

# The New Definitions of Intuitionistic and Belief-Plausibility Based Local Criteria With Interval and Fuzzy Inputs Applied to the Multiple Criteria Problem of a Raw Material Supplier Selection

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**ABSTRACT** The problems of the  $A - IFS$  and  $A - IVIFS$  theories are analyzed and to solve them, their Dempster-Shafer theory based redefinition is developed and a new concept called Belief-Plausibility number ( $BPN$ ) is introduced. The  $BPN$  can be applied beyond semantics of usual fuzzy sets and the  $A - IFS$ . Two main sources of uncertainty generating the  $A - IVIFS$  are considered: an uncertainty caused by the membership and non-membership functions multiplicity and an uncertainty of inputs. For the fuzzy inputs, the definition of the fuzzy-valued  $BPN$  is proposed. The developed approach to treat interval-valued intuitionistic fuzzy objects and their Belief-Plausibility re-definitions allowed us to introduce a set of useful definitions. It is important that some of the them were not earlier mentioned in the literature. The corresponding new arithmetical operations with such objects are proposed. The practical usefulness of the introduced approach is illustrated by the solution of the real-world Fuzzy-Valued Belief-Plausibility multiple criteria problem of the raw material suppliers selection on the steel rolling plant. The obtained results allowed us to state that the introduced Belief-Plausibility approach can be successfully used in the analysis and applications independently, without the use of the  $A - IFS$  concepts. This approach owns visible advantages in comparison with the  $A - IFS$  based one.

**INDEX TERMS** Belief-plausibility numbers, fuzzy-valued belief-plausibility multiple criteria problem of the raw material suppliers selection, interval valued and fuzzy valued belief-plausibility numbers, intuitionistic fuzzy values, new operational laws with belief-plausibility objects.

## I. INTRODUCTION

### A. ANALYSIS OF SOME MODERN THEORIES CONCERNED WITH THE CONCEPT OF INTUITIONISTIC FUZZY SETS

The classical definitions of the Intuitionistic fuzzy sets theory ( $A - IFS$ ) and the Interval-valued Intuitionistic fuzzy sets theory ( $A - IVIFS$ ) were introduced by K. Atanasov [1] and K. Atanassov and G. Gargov [2], respectively, and recognized to be practically useful extensions of the Fuzzy Set theory

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( $FST$ ) [3]. In the framework of the  $A - IFS$ , besides the degree of membership  $\mu$ , the non-membership degree  $\nu$  and the hesitant degree  $\pi$  are considered. It is postulated that  $0 \leq \mu, \nu, \pi \leq 1$  and  $\mu + \nu + \pi = 1$ . The  $A - IVIFS$  theory deals with interval extensions of  $\mu, \nu$  and  $\pi$ . Currently, the methods of these theories are actively applied for the solution of multiple criteria decision making problems ( $MCDM$ ) [4]–[7] and group  $MCDM$  problems [8] in different real-world areas. In this paper, we restrict ourselves to citing only several recently published papers in the field we deal with. We think they more than satisfactory present the nowadays

state of art in the field of the MCDM problems solution under intuitionistic uncertainty. But we do not intend to make a comprehensive analysis of the  $A-IFS$  and  $A-IVIFS$  practical applications.

However, there are some methodological problems of the  $A-IFS$  and  $A-IVIFS$  discussed in the literature.

The first of them is the difficulty that an expert experiences when assigning real values to  $\mu$  and  $\nu$ . It was noted by many scholars that sometimes experts insist on such values that their sum  $\mu + \nu$  occurred to be greater than 1 what is unacceptable in the framework of the intuitionistic theory.

To avoid or at least alleviate this problem, some extensions (corrections, modifications) of the  $A-IFS$  were developed.

The chronologically first of them is the so-called Neutrosophic Set theory ( $NST$ ) [9]–[12].

In terms of  $A-IFS$ , the basic assumptions of  $NST$  are presented in the form:

$0 \leq \mu + \nu + \pi \leq 3$  for the completely independent components ( $\mu, \nu, \pi \in [0, 1]$ ),  $0 \leq \mu + \nu + \pi \leq 1$  for the mutually dependent components ( $\mu, \nu, \pi \in [0, 1]$ ).

It is seen that the  $NST$  conceptually differs from the  $A-IFS$  by the accepted assumption that the components  $\mu, \nu, \pi$  are completely independent.

In our opinion, this assumption is not in line with reality and common sense although seems to be correct from the pure mathematical point of view.

This assumption in the  $NST$  means that the event  $\mu = 1, \nu = 1$  and  $\pi = 1$  is admissible and therefore the first constraint  $0 \leq \mu + \nu + \pi \leq 3$  is fulfilled. Let  $\mu, \nu$  and  $\pi$  be the degrees of truth, false and indeterminacy, respectively (in terms of the  $NST$ ). Therefore, in the case of complete truth ( $\mu = 1$ ), according to the formal logic and common sense, the false degree is equal to 0 ( $\nu=0$ ) with no indeterminacy ( $\pi=0$ ). Obviously, the similar conclusions can be inferred in the cases of complete false, complete indeterminacy and  $\mu = 0, \nu = 0, \pi = 0$ . In other words, we can state that the great truth is always accompanied by the small false and indeterminacy. Many of similar statements that will be in compliance with common sense and the formal logic can be claimed. So we can say that the hypothesis of the components  $\mu, \nu$  and  $\pi$  independence seems to be completely invalid in the framework of the problem we deal with and the assumption of mutually dependent components is directly originated from the formal logic and common sense.

It is somewhat surprising that the events  $\mu = 1, \nu = 1$  and  $\pi = 1$  and  $\mu = 0, \nu = 0, \pi = 0$  are qualified in [12] as a paradox and its formal definition is considered as an advantage of the  $NST$ . It is not only our opinion, but in any case, it seems preferable to use theories free of paradoxes, e.g. the  $A-IFS$  theory in the considered case. It is also important that described paradoxes occur in the two diametric asymptotic cases  $\mu = 1, \nu = 1$  and  $\pi = 1$  and  $\mu = 0, \nu = 0, \pi = 0$  achievable in the  $NST$ . Of course, the theory with so bad asymptotic properties is not a good choice if we deal with the real-world applications.

Formally, if the hypothesis of mutually dependent components is accepted, then the basic constrain of the  $NST$   $0 \leq \mu + \nu + \pi \leq 1$  is more general than that in the  $A-IFS$  ( $\pi + \mu + \nu = 1$ ). This fact was used in the so-called Picture fuzzy sets theory [13]. As a basic of this theory, the so-called refusal degree  $r$  such that  $\mu + \nu + \pi + r = 1$  was introduced. According to the Oxford Dictionary, the term “refusal” corresponds the rejection from something. Therefore, in considered case, it should be equivalent to the rejection from the analyzing the problem at hand. So, if this degree  $r$  is greater than 0, then we can not begin the studies at all and the refusal disappears when we are doing something, e.g. solving the  $MCDM$  problem. Therefore, the refusal degree has no sense even on the semantic level of analysis.

The operations defined on the mathematical objects of  $NST$  in [14] are shown to have some important drawbacks [15]. As a compromise, in [15] the more simple operations were proposed, but without analysis of their properties. The mathematical operations with the objects of  $NST$  developed in [14], [15] are obtained as simple mechanical extensions of operational laws earlier defined in the  $A-IFS$  theory. In our papers [7], [16], [17], we revealed 6 important drawbacks of these operations, which should be directly inherited in operations with objects of  $NST$ .

Finally we can say that although the  $NST$  perhaps can be interesting for pure mathematicians, it look as impractical extension of the  $A-IFS$  based on the rather artificial hypothesis.

The Pythagorean fuzzy set theory ( $PFSS$ ) was proposed in [18] based on the simple and clear reasoning [19]. Its somewhat modified version is presented here as follows.

Let  $\mu = 0.8$  be the membership degree and the  $\nu = 0.5$  be the corresponding non-membership degree. Since  $\mu + \nu > 1$ , we have no the intuitionistic fuzzy value. Then if in the considered example we will change  $\mu$  and  $\nu$  by  $\mu^2$  and  $\nu^2$ , respectively, we will get  $\mu^2 + \nu^2 < 1$ . So in accordance with the definition introduced in [19], in this example we have the truly intuitionistic fuzzy value.

Although such a reasoning seems to be acceptable, two obvious questions arise: Assuming that  $\mu$  is a membership degree, what is the sense of  $\mu^2$  in a natural language? What were logical arguments in favor of the squared degree, while the third, fourth, and so on degrees look to be even more attractive and promising?

Because we did not find answers on these questions, we can say that they may be treated as important vital methodological issues of the  $PFSS$ .

The obvious important practical question can be formulated as follows: does the  $PFSS$  always guarantee  $\mu^2 + \nu^2 < 1$ ? The answer is not positive, e.g. if  $\mu = 0.9$  and  $\nu = 0.5$ , then  $\mu + \nu > 1$  and  $\mu^2 + \nu^2 > 1$ .

To be fair, we should note here that the author of the  $PFSS$  said of only limited range of this theory applications: “... we observe that intuitionistic membership grades are all points under the line  $x + y \leq 1$  and the Pythagorean membership grades are all points with  $x^2 + y^2 \leq 1$ . We see then that the

Pythagorean membership grades allow for the representation on a larger body of nonstandard membership grades than intuitionistic membership grades” [19].

In our opinion, we can agree with this statement, but only if the reasonable answers on the above questions will be formulated. Also we can say that because the *PFSS* pretends to be an extension of the *A – IFS*, the use of its methods can only enhance the possibility of violating the basic constraint  $\mu + \nu \leq 1$ . Therefore, the practical usefulness of the *PFSS* seems to be very doubtful.

The Spherical fuzzy sets theory *SFSS* [20] is the formal extension of the *PFSS* since it is based on the introduced constrain  $\mu^2 + \nu^2 + \pi^2 \leq 1$ . It is seen that *SFSS* have all the negative properties of the *PFSS* additionally complicating the problem (see analysis in [12]).

The main problem of the *PFSS* and *SFSS* as well as of *A – IFS*, is the assigning of appropriate values to the components  $\mu$ ,  $\nu$  and  $\pi$  such that they provide the basic constrain ( $\mu + \nu \leq 1$ ) in the presence of objectively existing, but not explicitly defined (nearly undefined) mutual dependence between the components. Contrary to the *NST*, we are insisting that the dependence of components always takes place (may be, implicitly) and is originated from human thinking and logic.

It is not so difficult to fancy how important this problem is in practice for an expert who takes part in the solution of, e.g. the *MCDM* problems. The method for solution of this and some other collateral problems will be presented in this paper below.

In this paper, we will present a new method focused on the solution of this and the set of other collateral problems that is based on the redefinition of *A – IFS* and *A – IVIFS* in the framework of the Dempster-Shafer theory of evidence (*DST*).

Based on the provided analysis, we introduce a new mathematical object called Belief-Plausibility number (*BPN*), which can be reduced into its particular cases: an ordinary fuzzy number (the value of a membership function) and an intuitionistic fuzzy number. It is important that this object can be applied in an analysis completely separately, without using terms of ordinary fuzzy sets and the *A – IFS*, as it is typical for more general theories.

The above review concerns with the problems of intuitionistic fuzzy numbers.

Meanwhile, the second set of important problems we have revealed are typical in the membership  $\mu(x)$  and non-membership  $\nu(x)$  ( $x \subseteq X$ ) functions building. There are many approaches to the presentation of these functions proposed in the literature. In the papers [21]–[26], the triangular form of these functions was used and in [27] their interval valued triangular versions were proposed. Currently the most used are trapezoidal [28]–[34] and interval valued trapezoidal [35]–[41]  $\mu(x)$  and  $\nu(x)$  functions.

The natural limitations concerned with the too simple triangular and trapezoidal shapes of  $\mu(x)$  and  $\nu(x)$  functions are currently overcoming by the use of piece-wise linear functions consisting of different numbers of pieces. Most

known of them are Pentagonal [42]–[46], Hexagonal [47], Heptagonal [48]–[50], Octagonal [51], Nanogonal [52] and Hexadecagonal [53] intuitionistic fuzzy functions. Obviously, with the increasing the number of used pieces, such functions tend to be close to some smooth generally non-linear functions. The use of non-linear  $\mu(x)$  and  $\nu(x)$  functions is presented in [54], [55].

While analyzing the above papers, we have found that their authors do not care of the logically justified properties of these functions in the asymptotic cases when  $\mu(x) = 0$  or  $\nu(x) = 0$  or  $\mu(x) = 1$  or  $\nu(x) = 1$ . In these papers, we see that according to introduced mathematical presentations of  $\mu(x)$  and  $\nu(x)$  and even their graphical illustrations, we can observe in  $X$  the whole areas where, e.g.  $\mu(x) = 1$ , while  $\nu(x) > 0$  and even is rising or decreasing in such an area. We insist that this contradicts to common sense. Really, if  $\mu(x) = 1$ , then we have a complete (100%) certainty that some event has occurred. Therefore there is no place for the opposite event and hesitation, i.e. it should be  $\nu(x) = 0$  and  $\pi(x) = 0$ . The similar analyzes can easily be done for other asymptotic cases.

Let us consider the more meaningful confirmation of the above statements.

Let’s assume that we want to establish the truth of the statement that Admiral Nelson was a gentleman based on the judgments of individuals from a sample of 100 people. First consider the case when 20 people called Nelson a gentleman because he bravely fought for his Homeland, 30 people did not call him a gentleman because he was a pirate and the remaining 50 people found it difficult to assess. Then we can introduce the degree of Nelson membership in the “Club of gentleman” as  $\mu=0.2$ , the non-membership as  $\nu=0.3$  and hesitation degree as  $\pi=0.5$ .

Then let us suppose that all people in the sample refused to recognize Admiral Nelson as a gentleman. Obviously, we have  $\mu = 0$ . It is clear that in this case we have  $\nu = 1$  and  $\pi = 0$ , since we simply do not have people who could say anything else. Indeed, since everyone completely rejected Nelson’s belonging to gentlemen, then there are physically no those who would have hesitated in the assessment, hence  $\pi=0$ . At last, the complete rejection of Nelson’s belonging to gentlemen ( $\mu=0$ ) means the complete acceptance of the opposite hypothesis of Nelson not being a gentleman i.e.  $\nu=1$ . This can be inferred formally as follows: since by definition it always should be  $\mu + \nu + \pi=1$  and in our case we have  $\mu = \pi=0$ , then  $\nu=1$ .

We have found the only one paper [42], where the authors saying nothing about correct properties of the symmetrical pentagonal  $\mu(x)$  and  $\nu(x)$  functions in asymptotic cases, have constructed  $\mu(x)$  and  $\nu(x)$  in such a way that the desired properties were fulfilled. Unfortunately, in this paper, the functions  $\mu(x)$  and  $\nu(x)$  were defined to be equal to 0.5 in the points of their intersection. It is easy to see that in such a case, for piece-wise linear  $\mu(x)$  and  $\nu(x)$  we get  $\mu(x) + \nu(x) = 1$  for any  $x \subseteq X$ . This corresponds to the definition of the ordinary fuzzy set, not an intuitionistic one.

It is interesting that there are no definitions of asymptotic properties of the functions  $\mu(x)$  and  $\nu(x)$  in the classical Atanassov's works [1], [2], perhaps because they looked to be too obvious. Nevertheless, based on the above analysis, we can state the strong definitions of the discussed properties should be relevant in the basic definitions of the  $A - IFS$  and  $A - IVIFS$ . In the current paper, they will be included in these definitions.

We have not found in the literature clear and practical approaches to mathematical formulation of the functions  $\mu(x)$  and  $\nu(x)$  in such a way that all asymptotic properties and basic constraints  $\mu(x) + \nu(x) \leq 1$ ,  $\mu_U(x) + \nu_U(x) \leq 1$  ( $\mu_U(x)$  and  $\nu_U(x)$  are the upper bounds of corresponding interval-valued membership and non-membership functions) will be fulfilled jointly. In the current paper, such an approach will be proposed.

## B. ACTUAL PROPOSITIONS

In the literature, we have found only the examples of the use of Atanassov's definition of  $IVIFS$ , whereas in [17] we have revealed its limitations and drawbacks and proposed a new correct definition based on representation of the  $A - IVIFS$  in the framework of  $DST$ . Such a redefinition allowed us to develop new operational laws free of revealed drawbacks. Nevertheless, this definition needs upgrading taking into account the above analysis.

The main new results of our work are as follows:

- 1) Based on our experience, we can state that there are two main sources of uncertainty leading to the need for the use of  $A - IVIFS$ . The first of them is an interval uncertainty caused by the multiplicity of the functions  $\mu(x)$  and  $\nu(x)$ , e.g. proposed by different experts. The second source is the consequence of the interval uncertainty of input data, e.g. in the  $MCDM$ . And if input data we deal with are presented by fuzzy values, we meet the case of fuzzy-valued intuitionistic fuzzy set  $FVIFS$ .
- 2) The described above general approach to treat the interval-valued intuitionistic fuzzy objects and their Belief-Plausibility (based on the  $DST$ ) extensions allowed us to introduce a set of new fruitful definitions, some of the them, e.g. the definition of Fuzzy-Valued Belief-Plausibility value ( $FVBPV$ ) were not even mentioned in the literature in any form.
- 3) The corresponding new operational laws with these mathematical objects following from these definitions are proposed. They all and especially the definition of  $FVBPV$ , were thoroughly analyzed under general methodological principles and their practical usefulness is illustrated by the real-world case study based on the solution of the  $DST$  based uncertain  $MCDM$  problem of the raw material supplier selection on the steel rolling plant under triangular fuzzy input data.
- 4) The obtained results allow us to state that the introduced mathematical objects  $BPN$ ,  $IVBPN$  and  $FVBPN$  can be successfully used in the analysis and

applications independently, without the use of concepts of the  $A - IFS$  and such an approach has obvious advantages in comparison with the  $A - IFS$  based approach.

The rest of the paper is set out as follows.

In Section 2, we analyze the properties of the  $DST$  extension of  $A - IFS$  and its practical advantages for a decision maker in assigning certain real values to the parameters of  $A - IFS$ . As the result of provided studies, a new mathematical object called Belief-Plausibility number ( $BPN$ ) is introduced. It is shown that the  $BPN$  has at least two particular cases: usual fuzzy number (the value of a membership function) and intuitionistic fuzzy number. It is emphasized that the  $BPN$  can be successfully applied completely separately not using terms of usual fuzzy sets and the  $A - IFS$  and this is the typical feature of more general theories.

Section 3 is devoted to the justification of new definitions of membership and non-membership functions, their interval extensions inspired by two revealed sources of interval uncertainty. The case of fuzzy-valued input data is considered. It is shown that in such a case we should deal with fuzzy-valued intuitionistic fuzzy set  $FVIFS$ . The corresponding definitions in the spirit of  $BPN$ , i.e. Belief-Plausibility set ( $BPS$ ), Belief-Plausibility value ( $BPV$ ), Interval-Valued Belief-Plausibility value ( $IVBPV$ ) and Fuzzy-Valued Belief-Plausibility value ( $FVBPV$ ) are introduced. The new arithmetic operations with local criteria represented by  $BPV$ s and dependent on interval and fuzzy arguments are proposed.

In Section 4, we present the real-world case study. We have solved the Belief-Plausibility based  $MCDM$  problem of the raw material supplier selection on the steel-rolling plant, when Belief-Plausibility local criteria are dependent on fuzzy arguments. The problem formulation based on the  $FVBPV$  definition was chosen as the more correct, general and convenient than that in terms of the fuzzy-valued intuitionistic fuzzy set  $FVIFS$ .

It is worthy to note that in the current paper we consider only components of  $IFV$  and Belief-Plausibility values presented by real values, real valued interval and real valued fuzzy values. Therefore, the modern theories based on the so-called complex fuzzy sets, the relativistic fuzzy sets and so on, are out of scope of the current paper as based on the practically incomparable concepts.

Section 5 concludes the paper with some remarks.

## II. THE STUDIES OF THE $DST$ EXTENSION OF THE $A - IFS$ NUMBER. INTRODUCTION OF A NEW MATHEMATICAL OBJECT BELIEF-PLAUSIBILITY NUMBER

For the readers convenience, we present here some basic definitions of the Dempster-Shafer theory of evidence ( $DST$ ) [56], [57] in the extent needed for our analysis.

A Dempster-Shafer ( $DS$ ) belief structure is represented by a basic probability assignment ( $bpa$ ) which is a mapping  $m$  from subsets of  $X$  into a unit interval,  $m : 2^X \rightarrow [0, 1]$  such that  $m(\emptyset) = 0$ ,  $\sum_{A \subseteq X} m(A) = 1$ . The subsets of  $X$  with positive (non zero) values of  $m$  are called focal elements.



The measure of belief is defined as

$$Bel(Y) = \sum_{\emptyset \neq Z \subseteq Y} m(Z). \tag{1}$$

The measure of plausibility can be presented as

$$Pl(Y) = \sum_{Z \cap Y \neq \emptyset} m(Z). \tag{2}$$

It is seen that  $Bel(Y) \leq Pl(Y)$ .

A measure of ignorance about an event  $Y$  and its complementary  $\bar{Y}$  is treated as the length of the so-called belief interval  $[Bel(Y), Pl(Y)]$  ( $BI$ ). It can be considered also as imprecision of the “true probability” of  $Y$  [57].

A cornerstone of  $DST$  is the problem of combination of evidence having different origins.

This problem is very important and many different combination rules may be found in the literature.

The  $DST$  extension of  $A - IFS$  we have proposed in [58], [59], but here we will analyze it to expand its capabilities to solve the problems described in the previous section.

It was found in [59] that the set  $(\mu, \nu, \pi)$  can represent a mass (or probability) assignment in the framework of the  $DST$  [56], [57]. That can be clarified as follows.

It can be seen that when analyzing some object  $O$  in the framework of the  $A-IFS$ , we implicitly deal with the following propositions: *Yes*:  $x \in O$ , *No*:  $x \notin O$ , (*Yes, No*): no of the propositions  $x \in O$  and  $x \notin O$  cannot be rejected (the case of indeterminacy or hesitation).

Therefore  $\mu$  can be treated as the mass of evidence assigned to  $O$  or as the focal element of the basic assignment function:  $m(Yes) = \mu$ . Similarly we get  $m(No) = \nu$ . Since  $\pi$  is usually considered to be a hesitation degree, the evident proposition is  $m(Yes, No) = \pi$ . Because in the  $A-IFS$  theory we have  $\mu + \nu + \pi = 1$ , we can state that the triplet  $\mu, \nu$  and  $\pi$  represents an ordinary basic assignment function in the  $DST$ . Therefore, the  $DST$  values of belief and plausibility can be obtained as follows:  $Bel = m(Yes) = \mu$  and  $Pl = m(Yes) + m(Yes, No) = \mu + \pi = 1 - \nu$  and the belief interval is presented as  $BI = [Bel, Pl]$ . We can see that as the result of above redefinition, the basic components of the  $A - IFS$   $(\mu, \nu, \pi)$  transfer to the basic triplet of the  $DST$   $(m(Yes), m(No), m(Yes, No))$  or to the belief interval  $BI$ .

So based on such a redefinition of an intuitionistic fuzzy number  $\langle \mu, \nu \rangle$  in the framework of the  $DST$ , we obtain a new object, which informal presentation does not include the terms “intuitionistic” and “fuzzy”.

Therefore, to avoid possible confusions, here we introduce a new notion “Belief-Plausibility Number” ( $BPN$ ). Its formal definition formulated also taking into account our remarks concerned to the asymptotic properties (see previous section) is presented as follows:

*Definition 1:* A Belief-Plausibility Number is a mathematical object presented by the belief interval  $BI = [Bel, Pl]$ , where  $Bel = m(Yes)$ ,  $Pl = m(Yes) + m(Yes, No)$  and if  $Bel = 0$ , then  $Pl = 0$  or if  $Bel = 1$ , then  $Pl = 1$ .

In general, this definition looks as somewhat extended definition of a belief interval in the  $DST$ . Nevertheless, the Definition 1 is very important in context of further analysis from the methodological reasons.

First, it is remarkable that this definition does not contain any terms of the ordinary and intuitionistic fuzzy sets theories. So it can be used separately, based only on the mathematical tools of the  $DST$ . Meanwhile, always, when it is really needed, the  $BPN$  can be transformed to the  $A - IFS$  presentation of the problem at hand using the expressions  $Bel = \mu$  and  $Pl = 1 - \nu$ .

In the other asymptotic case, when  $Bel = Pl$ , the  $BPN$  reduces to the usual fuzzy number.

Obviously, based on the above analysis we can conclude that the direct use of the  $DST$  formalism seems to be more general approach than those based on the  $A - IFS$  or the ordinary fuzzy sets theory.

Nevertheless, one can say that as the intuitionistic fuzzy number  $A = \langle \mu, \nu \rangle$  can be converted to the object  $A = [\mu, 1 - \nu]$ , this last object can be naturally treated as a usual interval. But such an interval does not present any useful information with exception that some variable takes values in this interval. Besides, let  $A = \langle 0.6, 0.2 \rangle$  and after conversion we have the usual interval  $A = [0.6, 0.8]$ . Then using the corresponding interval arithmetic addition rule, we get  $A + A = [1.2, 1.6]$ . This interval is completely out of the unit interval and cannot be reconverted to an intuitionistic fuzzy number. On the other hand, using the idempotent addition operation with belief intervals defined in [16], we have no problems with the re-conversion. Therefore, we can say that the interpretation of intuitionistic fuzzy numbers as usual intervals is not useful and provides additional undesirable problems.

According to the commonly accepted generic scientific paradigm, the problem, which cannot be solved based on a given theory, may be solved by the use of a more general theory if such a theory exists. Therefore we should to aware that the  $DST$  can truly serve as a generalization of the  $A - IFS$ . We can see that only few of generic elements of the  $DST$  have been applied above for redefinition of all elements of the  $A - IFS$ . So the  $DST$  looks as a more general theory than the  $A - IFS$  and it should be expected that by applying the  $A - IFS$  objects in the framework of the  $DST$ , we can obtain new results that are not achievable in the  $A - IFS$  setting and we can solve problems unsolvable in the  $A - IFS$ . This is not especially new idea as the  $DST$  is a generic theory for some reputed theories that deal with the uncertainty modeling. It is proved that the probability and possibility theories are the specific asymptotic cases of the  $DST$ . In [16], [17], we showed that the  $DST$  generalizes the  $A - IFS$  and its interval-valued extension. In [60], we proved that the  $DST$  can serve as a fruitful generalization of the hesitant and interval-valued hesitant fuzzy sets. In [61], we presented the  $DST$  generalization of the rule-based evidential reasoning in the  $A - IFS$  environment. This approach we used in [62] for the type 2 diabetes diagnostic.

We used the *DST* extension of the  $A - IFS$  to generalize the *TOPSIS* method in [7]. There is especially close link between the *DST* and the rough set theory and it is really difficult to point out a more general theory between them [63]. The advantages of the *DST* were considered in [64], but we are aware that the *DST* is the more general theory as contrary to the rough set theory it origins the 2 set of derived theories.

Consider the advantages provided us by the *DST* redefinition of the  $A - IFS$ . The essential practical problem of the classical  $A - IFS$  is that decision makers or experts often experience considerable difficulties in assigning of such real values to  $\mu$  and  $\nu$  that always the inequality  $\mu + \nu \leq 1$  will be fulfilled.

When using the *DST* extension of  $A - IFS$ , we are only dealing with the belief interval  $BI = [Bel, Pl]$ , not considering  $\mu$  and  $\nu$  at all, but meanwhile we do not forget that initially we had based on the following representation:  $Bel = \mu$  and  $Pl = 1 - \nu = \mu + \pi$ .

So an expert or a decision maker is asked to assign to the  $Bel$  and  $Pl$ , the real values laying in the interval  $[0, 1]$ . If an expert has assigned a certain value to  $Bel$ , then he/she should inevitably assign a greater value to the plausibility  $Pl$ . This is the natural consequence of two premises: the definitions of belief  $Bel = \mu$  and plausibility  $Pl = \mu + \pi$  in context of the  $A - IFS$  and the semantics of a natural language reflecting the specificity of human thinking. For instance, imagine we consider the possibility of some future political event  $E$ . In this case,  $Bel(E)$  is our belief in the appearance of  $E$  (somehow guaranteed estimation) and  $Pl(E)$  is the plausibility of  $E$  appearance, which can be naturally treated as the degree to which the event  $E$  cannot be excluded or as a greatest admissible degree of belief tolerable in some specific favorable situations, and so on. It is easy to see that in the considered case, an expert should use only such degrees  $Bel$  and  $Pl$  that the constraint  $Bel \leq Pl$  should always been held.

This also follows from the semantics of a natural language and common sense and does not provide problems for an expert in assigning correct real values to  $Bel$  and  $Pl$ . The constraint  $Bel \leq Pl$  is the *DST* form of  $\mu \leq 1 - \nu$  and therefore of  $\mu + \nu \leq 1$ .

We can see that the analyzed practical problem of the classical  $A - IFS$  disappears in the framework of the *DST* extension. It is worthy to note here that using the proposed approach based on the defined above *BPN*, we do not need to use the somewhat artificial and heuristic Neutrosophic, Pythagorean and Spherical sets theories with their derivatives.

The other problem of the classical  $A - IFS$  is the drawbacks of commonly used mathematical operations with intuitionistic fuzzy numbers (*IFN*). In [16], using several persuasive examples we showed that the classical operations with *IFN*s possess 6 negative properties (two of them were firstly found in [65]), which may lead to unacceptable results of the real-world problems solutions (more information can be found in [16]). This is not so surprising as even the definitions

of these operations seem to be controversial, e.g. the multiplication by a scalar looks as completely unacceptable or wrong operation as it is directly defined by the power operation.

Therefore in [16], using the dual origin of a belief interval, which is an interval comprising a true power of some statement and at the same time is a usual interval, we inferred in the framework of the *DST* redefinition of the  $A - IFS$ , the set of operations with belief intervals representing corresponding *IFN*s. These operations are free of above mentioned drawbacks (this was proved formally with the use of relevant theorems). It is important that the introduced *DST* redefinition of the  $A - IFS$  generates new operations, particularly the power operation where both operands are belief intervals, which can be used for the extension of the Weighted Geometric operator, but cannot be defined within the framework of the classical  $A - IFS$ . This fact confirms the superiority of the *DST* based approach over the classical  $A - IFS$ , i.e. the *DST* redefinition is a more general theory in the relation to the  $A - IFS$ .

The *DST* extension of the interval-valued  $A - IFS$  was introduced in [17].

Here we present ones more important practical argument in favor of the separate use of the *BPN* (see Definition 1 above) without any terms from  $A - IFS$ . In communications with experts, we found that they easily and willingly set the values of the degree of membership. Very often in technological practice, particular criteria are formulated in the form of trapezoidal fuzzy number the top of which is the interval of the best values of the parameter and the bottom is the interval of acceptable values. Experts are used to this and are willing to use it. However, with the degree of non-membership, difficulties arise.

Let us say the expert sets  $\mu=0.2$ . Then he/she automatically sets  $\nu=0.8$ . Moreover, he/she considers other values of  $\nu$  as erroneous. It is quite possible that such a probabilistic style of thinking, which is a consequence of a good higher education, prevents the widespread use of methods of  $A - IFS$  to solve real-world problems, although there are other reasons of rather psychological nature. But in any case, this is the problem of the  $A - IFS$ .

At the same time, when the experts were asked to evaluate the *BPN*, there were no noticeable problems. It is characteristic that sometimes the experts offered their own interpretations of the belief interval  $BI$ . Among them the most successful and transparent was the interpretation of the  $BI = [Bel, Pl]$  as the interval between pessimistic and optimistic assessments of the membership degree.

It is worth noting that our approach is flexible enough. For example, if someone wants to use only the  $\mu$  and  $\nu$  formalism of the  $A - IFS$ , he/she can first obtain from experts the estimations of *BPN*s (avoiding the above discussed problems concerned with the estimation of  $\nu$ ) and then convert *BPN*s into  $\mu$  and  $\nu$  based on the expressions  $\mu = Bel$ ,  $\nu = 1 - Pl$ .

We should note here that in practice, we can meet the data available with such a structure that the  $\mu$  and  $\nu$  appear objectively without expert's judgments. (see examples in [61], [66]).

Summarizing we can state that the introduced *DST* redefinition of the  $A - IFS$  is fruitful and justified from both theoretical and practical points of view.

### III. NEW REPRESENTATIONS OF PEACE-WISE MEMBERSHIP AND NON-MEMBERSHIP FUNCTIONS, THEIR INTERVAL EXTENSIONS BASED ON THE REVEALED SOURCES OF INTERVAL AN FUZZY UNCERTAINTY

In the previous section, we considered only intuitionistic fuzzy numbers and Belief-Plausibility numbers without explicit pointing out their origins. Here we will consider objects characterizing by the functions  $\mu(x)$  and  $\nu(x)$ , i.e. if  $A \subset X$  then it is defined by the functions  $\mu_A(x)$  and  $\nu_A(x)$ . In the literature [21]–[41] the such type objects as  $A$  are called intuitionistic fuzzy numbers. Taking into account that in the previous section we called so (very reasonably) the objects of a different kind, here we will call objects of type  $A$  defined by  $\mu_A(x)$  and  $\nu_A(x)$ , intuitionistic fuzzy values (*IFV*). We are not going to introduce here any new terminology. Here we just use the terms “number” and “value” to indicate the differences between the two analyzed types of objects. Similarly, the objects defined by the  $Bel(x)$  and  $Pl(x)$  will be called Belief-Plausibility values (*BPV*).

We deliberately do not give strict definitions at the beginning of our analysis as they will appear later when all the necessary prerequisites are formulated.

#### A. DEFINITION AND BUILDING OF THE REAL-VALUED IFVs AND BPVs

As it was stated in the previous section, one of the problem of the known *IFV* definitions [21]–[41] are the controversial asymptotic properties.

Let us consider a typical case presented in Fig. 1a. We can see that in the interval  $[x_2, x_3]$  we have  $\nu(x) = 0$ , i.e. with the 100% of certainty ( $\pi(x) = 0$ ) we should exclude non-membership and therefore we have a case of complete membership ( $\mu(x) = 1$ ). But we have such a result only in one point in the  $[x_2, x_3]$ . In the interval  $[x_4, x_5]$ , we deal with the 100% of non-membership as  $\mu(x) = 0$ . So it should be  $\nu(x) = 1$  in this interval, but we see that  $0 < \nu(x) \leq 1$ .

Let us consider this problem from the more general point of view. Really, if  $\mu(x) = 1$ , then we have a complete (100%) certainty that some event has occurred. Therefore there is no place for the opposite event and hesitation, i.e. it should be  $\nu(x) = 0$  and  $\pi(x) = 0$ . The similar analyzes can easily be done for other cases.

At first glance, this problem can be solved by the small correction as in Fig. 1b), but however, this will not save the expert from the need to justify the feasibility of using such uncertain data (in the intervals  $[x_2, x_3]$  and  $[x_4, x_5]$ ) and this is also not always possible.

The next flaw of the known approaches is the lack of clear instructions on how to build the  $\mu(x)$  and  $\nu(x)$  functions so that it always executes  $0 \leq \mu(x) + \nu(x) \leq 1$ .

Let us consider the simple triangular functions  $\mu_A(x)$  and  $\nu_A(x)$ . It is easy to see that if they intersect in some  $x$

and  $0.5 < \mu_A(x) = \nu_A(x)$ , then  $1 < \mu_A(x) + \nu_A(x)$ ,  $x \in A$  and we have no an *IFV*. If  $\mu_A(x) = \nu_A(x) = 0.5$  we obtain an ordinary fuzzy value:  $\mu_A(x) + \nu_A(x) = 1$ ,  $x \in A$ . In the case of  $\mu_A(x) = \nu_A(x) < 0.5$ , we have  $\mu_A(x) + \nu_A(x) < 1$ ,  $x \in A$ , but the problems similar to those presented in Fig. 1a automatically arise. The similar situation we meet in the case of trapezoidal functions  $\mu_A(x)$  and  $\nu_A(x)$ .

For the sake of simplicity, hereinafter we will consider only triangular (not necessary simple) types of the functions  $\mu_A(x)$  and  $\nu_A(x)$ .

Although we showed that the simple triangular cannot provide acceptable non-controversial presentation of the functions  $\mu_A(x)$  and  $\nu_A(x)$ , the use of the small extension of triangular-the Pentagonal (piece-wise linear functions based on 5 reference points) *IFVs* [47]–[51] generally can solve this problem. In Fig. 2, we can see the example of such a presentation of the intuitionistic fuzzy value. It is seen that the lesser are the values of  $\mu_A(x)$  and  $\nu_A(x)$  in the point of intersection, the lesser is the uncertainty (hesitation degree) of the *IFV*. This may help in building the  $\mu_A(x)$  and  $\nu_A(x)$ . Also we can see that all reasonable asymptotic properties are fulfilled.

Of course, the proposed approach to building the functions  $\mu_A(x)$  and  $\nu_A(x)$ , which is focused on their intersection is not unique. Nevertheless, we recommend it as providing great convenience, when it is implemented as the graphical interface: by dragging and dropping with the mouse that pointed out on possible intersections and holding other reference points to be stable, we can easily establish suitable forms of  $\mu_A(x)$  and  $\nu_A(x)$ .

The above premises allow us to introduce some formal definitions.

**Theorem 1:** If the Pentagonal membership  $\mu_A(x)$  and non-membership  $\nu_A(x)$  functions such that  $\mu_A(x) = 1$ , when  $\nu_A(x) = 0$  and  $\mu_A(x) = 0$ , when  $\nu_A(x) = 1$ , intersect in the points where  $\mu_A(x) = \nu_A(x) < 0.5$ , then  $\mu_A(x) + \nu_A(x) < 1$ ,  $x \in A$ .

**Proof:** Follows directly from the geometric construction.  $\square$

Of course, the same theorems hold for piece-wise linear functions  $\mu(x)$  and  $\nu(x)$ , e.g., for the Hexagonal, Heptagonal, Octagonal *IFVs* and so on, as well as for any normalized convex  $\mu(x)$  and the corresponding concave  $\nu(x)$ .

**Definition 2:** The Pentagonal intuitionistic fuzzy value (*PIFV*) is the mathematical object defined by the pentagonal membership  $\mu_A(x)$  and non-membership  $\nu_A(x)$  functions intersecting in the points, where  $\mu_A(x) = \nu_A(x) < 0.5$  such that  $\mu_A(x) = 1$ , when  $\nu_A(x) = 0$  and  $\mu_A(x) = 0$ , when  $\nu_A(x) = 1$ .

Obviously, in any fixed  $x_1 \in A$  from *PIFV* we get the intuitionistic fuzzy number  $IFN = PIFV(x_1) = < \mu_A(x_1), \nu_A(x_1) >$ .

The Pentagonal intuitionistic fuzzy functions  $\mu_A(x)$  and  $\nu_A(x)$  can be presented as follows:

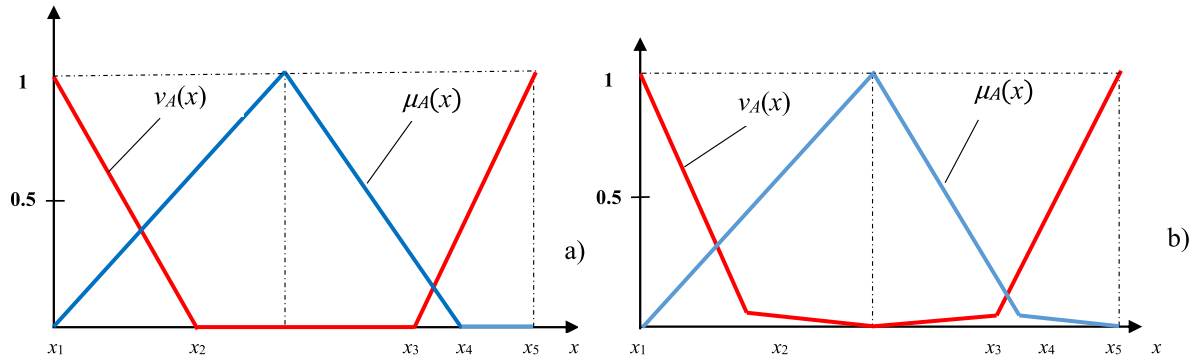


FIGURE 1. The controversial a) and the formally correct b) membership and non-membership functions.

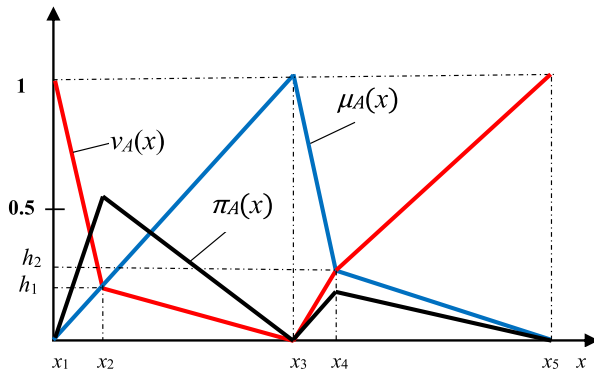


FIGURE 2. The example of the correct membership  $\mu_A(x)$ , non-membership  $\nu_A(x)$  and hesitation  $\pi_A(x)$  functions.

Let the interval  $[x_1, x_5]$  be the support of  $A \subset X$  (see Fig. 2). Then based on Fig. 2 we get:

$$\begin{aligned}
 \mu_A(x) &= h_1 \frac{(x - x_1)}{(x_2 - x_1)}, & \text{if } x \in [x_1, x_2]; \\
 \mu_A(x) &= h_1 + (1 - h_1) \frac{(x - x_2)}{(x_3 - x_2)}, & \text{if } x \in [x_2, x_3]; \\
 \mu_A(x) &= 1, & \text{if } x = x_3; \\
 \mu_A(x) &= 1 - (1 - h_2) \frac{(x - x_3)}{(x_4 - x_3)}, & \text{if } x \in [x_3, x_4]; \\
 \mu_A(x) &= h_2 \frac{(x_5 - x)}{(x_5 - x_4)}, & \text{if } x \in [x_4, x_5]; \\
 \nu_A(x) &= 1 - (1 - h_1) \frac{(x - x_1)}{(x_2 - x_1)}, & \text{if } x \in [x_1, x_2]; \\
 \nu_A(x) &= h_1 \frac{(x_3 - x)}{(x_3 - x_2)}, & \text{if } x \in [x_2, x_3]; \\
 \nu_A(x) &= 0, & \text{if } x = x_3; \\
 \nu_A(x) &= h_2 \frac{(x - x_3)}{(x_4 - x_3)}, & \text{if } x \in [x_3, x_4]; \\
 \nu_A(x) &= h_2 + (1 - h_2) \frac{(x_5 - x)}{(x - x_4)}, & \text{if } x \in [x_4, x_5], \\
 h_1, h_2 &\leq 0.5. & (3)
 \end{aligned}$$

Significantly less problems we meet when building the  $BPV_A(x) = [Bel_A(x), Pl_A(x)]$ , as in this case only what we should worried about of is the holding the constraint

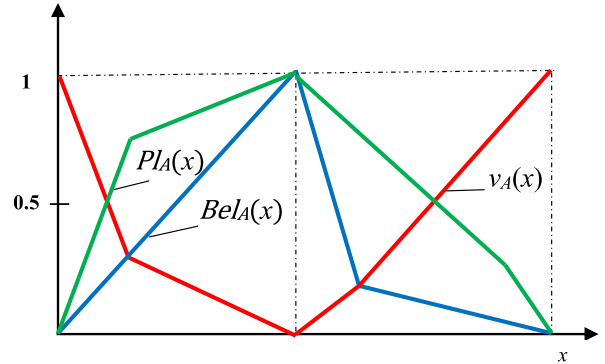


FIGURE 3. The example of the correct belief  $Bel_A(x)$ , plausibility  $Pl_A(x)$  and non-membership  $\nu_A(x)$  functions.

$Bel_A(x) \leq Pl_A(x)$  and asymptotic properties seem to be obvious and do not need additional explanations.

The formal definition of the  $BPV_A(x)$  is as follows:

**Definition 3:** The Belief-Plausibility value ( $BPV$ ) is the mathematical object defined by the functions  $Bel_A(x)$  and  $Pl_A(x)$  such that  $Pl_A(x) = 0$ , when  $Bel_A(x) = 0$ ,  $Pl_A(x) = 1$ , when  $Bel_A(x) = 1$  and  $0 \leq Bel_A(x) \leq Pl_A(x) \leq 1$ ,  $x \in A$ . For any fixed  $x_1 \in A$  from  $BPV$  we get the Belief-Plausibility number  $BPN = BPV(x_1) = [Bel_A(x_1), Pl_A(x_1)]$ .

It is remarkable that in comparison with the case of  $IFV$ , in this definition there is no need for any simplification, e.g. the use of only Pentagonal functions.

The example is shown in Fig. 3.

### B. INTERVAL-VALUED EXTENSIONS OF THE $IFV$ s AND $BPV$ s RESULTING FROM THE MULTIPLICITY OF MEMBERSHIP AND NON-MEMBERSHIP FUNCTIONS PRESENTATION

This is the case when, e.g., differed experts are asked to present own versions of functions  $\mu_A(x)$  and  $\nu_A(x)$ . The correct interval-valued Pentagonal Intuitionistic fuzzy value  $IVPIFV$  defined by the interval valued functions  $[\mu_A(x)]$  and  $[\nu_A(x)]$  in presented in Fig. 4. It is easy to see that to get a correct  $IVPIFV$  it is enough to take care that at the points of intersection of the upper bounds of the interval functions  $[\mu_A(x)]$  and  $[\nu_A(x)]$  it is always performed  $\mu_{AU}(x) = \nu_{AU}(x) \leq 0.5$ .



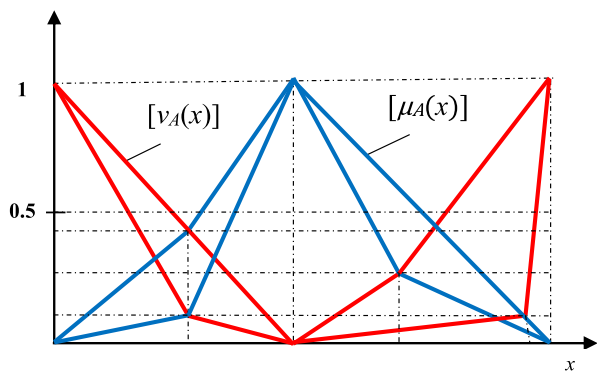


FIGURE 4. The correct interval membership  $[\mu_A(x)]$  and non-membership  $[v_A(x)]$  functions.

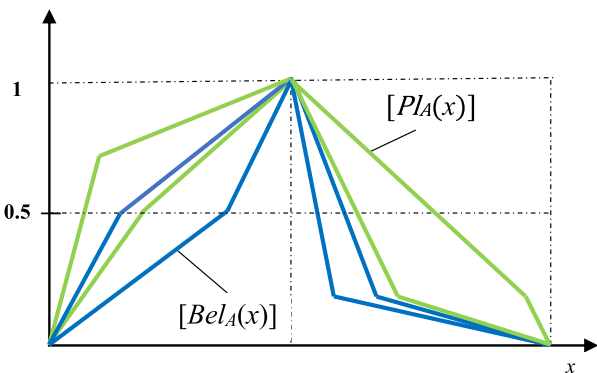


FIGURE 5. The example of the correct interval-valued belief  $[Bel_A(x)]$  and plausibility  $[Pl_A(x)]$  functions.

**Definition 4:** The Interval-Valued Pentagonal Intuitionistic fuzzy value  $IVPIFV$  is the  $IF$  object presented by the interval-valued membership and non-membership functions  $\langle [\mu_{AL}(x), \mu_{AU}(x)], [v_{AL}(x), v_{AU}(x)] \rangle$  such that if  $\mu_{AL}(x) = 0, \mu_{AU}(x) = 0$ , then  $v_{AL}(x) = 1, v_{AU}(x) = 1$  and if  $\mu_{AL}(x) = 1, \mu_{AU}(x) = 1$ , then  $v_{AL}(x) = 0, v_{AU}(x) = 0$ , while at the points of intersection of the upper bounds of the interval functions  $[\mu_A(x)]$  and  $[v_A(x)]$  it is always held  $\mu_{AU}(x) = v_{AU}(x) \leq 0.5$ .

For any fixed  $x_1 \in A$  from  $IVPIFV$  we obtain the interval-valued intuitionistic fuzzy number  $IVIFN = PIFV(x_1) = \langle [\mu_{AL}(x_1), \mu_{AU}(x_1)], [v_{AL}(x_1), v_{AU}(x_1)] \rangle$  consisting of two intervals, which may intersect or not (this will be studied below).

The interval-valued Belief-Plausibility value  $IVBPV_A(x)$  defined by the corresponding interval belief  $[Bel_A(x)]$  and plausibility  $[Pl_A(x)]$  functions is presented in Fig. 5.

The natural constraint should be held for the upper bounds:  $Bel_{AU}(x) \leq Pl_{AU}(x)$ . This constraint can be redefined in the framework of the  $A - IFS$  (-taking into account that  $Bel_{AU}(x) = \mu_{AU}(x)$  and  $Pl_{AU}(x) = 1 - v_{AL}(x)$ ) as follows:  $\mu_{AU}(x) + v_{AL}(x) \leq 1$ . This important constraint is absent in the Atanassov’s definition [2] and was first introduced in our paper [17].

But here we will call it “soft constraint” since it does not prevent the cases when  $Bel_{AU}(x) \geq Pl_{AL}(x)$  as in

the example in Fig. 5 (left side), which can be treated as controversial situations. Therefore, the “strong constraint”  $Bel_{AU}(x) \leq Pl_{AL}(x)$  that prevents the controversial situations like that in Fig. 5 (right side) reduces to its  $A - IFS$  form  $\mu_{AU}(x) + v_{AL}(x) \leq 1$ , which is the same as in [2].

It is easy to see that if the strong constraint is held, then the soft constraint is fulfilled as well, but not vice versa.

Besides, as we deal with interval extensions of belief and plausibility functions, the natural thing is the extension of the basic inequality  $Bel \leq Pl$  to its interval extension  $[Bel_A(x)] \leq [Pl_A(x)]$ , where inequality is treated in the interval form.

There are many methods for interval comparison proposed in the literature. The most popular ones were studied in [67], where it was found that the simplest method based of intervals midpoints comparison provides results not the worst than even the complex methods based on the probabilistic approach [68]. Then the above interval inequality can be presented as  $Bel_{AL}(x) + Bel_{AU}(x) \leq Pl_{AL}(x) + Pl_{AU}(x)$ .

Of course, having three possible constraints, we should present three different definitions of  $IVBPV_A(x)$ , but since the last of them seems to be the some compromise between two first considered constraints, we propose here the following definition:

**Definition 5:** The Interval-valued Belief-Plausibility value  $IVBPV_A(x)$  is the object represented by the interval-valued belief  $[Bel_A(x)]$  and plausibility  $[Pl_A(x)]$  functions such that if  $Bel_{AL}(x) = Bel_{AU}(x) = 0$ , then  $Pl_{AL}(x) = Pl_{AU}(x) = 0$  and if  $Bel_{AL}(x) = Bel_{AU}(x) = 1$ , then  $Pl_{AL}(x) = Pl_{AU}(x) = 1$ , while  $Bel_{AL}(x) + Bel_{AU}(x) \leq Pl_{AL}(x) + Pl_{AU}(x)$ .

For any  $x_1 \in A$  from  $IVBPV$  we obtain the Interval-Valued Belief-Interval number, which can be defined as follows:

**Definition 6:** The Interval-Valued Belief-Plausibility number under uncertainty caused by multiplicity of belief and plausibility functions is presented by the belief interval bounded belief interval:

$$IVBP = IVBPV(x_1) = [[Bel_{AL}(x_1), Bel_{AU}(x_1)], [Pl_{AL}(x_1), Pl_{AU}(x_1)]] \quad (4)$$

The introduced  $IVBPN$  is the belief interval bounded by two intersecting or not belief intervals. We can state that the intervals  $[Bel_{AL}(x_1), Bel_{AU}(x_1)]$  and  $[Pl_{AL}(x_1), Pl_{AU}(x_1)]$  can be formally treated as the belief intervals since in the Exp.(2) we have  $0 \leq Bel_{AL}(x_1) \leq Bel_{AU}(x_1) \leq 1$  and  $0 \leq Pl_{AL}(x_1) \leq Pl_{AU}(x_1) \leq 1$ . In the Definition 6, the presentation of  $IVBPN$  differs from that in [17], which in the current notation can be presented as follows:  $IVBPN = [[Bel_{AL}, Pl_{AL}], [Bel_{AU}, Pl_{AU}]]$ . We can say that a new Definition 6 may be treated as the upgraded version of that introduced in [17] as it is the direct interval extension of the real-valued  $BPN$  when the actual interval uncertainty is originated from multiplicity of belief and plausibility functions, whereas

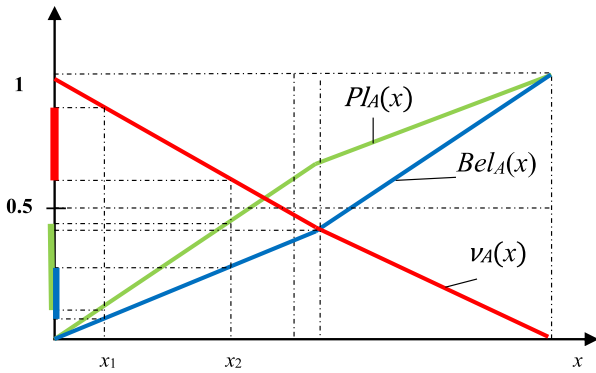


FIGURE 6. Calculation of  $Bel_A(x)$ ,  $Pl_A(x)$  and  $v_A(x)$  functions of interval argument  $[x_1, x_2]$ .

the definition from [17] is formulated without specifying the source of the interval uncertainty.

**C. INTERVAL-VALUED EXTENSIONS OF THE IFVs AND BPVs AS THE CONSEQUENCE OF THE INTERVAL NATURE OF INPUT DATA**

First we should say that introducing interval and fuzzy input data in the intuitionistic framework is not a purely mathematical exercise, but is justified by the specificity of real-world problems (see Section 4).

Suppose  $x$  be some parameter defining values of the corresponding local criterion represented by the  $IVIFV(x)$  or  $IVBPV(x)$ , where values of  $x$  belong to the interval of unstable measurements:  $x \in [x_1, x_2]$  as it is shown in Fig. 6. Generally for calculating the value of  $IVBPN$ , the following definition can be used:

*Definition 7:* The Interval-Valued Belief-Plausibility number under uncertainty coursed by interval nature of input data is the belief interval bounded interval number defined by the belief and plausibility function as follows:

$$\begin{aligned}
 IVBPN &= IVBPV([x_1, x_2]) \\
 &= [[ \inf_{x \in [x_1, x_2]} (Bel_A(x)), \sup_{x \in [x_1, x_2]} (Bel_A(x)) ], \\
 &\quad [ \inf_{x \in [x_1, x_2]} (Pl_A(x)), \sup_{x \in [x_1, x_2]} (Pl_A(x)) ]]. \quad (5)
 \end{aligned}$$

It is important that in this case, we obtain again the belief interval bounded by belief intervals, since in the Exp.(3) we have always  $0 \leq \inf Bel \leq \sup Bel \leq 1$  and  $0 \leq \inf Pl \leq \sup Pl \leq 1$ . So the Definition 7 is the upgraded version of that in [17] as it directly specifies the source of interval uncertainty. In the considered case (see Fig. 6), the general expression (3) can be simplified to  $IVBPN = IVBPV([x_1, x_2]) = [[Bel_A(x_1), Bel_A(x_2)], [Pl_A(x_1), Pl_A(x_2)]]$ . In this example, there are no intersections between the resulting intervals.

In the example presented in Fig. 7, the functions  $Bel_A(x)$  and  $Pl_A(x)$  have their maximum equal to 1 and the function  $v_A(x)$  has its minimum equal to 0 in  $[x_1, x_2]$ .

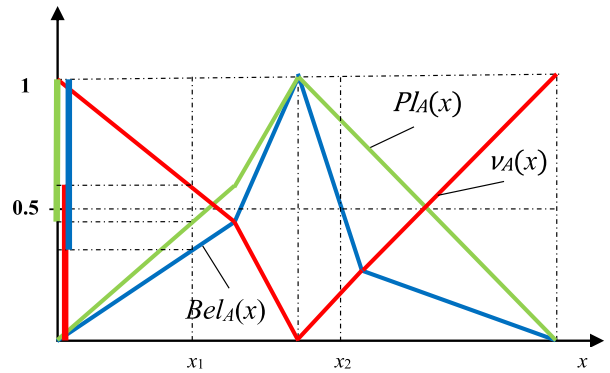


FIGURE 7. Calculation of  $Bel_A(x)$ ,  $Pl_A(x)$  and  $v_A(x)$  functions of interval  $[x_1, x_2]$ , where the  $\sup(Bel_A(x))$  and  $\sup(Pl_A(x))$  are achievable.

As the result, we obtain:  $IVBPN = IVBPV([x_1, x_2]) = [[Bel_A(x_1), 1], [Pl_A(x_1), 1]]$ . It is seen that in this case, all the resulting intervals are intersecting. It is shown in Fig. 7 and in Fig. 8 that all possible mutual locations of resulting intervals can be obtained. It is important that we often meet the intersecting intervals  $[Bel_A(x_1), Bel_A(x_2)]$  and  $[Pl_A(x_1), Pl_A(x_2)]$  such that  $Bel_A(x_2) > Pl_A(x_1)$  with the relatively wide area of intersection as in Fig. 7. Of course, we can complete eliminate this area redefining the results of the  $IVBPN$  calculation as follows:

$$IVBPN = IVBPV([x_1, x_2]) = [[Bel_A(x_1), Pl_A(x_1)],$$

$[Bel_A(x_2), Pl_A(x_2)]]$ . It is easy to see (Fig. 6) that in this case, we get two relatively small intervals  $[Bel_A(x_1), Pl_A(x_1)]$  and  $[Bel_A(x_2), Pl_A(x_2)]$  with the great gap between them. So instead of obtained in the direct calculations information we have the gap, which the width can serve as the measure of information lost. So such an approach seems to be at least unproductive with great information losses.

Therefore, we will use here the described above approach based on the Exp.(3) and its simplification.

It should be stressed that there are no any confrontational or debatable assumptions and methods in the Definition 7. So we should not reject from any information obtained with its use, but only properly treat and apply it in the analysis. For example, we have found that the width of the  $Bel$  and  $Pl$  intervals intersection is gradually decreasing along with lowering an interval uncertainty, i.e. the width of input interval. So this intersection area should be used in analysis. Also, it can be considered as an indeterminacy area, where nor the belief estimation and no the plausibility one cannot be excluded and this is entirely in the spirit of the Dempster-Shafer theory formalism.

It is important that although we have introduced here two different definitions 6 and 7 for the  $IVBPN$  and they somewhat differ from that in [17], the common thing in these definitions is the introduced concept of belief interval bounded by belief intervals. On the other hand, the Definitions 6 and 7 are the specified and upgraded versions of that in [17] as they

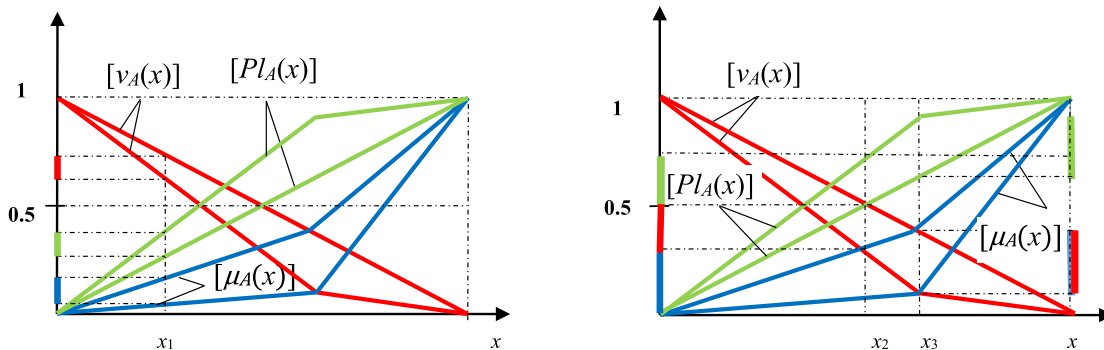


FIGURE 8. Calculations of the interval-valued  $[Bel_A(x)]$ ,  $[Pl_A(x)]$  and  $[v_A(x)]$  functions of real-valued arguments.

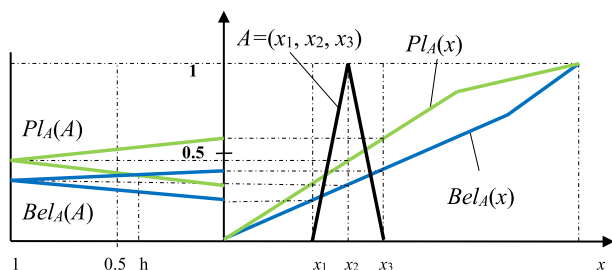


FIGURE 9. Calculating the triangular  $TFVBPn$ , which depends on the triangular fuzzy input variable.

originated from the different sources of interval uncertainty, while in the definition from [17] there are no any indications of uncertainty sources.

**D. TRIANGULAR FUZZY EXTENSION OF THE IFVs AND BPVs ORIGINATED FROM THE TRIANGULAR FUZZY NATURE OF INPUT DATA**

Suppose  $A = (x_1, x_2, x_3)$  is the simple triangular fuzzy value representing the range of measured values of the input parameter  $x$ , where  $[x_1, x_3]$  is the interval of all measurements and  $x_2$  is the point, around which the measurements are observed most frequently, so that in the point  $x = x_2$ , we have the top of  $A$ , i.e. the value of its membership function is equal to 1.

Let  $BPV_A(x)$  be the Belief-Plausibility value representing some criterion in the  $MCDM$  task by the corresponding belief and plausibility functions  $Bel_A(x)$  and  $Pl_A(x)$ . Then the value of this criterion for the fuzzy argument  $A = (x_1, x_2, x_3)$  can be obtained as the Triangular Fuzzy-Valued Belief-Plausibility Number  $TFVBPn_A = BPV_A(A)$ , which in our case of simple triangular  $A$ , can be presented by the mapping of  $A$  on the  $Bel_A(x)$  and  $Pl_A(x)$  as it is shown in Fig. 9.

We can see that the resulting  $TFVBPn_A$  is the belief interval bounded by the triangular fuzzy numbers  $Bel_A(A)$  and  $Pl_A(A)$  defined on the unit interval and such that  $Bel_A(A) \leq Pl_A(A)$ . So these triangles can be treated as the special Triangular Belief-Plausibility numbers. Therefore we can introduce the following definition:

*Definition 8:* The Triangular Belief-Plausibility Number is the belief interval bounded by the special Belief-Plausibility triangular fuzzy numbers defined on the unit interval.

In this case (see Fig. 9), the values of these bounding triangles can be calculated as follows:

$$Bel_A(A) = (Bel_A(x_1), Bel_A(x_2), Bel_A(x_3)), Pl_A(A) = (Pl_A(x_1), Pl_A(x_2), Pl_A(x_3)).$$

It is seen that  $Bel_A(A)$  and  $Pl_A(A)$  can intersect. In such cases, the small interval in the intersecting area, similarly to the considered in the above section interval extension, can serve as the measure of uncertainty caused by the uncertainty of the input triangular fuzzy value  $A = (x_1, x_2, x_3)$ . This is an interesting and useful feature of the developed approach, but now it is out of scope of this paper.

**E. THE ARITHMETIC OPERATION WITH LOCAL CRITERIA REPRESENTED BY BPNs DEPENDENT ON INTERVAL AND FUZZY ARGUMENTS**

The defined in the previous sections uncertain values  $IFV$ ,  $IVIFV$ ,  $BPV$ ,  $IVBPV$  based on the different types of uncertainty, can be directly used in the calculation, which usually are provided applying the  $\alpha$ -cuts representation of uncertain values. For example, this approach is justified in the solution of the Transportation Problem [47].

It is important that in this case, only the operations with uncertain values reduced to the operations with intervals on the  $\alpha$ -cuts are used, while the operations with uncertain numbers are not needed at all.

Therefore, our approach based on the introduction of the separate definitions and operational rules for uncertain values and uncertain numbers is methodologically justified.

When dealing with the  $MCDM$  problems, we usually deal with calculation of local criteria values based on some functions of the important parameters. Then we use these values in different aggregation modes. Therefore, if we have the  $MCDM$  problem under uncertainty, then we should use the such values as the numbers  $IFN$ ,  $IVIFN$ ,  $BPN$ ,  $IVBPN$ ,  $TFBPN$  defined above depending on the type of uncertainty occurred.

Then the corresponding operational laws with these numbers should be defined. In [16], [17], we showed that the classical arithmetical operations with the  $IFNs$  and  $IVIFNs$  are suffered from a set of important drawback that may lead to

unacceptable, illogical results. Therefore, in [16], [17] based on the redefinition of  $A - IFS$  and  $A - IVIFS$  in the framework of  $DST$ , the sets of new operations with  $BPNs$  and  $IVBPNs$  free of the revealed drawbacks of the classical operations with the  $IFNs$  and  $IVIFNs$  were proposed and justified.

Based on the above conclusions, we will use directly the numbers  $BPNs$ ,  $IVBPNs$  and  $TFBPNs$  to represent local criteria in the  $MCDM$  problem or the numbers obtained by the  $DST$  based redefinition (extension) of the problems firstly defined in context of the  $A - IFS$  or the  $A - IVIFS$ . These operations are originated from the dual nature of a belief interval, which simultaneously can be treated as an interval containing a power of some statement and as an ordinary interval included in the unit one.

So based on the first above treatment of a belief interval, the addition operation with belief intervals was inferred in such a way that it is idempotent one and provides a belief interval as the result. These demands can be provided only by the averaging operation. The other operations used in  $MCDM$  were defined as those in the classical interval arithmetic.

So for the real-valued  $BPN_1 = [Bel_1, Pl_1]$ ,  $BPN_2 = [Bel_2, Pl_2]$ , ...,  $BPN_n = [Bel_n, Pl_n]$  in [16], it was obtained:

$$BPN_1 \oplus BPN_2 \oplus \dots \oplus BPN_n = \left[ \frac{1}{n} \sum_{i=1}^n Bel_i, \frac{1}{n} \sum_{i=1}^n Pl_i \right]. \tag{6}$$

$$BPN_1 \otimes BPN_2 = [Bel_1 Bel_2, Pl_1 Pl_2]. \tag{7}$$

$$\alpha BPN = [\alpha Bel, \alpha Pl], \tag{8}$$

where  $\alpha$  is a real value. This operation is correct only for  $\alpha \leq 1$  since for  $\alpha > 1$  this operation cannot not always provide a correct belief interval. This restriction is admissible as  $\alpha$  is usually used in the operations with belief intervals to solve  $MCDM$  problems, where  $\alpha$  is less of 1 as it serves being the local criterion weight.

$$BPN^\alpha = [Bel(A)^\alpha, Pl(A)^\alpha], \tag{9}$$

where  $\alpha \geq 0$ .

$$BPN_1^{BPN_2} = [Bel_1^{Pl_2}, Pl_1^{Bel_2}]. \tag{10}$$

It is worth noting that the last operation cannot not be defined in the body of classical  $A - IFS$ .

With the use of corresponding theorems, we have proved in [16] that these operations have good algebraic properties and free of revealed drawbacks of the conventional operations defined in the framework of the  $A - IFS$ .

Here we present the more justified than in [17], upgraded set of arithmetical operations with  $IVBPNs$  obtained with the use of Definition 7 for the case of interval uncertainty of input data, as based on these operations the operation laws with  $TFVBPNs$  used in the case study in Section 4 were inferred.

Let

$$IVBPN_1 = IVBPV([x_{11}, x_{12}]) = [[Bel_1(x_{11}), Bel_1(x_{12})],$$

$$[Pl_1(x_{11}), Pl_1(x_{12})]] = [[Bel_{11}, Bel_{12}], [Pl_{11}, Pl_{12}]]$$

and

$$IVBPN_2 = IVBPV([x_{21}, x_{22}]) = [[Bel_2(x_{21}), Bel_2(x_{22})], [Pl_2(x_{21}), Pl_2(x_{22})]] = [[Bel_{21}, Bel_{22}], [Pl_{21}, Pl_{22}]].$$

Then

$$IVBPN_1 \oplus_{int} IVBPN_2 \oplus_{int} \dots \oplus_{int} IVBPN_n = \frac{1}{n} \left[ \left[ \sum_{i=1}^n Bel_{i1}, \sum_{i=1}^n Bel_{i2} \right], \left[ \sum_{i=1}^n Pl_{i1}, \sum_{i=1}^n Pl_{i2} \right] \right]. \tag{11}$$

$$IVBPN_1 \otimes_{int} IVBPN_2 = [[Bel_{11} Bel_{21}, Bel_{12} Bel_{22}], [Pl_{11} Pl_{21}, Pl_{12} Pl_{22}]]. \tag{12}$$

$$\alpha IVBPN_1 = [\alpha [Bel_{11}, Bel_{12}], \alpha [Pl_{11}, Pl_{12}]] = [[\alpha Bel_{11}, \alpha Bel_{12}], [\alpha Pl_{11}, \alpha Pl_{12}]], \alpha < 1. \tag{13}$$

$$IVBPN_1^\alpha = [[(Bel_{11})^\alpha, (Bel_{12})^\alpha], [(Pl_{11})^\alpha, (Pl_{12})^\alpha]]. \tag{14}$$

$$IVBPN_1^{BPN} = [[Bel_{11}, Bel_{12}]^{[Bel, Pl]}, [Pl_{11}, Pl_{12}]^{[Bel, Pl]}] = [[(Bel_{11})^{Pl}, (Bel_{12})^{Bel}], [(Pl_{11})^{Pl}, (Pl_{12})^{Bel}]], \tag{15}$$

$$IVBPN_1^{IVBPN_2} = \left[ [(Bel_{11})^{Pl_{22}}, (Bel_{12})^{Bel_{21}}], [(Bel_{12})^{Pl_{22}}, (Pl_{12})^{Bel_{21}}] \right]. \tag{16}$$

These operations have good algebraic properties and are free of the drawbacks of corresponding operations defined in the  $A - IVIFS$ . This is easily proved using the same theorems as in [17] replacing the  $Bel^L$  in them with  $Bel_{11}$ , the  $Pl^L$  with  $Bel_{12}$ , the  $Bel^U$  with  $Pl_{11}$  and  $Pl^U$  with  $Pl_{12}$  in the first operand and by making similar replacements in the second one.

Let

$$TFVBPN_1 = ((Bel_{11}, Bel_{12}, Bel_{13}), (Pl_{11}, Pl_{12}, Pl_{13})),$$

$$TFVBPN_1 = ((Bel_{21}, Bel_{22}, Bel_{23}), (Pl_{21}, Pl_{22}, Pl_{23})).$$

Then

$$TFVBPN_1 \oplus_{fuzzy} TFVBPN_2 \oplus_{fuzzy} \dots \oplus_{fuzzy} TFVBPN_n = \frac{1}{n} \left[ \left( \sum_{i=1}^n Bel_{i1}, \sum_{i=1}^n Bel_{i2}, \sum_{i=1}^n Bel_{i3} \right), \left( \sum_{i=1}^n Pl_{i1}, \sum_{i=1}^n Pl_{i2}, \sum_{i=1}^n Pl_{i3} \right) \right]. \tag{17}$$

$$TFVBPN_1 \otimes_{fuzzy} TFVBPN_2 = ((Bel_{11} Bel_{21}, Bel_{12} Bel_{22}, Bel_{13} Bel_{23}), (Pl_{11} Pl_{21}, Pl_{12} Pl_{22}, Pl_{13} Pl_{23})). \tag{18}$$

$$\alpha TFVBPN_1$$



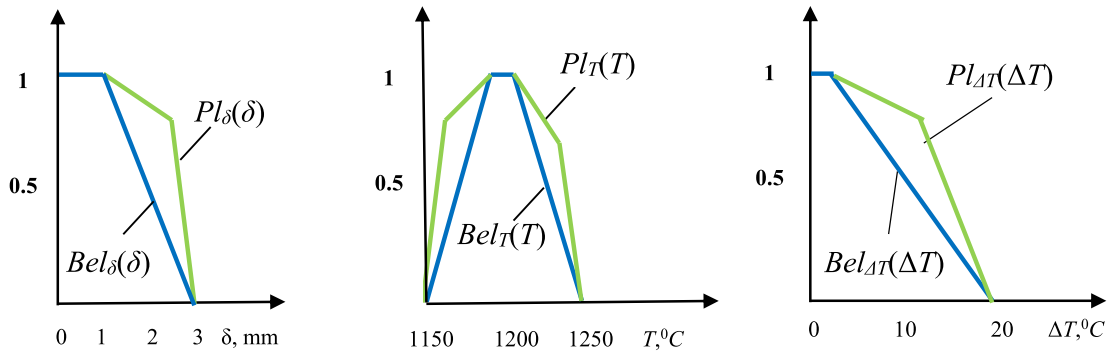


FIGURE 10. The local criteria of the steel ingots heating process quality before the hot rolling.

$$= ((\alpha Bel_{11}, \alpha Bel_{12}, \alpha Bel_{13}), (\alpha Pl_{11}, \alpha Pl_{12}, \alpha Pl_{13})), \quad \alpha < 1. \quad (19)$$

$$TFVBP_N_1^\alpha = (((Bel_{11})^\alpha, (Bel_{12})^\alpha, (Bel_{13})^\alpha), ((Pl_{11})^\alpha, (Pl_{12})^\alpha, (Pl_{13})^\alpha)). \quad (20)$$

$$TFVBP_N_1^{TFVBP_N_2} = ((Bel_{11}^{Pl_{23}}, Bel_{12}^{Pl_{22}}, Bel_{13}^{Pl_{21}}), (Pl_{11}^{Bel_{23}}, Pl_{12}^{Bel_{22}}, Pl_{13}^{Bel_{21}})). \quad (21)$$

The introduced operations with *TFVBP*Ns have good algebraic properties and are free of the drawbacks of those defined in the *A – IFS* for *INIFV*s. The corresponding theorems are obtained from those proved in [17] by simply replacing the notation and so they are not represented here.

#### IV. CASE STUDY: THE SOLUTION OF TRIANGULAR FUZZY BELIEF-PLAUSIBILITY MCDM PROBLEM OF THE RAW MATERIAL SUPPLIER SELECTION ON THE STEEL-ROLLING PLANT

In this section, we present the example of the practical application of the developed methods for the solution of the triangular fuzzy Belief-Plausibility *MCDM* problem of the raw material supplier selection for the Belorussian Mini Metallurgical Plant *BMMP* based on the quality estimation of the ingot heating process before the hot rolling on the rolling mill 850. The plant was erected and commissioned on a “turn-key” basis in 1984 by “Voest-Alpine”.

The basic raw material used in the *BMMP* is the metal scrap delivered by different suppliers and therefore characterizing by different chemical compositions affected the product quality. For example, if one scrap consist mainly of cutted ships and another one contains many pressed cars then to use them often different technologies should be applied.

In [69], we presented the solutions of two fuzzy multiple criteria optimization problems on the *BMMP*, but their applicability was restricted by the need to solve these problems for everyone new supplier.

Therefore, based on the compromise technology providing not the best, but at least acceptable product quality for the considerable group of suppliers, it was proposed to make the

multiple-criteria selection of them. The compromise technology was proposed by the Voest-Alpine, but not taking into account the specificity of the Belorussian suppliers.

The ingot heating process was chosen as a the basic one as it is the process immediately preceding the rolling and in a great extent determining its quality.

Therefore, the problem was formulated as follows: to make the selection of the metal scrap suppliers based on their ability to ensure high multiple criteria estimations of the ingot heating process quality.

In this process, the ingots are heated in methodical gas furnaces. The methodical furnace with walking beams of the mill 850 is designed for heating continuously cast blooms with a cross section of 250 × 300 and 300 × 400 mm with a length of 2.5 to 5.5 m with a layout step of 150 and 200 mm, respectively. The total length of the furnace is 23 m, according to the location of the burners, it is divided into seven sections, which can be divided into methodical, two the so-called welding and homogenizing zones.

Three parameters of the process quality were considered: the scale thickness  $\delta$  to be minimized as it directly represent the metal losses, the accuracy of heating  $T$  to be maximized, and the maximum temperature difference in the ingot  $\Delta T$  at the time of unloading to be minimized because it causes thermal stresses that can lead to the destruction of the ingot during rolling.

The corresponding membership functions  $\mu_\delta$ ,  $\mu_T$  and  $\mu_{\Delta T}$  (or  $Bel_\delta$ ,  $Bel_T$  and  $Bel_{\Delta T}$ ) representing the local criteria of the process quality are presented in Fig. 10. The membership functions of the criteria  $\mu_\delta$ ,  $\mu_T$  and  $\mu_{\Delta T}$  were proposed by the technologists based on the carried earlier industrial experiments, their experience and intuition. The criteria  $\mu_\delta$  and  $\mu_T$  were nearly directly based on the existing technological instructions. More details are presented in [69].

The crucial for the considered problem thing is that after defining the membership functions, the technologists refused to present the non-membership ones. They argued such a rejection insisting that presenting the criteria by membership functions they had used exhaustively all the information available and therefore there is no place for any non-membership functions, which in their opinion seem to be

something artificial senseless mathematical constructions. Perhaps this was caused by the psychological problem to give a negative assessment (non-membership), when a positive one (membership) is already established.

Therefore, we were forced to abandon the use of the *A-IFS* methods to solve the problem, at least explicitly.

Meanwhile, technologists experts were willing to answer the question: What would be your maximum permissible (optimistic) estimate of membership (plausibility, *PL*) if the  $Bel = \mu$  is a guaranteed minimum estimate? The obtained estimates of the plausibility degrees are approximately presented in Fig. 10 by the piece-wise linear functions.

The weights (relative importance) of the local criteria were calculated using the matrix of the linguistic pairwise comparisons of the criteria provided in the verbal form by experts. The method developed in [70] was used (see more details in [69]).

The following weights were obtained:  $w_\delta = 0.076$ ,  $w_T = 0.477$  and  $w_{\Delta T} = 0.447$  for the local criteria  $\mu_\delta$ ,  $\mu_T$  and  $\mu_{\Delta T}$ , respectively.

For all considered parameters of the process quality, their measurements were carried out for the sets of ingots produced from the scrap delivered by competing suppliers. Since the measurements of  $\delta$  and  $\Delta T$  required the use of very expensive methods based on the partial destruction of ingots it was not possible to obtain a sufficient number of measurements to apply rigorous statistical methods. Therefore, the results of measurement for each considered *i*th supplier were presented by triangular fuzzy numbers as follows:

$$A_i = (a_{1i}, a_{2i}, a_{3i}), B_i = (b_{1i}, b_{2i}, b_{3i}), C_i = (c_{1i}, c_{2i}, c_{3i}), \tag{22}$$

where  $A_i$ ,  $B_i$  and  $C_i$  are triangular fuzzy numbers representing the results of measurements of  $\delta$ ,  $T$  and  $\Delta T$ , respectively. For example, the interval  $[a_{1i}, a_{3i}]$  is the range of all obtained measurements of  $\delta$  and  $a_{2i}$  is the point of their visible concentration.

Then the generalized criterion  $GC_i$  should be presented as the convolution of local criteria with their weights. The weighted sum convolution was chosen as the more flexible than other popular aggregation modes as it provides the trade-off of low values of some criteria with high values of others. The *TFBPNs* representing the values of the local criteria for the corresponding triangular fuzzy arguments are presented in Fig. 11-13. The results of intermediate calculations for the four considered suppliers are presented in Table 1. The values of the obtained *TFBPNs* were calculated as follows:

$$\begin{aligned} Bel_{Tr\delta i} &= (Bel(a_{1i}), Bel(a_{2i}), Bel(a_{3i})), \\ Pl_{Tr\delta i} &= (Pl(a_{1i}), Pl(a_{2i}), Pl(a_{3i})), \\ Bel_{TrTi} &= (Bel(b_{1i}), Bel(b_{2i}), Bel(b_{3i})), \\ Pl_{TrTi} &= (Pl(b_{1i}), Pl(b_{2i}), Pl(b_{3i})), \\ Bel_{Tr\Delta Ti} &= (Bel(c_{1i}), Bel(c_{2i}), Bel(c_{3i})), \\ Pl_{Tr\Delta Ti} &= (Pl(c_{1i}), Pl(c_{2i}), Pl(c_{3i})). \end{aligned}$$

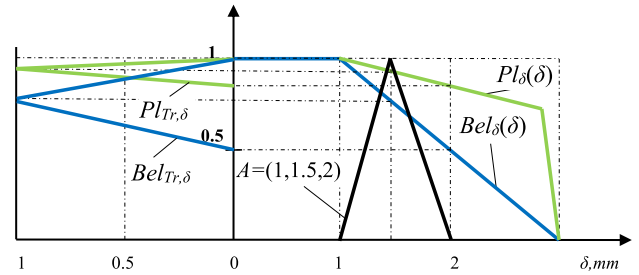


FIGURE 11. Calculating the triangular value of FVBP local criterion dependent on the triangular fuzzy  $\delta$ .

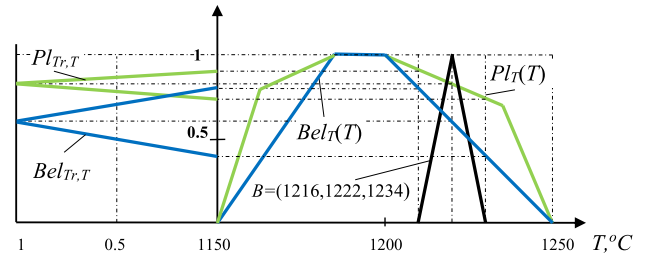


FIGURE 12. Calculating the triangular value of FVBP local criterion dependent on the triangular fuzzy  $T$ .

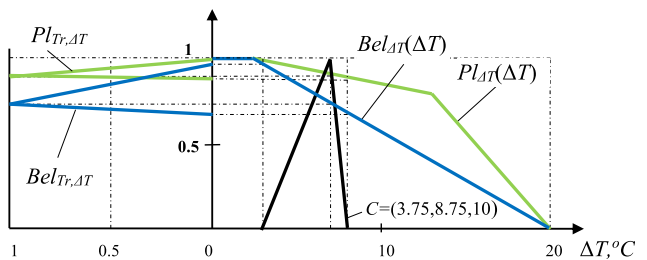


FIGURE 13. Calculating the triangular value of FVBP local criterion dependent on the triangular fuzzy  $\Delta T$ .

TABLE 1. The calculations of FVBPns.

Supplier's number	A B C	$Bel_{Tr\delta i}$ $Bel_{TrTi}$ $Bel_{Tr\Delta Ti}$	$Pl_{Tr\delta i}$ $Pl_{TrTi}$ $Pl_{Tr\Delta Ti}$
1	(1,1.5,2) (1211,1222,1233) (3.75,8.75,10)	(0.5,0.75,1) (0.32,0.55,0.75) (0.68,0.72,0.94)	(0.85,0.92,1) (0.7,0.8,0.9) (0.875,0.9,0.975)
2	(1.5,2,2.5) (1170,1180,1200) (8,10,12)	(0.25,0.5,0.75) (0.55,0.8,1) (0.45,0.55,0.675)	(0.8,0.85,0.9) (0.7,0.8,0.9) (0.8,0.85,0.875)
3	(1,2,2.5) (1200,1225,1250) (10,12,18)	(0.25,0.5,1) (0.0,0.475,1) (0.45,0.55,0.75)	(0.8,0.85,1) (0.775,1) (0.15,0.8,0.85)
4	(2,2.5,3) (1150,1160,1180) (4,10,14)	(0.0,0.25,0.5) (0.0,0.3,0.8) (0.34,0.56,0.9)	(0.0,0.8,0.85) (0.0,0.65,0.93) (0.64,0.84,0.95)

Then using the operations (15) and (17), we inferred the generalized criterion  $GC_i$  presentation for the considered problem as follow:

$$GC_i = (Bel_{GCi}, Pl_{GCi}) = ((w_\delta Bel_{Tr\delta i} + w_T Bel_{TrTi} + w_{\Delta T} Bel_{Tr\Delta Ti}), (w_\delta Pl_{Tr\delta i} + w_T Pl_{TrTi} + w_{\Delta T} Pl_{Tr\Delta Ti})).$$

The results are presented in Fig. 14 and Table 2.

Analyzing the results presented in Fig. 14 and Table 2, without additional calculations we can conclude that

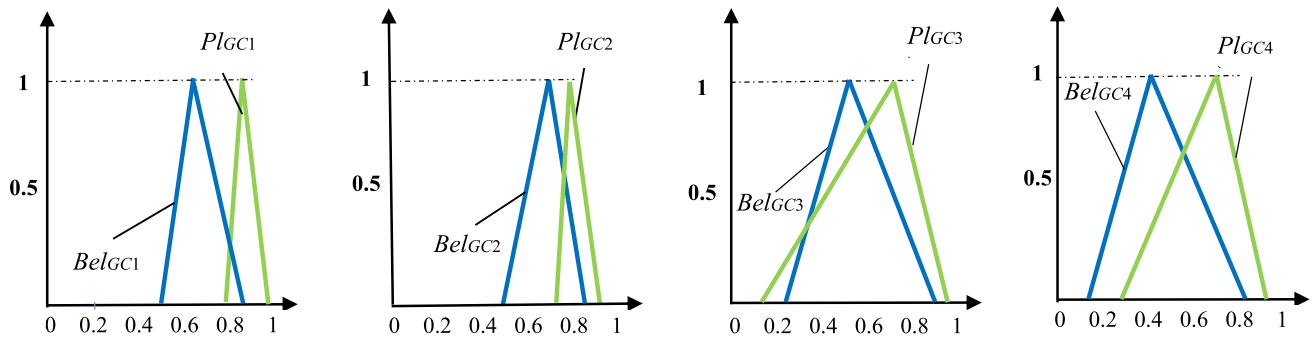


FIGURE 14. The TFVBNs representing the generalized criterion GC for the considered four suppliers.

TABLE 2. The calculations of FVBNs.

Supplier's number	$Bel_{GC_i}$	$Pl_{GC_i}$
1	(0.495,0.641,0.852)	(0.79,0.855,0.940)
2	(0.482,0.666,0.836)	(0.753,0.827,0.888)
3	(0.220,0.510,0.888)	(0.128,0.793,0.933)
4	(0.152,0.412,0.822)	(0.286,0.746,0.934)

$Bel_{GC_1} < Bel_{GC_2} < Bel_{GC_3} < Bel_{GC_4}$  and  $Pl_{GC_1} < Pl_{GC_2} < Pl_{GC_3} < Pl_{GC_4}$ . Therefore the final rating of the suppliers can be presented according to the rating of the generalized criterion of the heating process quality, which was obtained using the delivered by the competing suppliers raw materials as follows:  $GC_1 < GC_2 < GC_3 < GC_4$ .

Of course, in general, the comparison of TFVBNs is not so easy problem as above and the corresponding method for its solution should be developed. But this is out of scope of this paper and now it is in the final stage of elaboration.

The considered case study illustrates the ability of the proposed approach, based on the Belief-Plausibility formalism, to solve the real-world problems independently, without use the  $A-IFS$  concepts. It is shown that such an approach owns obvious advantages in comparison with the  $A-IFS$  based one.

## V. CONCLUSION

The contributions of this work into development of the methods for uncertainty modeling are as follows:

1. The several important internal theoretical and practical problems of the  $A-IFS$  and  $A-IVIFS$  theories are analyzed and approaches to their solutions based on the redefinition of them in the framework of the  $DST$  are developed. When analyzing the ability of the novel Neutrosophic, Picture, Pythagorean and Spherical sets theories to solve the problems of the classical  $A-IFS$  theory, some their own limitations and drawbacks are revealed.

2. It is shown that these drawbacks prevent the solution of the  $A-IFS$  problems, whereas the proposed approach based on the  $DST$  is free of them and performs better than these novel theories and the  $A-IFS$ .

3. The properties of the  $DST$  extension of the  $A-IFS$  and its practical advantages for a decision maker in assigning certain real values to the parameters of  $A-IFS$  are analyzed. As the result of provided studies, a new mathematical object

called Belief-Plausibility number ( $BPN$ ) is introduced. It is shown that the  $BPN$  has at least two particular cases: an usual fuzzy number (the value of a membership function) and an intuitionistic fuzzy number. It is emphasized that the  $BPN$  can be successfully applied completely separately not using terms of usual fuzzy sets and the  $A-IFS$  and this is the typical feature of more general theories.

4. We have not found in the literature clear and practical approaches to mathematical formulation of the membership  $\mu(x)$  and non-membership  $\nu(x)$  functions in such a way that all asymptotic properties and basic constraints  $\mu(x) + \nu(x) \leq 1$ ,  $\mu_U(x) + \nu_U(x) \leq 1$  ( $\mu_U(x)$  and  $\nu_U(x)$  are the upper bounds of corresponding interval-valued membership and non-membership functions) will be fulfilled jointly. Therefore, in the current paper, such an approach is proposed.

5. It was found that there are two main sources of uncertainty leading to the need for the use of  $A-IVIFS$ . The first of them is an interval uncertainty caused by the multiplicity of the functions  $\mu(x)$  and  $\nu(x)$ , e.g. proposed by different experts. The second is the consequence of interval uncertainty of input data, e.g. in the  $MCDM$ . And if input data we deal with are presented by fuzzy values, we meet the case of fuzzy-valued intuitionistic fuzzy set  $FVIFS$ .

6. The described above general approach to treatment of interval-valued intuitionistic fuzzy objects and their Belief-Plausibility (based on the  $DST$ ) extensions allowed us to introduce a set of new fruitful definitions, some of the them, e.g. the definition of Fuzzy-Valued Belief-Plausibility value ( $FVBPV$ ) were not even mentioned in the literature in any form.

7. The corresponding new arithmetical operations with such mathematical objects, following from these definitions, are proposed. They all, and especially the definition of  $FVBPV$ , were thoroughly analyzed under general methodological principles and their practical usefulness is illustrated by the real-world case study based on the solution of the  $DST$  based uncertain  $MCDM$  problem of the raw material supplier selection on the steel rolling plant under triangular fuzzy input data.

8. The obtained results allowed us to state that the introduced mathematical objects  $BPN$ ,  $IVBPN$  and  $FVBPV$  can be successfully used in the analysis and applications

independently, without the use of the  $A - IFS$  concepts and such an approach possesses obvious advantages in comparison with the  $A - IFS$  based approach.

The main directions of our future researches will be an extension of the proposed general approach to the cases when an uncertainty is originated jointly from the multiplicity of the  $Bel$  and  $Pl$  functions formulations and uncertainty of input data (functions arguments), and the development of methods for the comparison of the Belief-Plausibility based objects characterising by different uncertainty levels.

#### CREDIT AUTHORSHIP CONTRIBUTION STATEMENT

**Pavel Sevastjanov:** Conceptualization, Formal analysis, Investigation, Methodology, Resources, Supervision, Writing - original draft, Writing - review & editing. **Ludmila Dymova:** Data Curation, Formal analysis, Investigation, Methodology, Writing - original draft. **Krzysztof Kaczmarek:** Formal analysis, Investigation, Resources, Writing - original draft.

#### DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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