Modified Linear Technique for the Controllability and Observability of Robotic Arms

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ABSTRACT In this study, a modified linear technique is proposed for the controllability and observability of robotic arms, the modified linear technique consists of the following steps: a transformation is used to rewrite a nonlinear time-variant model as a quasi-linear time-variant model, this quasi-linear time-variant model is evaluated at origin to obtain a linear time-invariant model, and the rank condition tests the controllability and observability of the linear time-invariant model. The modified linear technique is better than the linearization technique because the modified linear technique does not use the Jacobian approximation, while the linearization technique needs the Jacobian approximation. The modified linear technique is better than the linear technique because the modified linear technique can be applied to robotic arms with rotational and prismatic joints, while the linear technique can only be applied to robotic arms with rotational joints.

INDEX TERMS Controllability, observability, modified, linear, linearization, robotic arms.

I. INTRODUCTION

The concept of controllability denotes the ability to move a robotic arm around in its entire configuration space using only certain admissible manipulations. The exact definition varies slightly within the framework or the kind of applied robotic arms [1]–[3]. Controllability and observability are dual aspects of the same problem.

Several authors have proposed interesting controllers or observers such as [4]–[7]; however, in most of the studies, the robotic arms are assumed to be controllable or observable without any proof. A technique for the controllability of a robotic arm is important because a controllable robotic arm can guarantee the existence of a controller to reach one of the objectives such as the regulation, tracking, disturbance rejection, etc. [8]–[11]. A technique for the observability of a robotic arm is important because an observable robotic arm can guarantee the existence of an observer to reach one of the objectives such as the states estimation, disturbance estimation, fault estimation, etc. Hence, it would be interesting

The associate editor coordinating the review of this manuscript and approving it for publication was Yangmin Li .

to study when the controllability and observability of robotic arms are not assumed.

The rank condition for the controllability and observability of linear time-invariant models is the obvious and precise option, the starting point in the difficulty of this study is when nonlinear time-variant models such as the robotic arms are considered, in this case, the rank condition for the controllability and observability can not be directly applied to robotic arms. Thus, other alternative techniques to obtain this objective should be studied.

There are several studies about the controllability and observability. In [12]–[15], authors are focused on the local, linear, global, and exact controllability and observability. In [16]–[18], authors discuss the controllability and observability of linear time-varying systems. In [19]–[21], authors use algorithms for the controllability and observability. In [22], [23], [24], authors use the geometric technique for the controllability and observability. In [25]–[27], authors consider the fact that robotic arms are changed from being linearly uncontrollable to linearly controllable when the equilibrium point is moved from the origin to a different one. In [28], [29], [30], authors propose linearization feedbacks of robotic

arms for the controllability. In [31]–[33], authors present controllability analysis proving that some robotic arms are not controllable. In [34]–[36], authors extract properties of the mechanical parameters to demonstrate that some robotic arms are controllable and observable. In [37], [38], [39], authors design elastic, flexible, or micro robotic arms which since their design are controllable. In [40], [41], authors find a transformation for which states have the greatest contribution in the controllability and observability.

From the aforementioned studies, the linearization technique has been frequently applied for the controllability and observability of robotic arms [25]–[28], [29]–[32], [33], the linearization technique consists of the following three steps: 1) the Jacobian approximation is used to approximate a nonlinear time-variant model as a quasi-linear time-variant model, 2) this quasi-linear time-variant model is evaluated at origin to obtain a linear time-invariant model, 3) the rank condition tests the controllability and observability of the linear time-invariant model. The linearization technique for the controllability and observability of robotic arms has the problem that it produces an undesired error caused by the Jacobian approximation. On the other hand, the linear technique also has been frequently applied for the controllability and observability of robotic arms [34]–[36], [37]–[40], [41], the linear technique consists of the following three steps: 1) if all the joints in the robotic arm are rotational, then mathematical operations are used to rewrite the nonlinear time-variant model as a quasi-linear time-variant model, 2) this quasi-linear time-variant model is evaluated at origin to obtain a linear time-invariant model, 3) the rank condition tests the controllability and observability of the linear time-invariant model. The linear technique for the controllability and observability of robotic arms has the problem that it can not be applied to robotic arms with prismatic joints. It would be interesting to propose a technique for the controllability and observability of robotic arms which evade the problems of the linearization and linear techniques.

In this study, a modified linear technique is proposed for the controllability and observability of robotic arms, the modified linear technique consists of the following three steps: 1) a transformation is used to rewrite a nonlinear time-variant model as a quasi-linear time-variant model, 2) this quasi-linear time-variant model is evaluated at origin to obtain a linear time-invariant model, 3) the rank condition tests the controllability and observability of the linear time-invariant model. Even the modified linear technique is similar to the linearization and linear techniques in the last step, it is very different in the first step, resulting on the following two advantages: 1) the modified linear technique is better than the linearization technique because the modified linear technique does not use the Jacobian approximation, while the linearization technique needs the Jacobian approximation, 2) the modified linear technique is better than the linear technique because the modified linear technique can be applied to robotic arms with rotational and prismatic joints, while

the linear technique can only be applied to robotic arms with rotational joints.

Finally, the modified linear technique is compared with the linearization and linear techniques for the controllability and observability of the scara and two links robotic arms. The scara and two links robotic arms are selected because they could be applied in pick and place, screwed, printed circuits boards, packaging and labeling, etc.

This paper is organized as follows. In section II, the linearization, the linear, and the modified linear techniques are detailed. In section III, the linearization, the linear, and the modified linear techniques are applied for the controllability and observability of the scara robotic arm. In section IV, the linearization, the linear, and the modified linear techniques are applied for the controllability and observability of the two links robotic arm. In section V, the conclusion and the forthcoming work are detailed.

II. THREE TECHNIQUES FOR THE CONTROLLABILITY AND OBSERVABILITY OF ROBOTIC ARMS

The main concern of this section is to present three techniques for the controllability and observability of robotic arms. First, the linearization technique for the controllability and observability of robotic arms will be presented [25]–[28], [29]–[32], [33]. Second, the linear technique for the controllability and observability of robotic arms will be presented [34]–[36], [37]–[40], [41]. And third, the modified linear technique for the controllability and observability of robotic arms will be presented as the main contribution of this study.

In this section, the nomenclature of the robotic arms models is shown in Table 1.

A. LINEARIZATION TECHNIQUE FOR THE CONTROLLABILITY AND OBSERVABILITY OF ROBOTIC ARMS

The linearization technique has been frequently applied for the controllability and observability of robotic arms [25], [26], [27]–[30], [31]–[33], the linearization technique consists of the following three steps: 1) the Jacobian approximation is used to approximate a nonlinear time-variant model as a quasi-linear time-variant model, 2) this quasi-linear time-variant model is evaluated at origin to obtain a linear time-invariant model, 3) the rank condition tests the controllability and observability of the linear time-invariant model.

Consider the n-link planar robotic arm having n rotational and prismatic joints, the robotic arm model is:

$$W(z_1)\dot{z_2} + V(z_1, z_2)z_2 + X(z_1) = u, \quad (1)$$

The fact that the derivative of the joint angles or link displacements $z_1 \in \Re^{n \times 1}$ is equal to the velocities $z_2 \in \Re^{n \times 1}$ is represented as follows:

$$\dot{z_1} = z_2, \quad (2)$$

TABLE 1. Nomenclature of the robotic arms.

Symbol	Description
$z_1 \in \Re^{n \times 1}$	joint angles or link displacements
$z_2 \in \Re^{n \times 1}$	velocities
$W(z_1) \in \Re^{n \times n}$	matrix with the inertia terms
$V(z_1, z_2) \in \Re^{n \times n}$	matrix with the Coriolis terms
$X(z_1) \in \Re^{n \times 1}$	vector with the gravity terms
$0_n \in \Re^{n \times 1}$	vector of zeros
$I_{n \times n} \in \Re^{n \times n}$	identity matrix
$0_{n \times n} \in \Re^{n \times n}$	matrix of zeros
$z \in \Re^{2n \times 1}$	the states
$y \in \Re^{n \times 1}$	the outputs
$F(z_1, z_2) \in \Re^{2n \times 1}$	nonlinear function vector
$A \in \Re^{2n \times 2n}$	time varying matrix
$B \in \Re^{2n \times n}$	time varying matrix
$D \in \Re^{n \times 2n}$	time varying matrix
$A_0 \in \Re^{2n \times 2n}$	constant matrix
$B_0 \in \Re^{2n \times n}$	constant matrix
$D_0 \in \Re^{n \times 2n}$	constant matrix
$g \in \Re$	gravity acceleration
$m_i \in \Re$	mass of the link i
$l_i \in \Re$	length of the link i
$l_{ci} \in \Re$	half length of the link i
$J_i \in \Re$	moment of inertia of the link i

Using (2), the robotic arm model (1) is rewritten as follows:

$$\dot{z}_1 - z_2 = 0_n,$$
$$W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) = u, \quad (3)$$

Considering (3), the nonlinear time-variant model (3) is:

$$F(z_1, z_2) = \tau, \quad y = Dz,$$
$$F(z_1, z_2) = \begin{bmatrix} \dot{z}_1 - z_2 \\ W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) \end{bmatrix},$$
$$F(z_1, z_2) = \begin{bmatrix} F_1 & F_2 & \cdots & F_{2n} \end{bmatrix}^T,$$
$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix},$$
$$\tau = \begin{bmatrix} 0_n & u \end{bmatrix}^T, \quad (4)$$

The nonlinear time-variant model (4) is linearized at origin by the Jacobian approximation to obtain the following linear time-invariant model:

$$\dot{z} = A_0 z + B_0 u, \quad y = D_0 z,$$
$$A_0 = \left. \frac{\partial F(z_1, z_2)}{\partial z} \right|_{z=0, u=0}, \quad B_0 = \left. \frac{\partial F(z_1, z_2)}{\partial u} \right|_{z=0, u=0},$$
$$A_0 = \begin{bmatrix} \frac{\partial F_1}{\partial z_{11}} & \frac{\partial F_1}{\partial z_{12}} & \cdots & \frac{\partial F_1}{\partial z_{nn}} \\ \frac{\partial F_2}{\partial z_{11}} & \frac{\partial F_2}{\partial z_{12}} & \cdots & \frac{\partial F_2}{\partial z_{nn}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_{2n}}{\partial z_{11}} & \frac{\partial F_{2n}}{\partial z_{12}} & \cdots & \frac{\partial F_{2n}}{\partial z_{nn}} \end{bmatrix}_{z=0, u=0},$$

$$B_0 = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_{2n}}{\partial u_1} & \frac{\partial F_{2n}}{\partial u_2} & \cdots & \frac{\partial F_{2n}}{\partial u_n} \end{bmatrix}_{z=0, u=0},$$
$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D_0 = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix}, \quad (5)$$

Since the linear time-invariant model has been obtained in (5), the rank condition of linear systems [32], [33] is used for the controllability and observability of robotic arms.

Using the linear time-invariant model (5) of the linearization technique, the controllability matrix is:

$$C_0 = \begin{bmatrix} B_0 & A_0 B_0 & A_0^2 B_0 & \cdots & A_0^{2n-1} B_0 \end{bmatrix}, \quad (6)$$

If the rank of the controllability matrix C_0 is equal to $2n$, then the robotic arm of the linearization technique is controllable at origin.

Using the linear time-invariant model (5) of the linearization technique, the observability matrix is:

$$O_0 = \begin{bmatrix} D_0 & D_0 A_0 & D_0 A_0^2 & \cdots & D_0 A_0^{2n-1} \end{bmatrix}^T, \quad (7)$$

If the rank of the observability matrix O_0 is equal to $2n$, then the robotic arm of the linearization technique is observable at origin.

Remark 1: The linearization technique for the controllability and observability of robotic arms has the problem that it produces an undesired error caused by the Jacobian approximation.

B. LINEAR TECHNIQUE FOR THE CONTROLLABILITY AND OBSERVABILITY OF ROBOTIC ARMS

The linear technique also has been frequently applied for the controllability and observability of robotic arms [34]–[36], [37]–[40], [41], the linear technique consists of the following three steps: 1) if all the joints in the robotic arm are rotational, then mathematical operations are used to rewrite the nonlinear time-variant model as a quasi-linear time-variant model, 2) this quasi-linear time-variant model is evaluated at origin to obtain a linear time-invariant model, 3) the rank condition tests the controllability and observability of the linear time-invariant model.

Consider the n-link planar robotic arm having n rotational joints of the Figure 1, the robotic arm model is:

$$W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) = u, \quad (8)$$

The nonlinear time-variant model, the matrices $W(z_1) \in \Re^{n \times n}$ and $V(z_1, z_2) \in \Re^{n \times n}$, and the vector $X(z_1) \in \Re^{n \times 1}$ are [34]–[36]:
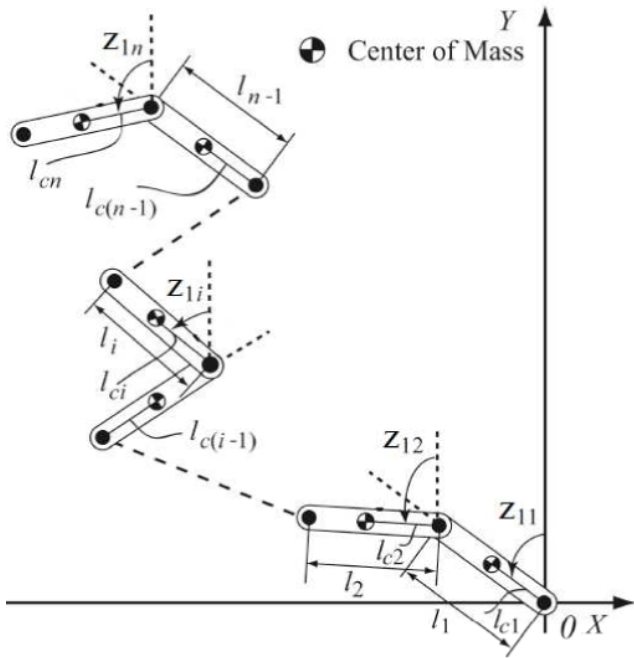
$$W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) = u,$$

FIGURE 1. n-link robotic arm with all rotational joints.

$$W(z_1) = [\alpha_{ij}C_{ij}], \quad V(z_1, z_2) = [-\alpha_{ij}z_2 S_{ij}],$$
$$X(z_1) = [-\beta_i S_i],$$
$$\alpha_{ii} = J_i + m_i l_{ci}^2 + l_i^2 \sum_{k=i+1}^{n} m_k, \quad 1 \le i \le n,$$
$$\alpha_{ij} = \alpha_{ji} = m_j l_i l_{cj} + l_i l_j \sum_{k=j+1}^{n} m_k, \quad 1 \le i \le j \le n,$$
$$\beta_i = \left(m_i l_{ci} + l_i \sum_{k=i+1}^{n} m_k \right) g, \quad 1 \le i \le n,$$
$$C_{ij} = \cos(z_{1j} - z_{1i}), \quad S_{ij} = \sin(z_{1j} - z_{1i})$$
$$S_i = \sin(z_{1i}), \quad m_i > 0, \quad l_i > 0, \quad l_{ci} > 0, \quad (9)$$

The parameters of (9) satisfy:

$$\alpha_{ii} g \le l_i \beta_i, \quad 1 \le i \le n,$$
$$\alpha_{ij} g \le l_i \beta_j, \quad 1 \le i \le j \le n, \quad (10)$$

Now, let us define the following matrices [42], [43]:

$$M_{2n} = \begin{bmatrix} m_1 & m_2 & \cdots & m_{2n} \\ 0 & m_2 & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{2n} \end{bmatrix},$$

$$L_{2n} = \begin{bmatrix} l_1 & 0 & \cdots & 0 \\ l_2 & l_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ l_{2n} & l_{2n} & \cdots & l_{2n} \end{bmatrix},$$

$$M_a = \begin{bmatrix} g\sum_{i=1}^{2n} m_i & 0 & \cdots & 0 \\ 0 & g\sum_{i=2}^{2n} m_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & gm_{2n} \end{bmatrix}, \quad (11)$$

Using (11), the nonlinear time-variant model (9) is rewritten as the following quasi-linear time-variant model [42], [43]:

$$\dot{z} = Az + Bu, \quad y = Dz,$$
$$A = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -(M_{2n}L_{2n})^{-1} M_a & 0_{n \times n} \end{bmatrix},$$
$$B = \begin{bmatrix} 0_{n \times n} \\ (M_{2n}L_{2n})^{-1} \end{bmatrix},$$
$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix}, \quad (12)$$

From (9), (10), (11), (12), let us consider the following equalities:

$$W(z_1) = M_{2n}L_{2n},$$
$$V(z_1, z_2) = M_a, \quad (13)$$

Using (13), (12), the nonlinear time-variant model (9) is rewritten as the following quasi-linear time-variant model [34]–[36]:

$$\dot{z} = Az + Bu, \quad y = Dz,$$
$$A = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -W(z_1)^{-1}V(z_1, z_2) & 0_{n \times n} \end{bmatrix},$$
$$B = \begin{bmatrix} 0_{n \times n} \\ W(z_1)^{-1} \end{bmatrix},$$
$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix}, \quad (14)$$

The quasi-linear time-variant model (14) is evaluated at origin to obtain the following linear time-invariant model:

$$\dot{z} = A_0 z + B_0 u, \quad y = D_0 z,$$
$$A_0 = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -W_0^{-1} V_0 & 0_{n \times n} \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0_{n \times n} \\ W_0^{-1} \end{bmatrix},$$
$$W_0 = W(z_1)|_{z=0} = [\alpha_{ij}],$$
$$V_0 = V(z_1, z_2)|_{z=0} = diag\{\beta_1, \ldots, \beta_n\},$$
$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D_0 = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix}, \quad (15)$$

Since the linear time-invariant model has been obtained in (15), the rank condition of linear systems [32], [33] is used for the controllability and observability of robotic arms.

Using the linear time-invariant model (15) of the linear technique, the controllability matrix is:

$$C_0 = \begin{bmatrix} B_0 & A_0 B_0 & A_0^2 B_0 & \cdots & A_0^{2n-1} B_0 \end{bmatrix}, \quad (16)$$

If the rank of the controllability matrix C_0 is equal to $2n$, then the robotic arm of the linear technique is controllable at origin.

Using the linear time-invariant model (15) of the linear technique, the observability matrix is:

$$O_0 = \begin{bmatrix} D_0 & D_0A_0 & D_0A_0^2 & \cdots & D_0A_0^{2n-1} \end{bmatrix}^T, \quad (17)$$

If the rank of the observability matrix O_0 is equal to $2n$, then the robotic arm of the linear technique is observable at origin.

Remark 2: The linear technique for the controllability and observability of robotic arms has the problem that it can not be applied to robotic arms with prismatic joints.

C. MODIFIED LINEAR TECHNIQUE FOR THE CONTROLLABILITY AND OBSERVABILITY OF ROBOTIC ARMS

A modified linear technique is proposed for the controllability and observability of robotic arms, the modified linear technique consists of the following three steps: 1) a transformation is used to rewrite a nonlinear time-variant model as a quasi-linear time-variant model, 2) this quasi-linear time-variant model is evaluated at origin to obtain a linear time-invariant model, 3) the rank condition tests the controllability and observability of the linear time-invariant model.

Consider the *n*-link planar robotic arm having *n* rotational and prismatic joints, the robotic arm model is:

$$W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) = u, \quad (18)$$

By using the fact that the inverse of the robotic arm inertia matrix $W(z_1) \in \Re^{n \times n}$ is well defined, the robotic arm model can be rewritten as:

$$\dot{z}_2 = -W^{-1}(z_1)V(z_1, z_2)z_2 - W^{-1}(z_1)X(z_1) + W^{-1}(z_1)u, \quad (19)$$

The fact that the derivative of the joint angles or link displacements $z_1 \in \Re^{n \times 1}$ is equal to the velocities $z_2 \in \Re^{n \times 1}$ is represented as follows:

$$\dot{z}_1 = z_2, \quad (20)$$

Using (19), (20), the nonlinear time-variant model is:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -W^{-1}(z_1)V(z_1, z_2)z_2 \\ &\quad -W^{-1}(z_1)X(z_1) + W^{-1}(z_1)u, \end{aligned} \quad (21)$$

Since the following transformation is true:

$$\left(z_1^T z_1\right)^{-1} z_1^T z_1 = 1,$$

$$\Rightarrow -W^{-1}(z_1)X(z_1) = -W^{-1}(z_1)X(z_1)\left(z_1^T z_1\right)^{-1} z_1^T z_1, \quad (22)$$

We can apply the transformation (22) to the second equation of the nonlinear time-variant model (21) to obtain a term depending of $z_1 \in \Re^{n \times 1}$ as follows:

$$\begin{aligned} \dot{z}_2 &= -W^{-1}(z_1)V(z_1, z_2)z_2 \\ &\quad -W^{-1}(z_1)X(z_1)\left(z_1^T z_1\right)^{-1} z_1^T z_1 + W^{-1}(z_1)u, \end{aligned} \quad (23)$$

Defining $H(z_1) \in \Re^{n \times 1}$, the second equation of the nonlinear time-variant model (21) is rewritten as the following transformation:

$$\begin{aligned} \dot{z}_2 &= -W^{-1}(z_1)V(z_1, z_2)z_2 \\ &\quad -W^{-1}(z_1)X(z_1)H(z_1)z_1 + W^{-1}(z_1)u, \\ H(z_1) &= \left(z_1^T z_1\right)^{-1} z_1^T, \end{aligned} \quad (24)$$

By using the transformation (24), the nonlinear time-variant model (21) is rewritten as the following quasi-linear time-variant model:

$$\begin{aligned} \dot{z} &= Az + Bu, \quad y = Dz, \\ A &= \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -W^{-1}(z_1)X(z_1)H(z_1) & -W^{-1}(z_1)V(z_1, z_2) \end{bmatrix}, \\ B &= \begin{bmatrix} 0_{n \times n} \\ W^{-1}(z_1) \end{bmatrix}, \quad H(z_1) = \left(z_1^T z_1\right)^{-1} z_1^T, \\ z &= \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix}, \end{aligned} \quad (25)$$

The quasi-linear time-variant model (25) is evaluated at origin to obtain the following linear time-invariant model:

$$\begin{aligned} \dot{z} &= A_0 z + B_0 u, \quad y = D_0 z, \\ A_0 &= \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ 0_{n \times n} & -W_0^{-1} V_0 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 0 \\ W_0^{-1} \end{bmatrix}, \\ W_0 &= W(z_1)|_{z=0}, \quad V_0 = V(z_1, z_2)|_{z=0}, \\ z &= \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D_0 = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix}, \end{aligned} \quad (26)$$

Since the linear time-invariant model has been obtained in (26), the rank condition of linear systems [32], [33] is used for the controllability and observability of robotic arms.

Using the linear time-invariant model (26) of the modified linear technique, the controllability matrix is:

$$C_0 = \begin{bmatrix} B_0 & A_0B_0 & A_0^2B_0 & \cdots & A_0^{2n-1}B_0 \end{bmatrix}, \quad (27)$$

If the rank of the controllability matrix C_0 is equal to $2n$, then the robotic arm of the modified linear technique is controllable at origin.

Using the linear time-invariant model (26) of the modified linear technique, the observability matrix is:

$$O_0 = \begin{bmatrix} D_0 & D_0A_0 & D_0A_0^2 & \cdots & D_0A_0^{2n-1} \end{bmatrix}^T, \quad (28)$$

If the rank of the observability matrix O_0 is equal to $2n$, then the robotic arm of the modified linear technique is observable at origin.

Remark 3: Even the modified linear technique is similar to the linearization and linear techniques in the last step,
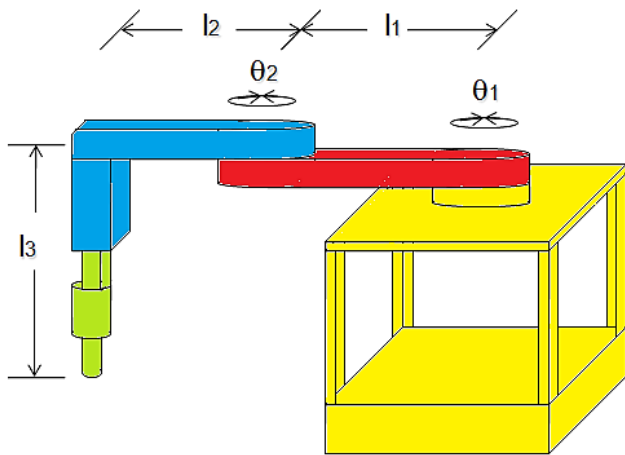
FIGURE 2. Scara robotic arm.

it is very different in the first step, resulting on the following two advantages: 1) the modified linear technique is better than the linearization technique because the modified linear technique does not use the Jacobian approximation, while the linearization technique needs the Jacobian approximation, 2) the modified linear technique is better than the linear technique because the modified linear technique can be applied to robotic arms with rotational and prismatic joints, while the linear technique can only be applied to robotic arms with rotational joints.

III. SCARA ROBOTIC ARM

In this section, we compare the three techniques for the controllability and observability in the scara robotic arm.

The scara robotic arm has three degrees of freedom, it has two rotational joints and two links configured in horizontal position, it has one prismatic joint and one link configured in vertical position. We express the scara robotic arm of the Figure 2, where θ_1, θ_2, are the angles of the joints one, two in rad, l_{c3} is the length of the link three, in m.

In this section, the nomenclature of the scara robotic arm is shown in Table 2.

A. LINEARIZATION TECHNIQUE

Now, the linearization technique of [25], [26], [27]–[30], [31]–[33] is applied to the scara robotic arm.

By using the linearization technique described in (4), the nonlinear time-variant model is:

$$F(z_1, z_2) = \tau, \quad y = Dz,$$
$$F(z_1, z_2) = \begin{bmatrix} z_2 \\ W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) \end{bmatrix},$$
$$F(z_1, z_2) = \begin{bmatrix} F_1 & F_2 & \cdots & F_6 \end{bmatrix}^T,$$
$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \end{bmatrix},$$
$$\tau = \begin{bmatrix} 0_3 & u \end{bmatrix}^T, \tag{29}$$

TABLE 2. Nomenclature of the scara robotic arm.

Symbol	Description
$z_2 \in \Re^{3\times1}$	velocities
$W(z_1) \in \Re^{3\times3}$	matrix with the inertia terms
$V(z_1, z_2) \in \Re^{3\times3}$	matrix with the Coriolis terms
$X(z_1) \in \Re^{3\times1}$	vector with the gravity terms
$0_2 \in \Re^{3\times1}$	vector of zeros
$I_{3\times3} \in \Re^{3\times3}$	identity matrix
$0_{3\times3} \in \Re^{3\times3}$	matrix of zeros
$z \in \Re^{6\times1}$	the states
$y \in \Re^{3\times1}$	the outputs
$F(z_1, z_2) \in \Re^{6\times1}$	nonlinear function vector
$g = 9.81$ m/s^2	gravity acceleration
$z_{11} = \theta_1, z_{12} = \theta_2$	angles of the joints 1, 2, in rad
$z_{13} = l_{c3}$	length of the link 3, in m
$m_1, m_2, m_3 = 0.3$ kg	masses of the links 1, 2, 3
$l_1 = l_2 = 0.3$ m	length of the link 1, 2
$l_{c1} = l_1/2, l_{c2} = l_2/2,$	half length of the link 1, 2
$J_1 = 0.0208$ kgm^2	moment of inertia of the link 1
$J_2, J_3 = 0.0127$ kgm^2	moment of inertia of the link 2, 3

where the robotic arm terms $F_1, F_2, \cdots, F_6$ of $F(z_1, z_2) \in \Re^{6\times1}$ are described as follows, (30), as shown at the bottom of the next page, $C_2 = \cos(z_{12})$, $S_2 = \sin(z_{12})$

By using the linearization technique described in (5), the linear time-invariant model is:

$$\dot{z} = A_0 z + B_0 u, \quad y = D_0 z,$$

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4.1331 & 0 & 0 \\ 0 & 21.529 & 0 \\ 0 & 0 & 3.3333 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{31}$$

By using the linearization technique described in (6), the controllability matrix is:

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & c_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{36} \\ c_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{63} & 0 & 0 & 0 \end{bmatrix}, \tag{32}$$

$c_{14} = c_{41} = 4.1331$, $c_{25} = c_{52} = 21.529$, $c_{36} = c_{63} = 3.3333$. Since the rank of the controllability matrix is 6, the scara robotic arm is controllable.

By using the linearization technique described in (7), the observability matrix is:

$$
O_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \tag{33}
$$

Since the rank of the observability matrix is 6, the scara robotic arm is observable.

B. LINEAR TECHNIQUE

The linear technique can not be applied to the scara robotic arm because the linear technique can only be applied to robotic arms with rotational joints and the scara robotic arm has one prismatic joint.

C. MODIFIED LINEAR TECHNIQUE

Now, the modified linear technique of this study is applied to the scara robotic arm.

By using the modified linear technique described in (21), the nonlinear time-variant model is:

$$
\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= -W^{-1}(z_1)V(z_1, z_2)z_2 \\
&\quad -W^{-1}(z_1)X(z_1) + W^{-1}(z_1)u,
\end{aligned} \tag{34}
$$

where the inertia terms $W_{11}, W_{12}, \cdots, W_{33}$ of $W(z_1) \in \Re^{3\times3}$, the centripetal and Coriolis terms $V_{11}, V_{12}, \cdots, V_{33}$ of $V(z_1, z_2) \in \Re^{3\times3}$, and the gravity terms X_1, X_2, X_3 of

$X(z_1) \in \Re^{3\times1}$ are described as follows:

$$
\begin{aligned}
W_{11} &= J_{13} + m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_2^2\right) \\
&\quad + m_3 \left(l_1^2 + l_2^2\right) + 2l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right), \\
W_{12} &= W_{21} = \left(m_2 l_{c2}^2 + m_3 l_2^2\right) + l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right), \\
W_{22} &= J_3 + \left(m_2 l_{c2}^2 + m_3 l_2^2\right), \quad W_{33} = m_3, \\
V_{11} &= -2l_1 S_2 \left(m_2 l_{c2} + m_3 l_2\right) z_{22}, \\
V_{12} &= -l_1 S_2 \left(m_2 l_{c2} + m_3 l_2\right) z_{22}, \\
V_{21} &= 2l_1 S_2 \left(m_2 l_{c2} + m_3 l_2\right) z_{21}, \\
X_3 &= -m_3 g, \tag{35}
\end{aligned}
$$

The other terms of $W(z_1)$, $V(z_1, z_2)$, $X(z_1)$ are zero, $C_2 = \cos(z_{12})$, $S_2 = \sin(z_{12})$.

By using the modified linear technique described in (25), the quasi-linear time-variant model is:

$$
\begin{aligned}
\dot{z} &= Az + Bu, \quad y = Dz, \\
A &= \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ -W^{-1}(z_1)X(z_1)H(z_1) & -W^{-1}(z_1)V(z_1, z_2) \end{bmatrix}, \\
B &= \begin{bmatrix} 0_{3\times3} \\ W^{-1}(z_1) \end{bmatrix}, \quad H(z_1) = \left(z_1^T z_1\right)^{-1} z_1^T, \\
z &= \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \end{bmatrix}, \tag{36}
\end{aligned}
$$

By using the modified linear technique described in (26), the linear time-invariant model is:

$$
\dot{z} = A_0 z + B_0 u, \quad y = D_0 z,
$$

$$
A_0 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
F_1 &= z_{21}, \quad F_2 = z_{22}, \quad F_3 = z_{23}, \quad F_4 \\
&= -\frac{\left[\left(m_2 l_{c2}^2 + m_3 l_2^2\right) + l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]}{\left[J_{13} + m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_2^2\right) + m_3 \left(l_1^2 + l_2^2\right) + 2l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]} \dot{z}_{22} \\
&\quad + \frac{\left[2l_1 S_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]}{\left[J_{13} + m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_2^2\right) + m_3 \left(l_1^2 + l_2^2\right) + 2l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]} z_{22} z_{21} \\
&\quad + \frac{\left[l_1 S_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]}{\left[J_{13} + m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_2^2\right) + m_3 \left(l_1^2 + l_2^2\right) + 2l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]} z_{22} z_{22} \\
&\quad + \frac{1}{\left[J_{13} + m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_2^2\right) + m_3 \left(l_1^2 + l_2^2\right) + 2l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]} u_1, \\
F_5 &= -\frac{\left[\left(m_2 l_{c2}^2 + m_3 l_2^2\right) + l_1 C_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]}{\left[J_3 + \left(m_2 l_{c2}^2 + m_3 l_2^2\right)\right]} \dot{z}_{21} \\
&\quad - \frac{\left[2l_1 S_2 \left(m_2 l_{c2} + m_3 l_2\right)\right]}{\left[J_3 + \left(m_2 l_{c2}^2 + m_3 l_2^2\right)\right]} z_{21} z_{21} + \frac{1}{\left[J_3 + \left(m_2 l_{c2}^2 + m_3 l_2^2\right)\right]} u_2, \\
F_6 &= [g] + \frac{1}{[m_3]} u_3, \tag{30}
\end{aligned}
$$

$$B_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 11.043 & -2.4567 & 0 \\ -2.4567 & 8.0055 & 0 \\ 0 & -0 & 3.3333 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad (37)$$

By using the modified linear technique described in (27), the controllability matrix is:

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & c_{14} & c_{15} & 0 \\ 0 & 0 & 0 & c_{24} & c_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{36} \\ c_{41} & c_{42} & 0 & 0 & 0 & 0 \\ c_{51} & c_{52} & 0 & 0 & 0 & 0 \\ 0 & -0 & c_{63} & 0 & 0 & 0 \end{bmatrix}, \quad (38)$$

$c_{15} = c_{51} = c_{24} = c_{42} = -2.4567$, $c_{14} = c_{41} = 11.043$, $c_{25} = c_{52} = 8.0055$, $c_{36} = c_{63} = 3.3333$. Since the rank of the controllability matrix is 6, the scara robotic arm is controllable.

By using the modified linear technique described in (28), the observability matrix is:

$$O_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (39)$$

Since the rank of the observability matrix is 6, the scara robotic arm is observable.

D. COMPARISON OF RESULTS

The linear technique can not be applied to the scara robotic arm because the linear technique can only be applied to robotic arms with rotational joints and the scara robotic arm has one prismatic joint. The linearization and modified linear techniques can be applied to the scara robotic arm where we obtained in both that the scara robotic arm is controllable and observable. It proves that the linearization and modified linear techniques are satisfactory options for the controllability and observability of the scara robotic arm. In this example, the three techniques obtain different numerical results.

IV. TWO LINK ROBOTIC ARM

In this section, we compare the three techniques for the controllability and observability in the two link robotic arm.

The two link robotic arm has two degrees of freedom, it has two rotational joints and two links configured in vertical position. We express the two link robotic arm of the Figure 3, where θ_1, θ_2 are the angles of the joints one, two in rad.

In this section, the nomenclature of the scara robotic arm is shown in Table 3.

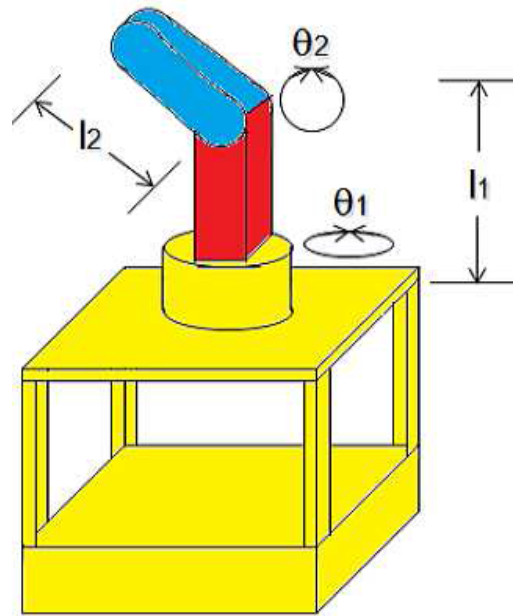FIGURE 3. Two link robotic arm.

TABLE 3. Nomenclature of the two link robotic arm.

Symbol	Description
$z_2 \in \Re^{2 \times 1}$	velocities
$W(z_1) \in \Re^{2 \times 2}$	matrix with the inertia terms
$V(z_1, z_2) \in \Re^{2 \times 2}$	matrix with the Coriolis terms
$X(z_1) \in \Re^{2 \times 1}$	vector with the gravity terms
$0_2 \in \Re^{2 \times 1}$	vector of zeros
$I_{2 \times 2} \in \Re^{2 \times 2}$	identity matrix
$0_{2 \times 2} \in \Re^{2 \times 2}$	matrix of zeros
$z \in \Re^{4 \times 1}$	the states
$y \in \Re^{2 \times 1}$	the outputs
$F(z_1, z_2) \in \Re^{4 \times 1}$	nonlinear function vector
$g = 9.81$ m/s^2	gravity acceleration
$z_{11} = \theta_1, z_{12} = \theta_2$	angles of the joints 1, 2, in rad
$m_2 = 0.34$ kg	is the mass of the link 2
$l_2 = 0.293$ m	length of the link 2
$l_{c2} = \frac{l_2}{2}$	half length of the link 2
$J_1 = 0.0208$ kgm^2	moment of inertia of the link 1
$J_2 = 0.0127$ kgm^2	moment of inertia of the link 2

A. LINEARIZATION TECHNIQUE

Now, the linearization technique of [25], [26], [27]–[30], [31]–[33] is applied to the two link robotic arm.

By using the linearization technique described in (4), the nonlinear time-variant model is:

$$F(z_1, z_2) = \tau, \quad y = Dz,$$

$$F(z_1, z_2) = \begin{bmatrix} z_2 \\ W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) \end{bmatrix},$$

$$F(z_1, z_2) = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix}^T,$$

$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \end{bmatrix},$$
$$\tau = \begin{bmatrix} 0_2 & u \end{bmatrix}^T, \tag{40}$$

where the robotic arm terms $F_1, F_2, \cdots, F_4$ of $F(z_1, z_2) \in \Re^{4\times1}$ are described as follows:

$$F_1 = z_{21}, \quad F_2 = z_{22},$$
$$F_3 = \frac{[m_2 l_{c2}^2 S_2 z_{21}]}{[J_{12} + m_2 l_{c2}^2 C_2]} z_{22} + \frac{1}{[J_{12} + m_2 l_{c2}^2 C_2]} u_1,$$
$$F_4 = -\frac{[m_2 l_{c2}^2 S_2 C_2 z_{21}]}{[J_2 + m_2 l_{c2}^2]} z_{21} - \frac{m_2 g l_{c2} C_2}{[J_2 + m_2 l_{c2}^2]} + \frac{1}{[J_2 + m_2 l_{c2}^2]} u_2, \tag{41}$$

$C_2 = \cos(z_{12}), S_2 = \sin(z_{12})$.

By using the linearization technique described in (5), the linear time-invariant model is:

$$\dot{z} = A_0 z + B_0 u, \quad y = D_0 z,$$
$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 24.512 & 0 \\ 0 & 50.007 \end{bmatrix},$$
$$D_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{42}$$

By using the linearization technique described in (6), the controllability matrix is:

$$C_0 = \begin{bmatrix} 0 & 0 & 24.512 & 0 \\ 0 & 0 & 0 & 50.007 \\ 24.512 & 0 & 0 & 0 \\ 0 & 50.007 & 0 & 0 \end{bmatrix}, \tag{43}$$

Since the rank of the controllability matrix is 4, the two link robotic arm is controllable.

By using the linearization technique described in (7), the observability matrix is:

$$O_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{44}$$

Since the rank of the observability matrix is 4, the two link robotic arm is observable.

B. LINEAR TECHNIQUE
Now, the linear technique of [34]–[36], [37]–[40], [41] is applied to the two link robotic arm.

By using the linear technique described in (8), the nonlinear time-variant model is:

$$W(z_1)\dot{z}_2 + V(z_1, z_2)z_2 + X(z_1) = u,$$
$$\alpha_1 = J_1,$$

$$\alpha_2 = J_2 + m_2 l_{c2}^2, \quad \alpha_3 = m_2 l_{c2}^2,$$
$$\beta_1 = 0, \quad \beta_2 = m_2 l_{c2} g, \tag{45}$$

where the inertia terms $W_{11}, W_{12}, \cdots, W_{22}$ of $W(z_1) \in \Re^{2\times2}$, the centripetal and Coriolis terms $V_{11}, V_{12}, \cdots, V_{22}$ of $V(z_1, z_2) \in \Re^{2\times2}$, and the gravity terms X_1, X_2 of $X(z_1) \in \Re^{2\times1}$ are described as follows:

$$W_{11} = \alpha_1 + \alpha_2 - \alpha_3 + \alpha_3 C_2, \quad W_{22} = \alpha_2,$$
$$V_{12} = -\alpha_3 S_2 z_{21}, \quad V_{21} = \alpha_3 S_2 C_2 z_{21},$$
$$X_2 = \beta_2 C_2, \tag{46}$$

The other terms of $W(z_1), V(z_1, z_2), X(z_1)$ are zero, $C_2 = \cos(z_{12}), S_2 = \sin(z_{12})$.

By using the linear technique described in (14), the quasi-linear time-variant model is:

$$\dot{z} = Az + Bu, \quad y = Dz,$$
$$A = \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ -W(z_1)^{-1}V(z_1, z_2) & 0_{2\times2} \end{bmatrix},$$
$$B = \begin{bmatrix} 0_{2\times2} \\ W(z_1)^{-1} \end{bmatrix},$$
$$z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \end{bmatrix}, \tag{47}$$

By using the linear technique described in (15), the linear time-invariant model is:

$$\dot{z} = A_0 z + B_0 u, \quad y = D_0 z,$$
$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 24.512 & 0 \\ 0 & 50.007 \end{bmatrix},$$
$$D_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{48}$$

By using the linear technique described in (16), the controllability matrix is:

$$C_0 = \begin{bmatrix} 0 & 0 & 24.512 & 0 \\ 0 & 0 & 0 & 50.007 \\ 24.512 & 0 & 0 & 0 \\ 0 & 50.007 & 0 & 0 \end{bmatrix}, \tag{49}$$

Since the rank of the controllability matrix is 4, the two link robotic arm is controllable.

By using the linear technique described in (17), the observability matrix is:

$$O_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{50}$$

Since the rank of the observability matrix is 4, the two link robotic arm is observable.

C. MODIFIED LINEAR TECHNIQUE

Now, the modified linear technique of this study is applied to the two link robotic arm.

By using the modified linear technique described in (21), the nonlinear time-variant model is:

$$
\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= -W^{-1}(z_1)V(z_1, z_2)z_2 \\
&\quad -W^{-1}(z_1)X(z_1) + W^{-1}(z_1)u,
\end{aligned} \tag{51}
$$

where the inertia terms $W_{11}, W_{12}, \cdots, W_{22}$ of $W(z_1) \in \Re^{2\times2}$, the centripetal and Coriolis terms $V_{11}, V_{12}, \cdots, V_{22}$ of $V(z_1, z_2) \in \Re^{2\times2}$, and the gravity terms X_1, X_2 of $X(z_1) \in \Re^{2\times1}$ are described as follows:

$$
\begin{aligned}
W_{11} &= J_{12} + m_2 l_{c2}^2 C_2, \quad W_{22} = J_2 + m_2 l_{c2}^2, \\
V_{12} &= -m_2 l_{c2}^2 S_2 z_{21}, \quad V_{21} = m_2 l_{c2}^2 S_2 C_2 z_{21}, \\
X_2 &= m_2 g l_{c2} C_2,
\end{aligned} \tag{52}
$$

The other terms of $W(z_1)$, $V(z_1, z_2)$, $X(z_1)$ are zero, $C_2 = \cos(z_{12})$, $S_2 = \sin(z_{12})$.

By using the modified linear technique described in (25), the quasi-linear time-variant model is:

$$
\begin{aligned}
\dot{z} &= Az + Bu, \quad y = Dz, \\
A &= \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ -W^{-1}(z_1)X(z_1)H(z_1) & -W^{-1}(z_1)V(z_1, z_2) \end{bmatrix}, \\
B &= \begin{bmatrix} 0_{2\times2} \\ W^{-1}(z_1) \end{bmatrix}, \quad H(z_1) = \left(z_1^T z_1\right)^{-1} z_1^T, \\
z &= \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad D = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \end{bmatrix},
\end{aligned} \tag{53}
$$

By using the modified linear technique described in (26), the linear time-invariant model is:

$$
\begin{aligned}
\dot{z} &= A_0 z + B_0 u, \quad y = D_0 z, \\
A_0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
B_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 24.512 & 0 \\ 0 & 50.007 \end{bmatrix}, \\
D_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{54}
$$

By using the modified linear technique described in (27), the controllability matrix is:

$$
C_0 = \begin{bmatrix} 0 & 0 & 24.512 & 0 \\ 0 & 0 & 0 & 50.007 \\ 24.512 & 0 & 0 & 0 \\ 0 & 50.007 & 0 & 0 \end{bmatrix}, \tag{55}
$$

Since the rank of the controllability matrix is 4, the two link robotic arm is controllable.

By using the modified linear technique described in (28), the observability matrix is:

$$
O_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{56}
$$

Since the rank of the observability matrix is 4, the two link robotic arm is observable.

D. COMPARISON OF RESULTS

The linearization, linear, and modified linear techniques can be applied to the two link robotic arm where we obtained in all that the two link robotic arm is controllable and observable. It proves that the linearization, linear, and modified linear techniques are satisfactory options for the controllability and observability of the two link robotic arm. In this example, the three techniques obtain the same numerical results. However, in general, the three techniques obtain different numerical results.

V. CONCLUSION

In this study, the modified linear technique is proposed for the controllability and observability of robotic arms. Some authors proposed a linearization technique and other proposed a linear technique, while this study considered a technique which has similarities and differences with the other two techniques. The numerical results showed that the modified linear technique can be applied to the two robotic arms, while the linear technique can only be applied to one of the two robotic arms. The proposed technique could be applied to any of the conventional structures of robotic arms. Since this study is mainly focused in a technique for the controllability and observability of robotic arms; in the forthcoming work, the proposed technique will be used to obtain a controller and an observer.

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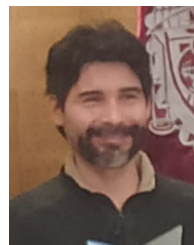
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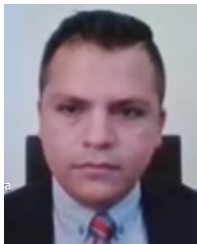
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