Received December 15, 2021, accepted December 25, 2021, date of publication December 30, 2021,
date of current version January 11, 2022.

Digital Object Identifier 10.1109/ACCESS.2021.3139564

New Constructions of Extended Sonar Sequences From Sidon Sets

LUIS MIGUEL DELGADO ORDOÑEZ, CARLOS ANDRES MARTOS OJEDA, AND CARLOS A. TRUJILLO
Departamento de Matemáticas, Universidad del Cauca, Popayán 190008, Colombia

Corresponding author: Luis Miguel Delgado Ordoñez (luismdelgado@unicauca.edu.co)

This work was supported in part by the Ministerio de Ciencia, Tecnología e Innovación MINCIENCIAS; in part by the Universidad del Cauca through the Projects "g-Golomb rulers and g-configurations" under Grant I.D 5367; and in part by the "Reglas Golomb generalizadas y la teoría de Ramsey" under Grant I.D 4938. The work of Luis Miguel Delgado Ordoñez was supported in part by the Ph.D. Program in Mathematical Sciences of Universidad del Cauca; and in part by the Ministerio de Ciencia, Tecnología e Innovación MINCIENCIAS. The work of Carlos Andres Martos Ojeda was supported by the Ministerio de Ciencia, Tecnología e Innovación MINCIENCIAS.

ABSTRACT A (m, n) sonar sequence is an $m \times n$ array with exactly one dot in each column and where all lines connecting two dots in the array are distinct as vectors. These arrays are known to have many applications such as sonar and radar detection and these are studied as a particular case of Golomb rectangles or two-dimensional Sidon sets. The main open problem for sonar sequences is: for fixed m, find the largest n for which there is an (m, n) sonar sequence, these sequences are called the best sonar sequences. The extended sonar sequences are generalizations of sonar sequences where each column has at most one dot, the motivation to study these arrays are the best results obtained when applied to radar and sonar detection. In this paper, we give the best sonar sequences with $m \leq 100$ obtained from an exhaustive computational search based on the Caicedo, Ruiz and Trujillo constructions and we present new constructions of extended sonar sequences that use Sidon sets.

INDEX TERMS Sonar sequences, extended sonar sequences, Sidon set.

I. INTRODUCTION

In this paper $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}_n$ and $\mathbb{F}_q$ denote the set of integers, positive integers, congruence classes modulo n and the finite field with q elements, respectively. For $n \in \mathbb{Z}^+$, let $[1, n] := \{1, 2, \ldots, n\}$.

A set of non-negative integers, in which all the sums of two different elements are distinct or equivalently with the property that all the differences of two elements are different, is called a Sidon set or Golomb ruler [1].

Sidon sets are important by their applications in different fields of engineering and communications, see [2]–[4]. Sidon sets are used to generate Optical Orthogonal Codes that are of great importance in communications [5].

Another application of Sidon sets is to study the B_h sets [6] and $g-$Golomb rulers [7]. There are also several generalizations of Golomb rulers or Sidon sets into two dimensions, one of the most general is a Golomb rectangle was given by Robinson [8] it is an $m \times n$ array with k dots such that all lines connecting two dots in the array are distinct as vectors, i.e. any two have either different lengths or slopes. An optimal Golomb rectangle is an array with the maximum number of points in a given rectangle, the problem of finding these optimal arrays has been widely studied; no general solution is known, but optimal rectangles have been determined for small orders see [9], [10]. The case where $m = n$ and each row and column has exactly one dot was first considered by Costas, they are known as Costas arrays. For complete information of Costas arrays see [12].

Sonar sequences are another class of Golomb rectangles, they were mentioned in [13]–[15], where $m \leq n$ and each column has exactly one dot. These arrays are applied as a solution to the sonar problem, in radar detection, and physical alignment. The fundamental problem in the study of the sonar sequences states that: "For a fixed m, find the largest n for which there exists an (m, n) sonar sequence". Equivalently investigate the following function:

$$S(m) = \max\{n : \text{there exist a } (m, n) \text{ sonar sequence}\}.$$

The associate editor coordinating the review of this manuscript and approving it for publication was Khmaies Ouahada.

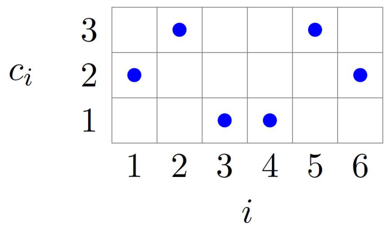


FIGURE 1. (3, 6) sonar sequence.

With the function $S(m)$ we can define the best sonar sequences.

Definition 1: Let $m, n \in \mathbb{Z}^+$. An (m, n) sonar sequence is called the best sonar sequence with parameters m and n, if and only if, $S(m) = n$.

For example, the (3, 6) sonar sequence given in Figure 1 where $m = 3$ is the best because $S(3) = 6$.

Moreno, Games and Taylor in [13] studied the function $S(m)$ and found the optimal value of n for m up to 100 with a computer search. Osorio et. al. [16] studied this function for small values of m and give the conjecture

$$\lim_{m\to\infty} \frac{S(m)}{m} = 1.$$

We can find a trivial upper bound counting the number of distinct differences of a sonar sequence with m rows, so

$$S(m) \leq 2m.$$

Some researchers achieved best the trivial upper bound, their results are the following

- $S(m) \leq m + 5m^{2/3}$, (Erdös, Graham, Ruzsa and Taylor [17]).
- $S(m) \leq m + 3,78m^{2/3} + 4,76m^{1/3} + 2$, (Caicedo [18]).

These upper bounds show that for large m, an (m, n) sonar sequence has to $S(m) = n$ closer to m than $2m$.

On the other hand in [19] Moreno, Golomb and Corrada studied a generalization of sonar sequence where $m \leq n$ and each column have at most one dot, they are called Extended Sonar Sequences (ESS) and has better results to be applied in the sonar detection because an ESS has the largest n that a sonar sequence using the same m rows. Moreno et. al. [19] also, show constructions of extended sonar sequences and a list of the best possible extended sonar sequences for m up to 10. Our motivation for considering these sequences come from new applications to study related concepts, for example, the ESS are used to obtain new constructions of resolvable Golomb rulers in [20], for study the Costas extended in [21], in the search of two-dimensional patterns with distinct differences and multiple target sonar [22], [23]. In this paper, we give the parameters of the best sonar sequences with $m \leq 100$ obtained from an exhaustive computational search based on the Caicedo, Ruiz and Trujillo constructions [24]. Also, we present new constructions of extended sonar sequences that use Sidon sets, these constructions can be considered as generalizations of those obtained by Caicedo, Ruiz and Trujillo in [24].

The rest of this paper is organized as follows. In section II we give the definitions of distinct difference property and sonar sequence, in addition, we describe the classic constructions of sonar sequences. In section III we review the constructions of modular Sidon sets, constructions of sonar sequences that use Sidon sets see [24] and the transformations that can be applied to sonar sequences to obtain better dimensions see [13], finally in this section we show the best sonar sequences with $m \leq 100$ (see Table 2), obtained from an exhaustive computational search based on the constructions of [24]. In section IV we turn our attention to circular extended sonar sequences that are used to obtain extended sonar sequences, also we give the formal definition of ESS. Finally, in section V, we give new constructions of extended sonar sequences that are a contribution to both the theory of Golomb rectangles and sonar detection.

II. CLASSIC SONAR SEQUENCES

In the study of the problem of radar detection, Costas was found that when sending a signal that comes from an array of frequencies that satisfy the Distinct Difference property, enough information is obtained to determine the speed and distance of the object.

The arrays found by Costas can be represented by matrices with entries in $\{0, 1\}$ or as sequences of integers whose elements must satisfy the following property:

Definition 2: The sequence of integers $[a_1, a_2, \ldots, a_n]$ has the Distinct Difference (DD) property if and only if $a_{i+h} - a_i = a_{j+h} - a_j$ with $1 \leq i < i + h \leq n$; $1 \leq j < j + h \leq n$, implies $i = j$.

In Costas arrays the frequencies can be sent just once and each time interval there must be only one frequency. Golomb and Taylor showed that the first condition is not necessary i.e., an array with one frequency in every time interval, where the frequencies can be used more than once and satisfy the DD property is useful in radar detection. Furthermore, these arrays provide enhanced results due to the fact that the received signal contains more information [15].

Definition 3: The sequence $[a_1, a_2, \ldots, a_n]$ is a sonar sequence (SS) with n elements over the set $[0, m - 1]$ (or an (m, n) SS) if it satisfies the DD property. If the differences are considered modulo m, then it is called an (m, n) modular sonar sequence (MSS).

In [13] Moreno et. al. constructed new and improve SS by applying multiplication, rotation and shearing transformation to Costas arrays and MSS. Table 1 shows the sonar sequences constructions given in [13], [14].

Using all these constructions in [13], Moreno, Games and Taylor give a table with the best dimensions of sonar sequences with up to 100 symbols.

III. KNOWN SONAR SEQUENCES CONSTRUCTIONS FROM SIDON SETS

One objective of this work is to present new constructions of extended sonar sequences, these constructions use special

TABLE 1. Parameters of modular sonar sequences. here p denotes a prime and q is a power of p.

Construction name	Parameters
Quadratic	$(p, p+1)$
Shift	$(q-1, q)$
Exponential Welch	$(p, p-1)$
Logarithmic Welch	$(p-1, p-1)$
Golomb	$(q-1, q-2)$

one-dimensional Sidon sets as provided by Bose [25] and Ruzsa [26].

Before we present our result we need to study the constructions of sonar sequences from Sidon sets. These constructions were presented by Caicedo, Ruiz and Trujillo in [24] which are different from the classical sonar sequences. This constructions allow us to state two constructions of extended SS as a consequence of Theorem 8.

First, we need to present two constructions of modular Sidon sets.

Theorem 1 (Bose Construction): Let $q = p^n$ be a power of a prime, α an algebraic element of degree 2 over $\mathbb{F}_q$ and θ a primitive element of $\mathbb{F}_{q^2}$. Then

$$B(q, \theta, \alpha) = \{\log_\theta(\alpha + a) : a \in \mathbb{F}_q\}, \quad (1)$$

is a Sidon set with q elements in $\mathbb{Z}_{q^2-1}$, and $\log_\theta(x)$ is the discrete logarithm.

The following proposition given in [24] establishes some properties of the Sidon set obtained by Bose's construction.

Proposition 1: The set $B(q, \theta, \alpha)$ given in (1) satisfy the following conditions:

- B1 $0 \notin B(q, \theta, \alpha) \,(\mathrm{mod}\, q+1) := \{b_i \,(\mathrm{mod}\, q+1) : b_i \in B(q, \theta, \alpha)\}$.
- B2 *If $b_i, b_j \in B(q, \theta, \alpha)$ with $i \neq j$, then, $b_i \not\equiv b_j \,(\mathrm{mod}\, q+1)$.*
- B3 $B(q, \theta, \alpha) \,(\mathrm{mod}\, q+1) = [1, q]$.

Now we present Ruzsa's construction.

Theorem 2 (Ruzsa's Construction): Let p be a prime number and θ a primitive element of the multiplicative group $\mathbb{F}_p^*$. Then the set

$$\{b_i \equiv ip - \theta^i(p-1) (\mathrm{mod}\, p^2 - p) : 1 \leq i \leq p-1\}, \quad (2)$$

which we denote by $R(\theta, p)$ is a Sidon set with $p-1$ elements in $\mathbb{Z}_{p^2-p}$.

The following proposition given in [24] presents some properties that $R(\theta, p)$ satisfies.

Proposition 2: Let $R(\theta, p)$ be the set given in (2):

- R1 $R(\theta, p) \,(\mathrm{mod}\, p) = [1, p-1]$.
- R2 $R(\theta, p) \,(\mathrm{mod}\, p-1) = [0, p-2]$.

The new constructions of sonar sequences evidenced in [24] employ Sidon sets, specifically the constructions of Bose and Ruzsa. Caicedo, Ruiz and Trujillo [24] obtain the following result.

Theorem 3 (Caicedo et al. Construction): Let $B = \{b_1, b_2, \ldots, b_n\}$ be a Sidon set in $\mathbb{Z}_{mb}$ such that $B \,(\mathrm{mod}\; b) = [1, n]$. If B is ordered in the form that $b_i \equiv i (\mathrm{mod}\; b)$, then the sequence defined by

$$a_i = \left\lfloor \frac{b_i}{b} \right\rfloor, \quad (3)$$

for $i \in [1, n]$ is an (m, n) modular sonar sequence.

Proof: For the proof, see [24]. □

As corollaries of Theorem III, in [24] are derived three constructions of sonar sequences.

Corollary 1 (Bose Sonar): Let $B(q, \theta, \alpha)$ be the Sidon set from Theorem 1. The sequence defined by

$$a_i = \left\lfloor \frac{b_i}{q+1} \right\rfloor, \quad (4)$$

where $b_i \in B(q, \theta, \alpha)$ is the only element such that $b_i \equiv i (\mathrm{mod}\, q+1)$, is a $(q-1, q)$ modular sonar sequence.

Example 1: Let $q = 11$, $\alpha = 4x$ an algebraic element of degree 2 over $\mathbb{F}_{11}$ and $\theta = 2x + 5$ a primitive element of $\mathbb{F}_{q^2}$. Then, from (1) we obtain the Sidon set $B(11, \theta, \alpha) = \{8, 13, 15, 19, 28, 29, 46, 54, 83, 86, 105\}$ in $\mathbb{Z}_{120}$. Now applying (4) of the Corollary 1 we have the following sequence $[1, 7, 1, 2, 2, 4, 1, 0, 8, 3, 6]$, which is a $(10, 11)$ modular sonar sequence.

Corollary 2 (Ruzsa Sonar 1): Let $R(\theta, p)$ be the Sidon set from Theorem 2. The sequence defined by

$$a_i = \left\lfloor \frac{b_i}{p} \right\rfloor, \quad (5)$$

where $b_i \in R(\theta, p)$ is the only element such that $b_i \equiv i (\mathrm{mod}\; p)$, is a $(p-1, p-1)$ modular sonar sequence.

Example 2: Let $p = 11$ and $\alpha = 2$ be a primitive root modulo 11. Then, from Theorem 2 we obtain the set $R(2, 11) = \{7, 39, 58, 63, 65, 86, 92, 100, 101, 104\}$ which is a Sidon set in $\mathbb{Z}_{110}$. Now applying (5) of the Corollary 2 we have the following $(10, 10)$ modular sonar sequence:

$$[9, 9, 5, 8, 9, 3, 0, 5, 7, 5].$$

Corollary 3 (Ruzsa Sonar 2): Let $R(\theta, p)$ be the Sidon set from Theorem 2. The sequence defined by

$$a_i = \left\lfloor \frac{b_i}{p-1} \right\rfloor, \quad (6)$$

where $b_i \in R(\theta, p)$ is the only element such that $b_i \equiv i (\mathrm{mod}\; p-1)$, is a $(p, p-1)$ modular sonar sequence.

Example 3: Let $p = 11$ and $\alpha = 2$ be a primitive root modulo 11. Then, from (2) we obtain the set $R(2, 11) = \{7, 39, 58, 63, 65, 86, 92, 100, 101, 104\}$ which is a Sidon set in $\mathbb{Z}_{110}$. Now applying (6) of the Corollary 3 we have the following $(11, 10)$ modular sonar sequence:

$$[10, 10, 9, 6, 10, 6, 8, 0, 5, 3].$$

We utilize the constructions Bose sonar, Ruzsa sonar 1 and Ruzsa sonar 2 to obtain the best sonar sequences from Sidon sets. After having used these constructions we did a computational search with the following transformations, see [13].

We denote the set of units of the group G with $U(G)$ and we refer as sonar array to the representation in an array of a sonar sequence:

1. Addition by a modulo m, $c_i = a_i + a \pmod m$. This be in tune to cyclically rotating the rows of the sonar arrays a units.
2. Multiplication by u, where $u \in U(\mathbb{Z}_m)$, $c_i = ua_i \pmod m$. This be in tune to a permutation of the rows of the sonar array.
3. Shearing by s modulo m, $c_i = a_i + si \pmod m$. This be in tune to shearing the columns of the sonar sequence by s units and reducing modulo m.

All these transformations produce an improved sonar sequence when the size of m can be reduced. Moreno, Games and Taylor in [13] showed that the transformations can be combined as in the following theorem.

Theorem 4: Let a_i be an (m, n) modular sonar sequence, $s, a \in \mathbb{Z}_m$ and $u \in U(\mathbb{Z}_m)$, then the sequence defined by $c_i \equiv ua_i + si + a \pmod m$ is an (m, n) modular sonar sequence.

Proof: For the proof, see [13]. □

Example 4 illustrates the application of these transformations.

Example 4: Given $p = 7$, from Construction Ruzsa sonar 1 we obtain the sequence $[b_i : 1 \leq i \leq 6] = [4, 1, 1, 3, 1, 2]$, which is a $(4, 6)$ modular sonar sequence. With the parameters $u = 1$, $s = 4$ and $a = 0$ we apply Theorem 4, so

$$c_i \equiv ub_i + si + a \pmod 6,$$

obtaining the $(3, 6)$ sonar sequence $c_i = [2, 3, 1, 1, 3, 2]$. The array of this sequence is shown in Figure 1. Note that the sequence c_i is better that b_i since it requires only three frequencies.

In Table 2 we give the best (m, n) sonar sequences result of the computational search with m up to 100 using the constructions Bose sonar, Ruzsa sonar 1, Ruzsa sonar 2 and Theorem 4.

TABLE 2. Best sonar sequences from Sidon sets with $m \leq 100$.

m	n	Base Construction	m	n	Base Construction
2	4	Bose	54	64	Bose
3	6	Ruzsa 1	55	64	Bose
4	8	Bose	56	67	Ruzsa 2
5	8	Bose	57	67	Bose
7	11	Bose	58	67	Bose
10	16	Bose	59	67	Bose
11	17	Bose	59	67	Bose
12	18	Ruzsa 2	60	71	Bose
13	19	Bose	62	73	Bose
14	19	Bose	63	73	Bose
16	23	Bose	64	73	Bose
17	23	Bose	65	73	Bose
18	23	Bose	67	79	Bose
19	25	Bose	68	79	Bose
21	28	Ruzsa 1	69	79	Bose
22	30	Ruzsa 1	70	79	Bose
23	31	Bose	72	83	Bose
24	32	Bose	73	83	Bose
25	32	Bose	74	83	Bose
26	32	Bose	75	83	Bose
27	32	Bose	76	88	Bose
28	37	Bose	77	89	Bose
29	37	Bose	78	89	Bose
30	37	Bose	79	89	Bose
31	37	Bose	80	89	Bose
32	41	Bose	81	89	Bose
33	41	Bose	82	89	Bose
34	43	Bose	83	96	Bose
35	43	Bose	85	97	Bose
36	43	Bose	86	97	Bose
37	46	Ruzsa 1	87	97	Bose
38	47	Bose	88	101	Bose
40	49	Bose	89	101	Bose
41	49	Bose	91	103	Bose
42	49	Bose	92	103	Bose
44	53	Bose	94	107	Bose
45	53	Bose	95	109	Bose
46	53	Bose	96	109	Bose
49	57	Bose	97	109	Bose
50	59	Bose	98	109	Bose
51	61	Bose	100	113	Bose
52	61	Bose			

IV. CIRCULAR EXTENDED SONAR SEQUENCES

When the sonar sequences were introduced by Golomb and Taylor it was allowed that a frequency is used more than once, which showed that the necessary characteristic that a sequence should have in the radar detection is that its elements satisfy the DD property. Later Moreno, Golomb and Corrada observed the behavior of the returning signal finding always blank time slots, so that the new idea was sending blank time intervals inside the sequences that satisfy the DD property to obtain better results, these sequences were called extended sonar sequences [19]. In the next sections we denote the blank time slots with the symbol $(*)$.

Definition 4: If the sequence $[a_1, a_2, \ldots, a_{n+k}]$ is contained in the set $[0, m-1] \cup \{*\}$, have k blanks $(*)$, and satisfy the DD property (where we consider only the differences between numbers). Then it is called an (m, n, k) extended sonar sequence (ESS).

The way to get ESSs depends on the constructions on circular extended sonar sequences that we define below.

Definition 5: Let $c \in \mathbb{Z}$, $d \in \mathbb{Z}^+$ and the following sequence

$$[a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n+1}]$$

where $a_{n+1} = *$ and $a_{k+n+1} \equiv a_k + c \pmod d$, for $1 \leq k \leq n$. Now consider the differences

$$a_{1+h} - a_1, a_{2+h} - a_2, \ldots, a_{n+h} - a_n$$

where $h \in [1, n]$ and where $* - a_i = *$. We call this a modular circular extended sonar sequence if these differences are always distinct modulo d, with exactly $n-1$ differences, and an $*$.

Now we present the constructions of modular circular extended sonar sequences given in [19].

Theorem 5 (Circular Extended Logarithmic Welch): Let p be a prime and α a primitive element of $\mathbb{F}_p$, then consider the *sequence defined by*

$$a_i = \begin{cases} \log_\alpha i, & i \in [1, p-1]. \\ *, & i = p. \\ \log_\alpha (i-p), & i \in [p+1, 2p-1]. \end{cases}$$

where $1 \leq a_i \leq p-1$. *Then the sequence is a modular circular extended sonar sequence, with* $n = p-1$, $d = p-1$ *and* $c = 0$.

Theorem 6 (Circular Extended Shift): Let p be a prime, α *a primitive element of* $\mathbb{F}_{p^{2r}}$ *and* β *a primitive element of* $\mathbb{F}_{p^r}$ *then consider the sequence for* $p = 2$ *defined by*

$$a_i = \begin{cases} \log_\beta((\alpha^i)^{p^r} + \alpha^i), & i \in [1, 2p^r + 1], i \neq p^r + 1. \\ *, & i = p^r + 1. \end{cases}$$

where $1 \leq a_i \leq p^r + 1$. *For p odd, define* a_i *similarly except that*

$$a_i = \begin{cases} \log_\beta((\alpha^i)^{p^r} + \alpha^i), & i \in \left[\frac{-(p^r-1)}{2}, \frac{3p^r+1}{2}\right] \\ & \text{and } i \neq \frac{p^r+1}{2}. \\ *, & i = \frac{p^r+1}{2}. \end{cases}$$

Then the sequence is a modular circular extended sonar sequence, with $n = p^r$, $d = p^r - 1$ *and* $c \neq 0$.

Theorem 7 (Circular Extended Golomb-Lempel): Let $p^r > 2$ *be a prime power,* α *and* β *primitive elements of* $\mathbb{F}_{p^r}$ *then consider the sequence defined by*

$$a_i = \begin{cases} j, & \text{iff } \alpha^i + \beta^j = 1 \text{ where } i \in [1, 2p^r - 3] \\ & i \neq p^r - 1. \\ *, & i = p^r - 1. \end{cases}$$

where $1 \leq a_i \leq p^r - 2$. *Then the sequence is a modular circular extended sonar sequence, with* $n = p^r - 2$, $d = p^r - 1$ *and* $c = 0$.

V. NEW CONSTRUCTIONS OF EXTENDED SONAR SEQUENCES FROM SIDON SETS

The extended sonar sequences were introduced in [19]. The signals provided for these sequences results be better as they used the same band or number of frequencies with more elements in the sequence. The new sequences have consequences in other related concepts as multiple target sonar and construction of Golomb rectangles [21], [23].

Theorem 8 presents the main result of this work that shows a new construction of modular circular extended sonar sequences using Sidon sets. Besides, by using this theorem it is possible to obtain ESSs, as we will show later.

Theorem 8: Let $B = \{b_1, b_2, \ldots, b_n\}$ *be a Sidon set in the additive group* $\mathbb{Z}_{m(n+1)}$ *such that* $B (\bmod\, n+1) = [1, n]$. *If B is ordered in the form that* $b_i \equiv i (\bmod\, n+1)$, *then the sequence defined by*

$$a_i = \begin{cases} \left\lfloor \frac{b_i}{n+1} \right\rfloor, & i \in [1, n]. \\ *, & i = n+1. \\ \left(\left\lfloor \frac{b_{i-(n+1)}}{n+1} \right\rfloor - 1\right) (\bmod\, m), & i \in [n+2, 2n+1]. \end{cases} \tag{7}$$

is a modular circular extended sonar sequence with $c = -1$ *and* $d = m$.

Proof: Let h, i, j be integers such that $1 \leq h, i, j \leq n$ with $i + h \neq n + 1$ and $j + h \neq n + 1$. Suppose that

$$a_{i+h} - a_i \equiv a_{j+h} - a_j \,(\bmod\, m).$$

We have the following cases:

1) If $i + h < n + 1$ and $j + h < n + 1$. We have from (7),

$$\left\lfloor \frac{b_{i+h}}{n+1} \right\rfloor - \left\lfloor \frac{b_i}{n+1} \right\rfloor \equiv \left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor - \left\lfloor \frac{b_j}{n+1} \right\rfloor \times (\bmod\, m).$$

So there exists an integer t such that

$$\left\lfloor \frac{b_{i+h}}{n+1} \right\rfloor - \left\lfloor \frac{b_i}{n+1} \right\rfloor = \left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor - \left\lfloor \frac{b_j}{n+1} \right\rfloor + tm.$$

Now, multiplying by $n + 1$, we obtain

$$(n+1)\left\lfloor \frac{b_{i+h}}{n+1} \right\rfloor - (n+1)\left\lfloor \frac{b_i}{n+1} \right\rfloor = (n+1)\left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor - (n+1)\left\lfloor \frac{b_j}{n+1} \right\rfloor + tm(n+1).$$

If we add $h = (i+h) - i = (j+h) - j$ to both sides of equation, we have

$$\left[(n+1)\left\lfloor \frac{b_{i+h}}{n+1} \right\rfloor + (i+h)\right] - \left[(n+1)\left\lfloor \frac{b_i}{n+1} \right\rfloor + i\right] = \left[(n+1)\left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor + (j+h)\right] - \left[(n+1)\left\lfloor \frac{b_j}{n+1} \right\rfloor + j\right] + tm(n+1).$$

$$b_{i+h} - b_i - b_{j+h} + b_j = tm(n+1),$$

this implies that

$$b_{i+h} + b_j \equiv b_{j+h} + b_i \,(\bmod\, m(n+1)).$$

Since B is a Sidon set in the additive group $\mathbb{Z}_{m(n+1)}$, then $\{i + h, j\} = \{j + h, i\}$, and thus $i = j$.

2) If $i + h > n + 1$ and $j + h < n + 1$. We have from (7),

$$\left\lfloor \frac{b_{i+h-(n+1)}}{n+1} \right\rfloor - 1 - \left\lfloor \frac{b_i}{n+1} \right\rfloor \equiv \left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor - \left\lfloor \frac{b_j}{n+1} \right\rfloor (\bmod\, m).$$

So there exists an integer t such that

$$\left\lfloor \frac{b_{i+h-(n+1)}}{n+1} \right\rfloor - 1 - \left\lfloor \frac{b_i}{n+1} \right\rfloor = \left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor \left\lfloor \frac{b_j}{n+1} \right\rfloor + tm.$$

Now, multiplying by $n+1$, we obtain

$$(n+1)\left(\left\lfloor \frac{b_{i+h-(n+1)}}{n+1} \right\rfloor - 1\right) - (n+1)\left\lfloor \frac{b_i}{n+1} \right\rfloor = (n+1)\left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor - (n+1)\left\lfloor \frac{b_j}{n+1} \right\rfloor + tm(n+1).$$

If we add $h = (i+h) - i = (j+h) - j$ to both sides of equation, we have

$$(n+1)\left(\left\lfloor \frac{b_{i+h-(n+1)}}{n+1} \right\rfloor - 1\right) + (i+h) - \left[(n+1)\left\lfloor \frac{b_i}{n+1} \right\rfloor + i\right] = (n+1)\left\lfloor \frac{b_{j+h}}{n+1} \right\rfloor + (j+h) - \left[(n+1)\left\lfloor \frac{b_j}{n+1} \right\rfloor + j\right] + tm(n+1).$$

Since $i+h > n+1$, then, replacing $i+h = k+(n+1)$.

$$(n+1)\left(\left\lfloor \frac{b_{k+(n+1)-(n+1)}}{n+1} \right\rfloor - 1\right) + k + (n+1) - b_i = b_{j+h} - b_j + tm(n+1).$$

$$(n+1)\left\lfloor \frac{b_k}{n+1} \right\rfloor + k - b_i = b_{j+h} - b_j + tm(n+1).$$

$$b_k - b_i - b_{j+h} + b_j = tm(n+1),$$

this implies that

$$b_k + b_j \equiv b_{j+h} + b_i \,(\mathrm{mod}\; m(n+1)).$$

Because B is a Sidon set in the additive group $\mathbb{Z}_{m(n+1)}$, then $\{k, j\} = \{j+h, i\}$, therefore $i = j$.

3) If $i+h < n+1$ and $j+h > n+1$. We have from (7) that $i = j$, by following the same steps as in the previous case.
4) If $i+h > q+1$ and $j+h > q+1$. We have from (7) that $i = j$, by following the same steps as in the two previous cases.

Therefore we have a modular circular extended sonar sequence. □

As a consequence of Proposition 1, Theorem 2 and Theorem 8, we have the following constructions of circular extended sonar sequences. Proofs follow immediately from Theorem 8.

Corollary 4 (Circular Extended Bose Sequence): Let $B(q, \theta, \alpha)$ be the Sidon set given in (1). Then the sequence defined by

$$a_i = \begin{cases} \left\lfloor \dfrac{b_i}{q+1} \right\rfloor, & i \in [1, q]. \\ *, & i = q+1. \\ \left\lfloor \dfrac{b_{i-(q+1)}}{q+1} \right\rfloor - 1 \,(\mathrm{mod}\; q-1), & i \in [q+2, 2q+1]. \end{cases} \tag{8}$$

where $b_i \in B(q, \theta, \alpha)$ is the unique element such that $b_i \equiv i \,(\mathrm{mod}\, q+1)$, is a modular circular extended sonar sequence, with $c = -1$, $d = q-1$ y $n = q$.

Corollary 5 (Circular Extended Ruzsa Sequence): Let $R(\theta, p)$ be the Sidon set given in (2). Then the sequence defined by

$$a_i = \begin{cases} \left\lfloor \dfrac{b_i}{p} \right\rfloor, & i \in [1, p-1]. \\ *, & i = p. \\ \left\lfloor \dfrac{b_{i-p}}{p} \right\rfloor - 1 \,(\mathrm{mod}\; p-1), & i \in [p+1, 2p-1]. \end{cases} \tag{9}$$

where $b_i \in R(\theta, p)$ is the unique element such that $b_i \equiv i \,(\mathrm{mod}\, p)$, is a modular circular sonar sequence with $c = -1$, $d = n = p-1$.

The Theorem 9 given in [19], allows us to display that from any circular extended Bose sequence or any circular extended Ruzsa sequence can be obtained n extended sonar sequences.

Theorem 9: Let $d \in \mathbb{Z}^+$ and $[a_1, a_2, \ldots, a_{2n+1}]$ be a modular circular extended sonar sequence with modulo d, then

$$[a_{k+1}, a_{k+2}, \ldots, a_{k+n+2}],$$

is a $(d, n+1, 1)$ extended sonar sequence for every $k \in [0, n-1]$.

As a consequence, we achieve the following two new constructions of extended sonar sequences:

Corollary 6 (Extended Bose Sequence): Let

$$[a_1, a_2, \ldots, a_q, a_{q+1}, a_{q+2}, \ldots, a_{2q+1}]$$

a circular extended Bose sequence as in Corollary V. Then every sequence

$$[a_{k+1}, a_{k+2}, \ldots, a_{k+q+2}],$$

be a $(q-1, q+1, 1)$ extended sonar sequence for $k \in [0, q-1]$.

Example 5: Let $q = 11$ and $B(11, \theta, \alpha)$ the Sidon set in $\mathbb{Z}_{120}$ of Example 1. Now applying (8) we have the following circular extended sonar sequence with $d = 10$, $m = 11$.

$$[1, 7, 1, 2, 2, 4, 1, 0, 8, 3, 6, *, 0, 6, 0, 1, 1, 3, 0, 9, 7, 2, 5].$$

Finally, from Corollary 6 we have the $(10, 12, 1)$ extended sonar sequences given in Table 3.

Corollary 7 (Extended Ruzsa Sequence): Let

$$[a_1, a_2, \ldots, a_{p-1}, a_p, a_{p+1}, \ldots, a_{2p-1}]$$

TABLE 3. Extended sonar sequences.

[1	7	1	2	2	4	1	0	8	3	6	$*$	0]
[7	1	2	2	4	1	0	8	3	6	$*$	0	6]
[1	2	2	4	1	0	8	3	6	$*$	0	6	0]
[2	2	4	1	0	8	3	6	$*$	0	6	0	1]
[2	4	1	0	8	3	6	$*$	0	6	0	1	1]
[4	1	0	8	3	6	$*$	0	6	0	1	1	3]
[1	0	8	3	6	$*$	0	6	0	1	1	3	0]
[0	8	3	6	$*$	0	6	0	1	1	3	0	9]
[8	3	6	$*$	0	6	0	1	1	3	0	9	7]
[3	6	$*$	0	6	0	1	1	3	0	9	7	2]
[6	$*$	0	6	0	1	1	3	0	9	7	2	5]

TABLE 4. Extended sonar sequences.

[9	9	5	8	9	3	0	5	7	5	$*$	8]
[9	5	8	9	3	0	5	7	5	$*$	8	8]
[5	8	9	3	0	5	7	5	$*$	8	8	4]
[8	9	3	0	5	7	5	$*$	8	8	4	7]
[9	3	0	5	7	5	$*$	8	8	4	7	8]
[3	0	5	7	5	$*$	8	8	4	7	8	2]
[0	5	7	5	$*$	8	8	4	7	8	2	9]
[5	7	5	$*$	8	8	4	7	8	2	9	4]
[7	5	$*$	8	8	4	7	8	2	9	4	6]
[5	$*$	8	8	4	7	8	2	9	4	6	4]

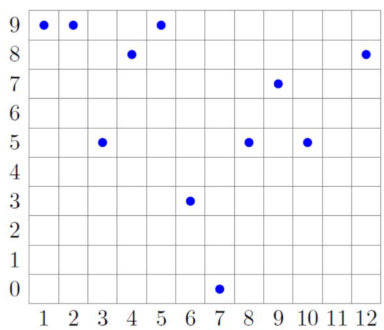


FIGURE 2. (10, 11, 1) extended sonar sequence.

a circular extended Ruzsa sequence as in the Corollary V. Then every sequence

$$[a_{k+1}, a_{k+2}, \ldots, a_{k+p+1}],$$

be a $(p-1, p, 1)$ *extended sonar sequence for* $k \in [0, p-2]$.

Example 6: Let $p = 11$ *and* $R(2, 11)$ *the Sidon set in* $\mathbb{Z}_{110}$ *of Example 2. Now applying (9) we have the following circular extended sonar sequence with* $d = m = 10$.

$$[9, 9, 5, 8, 9, 3, 0, 5, 7, 5, *, 8, 8, 4, 7, 8, 2, 9, 4, 6, 4].$$

Finally, from Corollary 7 we have the $(10, 11, 1)$ *extended sonar sequences given in Table 4.*

In Figure 2 we show the $(10, 11, 1)$ *extended sonar sequence given by* $[9, 9, 5, 8, 9, 3, 0, 5, 7, 5, *, 8]$.

VI. CONCLUSION

In this work we obtain two new constructions of extended sonar sequences using special one-dimensional Sidon sets, this contribution is a result for both the sonar detection and the theory of two-dimensional Sidon sets or Golomb rectangles. Through a computational search and using Sidon sets theory tools such as generations of disjoint Sidon sets and sonar sequences from Sidon sets we provide the best sonar sequences with $m \leq 100$ (see Table 2) these are particular near optimal Golomb rectangles. On the other hand, we consider important as future work use these new constructions in the study of resolvable Golomb rulers, Costas extended, two-dimensional patterns with distinct differences and multiple target sonar. Also the new constructions of extended sonar sequences can be used for search optimal Golomb rectangles because the Sidon sets theory is a powerful tool in the computational search.

ACKNOWLEDGMENT

The authors would like to thank Universidad del Cauca. This work was supported in part by the Ministerio de Ciencia, Tecnología e Innovación MINCIENCIAS; in part by the Universidad del Cauca through the Projects "g-Golomb rulers and g-configurations" under Grant I.D 5367; and in part by the "Reglas Golomb generalizadas y la teoría de Ramsey" under Grant I.D 4938. The work of Luis Miguel Delgado Ordoñez was supported in part by the Ph.D. Program in Mathematical Sciences of Universidad del Cauca; and in part by the Ministerio de Ciencia, Tecnología e Innovación MINCIENCIAS. The work of Carlos Andres Martos Ojeda was supported by the Ministerio de Ciencia, Tecnología e Innovación MINCIENCIAS.

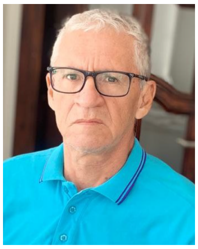
REFERENCES

[1] P. Erdös and P. Turán, "On a problem of Sidon in additive number theory, and on some related problems," *J. London Math. Soc.*, vol. 16, no. 4, pp. 212–215, Dec. 1941.
[2] M. Atkinson, N. Santoro, and J. Urrutia, "Integer sets with distinct sums and differences and carrier frequency assignments for nonlinear repeaters," *IEEE Trans. Commun.*, vol. 34, no. 6, pp. 614–617, Jun. 1986.
[3] W. T. Rankin, "Optimal Golomb rulers: An exhaustive parallel search implementation," M.S. thesis, Dept. Electron. Eng., Duke Univ., Durham, NC, USA, 1993.
[4] Y. Caicedo, C. Martos, and C. Trujillo, "*g*-Golomb rulers," *Revista Integración Temas Mat.*, vol. 33, no. 2, pp. 161–172, Nov. 2015.
[5] H. M. Ruiz, L. Delgado, and C. Trujillo, "A new construction of optimal optical orthogonal codes from Sidon sets," *IEEE Access*, vol. 8, pp. 100749–100753, 2020, doi: 10.1109/ACCESS.2020.2996813.
[6] C. Martos, L. Delgado, and C. Trujillo, "*Bh* sets as a generalization of Golomb rulers," *IEEE Access*, vol. 9, pp. 118042–118050, 2021, doi: 10.1109/ACCESS.2021.3106617.
[7] C. A. M. Ojeda, D. F. D. Urbano, and C. A. T. Solarte, "Near-optimal g-Golomb rulers," *IEEE Access*, vol. 9, pp. 65482–65489, 2021, doi: 10.1109/ACCESS.2021.3075877.
[8] J. Robinson, "Golomb rectangles," *IEEE Trans. Inf. Theory*, vol. 31, no. 6, pp. 781–787, Nov. 1985.
[9] J. P. Robinson, "Golomb rectangles as folded rulers," *IEEE Trans. Inf. Theory*, vol. 43, no. 1, pp. 290–293, Jan. 1997.
[10] Z. Shao, J. Zhou, M. Liang, F. Lang, and X. Xu, "Some new Golomb rectangles," *J. Comput. Theor. Nanosci.*, vol. 10, no. 1, pp. 66–68, Jan. 2013.
[11] J. Costas, "A study of a class of detection waveforms having nearly ideal range—Doppler ambiguity properties," *Proc. IEEE*, vol. 72, no. 8, pp. 996–1009, Aug. 1984.
[12] K. Taylor, K. Drakakis, and S. Rickard, "Costas arrays: Survey, standardization, and MATLAB toolbox," *ACM Trans. Math. Softw.*, vol. 37, no. 4, pp. 1–31, Feb. 2011.
[13] O. Moreno, R. Games, and H. Taylor, "Sonar sequences from Costas arrays and the best known sonar sequences with up to 100 symbols," *IEEE Trans. Inf. Theory*, vol. 39, no. 6, pp. 1985–1987, Nov. 1993.
[14] S. W. Golomb, "Algebraic constructions for Costas arrays," *J. Combinat. Theory A*, vol. 37, no. 1, pp. 13–21, 1983.
[15] S. Golomb and H. Taylor, "Two-dimensional synchronization patterns for minimum ambiguity," *IEEE Trans. Inf. Theory*, vol. 28, no. 4, pp. 600–604, Jul. 1982.

[16] R. Osorio, D. Ruiz, C. Trujillo, and C. Urbano, ''Sonar sequences and Sidon sets,'' *Revista de Ciencias*, vol. 18, no. 1, pp. 33–42, 2014.
[17] P. Erdős, R. Graham, I. Z. Ruzsa, and H. Taylor, ''Bounds for arrays of dots with distinct slopes or lengths,'' *Combinatorica*, vol. 12, no. 1, pp. 39–44, Mar. 1992.
[18] N. Y. Caicedo, ''Conjuntos de Sidon en dimensión dos,'' Ph.D. dissertation, Dept. Valle del Cauca, Univ. del Valle, Cali, Colombia, 2016.
[19] O. Moreno, S. Golomb, and C. Corrada, ''Extended sonar sequences,'' *IEEE Trans. Inf. Theory*, vol. 43, no. 6, pp. 1999–2005, Nov. 1997.
[20] M. Buratti and D. R. Stinson, ''On resolvable Golomb rulers, symmetric configurations and progressive dinner parties,'' *J. Algebr. Combinatorics*, pp. 1–16, Jan. 2021.
[21] O. Moreno, R. Omrani, and S. Maric, "A new construction of multiple target sonar and extended Costas arrays with perfect correlation,'' in *Proc. 40th Annu. Conf. Inf. Sci. Syst.*, Mar. 2006, pp. 512–517. [Online]. Available: https://www.researchgate.net/profile/O-Moreno/publication/224760621
[22] S. Blackburn, T. Etzion, K. Martin, and M. Paterson, ''Two-dimensional patterns with distinct differences—Constructions, bounds, and maximal anticodes,'' *IEEE Trans. Inf. Theory*, vol. 56, no. 3, pp. 1216–1229, Mar. 2010.
[23] C.-C. Yang, ''New extended Costas sequences with ideal auto-and cross-ambiguity functions,'' in *Proc. Int. Conf. Commun., Circuits Syst.*, Jul. 2009, pp. 156–159. [Online]. Available: https://ieeexplore.ieee.org/abstract/document/5250557
[24] D. F. Ruiz, C. A. Trujillo, and Y. Caicedo, ''New constructions of sonar sequences,'' 2013, *arXiv:1311.1679*.
[25] R. Bose, ''An affine analogue of Singer's theorem,'' *J. Indian Math. Soc.*, vol. 6, pp. 141–147, Mar. 1942.
[26] I. Ruzsa, ''Solving a linear equation in a set of integers I,'' *Acta Arithmetica*, vol. 65, no. 3, pp. 259–282, 1993.

LUIS MIGUEL DELGADO ORDOÑEZ received the B.S. degree in mathematics from the Universidad de Nariño, Colombia, in 2009, and the M.S. degree in mathematical sciences from Universidad del Cauca, Colombia, in 2018, where he is currently pursuing the Ph.D. degree. He is a member of the Research Group ALTENUA (Algebra, Teoria de Números y Aplicaciones). His research interests include additive number theory, Golomb rulers, and sonar sequences.

CARLOS ANDRES MARTOS OJEDA received the B.S. degree in mathematics and the M.S. degree in mathematical sciences from Universidad del Cauca, in 2010 and 2015, respectively, and the Ph.D. degree in mathematics from Universidad del Valle, in 2020. His research interests include additive number theory, Golomb rulers, Sidon sets, and mathematics of communications. He is currently a member of the Research Group ALTENUA (Algebra, Teoria de Números y Aplicaciones: ERM).



CARLOS A. TRUJILLO received the B.S. degree in mathematics from Universidad del Cauca, in 1978, the M.S degree in mathematical sciences from Universidad del Valle, in 1987, and the Ph.D. degree in mathematics from Universidad Politécnica de Madrid, in 1998. He is the Director of the Ph.D. Program in mathematical sciences with the Universidad del Cauca, and the Director of the Research Group ALTENUA (Algebra, Teoria de Números y Aplicaciones). His research interests include additive number theory, Sidon sets, coding theory, Golomb rulers, Sonar sequences, Costas arrays, and applications to the communications.