{p+1},$$

so the closed-loop system (1)–(4), (83) is globally asymptotically stable.

IV. EXAMPLES

Example 1: Consider a second-order system given by

$$\dot{x}_1 = x_2^3 \tag{86}$$
$$\dot{x}_2 = u \tag{87}$$
$$y = d_1 x_1 + d_2 \sin x_1 \tag{88}$$

where $2 < d_1 \leq 3, 0 < d_2 < 0.5$ are unknown constants.

The linearization of system (86)–(88) is given by $(A, B, C) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} d_1 + d_2 & 0 \end{bmatrix}\right)$ which is uncontrollable and unobservable. By Theorem 1, the dynamic output compensator is designed as follows

$$\dot{\hat{x}}_1 = \hat{x}_2^3 + L_1(x_1^3 - \hat{x}_1^3) \tag{89}$$
$$\dot{\hat{x}}_2 = u + L_1 L_2(x_1^3 - \hat{x}_1^3) \tag{90}$$
$$u = -(\beta_2(\hat{x}_2 + \beta_1 y))^3 \tag{91}$$

with a suitable choice of the parameters β_1, β_2, L_1 and L_2.

In the simulation, $d_1 = 2 + \text{rand}(1)$ and $d_2 = 0.5\text{rand}(1)$, using Matlab software, we have $\beta_1 = 1, \beta_2 = 2.2, L_1 = 36$, and $L_2 = 0.5$.

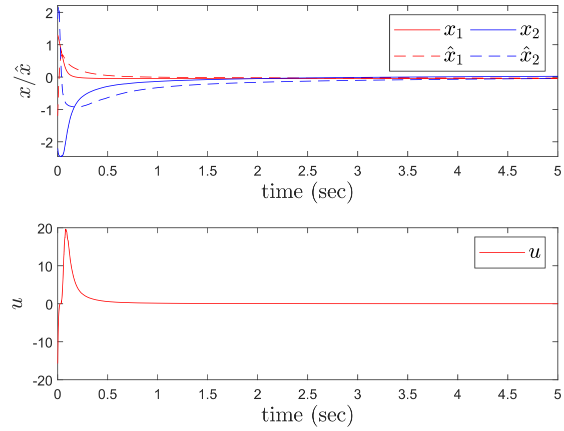

FIGURE 1. Time response of state and observer state, and the dynamic output compensator in Example 1.

From Fig. 1, it can be seen that the closed-loop system (86)–(88), (89)–(91) (with the initial condition $x_1(0) = 1.3$, $x_2(0) = -2.2$, and $\hat{x}_1(0) = -1.2$, $\hat{x}_2(0) = 1.8$) is globally asymptotically stable.

Example 2: Consider the electromechanical system in [12] as follows

$$M\ddot{q} + B\dot{q} + N\sin(q) = I \tag{92}$$
$$L\dot{I} = V_\varepsilon - RI - K_B\dot{q} \tag{93}$$

where $M = \frac{J}{K_\tau} + \frac{mL_0^2}{3K_\tau} + \frac{M_0L_0^2}{K_\tau} + \frac{2M_0R_0^2}{5K_\tau}$, $N = \frac{mL_0G}{2K_\tau} + \frac{M_0L_0G}{K_\tau}$, $B = \frac{B_0}{K_\tau}$. J is the rotor inertia, G is the gravity coefficient, K_τ is the coefficient which characterises the electromechanical conversion of armature current to torque. K_B is the back-emf coefficient, B_0 is the coefficient of viscous friction at the joint, m is the link mass, L_0 is the link length, M_0 is the load mass, R_0 is the radius of the load, $q(t)$ is the angular motor position, $I(t)$ is the motor armature current and L is the armature inductance, R is the armature resistance, and V_ε is the input control voltage.

The values of the parameters are given in [12] as $J = 1.625 \times 10^{-3}\text{kgm}^2$, $m = 0.506\text{kg}$, $R_0 = 0.023\text{m}$, $M_0 = 0.434\text{kg}$, $L_0 = 0.305\text{m}$, $B_0 = 16.25 \times 10^{-3}\text{Nms/rad}$, $L = 25.0 \times 10^{-3}\text{H}$, $R = 5.0\Omega$ and $K_\tau = K_B = 0.90\text{Nm/ A}$.

After some calculations, system (92), (93) can be expressed as follows

$$\dot{x}_1 = x_2 \tag{94}$$
$$\dot{x}_2 = x_3 - NL\sin(\frac{x_1}{ML}) - \frac{B}{M}x_2 \tag{95}$$
$$\dot{x}_3 = u - \frac{K_B}{ML}x_2 - \frac{R}{L}x_3 \tag{96}$$
$$y = \frac{1}{ML}x_1 \tag{97}$$

where $\phi_1(\cdot) = 0$, $\phi_2(\cdot) = -NL\sin(\frac{x_1}{ML}) - \frac{B}{M}x_2$ and $\phi_3(\cdot) = -\frac{K_B}{ML}x_2 - \frac{R}{L}x_3$. It is obvious that $\phi_1(\cdot)$, $\phi_2(\cdot)$ and $\phi_3(\cdot)$ satisfy Assumption 2. By Theorem 1, we construct the dynamic

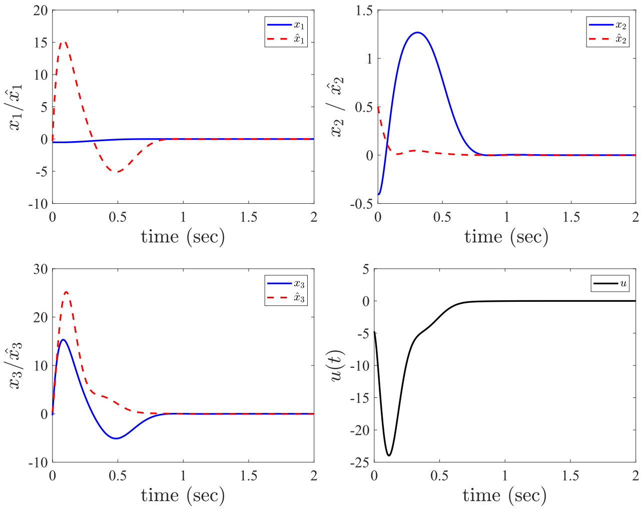

FIGURE 2. Time response of states and observer states of the closed-loop system and the control law in this paper.

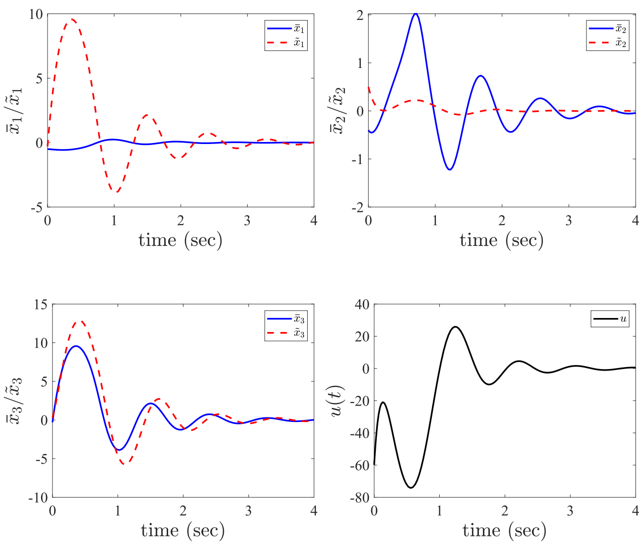

FIGURE 3. Time response of states and observer states of the closed-loop system and the control law in [12].

output compensator as

$$\dot{\hat{x}}_1 = \hat{x}_2 - L_1\hat{x}_1 \tag{98}$$

$$\dot{\hat{x}}_2 = \hat{x}_3 - NL\sin(\frac{\hat{x}_1}{ML}) - \frac{B}{M}\hat{x}_2 - L_1L_2\hat{x}_1 \tag{99}$$

$$\dot{\hat{x}}_3 = u - \frac{K_B}{ML}\hat{x}_2 - \frac{R}{L}\hat{x}_3 - L_1L_2L_3\hat{x}_1 \tag{100}$$

$$u = -\beta_3(\hat{x}_3 + \beta_2(\hat{x}_2 + \beta_1 y)) \tag{101}$$

with a suitable choice of the parameters $\beta_1, \beta_2, \beta_3, L_1, L_2$ and L_3. Using Matlab software, we get $\beta_1 = 0.4642$, $\beta_2 = 10.7$, $\beta_3 = 20$, $L_1 = 16$, $L_2 = 8$ and $L_3 = 0.5$. Simulation result is shown in Fig.2, with $(x_1(0), x_2(0), x_3(0), \hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-0.5, -0.4, -0.3, 0.5, 0.4, 0.3)$.

In [12], they choose parameters $a_1 = 6$, $a_2 = 11$, $a_3 = 6$, $\beta_1 = 1$, $\beta_2 = 1.2$, $\beta_3 = 1.5$ and $L = 2$. The simulation result is shown in Fig.3. The initial condition is the same as in Fig.2.

As can be seen from Fig. 2, the proposed output dynamic compensator makes the closed-loop system converge to zero quick, and the time required is less than 1 second. However, as shown in Fig. 3, the closed-loop system does not converge to zero until 4 seconds under the control law design in [12].

V. CONCLUSION

We investigate a class of high-order nonlinear systems whose output function and nonlinear terms are unknown. First, a smooth state feedback control law is designed by adding a power integrator technique. Next, we design a high-order observer to estimate the unmeasurable state by iteratively allocating gains of the observer. Finally, a dynamic output compensator is achieved such that the closed-loop system is globally asymptotically stable. Two examples are provided to demonstrate the effectiveness of the proposed method.

APPENDIX
PROOF OF THE INEQUALITY (14)

Since $|x_1| \geq 0$, from Assumption 1, we consider the definite integral of $\frac{\partial h(x)}{\partial x}$ from $x = 0$ to $x = x_1 \geq 0$, then

$$\int_0^{x_1} \underline{\theta} dx \leq \int_0^{x_1} \frac{\partial h}{\partial x} dx \leq \int_0^{x_1} \overline{\theta} dx. \tag{A.1}$$

Using the initial condition $h(0) = 0$, it follows that

$$\underline{\theta} x_1 \leq y = h(x_1) \leq \overline{\theta} x_1. \tag{A.2}$$

Therefore, we can obtain

$$|y|/\overline{\theta} \leq |x_1| \leq |y|/\underline{\theta}. \tag{A.3}$$

In the case when $x_1 < 0$, similar to A.1, we can obtain the following relation based on (5)

$$0 < -\underline{\theta} x_1 \leq \int_{x_1}^{0} \frac{\partial h}{\partial x} dx \leq -\overline{\theta} x_1 \tag{A.4}$$

with the initial condition $h(0) = 0$, it can be deduced that

$$\bar{\theta} x_1 \leq y = h(x_1) \leq \underline{\theta} x_1 < 0. \tag{A.5}$$

So we immediately obtain the inequality (A.3), which completes the proof.

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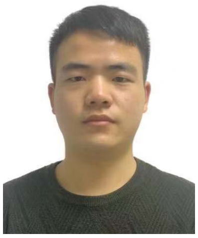

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