IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received November 11, 2021, accepted November 24, 2021, date of publication December 28, 2021,
date of current version January 10, 2022.

Digital Object Identifier 10.1109/ACCESS.2021.3139244

Stability Radii-Based Interval Observers for Discrete-Time Nonlinear Systems

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The work of Jesús D. Avilés was supported by the Facultad de Ciencias de la Ingeniería, Administrativas y Sociales, Universidad Autónoma de Baja California, Tecate, Mexico. The work of Jaime A. Moreno was supported by the DGAPA-UNAM, PAPIIT, under Project IN102121.

ABSTRACT In this paper, we investigate the interval observer problem for a class of discrete-time nonlinear systems, in absence or presence of external disturbances and parametric uncertainties. The interval observers depend on the design of two preserving order observers, providing lower and upper estimations of the state. The main objective is to apply the stability radii notions and cooperativity property in the estimation error systems in order to guarantee that the lower/upper estimation is always below/above the real state trajectory at each time instant from an appropriate initialization, and the estimation errors converge asymptotically towards zero when the disturbances and/or uncertainties are vanishing. For the disturbed case, the estimation errors practically converge to a vicinity of zero, while the lower/upper estimations preserve the partial ordering with respect to the state trajectory. The design conditions, that are valid for Lipschitz nonlinearities, can be expressed as Linear Matrix Inequalities (LMIs). A numerical simulation example is provided to verify the effectiveness of the proposed method.

INDEX TERMS Stability radii, interval observers, discrete-time systems, linear matrix inequalities.

I. INTRODUCTION

Observer design for nonlinear systems, which consists in reconstructing the values of the state from available measurements, is a fundamental problem in control [1]–[8]. There exist many kinds of observers, for example, Dissipative observers, Adaptive observers, Sliding-Mode observers, High-Gain observers, H_∞ observers, Unknown Input observers, etc. In recent years, the design of the so-called interval observers for nonlinear systems has been an attractive research topic in control theory for coping with uncertainties and disturbances [9]–[17]. These interval observers appeared as a solution of a highly uncertain bioreactor problem, reported in [9] and [18], providing the bounds (lower and upper estimations) of the uncertain state. The interval observers can be built by two preserving order observers, given as a lower and an upper observer, whose estimations satisfy (i) the preservation of a partial ordering with respect to the real state trajectory at each time instant from an appropriate initialization, and (ii) the assurance of a practical convergence to a vicinity of the real state values, given by the effects of bounded disturbances/uncertainties [12], [19]. When uncertainties and/or disturbances are absent in the system dynamics, the lower and upper estimations asymptotically converge to their real state values, preserving the partial ordering between the estimations and the state. Generally, the interval observer design methods are developed for continuous-time systems [9], [12], [18]–[23]. They are based on the properties of cooperative systems [24], [25] with the purpose of ensuring the partial ordering between the estimations and the state, taking into account the knowledge of bounds of uncertainties and disturbances. Such properties are considered in the estimation error dynamics. Since the cooperativity property depends on the coordinates, several methods as in [26]–[33] include state (similarity) transformations to relax the design conditions of the interval observers for some classes of continuous-time systems.

The associate editor coordinating the review of this manuscript and approving it for publication was Yangmin Li.

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Some works have been dedicated to the interval observer design for discrete-time systems [34]–[43]. They have been encouraged by the use of sample-data systems in presence of disturbances, for which the cooperativity property holds [44]. Most of these works are focused on the interval observer design for some classes of linear discrete-time systems. For instance, an interval observer design method is developed in [34] for Linear Time-Invariant (LTI) discrete-time systems, obtaining the observer gain by a solution of Linear Matrix Inequalities (LMIs). Additionally, a method makes use of a static transformation for designing interval observers for LTI Nonnegative discrete-time systems. These methods are extended in [35] for Linear Time-Varying discrete-time systems. In [37] an interval observer design approach is proposed for uncertain linear discrete-time systems, using the $\mathcal{H}_\infty$ theory to attenuate uncertainties, in order to obtain accurate interval estimation, formulating the design conditions in terms of LMIs. Moreover, the work [38], based on the Sylvester equation, has proposed an algebraic method to compute a constant state transformation and an observer gain, in order to design interval observers for linear descriptor discrete-time systems with both unknown inputs and measurement noise.

In general, the interval observer design for nonlinear discrete-time systems is clearly more difficult compared to the interval observer design theory for linear discrete-time systems. For example, one of the most crucial and difficult to check assumptions for the interval observer design deals with cooperativity property of the estimation error dynamics. The work [42] combines the dissipativity and cooperativity properties to design interval observers, based on the preserving order observers, for a class of nonlinear discrete-time systems with perturbations and uncertainties, requiring the solution of a finite set of LMIs, which depend on the types of the nonlinearities in the system.

From [12] and [42], the motivation of this study is to investigate the interval observer design problem for a class of nonlinear discrete-time systems in presence of disturbances and/or uncertainties. Specially, the results reported in [19] are extended to the discrete case. Thus, the main objective of this work is to design interval observers by making use of the stability radii notions and cooperativity property in the estimation error dynamics. The cooperativity property produces the partial ordering between the estimations and the state, while the stability radii theory defined for positive discrete-time systems in [45] and [46], determines the asymptotic convergence of the observer when the system disturbances are vanishing, through a simple formula that depends on the observer matrices. This design method is extended for the disturbed case, so the estimations (i) achieve practical convergence to their real values, and (ii) preserve the partial ordering with respect to the state trajectory. From the combined theories, a trade-off exists between preserving the partial order of the estimations with respect to the state trajectory and decreasing convergence velocity. The preserving order and interval observer design conditions, valid for Lipschitz nonlinearities, can be formulated as a finite set of iterative LMIs to find the observer matrices. The results of this work can be seen as a particular case of the proposed method in [42], since Lipschitz nonlinearities belonging to the $[-K, K]$-sector with $K = \gamma I$, considered in the present paper, can be always expressed by the dissipativity conditions used in [42], i.e. it is $(Q, S, R) - \mathrm{D} = (-I, 0, \gamma^2 I)$-Dissipative. Finally, our approach in the present paper has the benefit of being simpler, since it requires the calculation of only two observer matrices expressed from some LMIs for the design of preserving order and interval observers, instead of the five (matrix) variables needed in the nonlinear matrix inequalities in [42].

The remainder of the paper is structured as follows. In Section II we present the stability radii theory and cooperativity property for nonlinear discrete-time systems, and several results that will be useful in developing the approach in the following sections. The preserving order and interval observer design for a class of nominal systems is given in Section III, while the same design is extended for a class of disturbed/uncertain nonlinear discrete-time systems in Section IV. The design conditions are formulated in terms of LMIs in Section V. A numerical example is presented in Section VI to illustrate the behavior of the proposed observers and the solution algorithm. Section VII draws some final conclusions.

A. NOTATIONS

$\mathbb{R}^n$ represents the n-dimensional Euclidean space. We represent the partial ordering between two vectors/matrices through the symbol $\succeq$: For vectors $x,\ y \in \mathbb{R}^n$, if $x_i - y_i \geq 0$, $\forall i = 1, \ldots, n$, then $x \succeq y$, and for matrices $A,\ B \in \mathbb{R}^{n\times n}$ if $a_{ij} - b_{ij} \geq 0$, $\forall i, j = 1, \ldots, n$, then $A \succeq B$. In particular, we define the nonnegativity for vectors/matrices, for example, if $x \succeq 0$ then x is a nonnegative vector, i.e. $x_i \geq 0$, $\forall i = 1, \ldots, n$, and $A \succeq 0$ states a nonnegative matrix iff $a_{ij} \geq 0$ with $\forall i, j = 1, \ldots, n$. Similarly, it can expressed as $\mathbb{R}^n_+$ or $\mathbb{R}^{n\times m}_+$, which are given by the set of all vectors or matrices with nonnegative entries. It is important to discern that a positive definite matrix P (resp. positive semi-definite) is expressed by $P = P^T > 0$ (resp. $P = P^T \geq 0$). Additionally, $P = P^T < 0$ (resp. $P = P^T \leq 0$) is a negative definite matrix P (resp. negative semi-definite). For $x \in \mathbb{K}^n$ and $M \in \mathbb{K}^{n\times m}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the nonnegative vector $|x| \in \mathbb{R}^n_+$ and the nonnegative matrix $|M| \in \mathbb{R}^{n\times m}_+$ are defined as $|x| = (|x_i|)$ and $|M| = \left(|m_{ij}|\right)$, respectively. $\|\cdot\|$ is the Euclidean norm for the vector space $\mathbb{K}^n$. The induced matrix norm $\|M\|$ is given as the spectral norm. The spectral radius of A is expressed as $\rho(A) = \max\{|\lambda|; \lambda \in \sigma(A)\}$, where $\sigma(A) \subset \mathbb{C}$. I is the identity matrix with a proper dimension. $D_{x_k}f(x_k, u_k) = \partial f(x_k, u_k)/\partial x_k$ stands for the Jacobian matrix of $f(x_k, u_k)$ with respect to x_k, $D_{u_k}f(x_k, u_k) = \partial f(x_k, u_k)/\partial u_k$ is the Jacobian matrix with respect to u_k.

II. PRELIMINARIES

Before stating the main contributions of this work, we recall in this section some important notions on cooperativity and stability radii of positive discrete-time systems.

A. PARTIAL ORDERING ON THE STATE TRAJECTORIES

Cooperative systems define a partial ordering on the state and output trajectories at every time instant, based on the partial ordering on the inputs and the initial conditions [24], [25], [44]. We recall briefly some definitions and characterizations of cooperativity for discrete-time systems.

Definition 1 ([42]): Consider the following nonlinear discrete-time system

$$\Sigma_{\mathrm{NL}}:\begin{cases} x_{k+1}=f\left(x_k,\,u_k\right), & x\left(k_0\right)=x_{k0},\\ y_k=h\left(x_k,\,u_k\right), \end{cases} \tag{1}$$

where $x_k\in\mathbb{R}^n$ is the state vector, $u_k\in\mathbb{R}^m$ is the input, and $y_k\in\mathbb{R}^p$ is the measured output. The discrete-time system Σ_{NL} is cooperative *if whenever the initial conditions and inputs are partially ordered, i.e.,*

$$x_{k0}^2\succeq x_{k0}^1,\quad u_k^2\succeq u_k^1,\ \forall k\geq 0,$$

then the state and output trajectories preserve the partial ordering at every time instant, i.e., $\forall k\geq 0$

$$x\left(k,\,x_{k0}^2,\,u_k^2\right)\succeq x\left(k,\,x_{k0}^1,\,u_k^1\right)$$
$$y\left(k,\,x_{k0}^2,\,u_k^2\right)\succeq\left(k,\,x_{k0}^1,\,u_k^1\right).$$

The following propositions characterize cooperative discrete-time systems.

Proposition 2 ([42], [44]): The nonlinear discrete-time system Σ_{NL} in (1) *is* cooperative *iff the following conditions hold*:

1) $D_{x_k}f(x_k,\,u_k)\succeq 0$,
2) $D_{u_k}f(x_k,\,u_k)\succeq 0$, *and*
3) $D_{x_k}h(x_k,\,u_k)\succeq 0$.

Proposition 3 ([42], [44]): Consider the linear discrete-time system described as follows,

$$\Sigma_{\mathrm{L}}:\begin{cases} x_{k+1}=Ax_k+Bu_k, & x\left(k_0\right)=x_{k0},\\ y_k=Cx_k, \end{cases} \tag{2}$$

where the state, the input and the output are defined as $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}^m$, and $y_k\in\mathbb{R}^p$, respectively. Moreover, A, B, and C are constant matrices. Σ_{L} is a cooperative discrete-time linear system iff the following conditions hold:

1) $A\succeq 0$,
2) $B\succeq 0$, *and*
3) $C\succeq 0$.

Remark 4: Particularly, Σ_{L} in (2) *is a positive linear discrete-time system if the state and output trajectories are non-negative, i.e., $x\left(k,\,x_{k0},\,u_k\right)\succeq 0$ and $y\left(k,\,x_{k0},\,u_k\right)\succeq 0$ from the non-negativity on the initial conditions and inputs, i.e., $x_{k0}\succeq 0$ and $u_k\succeq 0$. It is important to mention that a positive linear discrete-time system shares the same conditions that have been provided for a cooperative linear discrete-time system (see Proposition 3).*

B. STABILITY RADII CONDITIONS FOR POSITIVE DISCRETE-TIME SYSTEMS

This method states the stability conditions for linear positive discrete-time systems taking into account linear or nonlinear disturbances [45], [46]. We present some important results that will be used in the design of the preserving order and interval observers, specially such outcomes will be applied in the estimation error dynamics, in the following sections.

1) POSITIVE DISCRETE-TIME SYSTEMS WITH LINEAR DISTURBANCES

Let the linear positive discrete-time system, given by the equations

$$\Gamma_{\mathrm{L}}:\left\{x_{k+1}=Ax_k,\quad x(k_0)=x_{k_0},\ k\in\mathbb{N},\right. \tag{3}$$

where $x_k\in\mathbb{R}^n$ is the state vector and $A\in\mathbb{R}_+^{n\times n}$ is a non-negative matrix. The positive discrete-time system Γ_{L} in (3) is Globally Asymptotically Stable (GAS) iff the matrix A is *Schur stable*, that is, all eigenvalues of the matrix A are located into the unit circle of the complex plane, i.e., $|\lambda_i(A)|<1$, $i=1,2,\ldots,n$, which can be also expressed as $\rho(A)<1$.

If the linear discrete-time system Γ_{L} in (3) is affected by a linear output feedback disturbance, $A\to A+B\Delta C$, the linear perturbed system is given as follows

$$\Gamma_{\mathrm{LD}}:\{x_{k+1}=(A+B\Delta C)x_k,\quad\|\Delta\|<\rho, \tag{4}$$

taking into account $B\succeq 0$, $C\succeq 0$, and Δ stands for an unknown disturbance matrix. Moreover, $\|\Delta\|$ determines the size of the perturbation and $\rho>0$ is a bound of the perturbation. We next recall some definitions and theorems of the stability radii for ensuring the asymptotic stability for Γ_{LD} in (4), which depend on the matrix space of Δ [45].

Definition 5: The stability radius for Γ_{LD} is defined by

$$r_{\mathbb{K}}\left(A;B,C\right)=\inf\left\{\|\Delta\|:\Delta\in\mathbb{K}^{m\times p},\,\rho\left(A+B\Delta C\right)\geq 1\right\} \tag{5}$$

where $\mathbb{K}=\mathbb{C},\,\mathbb{R},\,\mathbb{R}_+$ stands for the complex, real or nonnegative matrix space.

The stability radii $r_{\mathbb{K}}$ in (5) stand for the minimal bound of the (complex, real, nonnegative) disturbance Δ, which produce the destabilization for Γ_{LD}. Considering the matrix spaces in the stability radii $r_{\mathbb{K}}$, the next inequalities hold

$$0\leq r_{\mathbb{C}}\left(A;B,C\right)\leq r_{\mathbb{R}}\left(A;B,C\right)\leq r_{\mathbb{R}_+}\left(A;B,C\right).$$

In particular, the stability radii for the positive linear discrete-time system Γ_{LD} can be expressed by a unique formula.

Theorem 6 ([45]): The stability radii for the positive linear discrete-time system Γ_{LD} in (4) *are given by the formula*

$$r_{\mathbb{C}}\left(A;B,C\right)=\|\,G(1)\,\|^{-1}=\left\|C\left(I-A\right)^{-1}B\right\|^{-1}, \tag{6}$$

with $r_{\mathbb{C}}=r_{\mathbb{R}}=r_{\mathbb{R}+}$.

It is important to mention that the formula of theorem 6 is determined when $\|G(s)\|$ attains its maximum value on the unit circle at $s=1$, with $G(s)=C\left(sI-A\right)^{-1}B$.

2) POSITIVE DISCRETE-TIME SYSTEMS WITH NONLINEAR DISTURBANCES

The notion of stability radii is extended to the class of linear positive discrete-time systems connected in feedback with a static time-varying (disturbance) nonlinearity.

We consider the nonlinear discrete-time system

$$x_{k+1} = Ax_k + B\,\xi(Cx_k, k), \quad k \in \mathbb{N},$$

that is decomposed as a linear discrete-time subsystem connected, in feedback, with a static time-varying nonlinearity [46]. Then, this latter system can be written as follows

$$\Gamma_{\mathrm{ND}} : \begin{cases} x_{k+1} = Ax_k + Bu_k, \quad x(k_0) = x_{k_0}, \\ y_k = Cx_k, \\ u_k = \xi\,(y_k, k)\,, \end{cases} \tag{7}$$

where $\xi\,(y_k, k) : \mathbb{R}^p \times \mathbb{N} \rightarrow \mathbb{R}^m$ is the static time-varying nonlinear function that fulfills

1) $\xi\,(0, k) = 0, \quad k \in \mathbb{N}$, and
2) $\xi\,(y_k, k)$ is Lipschitz in y_k, such that

$$\|\xi\,(y_k, k)\| \leq \gamma\,\|y_k\|\,, \quad \forall y_k \in \mathbb{R}^p,\ \forall k \in \mathbb{N},\ \gamma > 0. \tag{8}$$

Remark 7: Note that the condition in (8) *can be equivalently expressed as belonging to the symmetric sector* $[-K,\ K]$, *with* $K = \gamma I \in \mathbb{R}^{m\times m}$, *i.e. the following inequality is fulfilled*

$$\gamma^2\, y_k^T\, y_k \;-\; \xi^T(y_k, k)\xi(y_k, k) \geq 0, \quad \forall k \in \mathbb{N}. \tag{9}$$

In addition, we consider that the size of the nonlinear function is given by

$$\|\xi\| = \inf\left\{\gamma \in \mathbb{R}_+;\ \forall y_k \in \mathbb{R}^p, k \in \mathbb{N} : \|\xi\,(y_k, k)\| \leq \gamma\,\|y_k\|\right\}.$$

Considering the previous suppositions, the following result provides the asymptotic stability conditions for Γ_{ND} in terms of the stability radii for the positive discrete-time systems [45], [46].

Lemma 8: Let Proposition 3 hold. Assume that $\rho\,(A) < 1$ *and* $\xi(y_k, k)$ *satisfies* (8). *The discrete-time system* Γ_{ND} *is Globally Exponentially Stable (GES) if the next inequality*

$$\rho \;<\; ||C\,(I - A)^{-1}\,B||^{-1} \tag{10}$$

holds with $\|\xi\| \leq \rho$. *In that case all solutions of the system satisfy*

$$\|x_k\| \;\leq\; \alpha\,\left\|x_{k_0}\right\|\exp(-\varrho k)\,,$$

for $k \in \mathbb{N}, \quad \alpha > 0, \quad \varrho > 0$.

Lemma 8 is a key result that will be used in the asymptotic stability analysis of the preserving order and interval observers for nominal nonlinear systems in Section III. A similar result can be found for the class of non-positive linear discrete-time systems.

Lemma 9 ([45]): Let the system Γ_{ND} *where the matrices* $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, *and* $|A|$ *is a Schur stable matrix. Iff* $f : \mathbb{K}^p \times \mathbb{N} \rightarrow \mathbb{K}^m$ *fulfills the following inequality*

$$\|f\,(y_k, k)\| < ||\,|C|\,(I - |A|)^{-1}\,|B|\,||^{-1}, \tag{11}$$

thus Γ_{NP} *in* (7) *is a GES discrete-time system.*

Next, we will develop the practical stability property for the class of discrete-time systems Γ_{ND} with an external input, connecting the stability radii theory and the Input-to-State-Stability (ISS).

Definition 10: Lets us consider the discrete-time system

$$\Gamma_{\mathrm{NE}} : \begin{cases} x_{k+1} = Ax_k + Bu_k + b_k, \quad x(k_0) = x_{k_0}, \\ y_k = Cx_k, \\ u_k = \xi\,(y_k, k)\,, \end{cases} \tag{12}$$

where b_k *represents the external input.* Γ_{NE} *is (globally)* Input-to-State Stable *(ISS) with respect to* b_k *if there exist a* $\mathcal{KL}$*-function* β *and* $\mathcal{K}$*-function* δ *such that, for each initial condition* $x_{k_0} \in \mathbb{R}^n$ *and each locally essentially bounded function* $b_k : \mathbb{N} \rightarrow \mathbb{R}^m$, *the next inequality is fulfilled*

$$||x_k|| \leq \beta(||x_{k_0}||, k) + \delta\left(\sup_{k\in\mathbb{N}}||b_k||\right),$$

where δ *is the ISS-gain for* Γ_{NE}.

The following lemma summarizes the result of the practical stability for Γ_{NE}.

Lemma 11: Let us suppose that the conditions of Lemma 8 are fulfilled, then the discrete-time system Γ_{NE} *is exponentially - ISS with respect to* b_k.

It is important to mention that Lemma 11 will be applied in the practical stability analysis of the preserving order and interval observers for uncertain/perturbed nonlinear systems in Section IV.

III. PRESERVING ORDER AND INTERVAL OBSERVERS FOR NOMINAL DISCRETE-TIME SYSTEMS

In this section we consider the problem of preserving order and interval observer design for a class of nonlinear discrete-time systems where the asymptotic convergence of the estimates is guaranteed through stability radii theory, while the partial ordering is stated by the cooperativity property. These conditions are applied to the estimation error systems.

Consider the following class of nonlinear discrete-time systems, described by

$$\Gamma_{\mathrm{S}} : \begin{cases} x_{k+1} = Ax_k + G\psi\,(\sigma_k; k, y_k, u_k) + \varphi\,(k, y_k, u_k)\,, \\ \sigma_k = Hx_k, \quad x\,(k_0) = x_{k_0}, \\ y_k = Cx_k, \end{cases} \tag{13}$$

where the state, the input, and the measurement output are defined as $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $y_k \in \mathbb{R}^q$. $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, $H \in \mathbb{R}^{r\times n}$ are known matrices and $\psi(\sigma_k; k, y_k, u_k) \in \mathbb{R}^m$ is a continuous function that depends on a linear function of the (unmeasured) state σ_k and on the measured vectors (y_k, u_k). Finally, $\varphi\,(k, y_k, u_k) \in \mathbb{R}^n$ is a nonlinear continuous function depending on (y_k, u_k).

Assumption 12: The initial state $x_0 \in \mathbb{R}^n$ *is bounded as follows*

$$x_{k0}^{+} \succeq x_{k0} \;\succeq\; x_{k0}^{-}, \tag{14}$$

where x_{k0}^{-} *and* x_{k0}^{+} *are known lower and upper bounds, respectively.*

Let us consider the following state observer for the system Γ_S, having the form

$$\Gamma_O : \begin{cases} \widehat{x}_{k+1} = A\widehat{x}_k + L(\widehat{y}_k - y_k) + \varphi(k, y_k, u_k) \\ \qquad + G\psi(\widehat{\sigma}_k + N(\widehat{y}_k - y_k); k, y_k, u_k), \\ \widehat{\sigma}_k = H\widehat{x}_k, \\ \widehat{y}_k = C\widehat{x}_k, \end{cases} \tag{15}$$

where the estimated state is denoted by $\widehat{x}_k$ with the initial condition $\widehat{x}(k_0) = \widehat{x}_{k_0}$. One has to select the observer matrices, $L \in \mathbb{R}^{n\times q}$ and $N \in \mathbb{R}^{r\times q}$, such that the estimated state $\widehat{x}_k$ converges asymptotically to x_k and preserves the partial ordering with respect to x_k. Notice that there are two injection terms in Γ_O: the linear one $L(\widehat{y}_k - y_k)$, is the original term used in the Luenberger observer, and another linear term $N(\widehat{y}_k - y_k)$ appears in the nonlinear function $\psi(\cdot)$.

The state estimation error is defined as $e_k = \widehat{x}_k - x_k$. The error dynamics can be computed as

$$\Gamma_E : \begin{cases} e_{k+1} = A_L e_k + G v_k, \\ z_k = H_N e_k, \\ v_k = -\xi(z_k, \sigma_k; k, y_k, u_k), \end{cases} \tag{16}$$

with the initial error $e(k_0) = e_{k0} = \widehat{x}_{k_0} - x_{k_0}$ and the matrices $A_L \triangleq A + LC$ and $H_N \triangleq H + NC$. Taking into account the relation, $\widehat{\sigma}_k + N(\widehat{y}_k - y_k) = Hx_k + (H + NC)(\widehat{x}_k) - x_k) = \sigma_k + H_N e_k = \sigma_k + z_k$ in $\xi(\cdot)$ of the observer, the incremental nonlinearity $\xi(\cdot)$ is defined by

$$\xi(z_k, \sigma_k; k, y_k, u_k) = \psi(\sigma_k; k, y_k, u_k) - \psi(\sigma_k + z_k; k, y_k, u_k). \tag{17}$$

It is our objective to design state observers for Γ_S, inspired by [12], [19], whose estimates converge to the actual state values and preserve the partial ordering with respect to the state trajectory. These conditions are established in the following definition.

Definition 13: Γ_O is a lower (upper) preserving order observer for Γ_S, if the next conditions are fulfilled:

(i) *The lower/upper estimated state $\widehat{x}_k$ asymptotically converges to the state trajectory x_k, i.e.,*

$$\lim_{k\to\infty} \|\widehat{x}_k - x_k\| \to 0, \quad \forall k \in \mathbb{N}.$$

(ii) *Given a partial ordering on the initial state, the (lower) upper estimated state $\widehat{x}_k$ preserves the partial ordering with respect to x_k, i.e.,*

$$\text{If } x_{k0} \succeq x_{k0}^- \succeq \widehat{x}_{k0} \left(\widehat{x}_{k0} \succeq x_{k0}^+ \succeq x_{k0}\right) \Rightarrow x_k \succeq \widehat{x}_k \left(\widehat{x}_k \succeq x_k\right), \quad \forall k \in \mathbb{N}.$$

Notice that a couple of observers, consisting of an upper and a lower preserving order observers, given by Γ_{O^+} and Γ_{O^-}, respectively, constitutes an *interval observer* for Γ_S, fulfilling

$$\widehat{x}_k^+ \succeq x_k \succeq \widehat{x}_k^-, \quad \forall k \in \mathbb{N}.$$

A. CONVERGENT OBSERVER

The following theorem gives sufficient conditions to guarantee the asymptotic convergence property of the state observer Γ_O, applying the stability radii theory to the system Γ_E.

Theorem 14: Let the observer Γ_O in (15) be applied to the system Γ_S in (13). Suppose that $G \succeq 0$ and $\xi(\cdot)$ in (17) fulfills (8). Assume that there exist matrices L and N such that the following conditions

$$\rho(A_L) < 1, \tag{18}$$
$$A_L \succeq 0, \tag{19}$$
$$H_N \succeq 0, \tag{20}$$
$$\rho < \|H_N (I - A_L)^{-1} G\|^{-1}, \tag{21}$$

are fulfilled. Then Γ_O is a globally and exponentially convergent observer for the nominal discrete-time system Γ_S in (13).

Proof: The proof of the observer convergence is the same as that for Lemma 8 and is omitted here. □

It is important to mention that the exponential convergence of the observer Γ_O, given by the Theorem 14, can be also obtained in terms of the dissipativity approach [8], making use of a storage function $V(e_k) = e_k^T P e_k$ with $P = P^T > 0$ and a quadratic supply rate, expressed as follows

$$\omega(v_k, z_k) = z_k^T Q z_k + 2 z_k^T S v_k + v_k^T R v_k$$

for $Q = Q^T \succeq 0$, $R = R^T \succeq 0$. The so-called discrete-time dissipativity inequality

$$V(e_{k+1}) - V(e_k) \leq -\epsilon V(e_k) + \omega(v_k, z_k)$$

is fulfilled by selecting $(Q, S, R) = (-R, S^T, -Q)$. Since $\xi(z_k, \sigma_k)$ fulfills the Lipschitz condition in (8), which can be similarly expressed as $(Q, S, R) = (-I, 0, \rho^2 I)$, then we write

$$\begin{aligned} \Delta V &= V(e_{k+1}) - V(e_k) \\ &= \begin{bmatrix} e_k \\ v_k \end{bmatrix}^T \begin{bmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB \end{bmatrix} \begin{bmatrix} e_k \\ v_k \end{bmatrix} \\ &\leq \begin{bmatrix} e_k \\ -\xi(\cdot) \end{bmatrix}^T \begin{bmatrix} -\rho^2 H_N^T H_N & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} e_k \\ -\xi(\cdot) \end{bmatrix} \\ &\quad - \epsilon V(e_k) \\ &\leq -\epsilon V(e_k). \end{aligned}$$

This result is similar to the one obtained by the stability radii in (43). The main difference is that the dissipativity approach requires some variables (L, N, ϵ, P, and θ) to check the exponential convergence of the observer Γ_O, while the stability radii theory only depends on the observer matrices (L, N).

Using the Lemma 9, the same exponential convergence for the class of non-positive estimation error systems Γ_E is provided in the next Theorem.

Theorem 15: Suppose that $(A_L, G, H_N) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times n}$. Additionally, assume that $|A_L|$ is a Schur stable matrix. If there exist matrices L and N, such that the incremental *nonlinearity satisfies the following inequality*

$$\|\psi(z_k,\sigma_k)\| < \left\| |H_N| \left(I - |A_L|\right)^{-1} |G| \right\|^{-1}, \tag{22}$$

then Γ_O *in* (15) *is a globally and exponentially convergent observer for* Γ_S.

Proof: Applying Lemma 9 in Γ_E, the exponential convergence is achieved in Γ_O. □

B. PRESERVING ORDER AND INTERVAL OBSERVERS DESIGN

In order to guarantee the partial ordering between the estimate $\widehat{x}_k$ and the state trajectory x_k, we apply the cooperativity property to the estimation error dynamics Γ_E using the Proposition 2.

In this case, Γ_E is a cooperative nonlinear discrete-time system if the Jacobian matrix

$$D_{e_k}f^*(e_k) = A_L - GD_{e_k}\xi(z_k,\sigma_k;\ k,\ y_k,\ u_k) \succeq 0$$

is nonnegative $\forall z_k \in \mathbb{R}^r, \forall k,\ \sigma_k,\ y_k,\ u_k$. This condition is equivalent to the Cooperativity Inequality [42]:

$$A_L - GD_{z_k}\xi(z_k,\sigma_k;\ k,\ y_k,\ u_k)H_N \succeq 0, \quad \forall z_k \in \mathbb{R}^r,$$

$\forall k,\ \sigma_k,\ y_k,\ u_k$. By setting $D_{z_k}\xi(z_k,\sigma_k;\ k,\ y_k,\ u_k) = -D_{\sigma_k}\psi(\sigma_k + z_k;\ k,\ y_k,\ u_k)$, thus the Cooperativity Inequality is written as follows

$$A_L + GD_{\sigma_k}\psi\left(\sigma_k + z_k;\ k,\ y_k,\ u_k\right)H_N \succeq 0, \tag{23}$$

$\forall z_k \in \mathbb{R}^r,\ \forall k, \sigma_k,\ y_k,\ u_k$.

The main result of this section is stated in the following theorem summarizing the design approach of preserving order and interval observers for the nominal discrete-time systems Γ_S.

Theorem 16: Consider the discrete-time nonlinear system Γ_S *in* (13) *under Assumption 12. Assume that* $G \succeq 0$. *Moreover,* $\psi(\cdot)$ *in* (17) *fulfills* (8). *If the matrices L and N exist, such that the following conditions*:

1) (18)-(21) *of the Theorem 14, and*
2) *the Cooperativity Inequality in* (23),

hold. Thus, the observer Γ_O *in* (15) *is said to be a* GES - (lower/upper) preserving order observer *for* Γ_S. *Furthermore, an* interval observer *is composed by a lower and an upper preserving order observer for* Γ_S.

Proof: Applying the Lemma 8 in Γ_E, it is guaranteed the exponential convergence property in Γ_O. Moreover, we use the Cooperativity Inequality condition (23) to ensure that the state trajectory is bounded by the lower/upper estimations. □

IV. INTERVAL OBSERVERS FOR DISTURBED SYSTEMS

In this section, we propose preserving order and interval observers for the nonlinear discrete-time system in presence of disturbances and/or uncertainties. The robust observers are also achieved by integrating of the radii stability approach with cooperativity property. The proposed design represents the extension of the continuous method in [19] to the discrete representation.

In this section, we assume that the nonlinear discrete-time system Γ_S is affected by uncertainties and/or disturbances. The system is now described by the set of equations

$$\Psi_S: \begin{cases} x_{k+1} = Ax_k + G\psi\left(\sigma_k; k, y_k, u_k\right) + \varphi\left(k, y_k, u_k\right) \\ \qquad + w(k, x_k, u_k), \quad x\left(k_0\right) = x_{k_0}, \\ \sigma_k = Hx_k, \\ y_k = Cx_k, \end{cases} \tag{24}$$

where $w_k \in \mathbb{R}^n$ is a disturbance/uncertainty term such that satisfies the next Assumption.

Assumption 17: The disturbance/uncertainty vector is bounded by the interval

$$w_k^+ = w^+(k,\ y_k,\ u_k) \succeq w_k \succeq w_k^- = w^-(k,\ y_k,\ u_k), \tag{25}$$

$\forall k \in \mathbb{N}$, *where* $w^+(k,\ y_k,\ u_k)$ *and* $w^-(k,\ y_k,\ u_k)$ *are known upper and lower (bound) functions.*

As pointed out in [12] the disturbances/uncertainties are not partially ordered inputs, we can then preserve the partial ordering between the estimates and the uncertain/disturbed state trajectory, incorporating a couple of observers for the disturbed discrete-time system Ψ_S:

$$\Psi_{O^+}: \begin{cases} \widehat{x}_{k+1}^+ = A\widehat{x}_k^+ + L^+\left(\widehat{y}_k^+ - y_k\right) + \varphi(k, y_k, u_k) \\ \qquad + G\psi\left(\widehat{\sigma}_k^+ + N^+\left(\widehat{y}_k^+ - y_k\right);\ k,\ y_k,\ u_k\right) \\ \qquad + w^+\left(k, y_k,\ u_k\right), \\ \widehat{\sigma}_k^+ = H\widehat{x}_k^+, \\ \widehat{y}_k^+ = C\widehat{x}_k^+, \end{cases} \tag{26}$$

$$\Psi_{O^-}: \begin{cases} \widehat{x}_{k+1}^- = A\widehat{x}_k^- + L^-\left(\widehat{y}_k^- - y_k\right) + \varphi(k, y_k, u_k) \\ \qquad + G\psi\left(\widehat{\sigma}_k^- + N^-\left(\widehat{y}_k^- - y_k\right);\ k,\ y_k,\ u_k\right) \\ \qquad + w^-\left(k, y_k,\ u_k\right), \\ \widehat{\sigma}_k^- = H\widehat{x}_k^-, \\ \widehat{y}_k^- = C\widehat{x}_k^-, \end{cases} \tag{27}$$

where $\widehat{x}_k^+$ and $\widehat{x}_k^-$ stand for the upper estimate and the lower estimate, respectively, with the initial conditions $\widehat{x}^+(k_0) = \widehat{x}_{k_0}^+ \succeq x_{k_0}^+$ and $\widehat{x}^-(k_0) = \widehat{x}_{k_0}^- \preceq x_{k_0}^-$. The matrices $L^+,\ L^- \in \mathbb{R}^{n\times q}$ and $N^+,\ N^- \in \mathbb{R}^{r\times q}$ are appropriately selected. Denoting $e_k^+ \triangleq \widehat{x}_k^+ - x_k$ and $e_k^- \triangleq x_k - \widehat{x}_k^-$, we can compute the upper and lower estimation error dynamics,

$$\Psi_{E^+}: \begin{cases} e_{k+1}^+ = A_L^+ e_k^+ + Gv_k^+ + b_k^+, \\ z_k^+ = H_N^+ e_k^+, \\ v_k^+ = -\xi^+\left(z_k^+, \sigma_k;\ k,\ y_k,\ u_k\right), \end{cases} \tag{28}$$

$$\Psi_{E^-}: \begin{cases} e_{k+1}^- = A_L^+ e_k^- + Gv_k^- + b_k^-, \\ z_k^- = H_N^- e_k^-, \\ v_k^- = -\xi^-\left(z_k^-, \sigma_k;\ k,\ y_k,\ u_k\right), \end{cases} \tag{29}$$

with the nonnegative initial errors $e^+(k_0) = e_{k_0}^+ \succeq 0$ and $e^-(k_0) = e_{k_0}^- \succeq 0$. $b_k^+ = w^+(k,\ y_k,\ u_k) - w(k,\ x_k,\ u_k) \succeq 0$ and $b_k^- = w(k,\ x_k,\ u_k) - w^-(k,\ y_k,\ u_k) \succeq 0$ are nonnegative external inputs that act in the systems

Ψ_{E^+} and Ψ_{E^-}. The incremental nonlinearities are described by $\xi^+\left(z_k^+, \sigma_k; k, y_k, u_k\right) = \psi(\sigma_k; k, y_k, u_k) - \psi(\sigma_k + z_k^+; k, y_k, u_k)$ and $\xi^-\left(z_k^-, \sigma_k; k, y_k, u_k\right) = \psi(\sigma_k - z_k^-; k, y_k, u_k) - \psi(\sigma_k; k, y_k, u_k)$.

The preserving order and interval observers design for the disturbed/uncertain discrete-time system Ψ_S in (24) is summarized in the following definition and theorem.

Definition 18: The system Ψ_{O^-} $\left(\Psi_{O^+}\right)$ is said to be a lower (upper) preserving order observer for Ψ_S, if the following conditions are fulfilled:

(i). *$w(k, x_k, u_k)$ is bounded as* (25),

(ii). *the partial ordering between the lower (upper) estimate $\widehat{x}_k^-$ $\left(\widehat{x}_k^+\right)$ and the state trajectory x_k is given as follows*

$$\begin{aligned} \text{If } x_{k0} \succeq x_{k0}^- \succeq \widehat{x}_{k0}^- \left(\widehat{x}_{k0}^+ \succeq x_{k0}^+ \succeq x_{k0}\right) \\ \Rightarrow x_k \succeq \widehat{x}_k^- \left(\widehat{x}_k^+ \succeq x_k\right), \quad \forall k \in \mathbb{N}. \end{aligned}$$

(iii). *the lower (upper) estimated state $\widehat{x}_k^-$ $\left(\widehat{x}_k^+\right)$ practically converges to a vicinity of x_k, i.e.,*

$$\lim_{k\to\infty} \|\widehat{x}_k^{\pm} - x_k\| \to \beta^{\pm}.$$

Additionally, Ψ_{O^+} and Ψ_{O^-} form an interval observer for Ψ_S. Hence, the partial ordering between the state trajectory and estimates is always preserved, satisfying,

$$\widehat{x}_k^+ \succeq x_k \succeq \widehat{x}_k^-, \quad \forall\, k \in \mathbb{N},$$

with the final bound

$$\lim_{k\to\infty} \|\widehat{x}_k^+ - \widehat{x}_k^-\| \to \beta.$$

The main result of our paper is established in the following Theorem for designing the preserving order and interval observers for the perturbed discrete-time systems Ψ_S, using the conditions of the Lemma 11 and Proposition 2 in the estimation error dynamics Ψ_{E^-} $\left(\Psi_{E^+}\right)$.

Theorem 19: Consider the perturbed discrete-time nonlinear system Ψ_S in (24) *under Assumption 12 and 17. Suppose that $\psi^-(\cdot)$ $\left(\psi^+(\cdot)\right)$ satisfies the Lipschitz condition in* (8). *Assume that $G \succeq 0$. If the matrices L^- and N^- (L^+, N^+) exist, such that the next conditions*:

- (18)-(21) *of the Theorem 14, and*
- *the Cooperativity Inequality in* (23),

hold. Hence, Ψ_{O^-} $\left(\Psi_{O^+}\right)$ is a globally ISS - lower (upper) preserving order observer for the disturbed system Ψ_S. Furthermore, Ψ_{O^-} and Ψ_{O^+} compose an interval observer *for Ψ_S.*

Notice that the interval observers can be obtained by the (lower or upper) preserving order observer design, taking the same matrices $L = L^+ = L^-$ and $N = N^+ = N^-$, which do not depend on the uncertainties/disturbances.

V. LMI FORMULATION

The results in LMIs formulation provide the advantage that can be easily solved by standard convex optimization algorithms. In this section, we formulate the design conditions of the interval and preserving order observers, presented in the Theorems 16 and 19, in terms of the LMI tools.

A. CONVERGENCE CONDITIONS

The convergence conditions of the interval and preserving order observers, for a class of nonlinear discrete-time systems, in presence or absence of disturbances-uncertainties, have been provided by stability radii theory in the Theorems 16 and 19. These conditions can be expressed, in some cases, by means of LMIs in the variables L and N.

We then analyze the conditions (18)-(21) of the Theorem 16 and 19 in the following paragraphs:

1) $\rho(A_L) < 1$ can be similarly rewritten as

$$A_L^T P A_L - P < 0,$$

which represents a quadratic Matrix Inequality (MI) in (P, L), adding the variable $P = P^T > 0$. Applying the Schur's complement, the previous MI becomes an LMI in the variables (PL, P), given as follows:

$$\begin{bmatrix} -P & -PA_L \\ -A_L^T P & -P \end{bmatrix} < 0, \tag{30}$$

Thus, the observer gain is computed as $L = P^{-1}PL$.

2) The non-negativity conditions: $A_L \succeq 0$ and $H_N \succeq 0$ are linear in L and N, respectively.

3) The condition $\rho < \left\|H_N(I - A_L)^{-1}G\right\|^{-1}$ becomes written as

$$\rho^{-2} I - G^T (I - A_L)^{-T} H_N^T H_N (I - A_L)^{-1} G > 0.$$

If we use the Schur's complement, we get the following MI:

$$\begin{bmatrix} \rho^{-2}\, I & H_N\ (I - A_L)^{-1}\ G \\ G^T\ (I - A_L)^{-T}\ H_N^T & I \end{bmatrix} > 0. \tag{31}$$

which is an LMI in N when L is fixed.

Based on the above conditions, we can construct an LMI algorithm to obtain convergent observers, which is given by the following schematic form.

Algorithm 1: The procedure to guarantee the convergence properties of the observers, in terms of LMIs, is summarized as follows

- ***Step 1:*** *The variable L can be found solving the following conditions*:
  - *(i). The expression in* (30) *is an LMI in* (PL, P).
  - *(ii). $A_L \succeq 0$ is a linear inequality in L.*
- ***Step 2:*** *The variable N can be found, fixing the matrix L, if the following conditions are simultaneously solved:*
  - *(i). $H_N \succeq 0$ is a linear inequality in N.*
  - *(ii). The inequality in* (31) *is an LMI in N.*

Notice that the expression in (30), taking the variables (P, L), is used as a substitute for (18) of the Theorem 14. In particular, P is not a required matrix for ensuring the convergence properties of the observers, but it can be utilized to find the matrices (L, N).

B. COOPERATIVITY CONDITION

The Cooperativity Inequality in (23), which has been considered in [42], determines the partial ordering between the estimated states and the real state trajectory. In general, the cooperativity condition in (23) can be expressed as a problem of solving an infinite number of LMIs in $(L,\ N)$, since they are checked for each value of the Jacobian matrix $D_\sigma\xi(\sigma+z;\ t,\ y,\ u)$. The main challenge in this problem is to find a finite set of LMIs, based on convex approximations, that verify the condition in (23). Let us recall the following useful results, taking into account the Lipschitz nonlinearities on the radii stability theory.

Let the following set:

$$\mathcal{J}=\left\{\Gamma\in\mathbb{R}^{m\times r}\,\middle|\,\Gamma=D_{\sigma_k}\psi\left(\sigma_k;\ k,\ y_k,\ u_k\right),\forall k,\sigma_k,y_k,u_k\right\}, \tag{32}$$

which includes all values of $D_\sigma\xi(\sigma+z;\ t,\ y,\ u)$. Hence, the inequality in (23) is now given by

$$A_L+GJH_N\succeq 0,\quad \forall\, J\in\mathcal{J}. \tag{33}$$

In addition, we consider that $\mathcal{J}$ is bounded, which implies that the Lipschitz-type nonlinearity $\psi\left(\sigma_k,\ z_k;\ k,\ y_k,\ u_k\right)$ is in the symmetric sector $[-K\ ,\ K]$ with $K=\gamma I$, satisfying

$$[\xi(\sigma,\ z;\ t,\ y,\ u)-(-K)\,z]^T\,[Kz-\xi(\sigma,\ z;\ t,\ y,\ u)]\geq 0. \tag{34}$$

Making use of the Mean Value Theorem, we can easily demonstrate that $\mathcal{J}$ is in the convex set Υ, expressed as follows

$$\Upsilon\triangleq\left\{\Gamma\in\mathbb{R}^{m\times r}\,\middle|\,[\Gamma-K]^T\,[K+\Gamma]\leq 0\right\}. \tag{35}$$

Due to the convexity of the function (33), we can evaluate the boundary points of $\mathcal{J}$ on (33), reducing significantly the number of LMIs that validate the cooperativity property in the estimation error dynamics. Sufficient LMI conditions which guarantee (33) will be provided in the next proposition.

Proposition 20: Let $\psi(z_k;k,y_k,u_k)\ :\ \mathbb{R}^r\ \rightarrow\ \mathbb{R}^m$. Suppose that $\xi\left(\sigma_k,\ z_k;k,\ y_k,\ u_k\right)$ is in the symmetric sector $[-K,\ K]$. The cooperativity condition in (23) *is satisfied $\forall z_k\in\mathbb{R}^r$, and for all $k,\ y_k,\ u_k$, if the following condition*:

$$A_L+G\Delta H_N\succeq 0 \tag{36}$$

hold for each point $\Delta\in\Upsilon_f$ of the boundary set

$$\Upsilon_f=\left\{\Gamma\in\mathbb{R}^{m\times r}\,\middle|\ \left(\Gamma+K_1\right)^T\left(K_2+\Gamma\right)=0\right\}. \tag{37}$$

Remark 21: If we consider the scalar case, i.e. $\sigma_k\ \rightarrow\ \xi(\sigma_k;\ k,\ y_k,\ u_k)\ :\ \mathbb{R}\ \rightarrow\ \mathbb{R}$, the matrix set (36) *is just given by two points of $\mathcal{J}\subset\Upsilon_f$, which are represented by the maximum and minimum values of the sector $\Upsilon_f=\{K_1,\ K_2\}$. The diagonal case, i.e. $\sigma_k\rightarrow\xi(\sigma_k;\ k,\ y_k,\ u_k):\mathbb{R}^m\rightarrow\mathbb{R}^m$, taking $\sigma_{k,i}\rightarrow\xi_i(\sigma_{k,i};\ k,\ y_k,\ u_k):\mathbb{R}\rightarrow\mathbb{R}$ for $i=1,\ldots,m$, defines a finite number of points of Υ_f.*

The detailed proof is omitted here for brevity, but it is analogously proved in Proposition 4 of [42]. Notice that the result in the proposition 20 roughly decrease the number LMIs, but in the general case they are still described by infinite LMIs. Making use of a geometrical (vectorial) representation with $r=1$, which is easily extended to matrix case, the set Υ can be characterized as an ellipsoid in $\mathbb{R}^m$ [47],

$$\Upsilon=\{\Gamma\in\mathbb{R}^m|\ \Gamma^TQ\Gamma-2\Gamma^TS+R\geq 0\} \tag{38}$$

with $Q<0$. Let us consider a pair of polytopes for Υ:

- An inscribed polytope in Υ

$$P_I=\left\{\sum_{i=1}^{k}\alpha_i\Delta_i\in\mathbb{R}^m\,\middle|\,\alpha_i\geq 0,\ \sum_{i=1}^{k}\alpha_i=1,\,i=1,\ldots,k\right\} \tag{39}$$

  where $\Delta_i\ \in\ \Upsilon_f$ are the vertices of P_I, which are increased for having an adequate approximation to Υ as in [42]
- A circumscribed polytope in Υ_f,

$$P_C=\left\{\sum_{i=1}^{\kappa}\alpha_i\Omega_i\in\mathbb{R}^m\,\middle|\,\alpha_i\geq 0,\ \sum_{i=1}^{\kappa}\alpha_i=1,\,i=1,\ldots,\kappa\right\} \tag{40}$$

  where $\Omega_i,\,i=1,\ldots,\kappa$ are the vertices of P_C, which can be depend on the vertices Δ_i..

In consequence, the cooperativity condition in (23) is fulfilled iff (33) is checked for a set of points, determined by the vertices of two polytopes P_I and P_C. This is stated in the next paragraph.

Remark 22: P_I and P_C define the necessary and sufficient conditions, respectively, for verifying (33). *Suppose that $P_I\ \subset\ \Upsilon$ and $\Upsilon\ \subset\ P_C$, the cooperativity inequality* (33) *is satisfied iff the next conditions hold*:

(i) *If $A_L+GJH_N\ \succeq\ 0$ is fulfilled $\forall J\ \in\ \mathcal{J}$, then $A_L+G\Delta_iH_N\succeq 0,\ i=1,\ldots,k$, is fulfilled for every Δ_i of P_I.*

(ii) *If $A_L+G\Omega_iH_N\succeq 0,\ i=1,\ldots,k$, is fulfilled of P_C, then $A_L+GJH_N\succeq 0$ is fulfilled $\forall J\in\mathcal{J}$.*

The cooperativity inequality (33) is then reduced to a finite set of LMI's in the design variables $L,\ N$. We can iteratively verify the cooperativity inequality.

VI. EXAMPLE

In this section, a numerical example is presented to show the validity and effectiveness of the proposed design approach. We consider the model for the electromechanical servo system [48], which can be used to control an inverted pendulum [49], described by the nonlinear discrete-time model with a sampling time of T_s 0.1 s,

$$\Gamma_G:\begin{cases}x_{k+1}=\begin{bmatrix}0.0468 & 0.1564\\ 0.2083 & 0.8154\end{bmatrix}x_k+\begin{bmatrix}39.2076\\ 11.5999\end{bmatrix}u_k\\ \qquad+\Phi\left(x_k\right)+w\left(k\right),\end{cases} \tag{41}$$

where $x_{1,\,k}$ is the load angular position, $x_{2,\,k}$ is the shaft speed, u_k is the input voltage, $\Phi\left(x_k\right)\ =\ \left[0,\ 0.005\sin(x_{1,\,k})\right]^\top$ is the nonlinear term that depends on the variable $x_{1,\,k}$, and

$w(k)$ is the bounded disturbance vector. Moreover, the output measurement is given by $y_k = x_{2,k}$.

The goal is to develop the interval observers, using the preserving order observer design given in the Theorems 16 and 19, for Γ_G in order to estimate the state from the measurement on $x_{2,k}$.

A. DESIGN FOR THE NOMINAL SYSTEM

In this case, we consider that $w(k) \equiv 0$. To use the proposed design approach, we write the discrete-time system Γ_G as in (13) with the matrices

$$A = \begin{bmatrix} 0.0468 & 0.1564 \\ 0.2083 & 0.8154 \end{bmatrix}, \quad G = C^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$\varphi(k, y_k, u_k) = \begin{bmatrix} 39.2076 \\ 11.5999 \end{bmatrix} u_k,$$

and $\psi(\sigma_k) = 0.005 \sin(\sigma_k)$. Furthermore, it is easy to verify that

$$\xi(z_k, \sigma_k) = 0.005 \left(\sin(\sigma_k) - \sin(\sigma_k + z_k)\right)$$

fulfills the Lipschitz condition in (8), and taking $\|\xi(\cdot)\| = 0.005$. According to Theorem 16, we get an interval observer with the following matrices

$$N = 0.075, \quad L = \begin{bmatrix} -0.1534 \\ -0.7452 \end{bmatrix},$$

and $\rho(A_L) = 0.0861$. In the simulation results, the proposed interval observer is compared with that presented in [42] using $\epsilon = 0.0303, \theta = 1, N = -1.6827 \times 10^{-4}$,

$$P = \begin{bmatrix} 1.7850 & -0.0002 \\ -0.0002 & 0.0004 \end{bmatrix}, \quad L = \begin{bmatrix} -0.1551 \\ -0.6770 \end{bmatrix}.$$

Figure 1 shows the interval estimation results provided by our approach $\widehat{x}_k^{\pm,sr}$ and the estimates given by the interval observer in [42] $\widehat{x}_k^{\pm,d}$ from the initial conditions $x_{k0} = [0, 0]^\top$, $\widehat{x}_{k0}^{\pm,sr} = [\pm 800, \pm 800]^\top$, and $\widehat{x}_{k0}^{\pm,d} = [\pm 800, \pm 800]^\top$. Both strategies present a similar behavior in the nominal case. It is evident that the estimations (bounds) preserve the partial ordering with respect to the state trajectory while they converge asymptotically to the real state values. The estimation errors converges towards zero. However, the calculations in the interval observer design proposed are easier than in the work [42].

B. DESIGN FOR THE DISTURBED SYSTEM

We now consider the interval observer design for system Γ_G affected by the bounded disturbance

$$w(k) = \begin{bmatrix} 39.2076 \\ 11.5999 \end{bmatrix} \left(\frac{w^- + w^+}{2} + \frac{w^+ - w^-}{2} \sin(15k) \right),$$

where the known upper and lower bounds are $w^+ = 2$ and $w^- = 1$, respectively. For this case, we use the same matrices of the above design, taking $L^+ = L^- = L$ and $N^+ = N^- = N$, such that the conditions of the Theorem 19 hold.

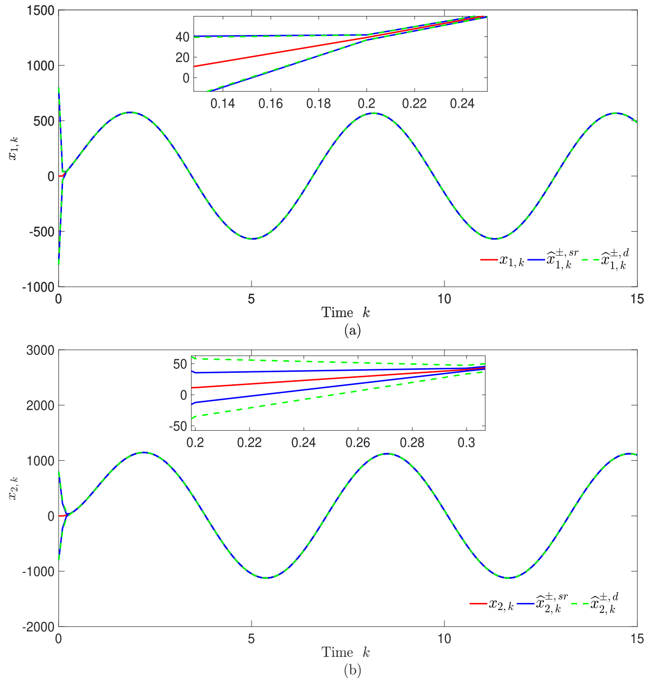

FIGURE 1. The trajectories of nominal system states x_k and its interval estimations of the states. (a) $x_{1,k}$, $\widehat{x}_{1,k}^{\pm,sr}$, and $\widehat{x}_{1,k}^{\pm,d}$. (b) $x_{2,k}$, $\widehat{x}_{2,k}^{\pm,sr}$, and $\widehat{x}_{2,k}^{\pm,d}$.

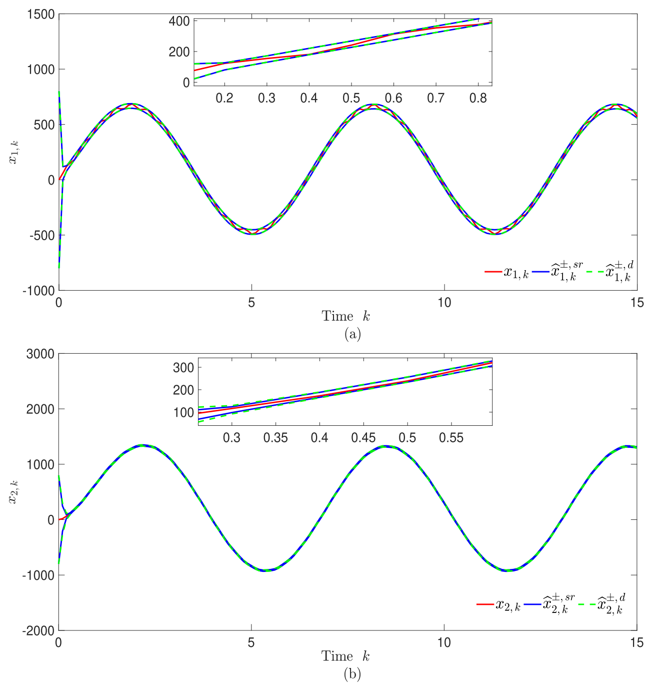

FIGURE 2. The trajectories of disturbed system states x_k and its interval estimations of the states. (a) $x_{1,k}$, $\widehat{x}_{1,k}^{\pm,sr}$, and $\widehat{x}_{1,k}^{\pm,d}$. (b) $x_{2,k}$, $\widehat{x}_{2,k}^{\pm,sr}$, and $\widehat{x}_{2,k}^{\pm,d}$.

Figure 2 provides the simulation results for disturbed states of the system Γ_G and their estimations proposed by the interval observers (Ψ_{O^+}, Ψ_{O^-}) and that presented in [42]. The initial conditions are taken from the nominal case. As mentioned in Definition 18, the estimations are converging to a neighborhood of the uncertain state, while they preserve the partial ordering with respect to the uncertain trajectory. It can be noted that the two interval observers achieve the objective of interval estimation, extending easily the proposed approach to the uncertain/disturbed case.

VII. CONCLUSION

We have addressed the design problem of interval observer and preserving order observers for a class of nonlinear discrete time systems in absence or presence of external disturbances and parametric uncertainties. The approach is based on the stability radii theory and the non-negativity conditions, which are applied to the estimation error dynamics. The proposed approach, applicable for Lipschitz nonlinearities, represents an extension of the method developed in Literature for the continuous-time systems. To derive design conditions in terms of the Linear Matrix Inequalities (LMIs), several mathematical tools are utilized to guarantee the convergence and cooperativity properties of the proposed observers for the discrete-time systems. A numerical example is included to illustrate the effectiveness of the proposed design. Future works will be devoted to the effect of measurement noise on the estimation of the preserving order and interval observers.

APPENDIX A
PROOF OF LEMMA 8

Proof: Consider that $\xi(\cdot)$ satisfies the Lipschitz condition in (8). Since $\rho < ||C(I-A)^{-1}B||^{-1}$, there exist a positive definite symmetric matrix $D = D^T < 0$ and a positive scalar $\epsilon > 0$ such that the so-called Discrete Algebraic Riccati Equation (DARE) is satisfied [45], [50]:

$$D - A^T DA + \epsilon I + \rho^2 C^T C + A^T DB(I + B^T DB)^{-1} B^T DA = 0. \quad (42)$$

By setting $P = -D$, $P = P^T > 0$, the inequality in (42) becomes

$$A^T PA - P + \epsilon I + \rho^2 C^T C - A^T PB(-I + B^T PB)^{-1} B^T PA = 0.$$

Using the Schur's complement result, we have

$$\Lambda \triangleq \begin{bmatrix} A^T PA - P + \epsilon I + \rho^2 C^T C & A^T PB \\ B^T PA & -I + B^T PB \end{bmatrix} \leq 0. \quad (43)$$

which implies the nonlinear discrete-time system Γ_{NE} is GES. Furthermore, we can easily recover the same stability result with a Lyapunov proof, taking a Lyapunov function candidate of the form $V_k \triangleq x_k^T P x_k$. Thus, the Lyapunov difference satisfies $\Delta V(x_k) \triangleq V(x_{k+1}) - V(x_k) = \tilde{e}_k^T \Lambda \tilde{e}_k \leq -\epsilon V(x_k)$ with $\|\xi\| \leq \rho$, where $\tilde{e}_k \triangleq [e_k^T, \xi(\cdot)^T]^T$. From the function difference, we can readily obtain

$$V(x_k) \leq (1-\epsilon)^k V(k_0),$$

for any $k \geq 0$, $0 < \epsilon < 1$. Using the Rayleigh inequality $\lambda_{\min}(P)\|x_k\|^2 \leq V(x_k) \leq \lambda_{\max}(P)\|x_k\|^2$, we write

$$\lambda_{\min}(P)\|x_k\|^2 \leq V(x_k) \leq (1-\epsilon)^k V(k_0) \leq \lambda_{\max}(P)(1-\epsilon)^k \|x_{k_0}\|, \quad (44)$$

where $\lambda_{\min,\max}(P)$ stand for the smallest and the greatest eigenvalues of the matrix solution P, respectively. Thus, the following inequality is satisfied

$$\|x_k\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|x_{k_0}\| \left(\frac{1}{\varsigma}\right)^{-\frac{k}{2}}$$

where $\varsigma = 1 - \epsilon$. This concludes the proof. □

APPENDIX B
PROOF OF LEMMA 11

Proof: Consider the disturbed nonlinear discrete-time system Γ_{NE}. By choosing the same Lyapunov candidate function $V_k \triangleq x_k^T P x_k$ for the proof of the Lemma 8, where P is a positive definite matrix, then its Lyapunov difference along the trajectories of the system Γ_{NE} is bounded as follows,

$$\Delta V_k \leq -\epsilon x_k^T P x_k + 2x_k^T A^T P b_k + b_k^T P b_k + 2\xi^T(k, y_k) B^T P b_k.$$

Since $||\xi(k, y_k)|| \leq \alpha \|y_k\| \leq \beta \|x_k\|$ for $\alpha > 0$, $\beta > 0$, we write

$$\Delta V(x_k) \leq -a\,\|x_k\|^2 - b\,\|x_k\|^2 + 2c\,\|x_k\|\,\|b_k\| + d\,\|b_k\|^2,$$

where $a = (1-\eta)\epsilon\lambda_{\min}(P)$, $b = \eta\epsilon\lambda_{\min}(P)$, $c = \|A^T P\| + \beta\,\|B^T P\|$, $d = \lambda_{\max}(P)$. Considering a sufficiently large value of $\|x_k\|$, we get

$$\Delta V(x_k) \leq -(1-\eta)\epsilon\lambda_{\min}(P)\,\|x_k\|^2, \quad \forall\, \|x_k\| \geq \gamma\,\|b_k\|,$$

where $\gamma = \frac{c+\sqrt{c^2+bd}}{b}$. Hence, Γ_{NE} is ISS with respect to b_k. This completes the proof. □

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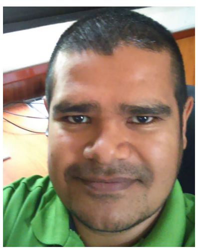

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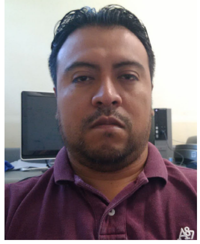

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