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Inverse and Direct Solutions of the Discrete Time Lyapunov Equation With System Matrix in Companion Form

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ABSTRACT The discrete time Lyapunov equation is used in many applications and there is interest in its inverse and direct solutions. New methods are proposed to obtain solutions for cases where the system matrix is in controllable canonical form. The approach is based on the relationship between the discrete Lyapunov equation and the entries of one of the stability tables presented by Jury. It is shown that the inverse solution, which is based on this stability table, can be obtained using LDL^t decomposition. Also the direct solution of the discrete Lyapunov equation can be obtained directly from the entries of this stability table. The proposed algorithms are illustrated by numerical examples.

INDEX TERMS Discrete systems, Lyapunon matrix equation, stability table, companion matrix form, Mansour matrix form.

I. INTRODUCTION

The Discrete Time Lyapunov Matrix Equation

$$AXA^t - X = -bUb^t \tag{1}$$

arises in many applications, for instance in the study of discrete time system stability [15], in covariance calculation [36], [37], in the iterative solution of the matrix Riccati equation [19] and in the optimal constant output feedback problem [11], [27]. Other relevant applications include, for instance statistics [20], [21], probability [2] and spectral analysis [12]. The equation is also a computational tool in the design of control systems [22] and in the coprime matrix fraction description of linear systems [49].

It is known that if the pair $\{A, b\}$ is controllable and U is any positive definite matrix, then the existence of a unique positive definite symmetric solution to (1) is sufficient and necessary for all eigenvalues of the matrix A to lie inside the unit disc. Given the state-space realization $x(k+1) = Ax(k) + bu(k)$ where $u(k)$ is a zero-mean stationary white random process with covariance U and A is stable, then the steady-state covariance defined by

$$X = E[x(k)x(k)^t] \tag{2}$$

satisfies the Lyapunov equation in (1).

Inverse problems are integrated in many applications and there is high research interest in these problems in last decades. The inverse solution of the Discrete time Lyapunov matrix equation has been considered in applications, such state-covariance assignment [39], the generation of q-Markov covers for discrete systems [40] and the design of Fuzzy controllers [7]. Few results for the inverse solution of the Lyapunov equation have been presented in the literature [39]. Hence there is interest in the derivation of new methods for the inverse solution of the Lyapunov equation. In this paper our aim is to propose a new method to obtain the inverse solution of the Discrete time Lyapunov matrix equation for the case where the system matrix is in controllable canonical form.

The relationship between stability properties and the solutions of the discrete time Lyapunov equation has been studied extensively and methods to obtain such solution have been presented in the literature [5], [6], [18], [30], [48]. Results for the solution of the continuous time Lyapunov equation

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have been presented in the literature as well [3], [4], [28], [40], [41], [47]. Whilst some proposed methods adopt system matrix in a specific way, other proposed methods rely on the Kronecker product. In [5] an elegant technique has been proposed for the solution of the discrete Lyapunov equation with the system matrix in controllable canonical form. It is shown in [5] that, in such a case, the solution of the discrete Lyapunov equation is the inverse of the Schur-Cohn matrix associated with the characteristic polynomial of the system matrix. Other interesting results for the solution of the Lyapunov equation have been presented in [10], [13], [14], [17], [23]–[26], [31]–[34], [42]–[46], [50], [51], [54], [55]. Excellent surveys about the Lyapunov equation have been presented in [8] and [35].

In this paper, new methods for obtaining the inverse and the direct solutions of the Discrete time Lyapunov matrix equation for linear discrete systems with the system matrix in controllable canonical form are presented. They are based on the properties of the Mansour matrix [1] and lead to solutions which are closely related to the Δ_j's, the entries of the stability table presented in [1], [16]. The proposed approach offers insight in the relationship between the discrete Lyapunov equation and the stability table test of the characteristic polynomial.

In section II some preliminary results on the Lyapunov equation and on the Mansour matrix form are presented. Also the stability table presented in [1], [16] is presented in detail. In section III the new methods are presented and illustrated with examples in section IV.

II. PRELIMINARIES

Consider a single-input linear discrete system described by

$$x(k+1) = Ax(k) + bu(k) \tag{3}$$

where $x(k) \in R^n$, $u(k) \in R$. The problem considered in this paper is to obtain the inverse and the direct solutions of the Lyapunov equation

$$AXA^t - X = -bb^t \tag{4}$$

assuming the system matrices A and b are in controllable canonical form [29].

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} \tag{5}$$

$$b_c = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^t \tag{6}$$

The problem for obtaining the inverse solution of the Lyapunov equation in (4) is stated as follows: Given the positive definite and symmetric matrix X find the system matrix A in the form of (5).The problem for obtaining the direct solution of the Lyapunov equation in (4) is stated as follows: Given the system matrices in (5), (6) find the positive definite and symmetric matrix X which satisfies (4). If the controllable-system matrix is not given in this form, it is easy to find a similarity transformation matrix that transform the given matrix to the controllable canonical form.

Consider now the basic statements of the proposed method. Let $f(z) = z^n + \sum_{i=1}^{n} \alpha_i z^{n-i}$ be a nth degree polynomial with real coefficients. A sequence of polynomials $f(z) = F_n(z), F_{n-1}(z), F_{n-2}(z), \ldots$ of degree, $n, n-1, n-2, \ldots$ is defined via the stability table presented in [1], [16]. Let

$$F_j(z) = \sum_{i=0}^{j} \alpha_{ij} z^{j-i},\ \alpha_{0j} = 1 \tag{7}$$

with the table

$$\left.\begin{array}{cccccc} 1 & \alpha_{1_n} = \alpha_1 & \alpha_{2_n} = \alpha_2 & \cdots & . & \alpha_{nn} = \alpha_n \\ \alpha_{nn} & \alpha_{n-1,n} & \alpha_{n-2,n} & \cdots & . & 1 \\ 1 & \alpha_{1,n-1} & \alpha_{2,n-1} & \cdots & \alpha_{n-1,n-1} & \\ \alpha_{n-1,n-1} & \alpha_{n-2,n-1} & \alpha_{n-3,n-1} & \cdots & 1 & \\ \vdots & & & & & \end{array}\right\} \tag{8}$$

The entries of any odd numbered row of the table are calculated from the entries of the previous two rows and given by the formula

$$\alpha_{i,j-1} = \frac{\alpha_{i,j} - \alpha_{jj}\alpha_{j-1,j}}{1 - |\alpha_{jj}|^2} \tag{9}$$

and equivalently to

$$F_{j-1}(z) = \frac{F_j(z) - \alpha_{jj} z^n F_j(z^{-1})}{z(1 - |\alpha_{jj}|^2)} \tag{10}$$

Define

$$\Delta_j = \alpha_{jj}, j = 1, 2, \cdots, n$$

It should be noted that the polynomials $F_j(z)$ can be obtained recursively using the Δ_j's. After manipulations it follows from (10) that

$$F_j(z) = zF_{j-1}(z) + \Delta_j z^{j-1} F_{j-1}(z^{-1}), F_0(z) = 1 \tag{11}$$

or equivalently the coefficients α_{ij} can be expressed as

$$\alpha_{ij} = \alpha_{i,j-1} + \Delta_j \alpha_{j-i,j-1}, 1 \le i \le j-1, \alpha_{0j} = 1, \alpha_{jj} = \Delta_j \tag{12}$$

Using the relation between the coefficients α_i of $f(z)$ and the quantities Δ_j's the Mansour form $\{\Sigma, b_\Sigma\}$ of the system has been defined and its similarity to a companion matrix has been established. The Mansour form $\{\Sigma, b_\Sigma\}$ of the system is represented by the matrices in (13), (14)

$$\Sigma = \begin{bmatrix} -\Delta_n\Delta_{n-1} & 1-\Delta_{n-1}^2 & 0 & \cdots & 0 \\ -\Delta_n\Delta_{n-2} & -\Delta_{n-1}\Delta_{n-2} & 1-\Delta_{n-2}^2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ -\Delta_n\Delta_1 & -\Delta_{n-1}\Delta_1 & \cdots & -\Delta_2\Delta_1 & 1-\Delta_1^2 \\ -\Delta_n & -\Delta_{n-1} & \cdots & -\Delta_2 & -\Delta_1 \end{bmatrix} \tag{13}$$

$$b_\Sigma = \begin{bmatrix} \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_1 & 1 \end{bmatrix} \tag{14}$$

The following theorem gives the similarity relation between the Mansour matrix and the matrix in companion form.

Theorem 1 [1]: Let A be the companion matrix associated with $f(z)$ as defined in (5); with the Δ_j as defined above, let the Mansour matrix associated with this system is given by (13) and with the α_{ij} as defined above, let

$$T = \begin{bmatrix} 1 & \alpha_{1,n-1} & \alpha_{2,n-1} & \cdots & \alpha_{n-1,n-1} \\ 0 & 1 & \alpha_{1,n-2} & \cdots & \alpha_{n-2,n-2} \\ 0 & 0 & 1 & \cdots & \alpha_{n-3,n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & \alpha_{11} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (15)$$

Then

$$TAT^{-1} = \Sigma, b_\Sigma = Tb_c \quad (16)$$

The necessary and sufficient conditions for the eigenvalues of the matrix A to be inside the unit disc (stability condition) is

$$|\Delta_j| < 1, \quad j = 1, 2, \cdots, n \quad (17)$$

where Δ_j are the appropriate entries of the stability table. The controllability Gramian P of a discrete system in Mansour form has some interesting properties which are summarized in the following lemma.

Lemma 1: The solution P of the Lyapunov equation

$$\Sigma P \Sigma^t - P = -b_\Sigma b_\Sigma^t \quad (18)$$

where $\{\Sigma, b_\Sigma\}$ is in Mansour form can be given by the diagonal matrix

$$P = \text{diag}\{p_1, p_2, \cdots, p_n\} \quad (19)$$

with

$$p_1 = (1 - \Delta_n^2)^{-1} \quad (20)$$

$$p_2 = (1 - \Delta_n^2)^{-1}(1 - \Delta_{n-1}^2)^{-1} \quad (21)$$

$$\cdots$$

$$p_n = \prod_{j=0}^{n-1} (1 - \Delta_{n-j}^2)^{-1} \quad (22)$$

Proof: The proof of this lemma is consequence of the discussions in [1].

In this paper the LDL^t decomposition is an important method. Let X be a positive-definite symmetric matrix. Then X has unique decomposition $X = LDL^t$ where:

- L is unit lower triangular matrix,
- D is diagonal matrix with positive diagonal entries.

This decomposition is discussed in detail in [9].

III. THE INVERSE AND DIRECT SOLUTIONS OF THE DISCRETE TIME LYAPUNOV EQUATION

The inverse and the direct solutions of the Lyapunov equation proposed in this section are based on the properties of the Mansour form summarized in the previous section.

A. THE INVERSE SOLUTION OF THE LYAPUNOV EQUATION

Consider a single-input discrete system $\{A_c, b_c\}$ in controllable canonical form and the positive definite and symmetric matrix X_c which is the solution of the Lyapunov equation

$$A_c X_c A_c^t - X_c = -b_c b_c^t \quad (23)$$

The problem to find the inverse solution of the Lyapunov equation in (23) was also solved in [39]. In this section it will be solved using the facts (presented in lemma 1) that the controllability Gramian in Mansour form is diagonal and that these diagonal elements are related to the $\Delta_j, j = 1, 2, \cdots, n$ with simple expressions.

Consider the LDL^t decomposition [9] of X_c i.e

$$X_c = LPL^t \quad (24)$$

where L is a lower triangular matrix with 1's in the diagonal and P is a diagonal matrix with positive entries. Using (24) one obtains the following equation from (23)

$$\Sigma P \Sigma^t - P = -b_\Sigma b_\Sigma^t \quad (25)$$

It can be easily seen that L^{-1} corresponds to the transformation matrix in lemma 1.

$$\Sigma = L^{-1} A_c L \quad (26)$$

$$b_\Sigma = L^{-1} b_c \quad (27)$$

The matrix P is diagonal and represents the controllability Gramian in Mansour form given in lemma 1 and therefore, the diagonal elements of the matrix P are related to the quantities $\Delta_j, j = 1, 2, \cdots, n$. This implies that the $\Delta_j, j = 1, 2, \cdots, n$ can be computed from the diagonal matrix P obtained from the the LDL^t decomposition of the matrix X_c. Further, the entries of the transformation matrix T are also functions of $\Delta_j, j = 1, 2, \cdots, n-1$. The last column of the matrix T is given by $[\Delta_{n-1}\Delta_{n-2} \cdots \Delta_2\Delta_1 1]^t$. Therefore the $\Delta_j, j = 1, 2, \cdots, n-1$ can be also obtained from this column of the matrix T. Using (20) one obtains the Δ_n and obviously there are two solutions for Δ_n. Consequently this leads to two solutions of this problem. Once the $\Delta_j, j = 1, 2, \cdots, n$ are known, the system representation $\{\Sigma, b_\Sigma\}$ can be obtained directly using lemma 1.

These observations lead to the following algorithm:

1) THE ALGORITHM FOR THE INVERSE SOLUTION OF THE LYAPUNOV EQUATION

Given the positive definite and symmetric matrix X_c which is the solution of the Lyapunov equation in (23) and assuming the single-input discrete system $\{A_c, b_c\}$ in controllable canonical form, the inverse solution of the Lyapunov equation can be obtained as follows:

1) Compute the lower triangular matrix L with 1's in the diagonal using LDL^t decomposition [9] so that

$$X_c = LPL^t \quad (28)$$

$$P = \text{diag}\{p_1, p_2, \cdots, p_n\} \quad (29)$$

2) Find Δ_n from

$$p_1 = (1 - \Delta_n^2)^{-1} \tag{30}$$

This will lead to two solutions for Δ_n.

3) Compute the $\Delta_j, j = 1, 2, \ldots, (n-1)$ using the following

$$p_2 = (1 - \Delta_n^2)^{-1}(1 - \Delta_{n-1}^2)^{-1} \tag{31}$$

$$\ldots$$

$$p_n = \prod_{j=0}^{n-1}(1 - \Delta_{n-j}^2)^{-1} \tag{32}$$

These equations lead to a positive and negative solution for Δ_j. The sign of Δ_j is decided from the last column of matrix T as discussed before.

4) Compute the system matrix representation $\{A_c, b_c\}$ in controllable form using

$$A_c = L\Sigma L^{-1} \tag{33}$$

$$b_c = Lb_\Sigma = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^t \tag{34}$$

Since there are two solutions for Δ_n, this will also lead to two solutions for $\{A_c, b_c\}$.

It should be mentioned that for the inverse solution of the discrete Lyapunov equation few results have been presented in the literature [39]. The method presented there involves the solution of linear systems of equations. The method for the inverse solution of the Lyapunov equation proposed here is novel and based on the relation between the discrete Lyapunov equation and the stability table and can be obtained using LDL^t decomposition.

Remarks:

1. It can be easily seen that $\Delta_j, j = 1, 2, \ldots, (n-1)$ can be also obtained from the last column of the matrix T. However using the matrix P instead of matrix T is numerically more accurate.

2. It can be easily seen that the two systems $\{\Sigma_1, b_\Sigma\}$ and $\{\Sigma_2, b_\Sigma\}$ with the same $\Delta_j, j = 1, 2, \ldots, (n-1)$ and (Δ_n for $\Sigma_1) = -(\Delta_n$ for Σ_2) are related by

$$\Sigma_1 = \Sigma_2 \mathrm{diag}[-1, 1, \cdots, 1] \tag{35}$$

and

$$\Sigma_1 P \Sigma_1^t - P = \Sigma_2 P \Sigma_2^t - P = -b_\Sigma b_\Sigma^t \tag{36}$$

B. THE DIRECT SOLUTION OF THE LYAPUNOV EQUATION

Consider a single-input discrete system $\{A_c, b_c\}$ in controllable canonical form. Our aim is to obtain the matrix X_c which is the solution of the Lyapunov equation

$$A_c X_c A_c^t - X_c = -b_c b_c^t \tag{37}$$

The solution of the equation is a symmetric matrix withe real entries if and only if the system matrix A has no reciprocal eigenvalues,i.e. $\lambda_i\lambda_j \neq 0,\ i, j \in 1, 2, \cdots, n$. Also the solution is unique and has the so-called Toeplitz symmetric structure. [52], [53].

The system in controllable canonical form $\{A_c, b_c\}$ can be transformed to the Mansour form $\{\Sigma, b_\Sigma\}$ in (13), (14) using the transformation matrix T in (15). Then, the matrix X_c can be expressed as

$$X_c = T^{-1} P T^{-t} \tag{38}$$

where T and P are given in (15), (19)-(22) respectively. The entries of the matrix T and the entries of the controllability Gramian matrix P of a system in Mansour form can be expressed as functions of the Δ_j's of the stability table. This implies that the matrix X_c can be computed directly from the $\Delta_j, j = 1, 2, \cdots, n$. These observations lead to the following algorithm:

1) THE ALGORITHM FOR THE DIRECT SOLUTION OF THE LYAPUNOV EQUATION

Given a single-input discrete system $\{A_c, b_c\}$ in controllable canonical form, the symmetric matrix X_c which is the solution of the Lyapunov equation in (4) can be obtained as follows:

1) Find the quantities $\Delta_j, j = 1, 2, \cdots, n$ using the stability table in [1], [15].

2) Form the matrices T and P using eqs. (15), (19)-(22).

3) Compute the solution of the Lyapunov equation for the discrete system in controllable canonical form as

$$X_c = T^{-1} P T^{-t}$$

Clearly, the matrix P is positive definite if the matrix A is stable.

As mentioned before some methods for the direct solution of the Lyapunov equation with the system matrix in general form are based on the Kronecker product while others require the solution of a linear system of equations. The solution of the discrete Lyapunov equation with the system matrix in companion form has been studied in [4] and [5]. In [4] the solution involves the solution of a linear system of equations and the computational cost requires in general n^3 flops. In [5] the solution of the discrete Lyapunov equation is the inverse of the Schur-Cohn matrix associated with the characteristic polynomial of the system matrix. In the method proposed in this paper the solution of the Lyapunov equation is the product of three matrices, a diagonal P and two other matrices T and T^{-t} which are triangular with 1's in the diagonal. The solution involves the inversion of a triangular matrix and the computational cost for that requires n^2 flops. The proposed method requires obtaining the matrices T and P from the entries of the stability table and inverting the upper triangular matrix T with 1's in the diagonal which is simpler than inversion of general and/or positive definite matrices.

The proposed algorithms will be illustrated with two examples in the next section.

IV. EXAMPLES

Example 1: In this example the inverse solution of the Lyapunov equation is studied. Consider the fourth-order system in controllable canonical form described by the state equation

(3) and the solution of the Lyapunov equation is given by

$$X_c = \begin{bmatrix} 8 & -7 & 6 & -5 \\ -7 & 8 & -7 & 6 \\ 6 & -7 & 8 & -7 \\ -5 & 6 & -7 & 8 \end{bmatrix}$$

The LDL^t decomposition for the above matrix X_c yields

$$L = \begin{bmatrix} 1 & -0.9284 & 0.8667 & -0.6250 \\ 0 & 1 & -0.9333 & 0.75 \\ 0 & 0 & 1 & -0.875 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1.8574 & 0 & 0 & 0 \\ 0 & 1.8669 & 0 & 0 \\ 0 & 0 & 1.875 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

The transformation matrix T is given by $T = L^{-1}$

$$T = \begin{bmatrix} 1 & 0.9284 & -0.0002 & -0.0715 \\ 0 & 1 & 0.9333 & 0.0666 \\ 0 & 0 & 1 & 0.875 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The quantities $\Delta_j, j = 1, 2, 3, 4$ are given by

$$\Delta_1 = 0.875,\ \Delta_2 = 0.065727,\ \Delta_3 = -0.0713348,$$
$$\Delta_4 = \pm 0.6794$$

The two systems in the Mansour form are given by

$$\Sigma_1 = \begin{bmatrix} 0.048466 & 0.9949 & 0 & 0 \\ -0.044656 & 0.0047 & 0.99568 & 0 \\ -0.594493 & 0.062418 & -0.05751 & 0.2344 \\ -0.6794 & 0.071335 & -0.065727 & -0.875 \end{bmatrix}$$
$$\Sigma_2 = \begin{bmatrix} -0.048466 & 0.9949 & 0 & 0 \\ 0.044656 & 0.0047 & 0.99568 & 0 \\ 0.594493 & 0.062418 & -0.05751 & 0.2344 \\ 0.6794 & 0.071335 & -0.065727 & -0.875 \end{bmatrix}$$

The system matrices in the companion form are given by

$$A_{c1} = T^{-1}\Sigma_1 T$$
$$A_{c2} = T^{-1}\Sigma_2 T$$
$$A_{c1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.6794 & -0.5594 & 0.0010 & -0.8794 \end{bmatrix}$$
$$A_{c2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.6794 & 0.7021 & 0.0007 & -0.9763 \end{bmatrix}$$

Example 2: In this example the direct solution of the Lyapunov equation is studied. Consider the third-order system in controllable canonical form described by the state equation (3) where

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -0.625 & -0.75 \end{bmatrix} \text{ and } b_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The quantities $\Delta_j, j = 1, 2, 3$ associated with the Jury's stability table are given as

$$\Delta_1 = 0.4375,\ \Delta_2 = 0.3333,\ \Delta_3 = 0.5$$

The transformation matrix T is given by

$$T = \begin{bmatrix} 1 & \Delta_1 + \Delta_1\Delta_2 & \Delta_2 \\ 0 & 1 & \Delta_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5833 & 0.3333 \\ 0 & 1 & 0.4375 \\ 0 & 0 & 1 \end{bmatrix}$$

and the matrix P which is the controllability Gramian in the Mansour form is given by

$$P = \mathrm{diag}\{p_1, p_2, , p_3\}$$

with

$$p_1 = (1 - \Delta_3^2)^{-1}$$
$$p_2 = (1 - \Delta_3^2)^{-1}(1 - \Delta_2^2)^{-1}$$
$$p_3 = (1 - \Delta_3^2)^{-1}(1 - \Delta_2^2)^{-1}(1 - \Delta_1^2)^{-1}$$

and

$$P = \begin{bmatrix} 1.3335 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.8551 \end{bmatrix}$$

Finally, the matrix X_c which is the solution of the Lyapunov equation with the system matrices in controllable form is given by

$$X_c = \begin{bmatrix} 1.8551 & -0.8116 & -0.1449 \\ -0.8116 & 1.8551 & -0.8116 \\ -0.1449 & -0.8116 & 1.8551 \end{bmatrix}$$

V. CONCLUSION

In this paper, simple methods for obtaining the inverse and the direct solutions of the Lyapunov equation for discrete systems with the system matrix in companion form have been proposed. The inverse solution of the Lyapunov equation is based on the relationship between the Lyapunov equation and the entries of the stability table can be obtained using LDL^t decomposition. It was shown that the proposed method for the direct solution of the Lyapunov equation involved only the evaluation of the entries of the stability table and did not involve solution of a linear system of equations. The proposed methods offer a computational efficient alternative to existing methods for system matrices in a companion form.

REFERENCES

[1] B. D. O. Anderson, E. I. Jury, and I. Mansour, "Schwarz matrix properties for continuous and discrete time systems," *Int. J. Control*, vol. 23, no. 1, pp. 1–16, Jan. 1976.

[2] A. D. Barbour and S. Utev, "Solving the stein equation in compound Poisson approximation," *Adv. Appl. Probab.*, vol. 30, no. 2, pp. 449–475, Jun. 1998.

[3] S. Barnett and C. Storey, *Matrix Methods in Stability Theory*. Nashville, TN, USA: Thomas Nelson and Sons, 1970.

[4] A. Betser, N. Cohen, and E. Zeheb, "On solving the Lyapunov and stein equations for a companion matrix," *Syst. Control Lett.*, vol. 25, no. 3, pp. 211–218, Jun. 1995.

[5] R. R. Bitmead and H. Weiss, "On the solution of the discrete-time Lyapunov matrix equation in controllable canonical form," *IEEE Trans. Autom. Control*, vol. AC-24, no. 3, pp. 481–482, Jun. 1979.

[6] R. R. Bitmead, "Explicit solutions of the discrete-time Lyapunov matrix equation and Kalman–Yakubovich equations," *IEEE Trans. Autom. Control*, vol. AC-26, no. 6, pp. 1291–1294, Dec. 1981.

[7] W.-J. Chang, "Fuzzy controller design via the inverse solution of Lyapunov equations," *J. Dyn. Syst., Meas., Control*, vol. 125, no. 1, pp. 42–47, Mar. 2003.

[8] Z. Gajic and M. T. J. Qureshi, *Lyapunov Matrix Equation in System Stability and Control*. Mineola, NY, USA: Dover, 2008.

[9] G. H. Golub and C. F. Van Loan, *Matrix Computations*. Baltimore, MD, USA: John Hopkins Univ. Press, 1983.

[10] M. Hajarian, "Finite algorithms for solving the coupled Sylvester-conjugate matrix equations over reflexive and Hermitian reflexive matrices," *Int. J. Syst. Sci.*, vol. 46, no. 3, pp. 488–502, Feb. 2015.

[11] H. Horisberger and P. R. Belanger, "Solution of the optimal constant output feedback problem by conjugate gradients," *IEEE Trans. Autom. Control*, vol. AC-19, no. 4, pp. 434–435, Aug. 1974.

[12] G.-D. Hu and Q. Zhu, "Bounds of modulus of eigenvalues based on Stein equation," *Kybernetika*, vol. 46, no. 4, pp. 655–664, 2010.

[13] K. Jbilou, "Low rank approximate solutions to large Sylvester matrix equations," *Appl. Math. Comput.*, vol. 177, no. 1, pp. 365–376, Jun. 2006.

[14] I. Jonsson and B. Kågström, "Recursive blocked algorithms for solving triangular systems—Part II: Two-sided and generalized Sylvester and Lyapunov matrix equations," *ACM Trans. Math. Softw.*, vol. 28, no. 4, pp. 416–435, Dec. 2002.

[15] E. I. Jury, *Inners and Stability of Dynamics Systems*. New York, NY, USA: Wiley, 1974.

[16] E. I. Jury, *Theory and Application of the Z-Transform Method*. New York, NY, USA: Wiley, 1964.

[17] E. Kaszkurewicz and A. Bhaya, *Matrix Diagonal Stability in Systems and Computation*. New York, NY, USA: Springer, 2012.

[18] G. Kitagawa, "An algorithm for solving the matrix equation $X = FXF^T + S$," *Int. J. Control*, vol. 25, no. 5, pp. 745–753, 1977.

[19] D. Kleinman, "On an iterative technique for Riccati equation computations," *IEEE Trans. Autom. Control*, vol. AC-13, no. 1, pp. 114–115, Feb. 1968.

[20] A. Klein and P. Spreij, "On Stein's equation, Vandermonde matrices and Fisher's information matrix of time series processes—Part I: The autoregressive moving average process," *Linear Algebra Appl.*, vol. 329, nos. 1–3, pp. 9–47, May 2001.

[21] A. Klein and P. Spreij, "On the solution of Stein's equation and Fisher's information matrix of an ARMAX process," *Linear Algebra Appl.*, vol. 396, pp. 1–34, Feb. 2005.

[22] H. Kong, B. Zhou, and M.-R. Zhang, "A stein equation approach for solutions to the diophantine equations," in *Proc. Chin. Control Decis. Conf.*, May 2010, pp. 3024–3028.

[23] D. Kressner, "Block variants of Hammarling's method for solving Lyapunov equations," *ACM Trans. Math. Softw.*, vol. 34, no. 1, pp. 1–15, Jan. 2008.

[24] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*. Oxford, U.K.: Oxford Univ. Press, 1995.

[25] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed. New York, NY, USA: Academic Press, 1985.

[26] L. Lerer and A. C. M. Ran, "A new inertia theorem for stein equations, inertia of invertible Hermitian block Toeplitz matrices and matrix orthogonal polynomials," *Integral Equ. Operator Theory*, vol. 47, no. 3, pp. 339–360, Nov. 2003.

[27] W. Levine and M. Athans, "On the determination of the optimal constant output feedback gains for linear multivariable systems," *IEEE Trans. Autom. Control*, vol. AC-15, no. 1, pp. 44–48, Feb. 1970.

[28] B. P. Molinari, "Algebraic solution of matrix linear equations in control theory," *Proc. Inst. Elect. Eng.*, vol. 116, no. 10, pp. 1748–1754, Oct. 1969.

[29] K. Ogata, *Discrete-Time Control Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1987.

[30] P. C. Parks, "Lyapunov and the Schur–Cohn criterion," *IEEE Trans. Autom. Control*, vol. 9, p. 121, 1964.

[31] B. Porat and M. Morf, "Efficient solution of Lyapunov equation for matrix autoregressive models and its application to the inverse Levinson problem," in *Proc. 20th IEEE Conf. Decis. Control Including Symp. Adapt. Processes*, Dec. 1981, pp. 1070–1074.

[32] M. A. Ramadan, M. A. A. Naby, and A. M. E. Bayoumi, "On the explicit solutions of forms of the Sylvester and the Yakubovich matrix equations," *Math. Comput. Model.*, vol. 50, nos. 9–10, pp. 1400–1408, 2009.

[33] F. C. Silva and R. Simoes, "On the Lyapunov and Stein equations I," *Linear Algebra Appl.*, vol. 420, nos. 2–3, pp. 329–338, 2007.

[34] F. C. Silva and R. Simoes, "On the Lyapunov and stein equations, II," *Linear Algebra Appl.*, vol. 426, nos. 2–3, pp. 305–311, Oct. 2007.

[35] V. Simoncini, "Computational methods for linear matrix equations," *SIAM Rev.*, vol. 58, no. 3, pp. 377–441, Jan. 2016.

[36] R. E. Skelton, *Dynamic Systems Control: Linear Systems Analysis and Synthesis*. New York, NY, USA: Wiley, 1988.

[37] R. E. Skelton, T. Iwasaki, and D. E. Grigoriadis, *A Unified Algebraic Approach to Control Design*. Boca Raton, FL, USA: CRC Press, 1997.

[38] V. Sreeram and P. Agathoklis, "Solution of Lyapunov equation with system matrix in companion form," *IEE Proc. D Control Theory Appl.*, vol. 138, no. 6, pp. 529–534, Nov. 1991.

[39] V. Sreeram and P. Agathoklis, "On the theory of state-covariance assignment for single-input linear discrete systems," *IEEE Trans. Autom. Control*, vol. 38, no. 7, pp. 1111–1115, Jul. 1993.

[40] V. Sreeram, P. Agathoklis, and M. Mansour, "The generation of discrete-time Q-Markov covers via the inverse solution of Lyapunov equation," in *Proc. 30th IEEE Conf. Decis. Control*, Dec. 1991, pp. 1613–1618.

[41] B. Dickinson, "Analysis of the Lyapunov equation using generalized positive real matrices," *IEEE Trans. Autom. Control*, vol. AC-25, no. 3, pp. 560–563, Jun. 1980.

[42] A. G. Wu, G. R. Duan, and Y. Xue, "Kronecker maps and Sylvester-polynomial matrix equations," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 905–910, May 2007.

[43] A. G. Wu, G. R. Duan, and B. Zhou, "Solution to generalized Sylvester matrix equations," *IEEE Trans. Autom. Control*, vol. 53, no. 3, pp. 811–815, Apr. 2008.

[44] A. G. Wu, Y. M. Fu, and G. R. Duan, "On solutions of matrix equations $V - -AVF = BWandV - -A\bar{V}F = BW$," *Math. Comput. Model.*, vol. 47, nos. 11–12, pp. 1181–1197, 2008.

[45] A.-G. Wu, G. Feng, G.-R. Duan, and W.-J. Wu, "Closed-form solutions to Sylvester-conjugate matrix equations," *Comput. Math. With Appl.*, vol. 60, no. 1, pp. 95–111, Jul. 2010.

[46] A. G. Wu, "Explicit solutions to the matrix equation $EXF - AX = C$, *IET Control Theory Appl.*, vol. 7, no. 12, pp. 1589–1598, 2013.

[47] C.-S. Xiao, Z.-M. Feng, and X.-M. Shan, "On the solution of the continuous-time Lyapunov matrix equation in two canonical forms," *IEE Proc. D Control Theory Appl.*, vol. 139, no. 3, pp. 286–290, May 1992.

[48] N. J. Young, "Formulae for the solution of Lyapunov matrix equations," *Int. J. Control*, vol. 31, no. 1, pp. 159–179, Jan. 1980.

[49] B. Zhou, G.-R. Duan, and Z.-Y. Li, "Gradient based iterative algorithm for solving coupled matrix equations," *Syst. Control Lett.*, vol. 58, no. 5, pp. 327–333, May 2009.

[50] P. Suchomski, "Structural properties of solutions of continuous-time and discrete-time matrix Lyapunov equations in controllable form," *IEE Proc. Control Theory Appl.*, vol. 146, no. 5, pp. 477–483, Sep. 1999.

[51] C. S. Berger, "A numerical solution of the matrix equalion $P = \Phi P \Phi^t + S$," *IEEE Trans. Autom. Control*, vol. AC-15, no. 4, pp. 381–382, Aug. 1971.

[52] E. H. Petkov, N. D. Christow, and M. M. Konstantinov, *Computational Methods for Linear Control Svstems*. New York, NY, USA: Prentice-Hall, 1991.

[53] A. Y. Barraud, "A numerical algorithm to solve $A^T XA - -X = Q$," *IEEE Trans. Autom. Control*, vol. AC-22, no. 5, pp. 883–885, Dec. 1977.

[54] V. Sreeram, "Recursive technique for computation of Grammians," *IEE Proc. D-Control Theory Appl.*, vol. 140, no. 3, pp. 160–166, 1993.

[55] H.-C. Kim and C.-H. Choi, "Closed-form solution of the continuous-time Lyapunov matrix equation," *IEE Proc. Control Theory Appl.*, vol. 141, no. 5, pp. 350–356, Sep. 1994.

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