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Achieving the Sum Rate Capacity of MIMO Downlink Communications With Better Fairness

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ABSTRACT Transmission schemes taking both sum rate and fairness into account for dirty paper coding (DPC) based MIMO downlink communications are investigated in this paper. In contrast to existing works which have mostly focused on maximizing the sum rate, we first investigate the problem of finding the maximal sum rate achieved by DPC when the qualitative notions of fairness such as max-min fairness and proportional fairness are employed. This corresponds to a nonconvex problem and cannot be solved by usual weighted sum rate techniques. Several efficient methods for finding the optimal solutions are presented in this paper when the order of users is adjustable during DPC encoding. Simulation results show surprisingly and impact greatly on the design of practical systems that it is often possible to achieve the sum rate capacity with absolute fairness, i.e., an equal rate for each user, when multiple encoding orders of users are used during transmission. When sum rate capacity and absolute fairness cannot be achieved at the same time, the optimal tradeoff between sum rate and fairness is also provided for a general class of quantitative fairness measures.

INDEX TERMS Dirty paper coding, fairness, max-min fairness, MIMO downlink communications, proportional fairness.

I. INTRODUCTION

In multiple-input-multiple-output (MIMO) downlink communications (also known as *broadcast channels* (BC) [3], [10]), the base station uses multiple transmit antennas to send pieces of statistically independent information simultaneously to a group of multi-antenna, noncooperative users. Assuming that each user knows perfectly the channel state information of his/her incoming channel and that the base station has the complete knowledge of channel state information of all users, the dirty paper coding (DPC) [11] is the only known scheme that can achieve the maximal sum rate of users (also known as the *sum rate capacity*) of this MIMO BC [4], [10]. Several practical implementations of DPC have been proposed in recent years [12]–[16]. However, finding the numerical value of sum rate capacity and the corresponding optimal input distribution of transmitted signals of a given MIMO BC are not straightforward. Such problems are generally nonconvex and call for the use of a duality [3] between uplink and downlink communications. Using this duality, the problem of sum rate maximization for DPC-based MIMO BC can be converted to another problem of sum rate maximization for MIMO uplink communications. The new problem then has a convex objective function and convex constraints, hence it can be solved using standard convex optimization techniques. Faster, waterfilling-based, iterative algorithms for solving the dual problem are also available [4]. Once the optimal solutions are found, they can be converted, using the same duality, back to the DPC-based MIMO BC to yield the optimal input distributions of transmitted signals in the original problem and give the numerical value of sum rate capacity.

Many existing works on DPC-based MIMO BC [3]–[6] have focused on maximizing the sum rate of all users and have not fully investigated how the rates are distributed among them, i.e., the issue of fairness. For non-capacity achieving and suboptimal coding strategies, some works are available from literature that do take fairness into account. For instance,

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TABLE 1. Comparison of designs of mimo downlink communication systems with fairness considerations.

Type of BC: Transmission Schemes	Fairness Consideration	Optimality in Capacity Achieving	References
MISO-BC: zero forcing	proportional fairness	suboptimal	[1]
MISO-BC: ZFDPC	max-min fairness proportional fairness harmonic fairness	suboptimal	[2]
MIMO-BC: DPC	max sum rate, no fairness	optimal	[3], [4], [5], [6]
MIMO-BC: PSK	max-min fairness	suboptimal	[7]
MIMO-BC: SZFDPC	max-min fairness proportional fairness	suboptimal	[8], [9]
MIMO-BC: SZFDPC	quantitative fairness measures	suboptimal	[9]
MIMO-BC: DPC	max-min fairness proportional fairness quantitative fairness measures	optimal	this work

Salem *et al.* [7] considered the max-min fairness when PSK signaling and zero-forcing precoding are used for MIMO BC. Tran *et al.* [2] investigated the designs of zero-forcing DPC (ZFDPC) for the multiple-input-single-output (MISO) BC when the notions of fairness such as max-min fairness, proportional fairness and harmonic fairness are employed. Bayesteh *et al.* [17] considered a random beamforming based MISO BC and showed that a certain scheduling of single-antenna users can achieve maximal average sum rate and maximum fairness (defined in terms of network scheduling) at the same time when the number of users goes to infinity. Wang *et al.* [1] also considered the MISO BC but used zero forcing beamforming. They showed that the problem of user selection subject to the proportional fairness constraint can be reduced to the maximization of a weighted sum rate. In summary, the comparison of designs of MIMO BC with fairness considerations is summarized in Table 1.

In this paper, we will focus on the use of optimal coding strategy, i.e., DPC for the best possible sum rate and system performance. To take fairness into account, we will first adopt the notions of max-min fairness and proportional fairness, which have been widely used in many areas of communications [1], [2], [9], [18]–[23]. For a given order of users for DPC encoding, it will be seen that the problems of sum rate maximization for DPC-based MIMO BC subject to either max-min fairness or proportional fairness are generally nonconvex and are extremely difficult for solving even with the uplink-downlink duality. The commonly used technique of weighted sum rate would fail in the present problem either, since 1) it is completely unknown which set of weights should be used to yield max-min fairness, proportional fairness, etc. and there are infinitely many choices of weights to be considered, 2) the weights might not even exist as the achievable rate regions for max-min fairness and proportional fairness might not be convex, and 3) the rate-tuples corresponding to max-min fairness, proportional fairness, etc. could be some interior points of the achievable rate region while the weighted sum rate approach focuses only on boundary points.

To overcome the above difficulties, rather than considering only a fixed encoding order for DPC, we will include all possible orders of DPC encoding into consideration. It will be seen in Section III-A that our new approach not only simplifies the original problems by making them become convex but also yields a much larger sum rate than that from a fixed encoding order. A low-complexity algorithm for reducing the number of required encoding orders is also provided in Section III-B. Furthermore, we will apply this method to find the tradeoff between sum rate and fairness for DPC-based MIMO BC, when the notion of fairness is replaced by the quantitative fairness measures such as the recent ℓ_1-norm based index [9], [23], Jain's fairness index [24], [25] or entropy-based index [26]. It will be seen from simulations in Section IV that our new approach of using multiple encoding orders for DPC can often achieve the sum rate capacity of MIMO BC with absolute fairness, i.e., an equal rate for every user. This is very surprising and is thought not possible before. Compared to the work by Bayesteh *et al.* [17], our new approach achieves not only the true maximum sum rate and exact maximum fairness with equal rate at the same time for MIMO BC but also in finite number of multi-antenna users. This also leads to a huge impact on the design of practical MIMO BC as now it is possible to achieve max-min fairness without any loss of sum rate capacity.

The major contributions of this paper include:

1) new methods (cf. Theorems 2 and 3) for finding the maximal sum rate of DPC-based MIMO BC subject to either max-min fairness or proportional fairness,
2) a new transmission scheme that uses multiple encoding orders of users for DPC and often achieves the sum rate capacity of MIMO BC with an absolutely fair distribution of rates among users,
3) an efficient, low-complexity algorithm (cf. Algorithm 1) that minimizes the number of required encoding orders,
4) a complete characterization on the optimal tradeoff between the sum rate and fairness (cf. Theorem 4) that gives system designers a total flexibility in selecting the operating sum rate and fairness for DPC-based MIMO downlink communications, when sum rate capacity and absolute fairness cannot be achieved at the same time, and

5) a surprising finding that is quite opposite to the well-known result of a negligible performance gap [10], [27] between DPC and its simplified version, the *successive zero forcing DPC* [2], [8], [28]. Our simulation results show that the latter suffers a huge performance loss and the performance gap is enormous, when the issue of fairness is taken into consideration.

The following notations will be used in this paper. Underlined lowercase letter $\underline{x}$ represents a vector, and uppercase letter A denotes a matrix of certain size. $A^\dagger$ (resp. $A^\top$) denotes the Hermitian transpose (resp. transpose) of matrix A. $|A|$ is the determinant of a square matrix A. 0 and I are the all-zero and identity matrices of proper sizes, respectively. By $\underline{0}$ we mean the all-zero vector, and $\underline{1}$ is the all-one vector of proper length. Matrix inequalities such as $\succeq$ and $\preceq$ are the partial orderings of positive semidefinite matrices. We will write $\underline{x} \sim \mathbb{C}\mathcal{N}\left(\underline{m}, \Omega\right)$ when $\underline{x}$ is a circularly symmetric complex Gaussian random vector with mean vector $\underline{m}$ and covariance matrix $\Omega \succeq 0$.

II. DPC-BASED MIMO DOWNLINK COMMUNICATIONS

Consider a MIMO BC, where a base station uses N transmit antennas to send pieces of statistically independent information simultaneously to a group of K users. Assuming m_k receive antennas for user k, where $k = 1, 2, \ldots, K$, the signal received by user k is a length-m_k vector given by

$$\underline{y}_k = H_k \underline{x} + \underline{w}_k, \tag{1}$$

where $H_k \in \mathbb{C}^{m_k \times N}$ is the channel matrix of user k with arbitrary statistics including the ones from millimeter wave (mmWave) communications. The vector $\underline{w}_k \sim \mathbb{C}\mathcal{N}\left(\underline{0}, I\right)$ is the additive noise. The transmitted signal vector $\underline{x} \in \mathbb{C}^N$ takes the following form

$$\underline{x} = \underline{x}_1 + \underline{x}_2 + \cdots + \underline{x}_K, \tag{2}$$

where $\underline{x}_k \sim \mathbb{C}\mathcal{N}\left(\underline{0}, Q_k\right)$ is the component signal vector containing the information intended only for user k. The covariance matrices Q_k are constrained by the sum power, i.e., we have $\sum_k \mathrm{tr}(Q_k) \leq P$ for some nonnegative power P.

For an encoding order of $1, 2, \ldots, K$, the DPC encoder first encodes the information of user 1 and produces the component signal $\underline{x}_1$. Then for $k = 2, \ldots, K$, the DPC encoder iteratively encodes the information of user k by taking into account the signals of preceding users such that the interferences caused by $\underline{x}_1, \ldots, \underline{x}_{k-1}$ can be properly eliminated at the receiver of user k. The reader is referred to [5], [6], [16], [29] for further understanding of DPC.

In a nutshell, the signal received by user k in DPC-based MIMO downlink communications is equivalent to

$$\hat{\underline{y}}_k = H_k \underline{x}_k + H_k \sum_{j>k} \underline{x}_j + \underline{w}_k. \tag{3}$$

Thus, the achievable rate (in bits per channel use) for user k using DPC is

$$R_k = \log_2 \frac{\left|I + H_k \sum_{j \geq k} Q_j H_k^\dagger\right|}{\left|I + H_k \sum_{j > k} Q_j H_k^\dagger\right|}. \tag{4}$$

Consequently, the maximal achievable sum rate of all users using DPC is

$$\begin{aligned} R_{\mathrm{ms}} := &\ \text{maximize } R_1 + \ldots + R_K \\ &\ \text{subject to } Q_k \succeq 0, \sum_k \mathrm{tr}(Q_k) \leq P \end{aligned} \tag{5}$$

which turns out to be the sum rate capacity [10], [30] of MIMO BC specified in (1). To solve the optimization problem (5), the common approach is to invoke the uplink-downlink duality [3] and convert the sum rate maximization problem for downlink communications to that for uplink communications. Specifically, it has been shown [3] that the following equality holds for the sum rate capacity R_{ms} of MIMO BC

$$\begin{aligned} R_{\mathrm{ms}} = &\ \text{maximize } \log_2 \left|I + \sum_k H_k^\dagger P_k H_k\right| \\ &\ \text{subject to } P_k \succeq 0, \sum_k \mathrm{tr}(P_k) \leq P \end{aligned} \tag{6}$$

Note that the right-hand side (RHS) of (6) is the maximal sum rate of uplink communications when the users simultaneously transmit signals to the base station with signal covariance matrices P_k and channel matrices $H_k^\dagger$ for $k = 1, \ldots, K$. A similar equality holds for the individual rates as well [3], i.e., we have

$$R_k^\star = \log_2 \frac{\left|I + H_k \sum_{j \geq k} Q_j^\star H_k^\dagger\right|}{\left|I + H_k \sum_{j > k} Q_j^\star H_k^\dagger\right|} = \log_2 \frac{\left|I + \sum_{i \leq k} H_i^\dagger P_i^\star H_i\right|}{\left|I + \sum_{i < k} H_i^\dagger P_i^\star H_i\right|} \tag{7}$$

where $Q_k^\star$ and $P_k^\star$ are optimal solutions to problems (5) and (6), respectively, and $R_k^\star$ is the rate of user k satisfying $\sum_k R_k^\star = R_{\mathrm{ms}}$. Note that the middle term of (7) is the rate of user k for the DPC-based MIMO BC when the encoding order $1, 2, \ldots, K$, is employed, and the RHS of (7) is the rate of user k for the MIMO uplink communication when a successive interference cancellation (SIC) decoder is used at base station with decoding ordering $K, K-1, \ldots, 1$. Since the objective function of (6) is concave in covariance matrices P_k and the constraints are affine, the problem (6) can be solved using standard convex optimization methods [31]. Faster algorithms that make use of iterative waterfilling for solving (6) are available in [4]. Once the optimal solutions $P_k^\star$ for (6) are obtained, they can be converted iteratively using (7) to yield the corresponding $Q_k^\star$'s for $k = K, K-1, \ldots, 2, 1$, see [3] for details.

III. THE NEW CHALLENGES: SUM RATE AND FAIRNESS

Besides sum rate maximization, other objectives for DPC-based MIMO BC such as max-min fairness, proportional fairness, etc. have not been investigated in the past. The max-min fairness [2], [23] aims to maximize the minimal normalized rate of all users, and the proportional fairness is derived from the Nash standard of comparison [21], [32] that a transfer of resources (i.e., transmit powers in our case) among users is favorable and fair if the sum of the percentage increases of each user's rate is positive. Specifically, the max-min fairness for DPC-based MIMO BC seeks optimal solutions to the following problem

$$\begin{aligned} &\text{maximize } \min_k \frac{1}{m_k} \log_2 \frac{\left|I + H_k \sum_{j \geq k} Q_j H_k^\dagger\right|}{\left|I + H_k \sum_{j > k} Q_j H_k^\dagger\right|} \\ &\text{subject to } Q_k \succeq 0, \sum_k \operatorname{tr}(Q_k) \leq P \end{aligned} \tag{8}$$

and the problem of proportional fairness can be formulated as [23]

$$\begin{aligned} &\text{maximize } \sum_k \log\left(\frac{1}{m_k} \log_2 \frac{\left|I + H_k \sum_{j \geq k} Q_j H_k^\dagger\right|}{\left|I + H_k \sum_{j > k} Q_j H_k^\dagger\right|}\right) \\ &\text{subject to } Q_k \succeq 0, \sum_k \operatorname{tr}(Q_k) \leq P \end{aligned} \tag{9}$$

Note that in both problems (8) and (9) each individual rate R_k is normalized by the corresponding number of receive antennas such that the fairness is ensured among all transmit beams [9].

Unfortunately, neither (8) nor (9) is convex, and solving these non-convex optimization problems can be highly challenging. One trial approach is to invoke the uplink-downlink duality and convert problems (8) and (9) to those for uplink communications. For instance, we could reformulate (8) as

$$\begin{aligned} &\text{maximize } \min_k \frac{1}{m_k} \log_2 \frac{\left|I + \sum_{i \leq k} H_i^\dagger P_i H_i\right|}{\left|I + \sum_{i < k} H_i^\dagger P_i H_i\right|} \\ &\text{subject to } P_k \succeq 0, \sum_k \operatorname{tr}(P_k) \leq P \end{aligned} \tag{10}$$

Unfortunately, the new problem remains as difficult as before and the sum power iterative waterfilling algorithms [4] seems powerless here.

A. THE PROPOSED APPROACH

To solve problems (8), (9) and many others in general, we propose to include all possible orders of users in consideration when performing the DPC encoding. To this end, let $\pi : \{1, \ldots, K\} \to \{1, \ldots, K\}$ be a permutation of users and let

$$R_{\pi(k)} = \log_2 \frac{\left|I + H_{\pi(k)} \sum_{j \geq k} Q_{\pi(j)} H_{\pi(k)}^\dagger\right|}{\left|I + H_{\pi(k)} \sum_{j > k} Q_{\pi(j)} H_{\pi(k)}^\dagger\right|}, \tag{11}$$

be the rate of user $\pi(k)$ when the DPC encoding order $\pi(1), \ldots, \pi(K)$ is employed. The achievable rate region of DPC is thus the convex hull of the union of rate vectors taken over all permutations π subject to the sum power constraint, i.e.,

$$\begin{aligned} \mathcal{C}_{\mathrm{DPC}} := \operatorname{Conv}\Bigg(\bigcup_\pi \Big\{&(R_1, \ldots, R_K) : \ R_k \text{ given by (11)}, \\ &Q_k \succeq 0, \sum_k \operatorname{tr}(Q_k) \leq P\Big\}\Bigg). \end{aligned} \tag{12}$$

It is known [3, Theorem 2] that the region $\mathcal{C}_{\mathrm{DPC}}$ is equal to the capacity region of dual MIMO uplink communications with sum power constraint P, when for each permutation π of users the order $\pi(K), \ldots, \pi(1)$ is used for SIC decoding. The result is reproduced below.

Theorem 1 (Vishwanath, Jindal, Goldsmith [3]):

$$\begin{aligned} \mathcal{C}_{\mathrm{DPC}} = \bigcup_{\{P_k\}} \Big\{&(R_1, \ldots, R_K) : \\ &\sum_{k \in \mathcal{S}} R_k \leq \log_2 \left|I + \sum_{k \in \mathcal{S}} H_k^\dagger P_k H_k\right| \\ &\text{for all } \mathcal{S} \subseteq \{1, \ldots, K\}\Big\}. \end{aligned} \tag{13}$$

■

The union in the RHS of (13) is taken over all possible covariance matrices $P_k \succeq 0$ subject to the sum power constraint $\sum_k \operatorname{tr}(P_k) \leq P$. In particular, for each choice of P_k's the region

$$\begin{aligned} \Big\{(R_1, \ldots, R_K) : \sum_{k \in \mathcal{S}} R_k \leq \log_2 \left|I + \sum_{k \in \mathcal{S}} H_k^\dagger P_k H_k\right| \\ \text{for all } \mathcal{S} \subseteq \{1, \ldots, K\}\Big\} \end{aligned} \tag{14}$$

is a K-dimensional polytope with $K!$ corner points on the face intersecting the hyperspace of total sum rate constraint, and each corner point corresponds to one of the possible $K!$ possible decoding orders of SIC decoder in the uplink communications. Best of all, it is known from [33] that the region $\mathcal{C}_{\mathrm{DPC}}$ is convex.

Armed with the above results, we can formulate the following new problem for max-min fairness that seeks a vector $(R_1, \ldots, R_K) \in \mathcal{C}_{\mathrm{DPC}}$ having the maximal normalized minimal rate

$$\begin{aligned} &\text{maximize } t \\ &\text{subject to } R_k \geq m_k t \\ &\qquad \sum_{k \in \mathcal{S}} R_k \leq \log_2 \left|I + \sum_{k \in \mathcal{S}} H_k^\dagger P_k H_k\right|, \\ &\qquad \text{for all } \mathcal{S} \subseteq \{1, \ldots, K\}, P_k \succeq 0, \sum_k \operatorname{tr}(P_k) \leq P, \end{aligned} \tag{15}$$

Since the objective function of (15) is affine, the second constraint is affine in the rate R_k and concave in the covariance matrix P_k, and since the remaining constraints are all affine, the problem (15) can be efficiently solved by standard convex optimization methods. We summarize the above approach in the theorem below.

Theorem 2: The maximal sum rate R_{mm} achieved by max-min fairness for a given DPC-based MIMO BC is

$$R_{\text{mm}} := \sum_k m_k t^\star = Mt^\star, \tag{16}$$

where $M := \sum_k m_k$ is the total number of receive antennas of all users and $t^\star$ is the optimal value of (15). Moreover, the optimal solutions of individual rates $R_k^\star$ to (15) occur when the normalized individual rates satisfy $R_k^\star / m_k = t^\star$ for all users $k = 1, \ldots, K$.

Proof: See Appendix A. ■

It should be noted that the optimal covariance matrices $P_k^\star$ obtained from (15) are not necessarily the optimal solutions to problems (8) and (10). Very often, they are much better solutions in the sense that the maximal sum rate R_{mm} obtained from (16) is much larger than those derived from the optimal solutions to problems (8) and (10). To see this, recall that the optimal rate vector $\underline{R}^\star = [R_1^\star, \ldots, R_K^\star]^\top = (m_1 t^\star, \ldots, m_K t^\star)$ of (15) is a vector within the region $\mathcal{C}_{\text{DPC}}$, which includes all possible permutations π of users in (12). Optimal solutions to (8) and (10), on the other hand, are obtained subject to a fixed encoding order $\pi(k) = k$ in (11) for DPC. Thus, the proposed approach not only solves the problem of sum rate maximization for max-min fairness but also yields a much larger optimal value.

Secondly, let the optimal individual rates derived from the max-min fairness be $R_k^\star$, which are given by the optimal solutions to problem (15). By Theorem 2 we have $R_k^\star = m_k t^\star$, and the individual rates $R_k^\star$ are linearly proportional to the numbers of receive antennas m_k for each user $k = 1, \ldots, K$. In other words, the user k, who has m_k receive antennas, is given m_k signal beams during the downlink communication. Each signal beam of each user has the same transmission rate in the max-min fairness, showing an absolutely fair manner. This motivates the following definition of absolute fairness.

Definition 1: Let R_k be the rate of user k in a MIMO downlink communication for $k = 1, \ldots, K$. Let m_k be the number of receive antennas of user k. We say the distribution of rates $\{R_k\}$ is absolutely fair if R_k / m_k is a constant for all k. ■

When the rates are distributed in the absolutely fair manner, the users who have the latest communication equipments with more receive antennas, will be given higher rates during downlink communications.

B. FINDING THE REQUIRED PERMUTATIONS OF USERS

Solving the optimization problem (15) only gives the maximal sum rate R_{mm} of DPC-based MIMO BC subject to max-min fairness, which is from the perspective of dual MIMO MAC. In this section we will show how to design DPC-based coding scheme that can achieve the value R_{mm} in MIMO BC. Recall that the optimal rate vector $\underline{R}^\star = [R_1^\star, \ldots, R_K^\star]^\top$ of individual rates is a certain convex combination of some of the $K!$ corner points of a $(K-1)$-dimensional polytope, given by intersecting the K-dimensional polytope $\mathcal{C}_{\text{DPC}}$ with the hyperspace satisfying $\sum_k R_k = R_{\text{mm}}$. Each corner point on the intersecting facet corresponds to one possible decoding order of SIC decoder, or equivalently an encoding order of DPC. To be more specific, let $P_k^\star$ be the optimal covariance matrices from problem (15) and let $A = [\underline{a}_1, \ldots, \underline{a}_{K!}]$ be a matrix of size $(K \times K!)$, where each column $\underline{a}_\ell$ corresponds to a corner point of the $(K-1)$-dimensional polytope, or equivalently one possible permutation π_ℓ of $\{1, \ldots, K\}$. The $\pi_\ell(k)$-th entry of $\underline{a}_\ell$ is the rate of user $\pi_\ell(k)$

$$R_{\pi_\ell(k)} = \log_2 \frac{\left| I + \sum_{i \le k} H_{\pi_\ell(i)}^\dagger P_{\pi_\ell(i)}^\star H_{\pi_\ell(i)} \right|}{\left| I + \sum_{i < k} H_{\pi_\ell(i)}^\dagger P_{\pi_\ell(i)}^\star H_{\pi_\ell(i)} \right|}, \tag{17}$$

when the order of $\pi_\ell(K), \ldots, \pi_\ell(1)$ is used for SIC decoding. The optimal rate vector $\underline{R}^\star = [R_1^\star, \ldots, R_K^\star]^\top$ is then a certain convex combination of columns of matrix A, i.e., we have

$$\underline{R}^\star = \sum_{\ell=1}^{K!} q_\ell \underline{a}_\ell = A\underline{q} \tag{18}$$

for some $\underline{q} = [q_1, \ldots, q_{K!}] \succeq \underline{0}$ satisfying $\underline{1}^\top \underline{q} = 1$. The coefficient q_ℓ associated to the rate vector $\underline{a}_\ell$ lies between zero and one and can be interpreted as how often the encoding order $\pi_\ell(1), \ldots, \pi_\ell(K)$ should be used for DPC encoding. In other words, the value q_ℓ represents the probability of order π_ℓ used for DPC encoding when seeking to achieve the optimal rate vector $\underline{R}^\star$.

Obviously, for ease of system implementation, we shall aim to minimize the number of required encoding orders. This problem can be formulated as follows

$$\begin{aligned} &\text{minimize } \left\| \underline{q} \right\|_0 \\ &\text{subject to } A\underline{q} = \underline{R}^\star, \ \ \underline{q} \succeq \underline{0}, \ \ \underline{1}^\top \underline{q} = 1 \end{aligned} \tag{19}$$

where $\left\| \underline{q} \right\|_0$ is usual 0-norm of vector $\underline{q}$. Unfortunately, this problem seeks the sparest solution to an underdetermined linear system and is known to be extremely difficult and NP-hard in general [34]. While some suboptimal approaches have been proposed for solving (19), such as the orthogonal matching pursuit [35], basis pursuit [36], and other algorithms from compressed sensing [37], these algorithms require a priori the knowledge of matrix A, which has $K!$ columns and can be extremely difficult to obtain when K is large.

To overcome the difficulty, below we propose an efficient method that can produce the required encoding orders $\pi_1, \ldots, \pi_n$ and the corresponding probabilities $q_1, \ldots, q_n$, with $n \le K$. It should be noted that our method achieves a

Algorithm 1 The Proposed Algorithm for Finding the Coefficients of Convex Combination

Input: Covariance matrices $P_k^\star$ and rate vector $\underline{R}^\star = [R_1^\star, \ldots, R_K^\star]^\top$

Output: An ordered set $\mathcal{P} = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and a vector $\underline{q} = [q_1, \ldots, q_n]^\top$, where each π_i is a permutation of $\{1, \ldots, K\}$ for the order of DPC encoding, and where $q_i > 0$ with $\sum_i q_i = 1$.

1: Initialize: corner-flag$\leftarrow$true, π an arbitrary permutation of $\{1, \ldots, K\}$ and $\mathcal{S} \leftarrow \emptyset$
2: **while** $|\mathcal{S}| < K$ and corner-flag=true **do**
3: **if** exists $k \in \{1, \ldots, K\} \setminus \mathcal{S}$ such that

$$R_k^\star + \sum_{i \in \mathcal{S}} R_i^\star = \log_2 \left| I + \sum_{i \in \mathcal{S} \cup \{k\}} H_i^\dagger P_i^\star H_i \right| \tag{20}$$

then
4: Update $\mathcal{S} \leftarrow \mathcal{S} \cup \{k\}$ and set $\pi(|\mathcal{S}|) \leftarrow k$
5: **else**
6: Set corner-flag=false
7: **end if**
8: **end while**
9: **if** corner-flag=true **then**
10: Set $\mathcal{P} = \{\pi\}$ and $q = 1$
11: **else**
12: Initialize: found-flag=false, $\mathcal{P} = \emptyset$, A an empty matrix
13: **while** found-flag=false **do**
14: Choose a permutation π of $\{1, \ldots, K\}$ with $\pi \notin \mathcal{P}$
15: Compute $\underline{R} = [R_1, \ldots, R_K]^\top$ with entry $R_{\pi(k)}$ according to (17) for $k = 1, \ldots, K$
16: Update $A \leftarrow [A, \underline{R}]$ and $\mathcal{P} \leftarrow \mathcal{P} \cup \{\pi\}$
17: **if** the following linear programming is feasible

$$\begin{aligned}&\text{minimize } 0\\ &\text{subject to } A\underline{q} = \underline{R}^\star,\ \underline{q} \succeq \underline{0},\ \underline{1}^\top \underline{q} = 1\end{aligned}$$

then
18: Set found-flag$\leftarrow$true and let $\underline{q}^\star$ be a solution
19: **end if**
20: **end while**
21: **while** exists $\underline{u} \neq \underline{0}$ such that $A\underline{u} = \underline{0}$ and $\underline{1}^\top \underline{u} = 0$ **do**
22: Find the index $k = \arg\min\{q_i^\star / u_i : u_i > 0, i = 1, \ldots, |\mathcal{P}|\}$ and set $\alpha = q_k^\star / u_k$
23: Delete entry $q_k^\star$ from $\underline{q}^\star$, u_k from $\underline{u}$, column $\underline{a}_k$ from A and the k-th entry from ordered set $\mathcal{P}$
24: Update $\underline{q}^\star \leftarrow \underline{q}^\star - \alpha\, \underline{u}$
25: **end while**
26: **end if**

density $n/K! \leq 1/(K-1)!$, which is extremely small for large K.

The proposed method consists of three major components and is given in Algorithm 1. For an overview, the first component aims to check efficiently whether the optimal rate vector $\underline{R}^\star$ belongs to one of the $K!$ corner points. If the answer is true, then we are done and we simply return the encoding order associated with the corner point. If otherwise, we continue to the second component and seek a set of corner points such that the rate vector $\underline{R}^\star$ is a certain convex combination of these corner points. The last component is to reduce the number of corner points required for the convex combination. This would in turn reduce the number of orders required for DPC encoding.

a: THE FIRST COMPONENT (STEPS 1-8)

This component checks efficiently whether the optimal rate vector $\underline{R}^\star$ belongs to one of the $K!$ corner points of the $(K-1)$-dimensional polytope in roughly $K^2/2$ trials. It is based on the observation that if there exists a permutation π such that $R_{\pi(K)}^\star, \ldots, R_{\pi(1)}^\star$ are the rates derived from the decoding order $\pi(K), \ldots, \pi(1)$ in SIC decoding, then by (17) the last decoded user $\pi(1)$ at the SIC decoder must be interference free and the following equality holds

$$R_{\pi(1)}^\star = \log_2 \left| I + H_{\pi(1)}^\dagger P_{\pi(1)}^\star H_{\pi(1)} \right|. \tag{21}$$

Hence, it suffices to check only whether there exists a user such that the above holds. If true, then we can check whether a similar equality holds for the sum rate of the last two decoded users at SIC decoder and so on and so forth. This is the main doing of (20).

If the first component succeeds, then the optimal rate vector $\underline{R}^\star$ is from one of the $K!$ corner points of the $(K-1)$-dimensional polytope. This shows that the only required number of encoding orders is $n = 1$ and we are done; otherwise the vector $\underline{R}^\star$ is either a boundary or an interior point of $(K-1)$-dimensional polytope.

b: THE SECOND COMPONENT (STEPS 12-20)

The second component seeks to build the matrix A incrementally by adding one randomly chosen permutation π at a time until a valid convex combination $\underline{q}$ with $A\underline{q} = \underline{R}^\star$ is found. This is done is step 17, where a simple linear programming is used to determine whether enough permutations π have been found to construct $\underline{R}^\star$.

c: THE LAST COMPONENT (STEPS 21-25)

It aims to reduce the number of permutations found in the second component. As $\underline{R}^\star \in \mathbb{R}^K$, Carathéodory's theorem [38] asserts that $\underline{R}^\star$ is a convex combination of at most $K+1$ corner points. In the proposed Algorithm 1, we further reduce the number of required permutations to be no larger than K. The proof for this claim is given in Appendix V, and the proof for steps 22-24 is relegated to Appendix V.

Once the ordered set $\mathcal{P} = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and the corresponding vector $\underline{q} = [q_1, \ldots, q_n]^\top$ are obtained from Algorithm 1, we apply the uplink-downlink duality [3] and use (7) to find for each permutation $\pi_i \in \mathcal{P}$ the optimal covariance matrices $Q_{\pi_i(k)}^\star$ for downlink communications from the uplink covariance matrices $P_k^\star$. Then, we randomly select a permutation π_i with probability q_i from set $\mathcal{P}$ and use DPC to iteratively encode user $\pi_i(k)$'s information with covariance matrix $Q^\star_{\pi_i(k)}$, taking into account the messages from users $\pi_i(1), \ldots, \pi_i(k-1)$, for $k = 1, 2, \ldots, K$. It then follows from (7) that the average achievable rate of user k equals exactly $R_k^\star$ and the overall sum rate for the above scheme equals exactly R_{mm}.

C. COMPLEXITY ANALYSIS

To achieve max-min fairness for DPC-based MIMO BC, the proposed approach consists of two parts, namely, solving the convex optimization problem (15) for the maximal sum rate R_{mm} and finding the coefficients for the convex combination of DPC encoding orders using Algorithm 1. For the first part, the complexity of standard interior point methods is at most cubic with respect to the dimensionality of input space [31]. It follows that the complexity for solving (15) is at most $O\left(\left(K + \sum_k m_k^2\right)^3\right)$.

For the second part, there are three major components in Algorithm 1. The major complexity of the first component (steps 1-8) comes from the evaluation of log-determent of matrices in (20), which has complexity $O\left(\sum_k n_t m_k^2\right)$ for matrix multiplication and $O(n_t^3)$ for computing the determinant. Since (20) is computed in no more than $K^2/2$ times, the overall complexity of steps 1-8 of Algorithm 1 is at most $O\left(K^2\left(n_t^3 + n_t \sum_k m_k^2\right)\right)$.

For the second component (steps 12-20) of Algorithm 1, the most complexity comes from step 15 for evaluating (17). Similar to (20), it has complexity equal to $O\left(K\left(n_t^3 + n_t \sum_k m_k^2\right)\right)$, due to the computations of $R_{\pi_\ell(1)}, \ldots, R_{\pi_\ell(K)}$. Step 17 is simply a linear programming and has complexity at most $O(n^2)$ [39], where n is the length of vector $\underline{q}$ and equals the number of permutations in step 14.

The last component (steps 21-25) of Algorithm 1 is to solve a system of linear equation. Since there are at most n variables in step 21, the complexity is at most $O(n^3)$. Steps 21-25 are performed repeatedly until the number of required orders is no larger than K. Hence, the complexity of this component is at most $O(n^4)$.

D. OTHER NOTIONS OF FAIRNESS

The proposed approach (15) and Algorithm 1 for finding the maximal sum rate of max-min fairness and the corresponding DPC encoding orders are very general and extremely powerful. They can be applied to solve many similar problems after slight modifications. For instance, the problem for sum rate maximization subject to proportional fairness can be solved using the method below.

Theorem 3: The maximal sum rate achieved by proportional fairness for DPC-based MIMO BC is

$$R_{\text{pf}} := R_1^\star + \ldots + R_K^\star, \tag{22}$$

where $R_k^\star$ are optimal solutions to the following convex optimization problem

$$\begin{aligned} &\text{maximize } \sum_k \log(R_k/m_k) \\ &\text{subject to } \sum_{k\in\mathcal{S}} R_k \le \log_2\left|I + \sum_{k\in\mathcal{S}} H_k^\dagger P_k H_k\right|, \\ &\qquad \text{for all } \mathcal{S} \subseteq \{1,\ldots,K\}, P_k \succeq 0, \sum_k \operatorname{tr}(P_k) \le P \end{aligned} \tag{23}$$

Moreover, the sum rate R_{pf} can be achieved by methods outlined in Algorithm 1.

Proof: The objective function in (23) is a concave function in the rate R_k and is derived from the formulation of proportional fairness in (9). The first constraint in (23) is affine in R_k and concave in the covariance matrix P_k. The remaining constraints are all affine. Hence, the problem (23) belongs to the class of convex optimization problems. The optimal solutions $R_k^\star$ to problem (23) constitute the optimal rate distribution for proportional fairness. ■

Another important application of the present work is to find the optimal tradeoff between sum rate and fairness for DPC-based MIMO BC. While the max-min fairness and proportional fairness are based on some qualitative interpretations of fairness [23], the notion of fairness can also be measured quantitatively using for example, Jain's index [24], entropy index [26] or a more recent ℓ_1-norm based index [23]. In particular, the latter defines the value of fairness as

$$f(\underline{R}) := 1 - c\sum_{k=1}^K \left|\frac{R_k}{R_{\text{sum}}} - \frac{m_k}{M}\right|, \tag{24}$$

for a given rate vector $\underline{R} = [R_1, \ldots, R_K]^\top$, where $M = \sum_k m_k$ is the total number of receive antennas, $R_{\text{sum}} = \sum_k R_k$ is the sum rate, and $c := \left(2 - \frac{2}{M}\min_k m_k\right)^{-1}$ is a constant for $K \ge 2$, such that for any nonnegative rate vector $\underline{R}$ with sum rate R_{sum} the value $f(\underline{R})$ has a tight upper bound 1 (meaning *absolutely fair*) and a tight lower bound 0 (meaning the most unfair). For instance, consider the max-min fairness, where we have $R_{\text{sum}} = R_{\text{mm}}$ and $R_k/m_k = t$ for some constant t. The ℓ_1-norm based index (24) then tells that max-min fairness achieves the 100% fairness value, since we have $R_k/R_{\text{sum}} = m_k/M$ for all k.

If we seek to achieve a sum rate larger than R_{mm}, then a certain level of fairness might have to be sacrificed. In general, the larger the value of sum rate is, the smaller the value of fairness becomes. To find the efficient frontier between sum rate R_{sum} and fairness $f(\underline{R})$, we propose the following approach.

Theorem 4: Given the target sum rate $R_{\text{sum}} \in [R_{\text{mm}}, R_{\text{ms}}]$, the maximal achievable fairness value for DPC-based MIMO BC subject to the ℓ_1-norm fairness index is the optimal value of the following convex optimization problem

$$\begin{aligned} F(R_{\text{sum}}) := \text{maximize } & 1-c\sum_{k=1}^{K}\left|\frac{R_k}{R_{\text{sum}}}-\frac{m_k}{M}\right| \\ \text{subject to } & R_k \geq 0, P_k \succeq 0, \\ & \sum_k R_k = R_{\text{sum}}, \sum_k \text{tr}(P_k) \leq P \\ & \sum_{k\in\mathcal{S}} R_k \leq \log_2\left|I+\sum_{k\in\mathcal{S}} H_k^{\dagger}P_kH_k\right| \\ & \text{for all } \mathcal{S}\subseteq\{1,\ldots,K\} \end{aligned} \tag{25}$$

Proof: Note that the objective function of (25) is concave in R_k's and is given by the ℓ_1-norm based fairness index defined in (24). The first four constraints are affine in R_k's and P_k's. The fifth constraint is affine in R_k's and concave in P_k's. Hence, the problem (25) belongs to the class of convex optimization problems. Its optimal solutions R_k's and P_k's maximize the value of ℓ_1-norm based fairness index. ■

Since the problem (25) is convex, it can be efficiently solved by using standard convex optimization methods. Same as before, the optimal solutions $R_k^{\star}$'s of (25) are certain convex combinations of the $K!$ corner points on the intersecting facet of a K-dimensional polytope and the hyperspace formed by $\sum_k R_k = R_{\text{sum}}$. We can apply Algorithm 1 to find one such convex combination, which will in turn give the desired set of encoding orders for DPC and the corresponding relative frequencies for using each order.

The optimal sum rate-fairness tradeoff $F(R_{\text{sum}})$ gives the designer of MIMO downlink communications a total flexibility in selecting the values of sum rate and fairness for system operation. Other than following the common objectives of max sum rate, max-min fairness and proportional fairness, the system designer now can use $F(R_{\text{sum}})$ to adjust freely and optimally the sum rate of downlink communication according to the required fairness among users and vice versa.

While Theorem 4 aims for finding the optimal tradeoff between sum rate and the ℓ_1-norm fairness index for DPC-based MIMO downlink communications, we remark that the same approach can be applied to other quantitative fairness measures such as Jain's index, entropy index, etc. These fairness indices are all concave in R_k's [26]. Simply replace the objective function of (25) with the desired fairness measure, and the resulting optimal sum rate-fairness tradeoff can be similarly found.

IV. SIMULATION RESULTS AND DISCUSSIONS

In this section we provide simulation results of performances achieved by max sum rate, max-min fairness and proportional fairness and the optimal sum rate-fairness tradeoff (25). We first consider a MIMO BC with $N = 10$ transmit antennas at base station and a group of $K = 3$ users, having $m_1 = 2$, $m_2 = 4$, $m_3 = 8$ receive antennas, respectively. The channel matrices H_k are randomly and independently generated using a geometry-based mmWave

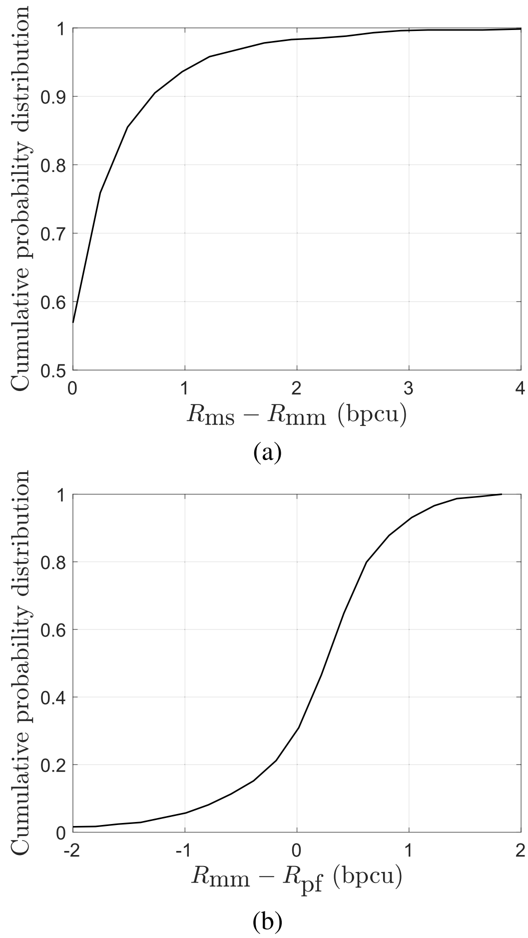

FIGURE 1. Probability distributions for the values of (a) ($R_{\text{ms}} - R_{\text{mm}}$) and (b) ($R_{\text{mm}} - R_{\text{pf}}$) for the DPC-based MIMO BC with $P = 20$ dB, $N = 10$ transmit antennas and a group of $K = 3$ users having $m_1 = 2$, $m_2 = 4$ and $m_3 = 8$ receive antennas, respectively.

channel model [40] assuming 1) sixteen effective channel paths between base station and each user, 2) uniform linear arrays with half wavelength spacing for transmit and receive antennas, 3) uniformly distributed angles of departure within $[-\frac{\pi}{6}, \frac{\pi}{6}]$, 4) uniformly distributed angles of arrival within $[-\pi, \pi]$, and 5) independently Gaussian distributed complex gain associated with each propagation path with the sum of variances normalized to 1.

The simulations are performed at a total power $P = 20$ dB and over 10,000 independent channel realizations. We first investigate the difference $(R_{\text{ms}} - R_{\text{mm}})$ among various channel realizations. This value tells the loss of sum rate if the design objective is changed from the traditional sum rate maximization to max-min fairness for achieving an absolute fairness among users. The results are given in Fig. 1(a). Surprisingly, in roughly 5,500 channel realizations, or equivalently with a probability roughly 0.55, we have $R_{\text{ms}} = R_{\text{mm}}$, i.e., there is no loss in sum rate at all. This happens only when the $(K-1)$-dimensional polytope formed by the intersecting facet of $\mathcal{C}_{\text{DPC}}$ and the hyperspace $\sum_k R_k = R_{\text{ms}}$ intersects the ray vector $\underline{R} = [tm_1/M, \ldots, tm_K/M]^{\top}$ for some $t > 0$. In other

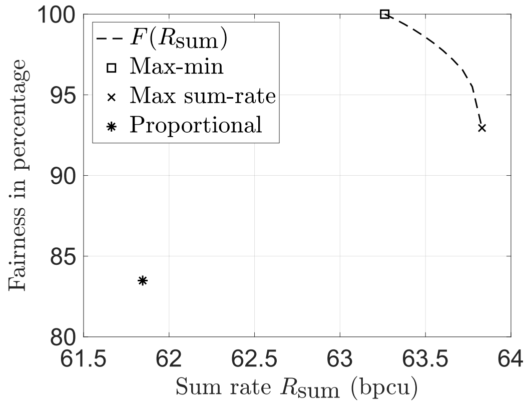

FIGURE 2. **The optimal sum rate-fairness tradeoff and the performances of max sum rate, max-min fairness and proportional fairness for a DPC-based MIMO BC with $P = 20$ dB, $N = 10$ transmit antennas and a group of $K = 3$ users having $m_1 = 2$, $m_2 = 4$ and $m_3 = 8$ receive antennas, respectively.**

words, while it is commonly believed that one cannot achieve sum rate capacity and absolute fairness at the same time for MIMO downlink communications, it is often the contrary if our new approach of multiple encoding orders of users is employed. Thus, other than following the common objective of sum rate maximization when designing MIMO downlink communication systems, we can replace the design objective with (15) for max-min fairness, and Fig. 1(a) shows that for more than half of the time, we can achieve the same max sum rate with 100% absolute fairness value.

Fig. 1(b) compares the performances of max-min fairness and proportional fairness. It can be seen that for more than 70% of channel realizations, the popular proportional fairness results in a sum rate smaller than R_{mm} of max-min fairness and has a small fairness value. The optimal sum rate-fairness tradeoff of one particular channel realization (one of the remaining 4,500 realizations) is given in Fig. 2. It is seen that the max sum rate (cf. (6)) achieves the sum rate $R_{\text{ms}} = 63.83$ bits per channel use (bpcu) with fairness value 93%, while the max-min fairness achieves 100% fairness value at a small cost of 0.57 bpcu in overall sum rate. The proportional fairness, however, achieves only a sum rate of 61.85 bpcu and a much smaller fairness value of 83.5%.

Further simulations are performed for the DPC-based MIMO BC with $P = 20$ dB, $N = 16$ transmit antennas and a group of $K = 4$ users with $m_1 = m_2 = 2$, $m_3 = 4$ and $m_4 = 8$ receive antennas, respectively. The results are given in Fig. 3 and 4. It can be seen similarly that in more than half of channel realizations we can achieve max-sum rate with 100% fairness value and in roughly 75.4% of channel realizations the proportional fairness has a sum rate smaller than max-min fairness. For one particular channel realization, the optimal sum rate-fairness tradeoff is given in Fig. 4. It is seen that the max sum rate (cf. (6)) achieves the sum rate $R_{\text{ms}} = 78.6$ bpcu with fairness value 95.7%, while the max-min fairness achieves 100% fairness value at a small cost of 0.5 bpcu in overall sum rate. The proportional fairness,

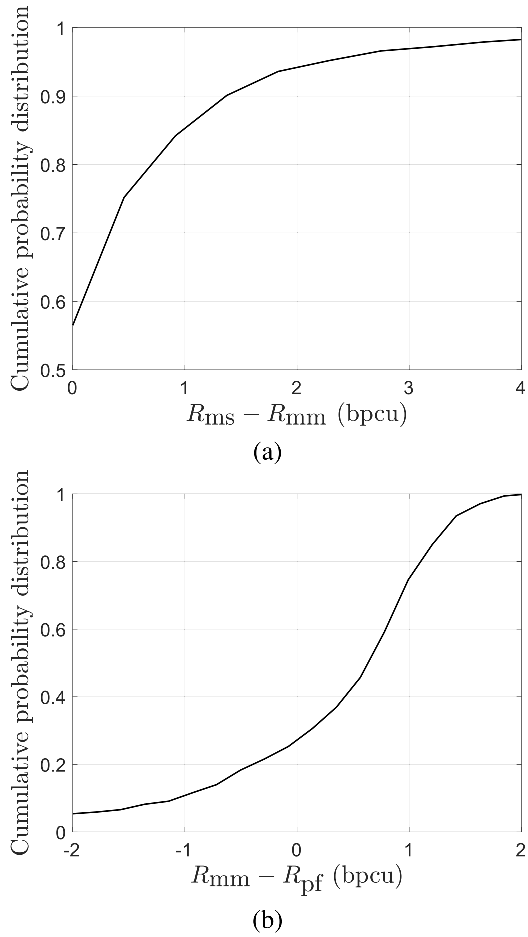

FIGURE 3. **Probability distributions for the values of (a) ($R_{\text{ms}} - R_{\text{mm}}$) and (b) ($R_{\text{mm}} - R_{\text{pf}}$) for the DPC-based MIMO BC with $P = 20$ dB, $N = 16$ transmit antennas and a group of $K = 4$ users having $m_1 = m_2 = 2$, $m_3 = 4$ and $m_4 = 8$ receive antennas, respectively.**

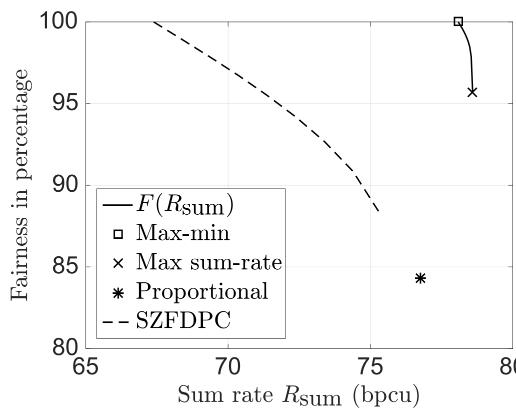

FIGURE 4. **The optimal sum rate-fairness tradeoffs and the performances of max sum rate, max-min fairness and proportional fairness for DPC-based and SZFDPC-based MIMO BC [9] with $P = 20$ dB, $N = 16$ transmit antennas and a group of $K = 4$ users having $m_1 = m_2 = 2$, $m_3 = 4$ and $m_4 = 8$ receive antennas, respectively.**

however, achieves only a sum rate of 76.7 bpcu and a much smaller fairness value of 84%.

The result is also compared to the optimal sum rate-fairness tradeoff when the encoding method of DPC is replaced by

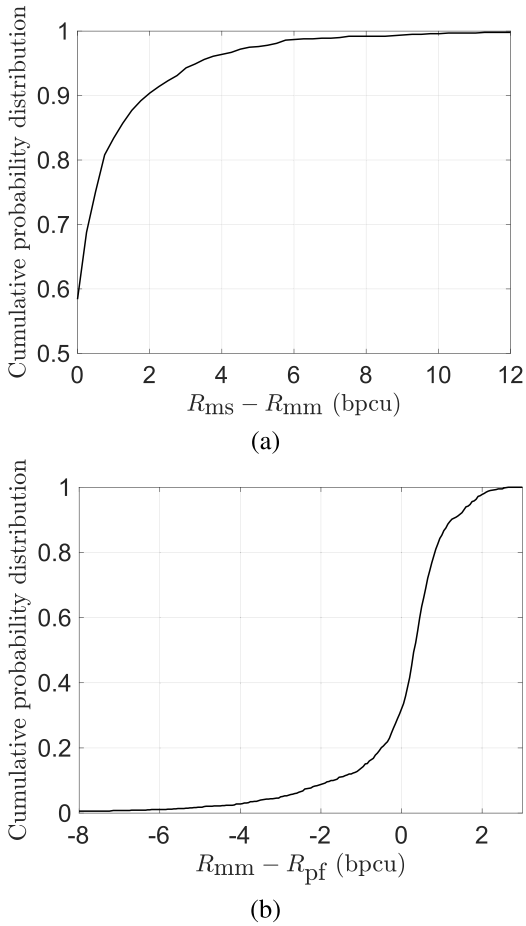

FIGURE 5. Probability distributions for the values of (a) $(R_{\mathrm{ms}} - R_{\mathrm{mm}})$ and (b) $(R_{\mathrm{mm}} - R_{\mathrm{pf}})$ for the DPC-based MIMO BC with $P = 10$ dB, $N = 256$ transmit antennas and a group of $K = 3$ users having $m_1 = 16$, $m_2 = 32$ and $m_3 = 32$ receive antennas, respectively.

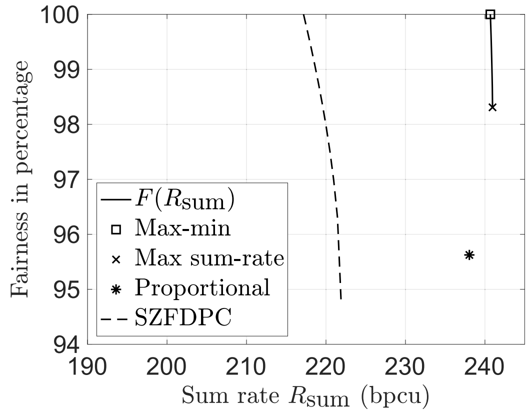

FIGURE 6. The optimal sum rate-fairness tradeoffs and the performances of max sum rate, max-min fairness and proportional fairness for DPC-based and SZFDPC-based MIMO BC [9] with $P = 10$ dB, $N = 256$ transmit antennas and a group of $K = 3$ users having $m_1 = 16$, $m_2 = 32$ and $m_3 = 32$ receive antennas, respectively.

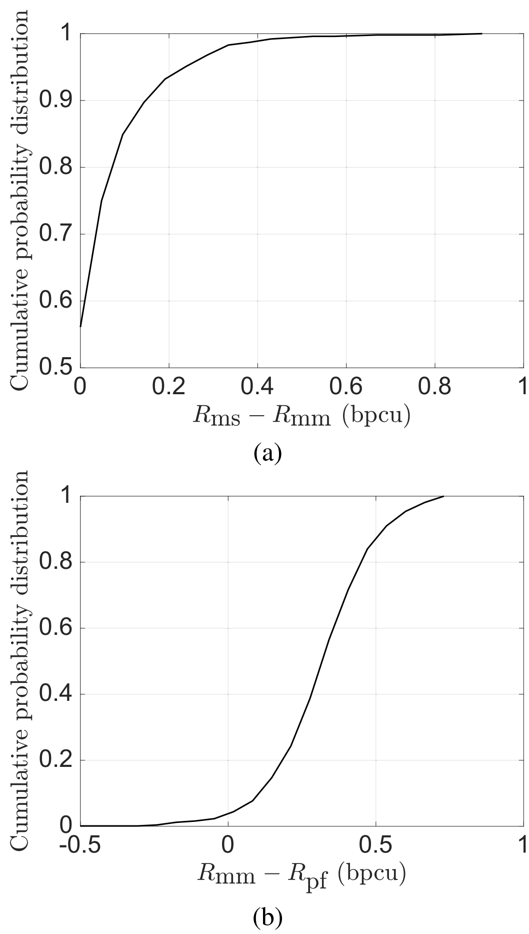

FIGURE 7. Probability distributions for the values of (a) $(R_{\mathrm{ms}} - R_{\mathrm{mm}})$ and (b) $(R_{\mathrm{mm}} - R_{\mathrm{pf}})$ for the rich-scattered, DPC-based MIMO BC with $P = 10$ dB, $N = 10$ transmit antennas and a group of $K = 4$ users having $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ and $m_4 = 4$ receive antennas, respectively.

the suboptimal successive zero-forcing DPC (SZFDPC) [2], [9], [23] which can be employed only when the total number of receive antennas of all users is no larger than the number of transmit at base station. This comparison leads to another surprising finding of the present work. While it is well known that the SZFDPC can approach the sum rate capacity of MIMO BC within a small gap [10], [27], the loss can be significantly huge when the fairness is taken into account. The dashed line in Fig. 4 corresponds to the optimal sum rate-fairness tradeoff of SZFDPC-based MIMO BC. It shows that SZFDPC achieves a maximal sum rate of 75.7 bpcu and has a gap of roughly 2.9 bpcu to the sum rate capacity. However, if the notion of max-min fairness is taken into account, the SZFDPC can offer only 67.3 bpcu in sum rate, which corresponds to a huge loss of 10.8 bpcu compared to the present DPC-based approach.

For mmWave system with large antenna arrays, we consider a MIMO BC with $N = 256$ transmit antennas and a group of $K = 3$ users having $m_1 = 16$, $m_2 = 32$ and $m_3 = 32$ receive antennas, respectively. The simulation results are presented in Fig. 5 and Fig. 6 for $P = 10$ dB. It is seen from Fig. 5 that the proposed approach (15) can achieve the sum rate capacity with 100% fairness value in roughly 58.4% of channel realizations and outperforms proportional fairness in roughly 69.2% of

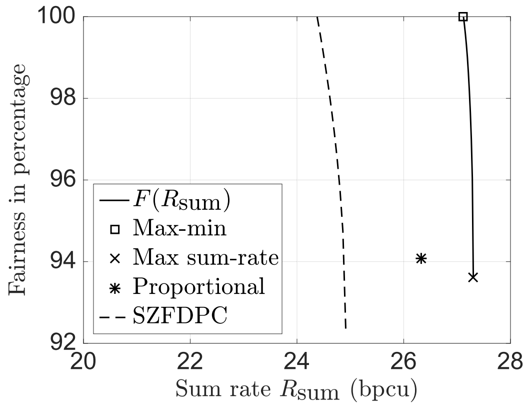

FIGURE 8. The optimal sum rate-fairness tradeoffs and the performances of max sum rate, max-min fairness and proportional fairness for the rich-scattered, DPC-based and SZFDPC-based MIMO BCs [9] with $P = 10$ dB, $N = 10$ transmit antennas and a group of $K = 4$ users having $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ and $m_4 = 4$ receive antennas, respectively.

channel realizations. The optimal sum rate-fairness tradeoff for a particular channel realization is given in Fig. 6. It is seen that the proposed approach achieves 100% fairness value at an almost negligible cost of 0.27 bpcu in overall sum rate from the sum rate capacity. The proportional fairness has a sum rate smaller than max-min fairness with a gap of 2.64 bpcu and has a smaller fairness value of 95.6%. The SZFDPC-based MIMO BC performs much worse in this case. Its maximal sum rate is 221.92 bpcu and has a relatively large gap of 19.03 bpcu from the sum rate capacity. To achieve 100% fairness value, the SZFDPC offers only 217.17 bpcu in sum rate, which corresponds to a huge loss of 23.5 bpcu compared to the present DPC-based approach.

For conventional sub-6GHz channels with rich scattering, we consider a MIMO BC with $N = 10$ transmit antennas and a group of $K = 4$ users having $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ and $m_4 = 4$ receive antennas, respectively. The entries of channel matrices H_k are modeled as independent and identically distributed $\mathbb{CN}(0, 1)$ random variables for quasi-static Rayleigh fading. At $P = 10$ dB, it is seen from Fig. 7 that the proposed approach achieves the sum rate capacity in roughly 56.1% of channel realizations and has a sum rate larger than proportional fairness in 97.7% of channel realizations. For a particular channel realization, Fig. 8 shows that the proposed approach achieves 100% fairness value at a negligible cost of 0.18 bpcu from the sum rate capacity and outperforms the SZFDPC-based MIMO BC by more than 2.8 bpcu in overall sum rate.

V. CONCLUSION

In this paper we have presented several efficient algorithms for finding the maximal sum rate achieved by DPC-based MIMO BC when the qualitative notions of fairness such as max-min fairness and proportional fairness are employed and when the order of users can be adjustable during DPC encoding. Our new approach can often achieve the sum rate capacity of MIMO BC with absolute fairness in finite number of users. This brings a huge impact on the design of practical MIMO BC as now the rates can be completely fairly distributed among the users without any loss on the sum rate capacity. Furthermore, in sharp contrast to common belief, simulation results show that there is an enormous advantage of DPC over SZFDPC when the issue of fairness is taken into consideration. We have also provided a complete characterization on the optimal tradeoff between the sum rate and fairness that gives system designers a total flexibility in selecting the operating sum rate and fairness for DPC-based MIMO downlink communications.

As for future works, we remark that the questions of finding optimal input distributions for DPC-based MIMO BC with a fixed encoding order subject to the criterion of max-min fairness or proportional fairness remain open. How to solve efficiently the non-convex problems (8) and (9) calls for further works. Although (8) and (9) could have smaller sum rates and fairness values than the present work, they use only one fixed DPC encoding order and are therefore simpler for implementation. Another direction for future work is to investigate the dramatic loss of SZFDPC to DPC in sum rate when the issue of fairness is taken into account. Equivalently, the simulation results show that there is plenty of room for improving the performance of SZFDPC-based MIMO BC and suggest a great challenge of designing SZFDPC-based MIMO BC for future work.

APPENDIX A
PROOF OF THEOREM 2

The optimization problem (15) is equivalent to the following

$$\begin{aligned} &\text{minimize} \quad -t \\ &\text{subject to} \quad \frac{R_k}{m_k} \geq t \\ &\qquad \sum_{k\in\mathcal{S}} R_k \leq \log_2 \left| I + \sum_{k\in\mathcal{S}} H_k^\dagger P_k H_k \right|, \\ &\qquad \text{for all } \mathcal{S} \subseteq \{1, \ldots, K\}, P_k \succeq 0, \sum_k \mathrm{tr}(P_k) \leq P, \end{aligned} \tag{26}$$

which has the following Lagrangian

$$\begin{aligned} L = &-t + \sum_{k=1}^{K} \lambda_k \left(t - \frac{R_k}{m_k} \right) \\ &+ \sum_{\mathcal{S}\subseteq\{1,\ldots,K\}} \lambda_{\mathcal{S}} \left(\sum_{k\in\mathcal{S}} R_k - \log_2 \left| I + \sum_{k\in\mathcal{S}} H_k^\dagger P_k H_k \right| \right) \\ &+\lambda \left(\sum_{k=1}^{K} \mathrm{tr}(P_k) - P \right) \end{aligned} \tag{27}$$

where $\lambda_k \geq 0$, $\lambda_{\mathcal{S}} \geq 0$ and $\lambda \geq 0$ are the Lagrange multipliers for the three sets of constraints in (26), respectively. The Kuhn-Kuhn-Tucker (KKT) conditions [31] for problem (26) are

$$\frac{\partial L}{\partial t} = -1 + \sum_k \lambda_k = 0 \tag{28}$$

$$\frac{\partial L}{\partial R_k} = -\frac{\lambda_k}{m_k} + \sum_{\mathcal{S}\in\mathcal{S}(k)} \lambda_{\mathcal{S}} = 0 \tag{29}$$

$$-\sum_{\mathcal{S}\in\mathcal{S}(k)} \lambda_{\mathcal{S}} H_k \left(I + \sum_{i\in\mathcal{S}} H_i^\dagger P_i H_i\right)^{-1} H_k^\dagger + \lambda I = 0 \tag{30}$$

where the last condition is the gradient of L with respect to the positive semidefinite matrix P_k and where $\mathcal{S}(k)$ is the collection of subsets of $\{1,\ldots,K\}$ containing k, i.e.

$$\mathcal{S}(k) := \{\mathcal{S} : k \in \mathcal{S} \subseteq \{1,\ldots,K\}\}. \tag{31}$$

If $\lambda_k = 0$ for some k, then by (29) we have $\lambda_{\mathcal{S}} = 0$ for all $\mathcal{S} \in \mathcal{S}(k)$ since $\lambda_{\mathcal{S}}$ is nonnegative. It then follows from (30) that $\lambda = 0$. Now, if there exists $k' \neq k$ with $\lambda_{k'} > 0$, the KKT condition for the gradient of L with respect to $P_{k'}$ (cf. (30)) reduces to

$$-\sum_{\mathcal{S}\in\mathcal{S}(k')} \lambda_{\mathcal{S}} H_{k'} \left(I + \sum_{i\in\mathcal{S}} H_i^\dagger P_i H_i\right)^{-1} H_{k'}^\dagger = 0 \tag{32}$$

since $\lambda = 0$.

Note that the matrices $H_{k'}\left(I + \sum_{i\in\mathcal{S}} H_i^\dagger P_i H_i\right)^{-1} H_{k'}^\dagger$ are all positive semidefinite, and (32) holds if and only if $\lambda_{\mathcal{S}} = 0$ for all $\mathcal{S} \in \mathcal{S}(k')$. This in turn implies that the partial derivative of L with respect to $R_{k'}$ must satisfy

$$0 = \frac{\partial L}{\partial R_{k'}} = -\frac{\lambda_{k'}}{m_{k'}} + \sum_{\mathcal{S}\in\mathcal{S}(k')} \lambda_{\mathcal{S}} = -\frac{\lambda_{k'}}{m_{k'}} \tag{33}$$

due to KKT conditions. The above then shows $\lambda_{k'} = 0$, a contradiction. Hence, if there exists k such that $\lambda_k = 0$, then we must have $\lambda_{k'} = 0$ for all k'. But this contradicts to the first KKT condition (28). As a result, we must have $\lambda_k > 0$ for all $k = 1,\ldots,K$. Finally, by the complementary slackness of KKT conditions $\lambda_k\left(t - \frac{R_k}{m_k}\right) = 0$ for the optimal solutions R_k and optimal value t of convex optimization problem (26), we conclude that $t - \frac{R_k}{m_k} = 0$ for all k.

APPENDIX B
REDUCING THE NUMBER OF REQUIRED ORDERS TO AT MOST *K*

Let A be the matrix of size $K \times |\mathcal{P}|$ obtained from the second component of Algorithm 1. Step 21 in Algorithm 1 then seeks a vector $\underline{u}$ such that it is a nontrivial solution to the following system of linear equations

$$\underbrace{\begin{bmatrix} \underline{1}^\top \\ A \end{bmatrix}}_{:=\bar{A}} \underline{u} = \underline{0}. \tag{34}$$

Note that the matrix $\bar{A}$ is of size $(K+1)\times|\mathcal{P}|$, hence a nontrivial $\underline{u}$ exists whenever $|\mathcal{P}| > K+1$. Thus, it suffices to consider the case when $|\mathcal{P}| = K+1$ and $\bar{A}$ is a square matrix of size $(K+1)\times(K+1)$. Note that by the second component of Algorithm 1 the columns of matrix A have the same column sum equal to the sum rate $\underline{1}^\top \underline{R}^\star$, i.e., we have $\underline{1}^\top A = \left(\underline{1}^\top \underline{R}^\star\right)\underline{1}^\top$. It follows that the square matrix $\bar{A}$ must be singular since we have

$$\left[-\left(\underline{1}^\top \underline{R}^\star\right)\ \underline{1}^\top\right]\bar{A} = \underline{0}^\top. \tag{35}$$

Hence, a nontrivial solution $\underline{u}$ for $\bar{A}\underline{u} = \underline{0}$ must exist, and the while loop of steps 21-25 proceeds whenever $|\mathcal{P}| \geq K$.

APPENDIX C
PROOF OF STEPS 22-24 IN ALGORITHM 1

Here we will give a short proof to verify the steps 22-24 in Algorithm 1. Note that if there exists $\underline{u} \neq \underline{0}$ with $\underline{1}^\top \underline{u} = 0$, then the nonzero entries of $\underline{u}$ cannot be all negative and the index k of step 22 must exist, since A is a nonnegative matrix. By the choice of α and k in step 22, it is clear that $\underline{q}^\star \succeq \alpha\underline{u}$. Moreover, we have $\underline{1}^\top(\underline{q}^\star - \alpha\underline{u}) = \underline{1}^\top\underline{q}^\star - \alpha\underline{1}^\top\underline{u} = 1 - 0 = 1$. Thus, the vector $(\underline{q}^\star - \alpha\underline{u}) \succeq \underline{0}$ is a valid solution and satisfies $A(\underline{q}^\star - \alpha\underline{u}) = A\underline{q}^\star - \alpha A\underline{u} = \underline{R}^\star$. The k-th entry of $(\underline{q}^\star - \alpha\underline{u})$ is zero and can be removed in step 23. Now replacing $\underline{q}^\star$ by $(\underline{q}^\star - \alpha\underline{u})$ decreases the vector-length by 1.

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