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Complex Dynamics of a Dysentery Diarrhoea Epidemic Model With Treatment and Sanitation Under Environmental Stochasticity: Persistence, Extinction and Ergodicity

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ABSTRACT To understand the transmission dynamics of diarrhea in random environment, in this paper we propose a stochastically perturbed dysentery diarrhoea epidemic model with treatment and sanitation. Using the theory of stopping time, we first show the existence of global positive solution of the model. Then, we study the stochastic dynamics of the model and present a stochastic threshold $\mathscr{R}_0^S$ which determines the extinction and persistence of the disease. Based on Khasminskii's theory, we further prove that the model has a unique ergodic stationary distribution under the condition of $\mathscr{R}_0^S > 1$. Numerical simulations are carried out to verify the analytical results, showing that the white noise, and the constant treatment and sanitation may have certain inhibitory effects on disease transmission. Lastly, the model is further extended to include colored noise and seasonal fluctuation to study the long-term transmission dynamics of disease. It is found that the method proposed in this paper is universal.

INDEX TERMS Stochastic dysentery model, threshold dynamics, persistence and extinction, ergodicity.

I. INTRODUCTION

Diarrhoea is an ancient disease that continues to cause epidemics despite ongoing efforts to limit its spread [1]–[4]. It is well known that dysentery is a typical diarrhoeal disease caused by Shigella (S. flexneri and S. dysenteriae), which is associated with contaminated water and poor sanitation. As a result, the disease often occurs in refugee camps. The elderly, the weak and the malnourished are particularly vulnerable to this disease and cause serious death. According to the World Health Organization (WHO) [5], diarrhea is the second leading cause of death in children under five years old, killing around 525000 children every year, of which dysentery and cholera contribute most of the cases. Available information indicates that about 15% of diarrhoea under five years old is dysentery, but up to 25% of all diarrhoea deaths [6].

Over the past few decades, WHO has launched a series of measures to prevent and control dysentery [7], such as advocating exclusive breastfeeding, using improved sanitation, rotavirus immunization, good personal and food hygiene, health education about how infections spread, etc., and has made remarkable achievements. However, for developing countries, dysentery remains a significant public health burden due to economic constraints and continues to receive worldwide attention [8]–[14]. Stemming from this motivation, to understand the transmission dynamics of dysentery Berhe *et al.* [15] proposed the following dysentery diarrhoea epidemic model with treatment and sanitation:

$$\begin{cases} \frac{dS}{dt} = \Lambda + \gamma(1-\rho_1)I + e_1u_1(1-\rho_2)I + \alpha R \\ \qquad\quad - \left(\frac{\beta_1 B}{K+B} + \beta_2 I + \mu\right)S, \\ \frac{dI}{dt} = \left(\frac{\beta_1 B}{K+B} + \beta_2 I\right)S - (\mu + d + \gamma + e_1u_1)I, \\ \frac{dR}{dt} = (\gamma\rho_1 + e_1u_1\rho_2)I - (\mu+\alpha)R, \\ \frac{dB}{dt} = kI - (\delta + e_2u_2)B, \end{cases} \tag{1}$$

where S, I, R stand for the susceptible, infected and recovered individuals respectively, and B is the concentration

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of pathogen population (or concentration of shigella in the environment). Model (1) is based on two assumptions, namely that disease transmission is multiple pathway and the population is homogeneously mixed. As a result, the incidence is composed of two parts: one is modeled by a logistic response curve $\frac{\beta_1 B}{K+B}$ representing the environment-to-human transmission and the other is modeled by $\beta_2 I$ representing the human-to-human interaction. All parameters in model (1) are positive and have the following biological significance: β_1 and β_2 denote the rates of ingesting shigella from a contaminated environment and through human-to-human interaction respectively; K is the pathogen concentration that yields 25-50% chance of catching dysentery diarrhoea [16]; Λ denotes the recruitment rate of susceptible humans; μ represents the natural death rate of all human classes; k is the rate of infected individuals contribution to shigella; γ is the natural recovery rate of diarrhea; α is the relapse rate of the recovered humans to the susceptible class; d is the disease-induced death rate; δ is the net death rate of the pathogen population in the environment; u_1 is the rate of treatment; e_2u_2 is the rate of sanitation; e_i $(i = 1, 2)$ are the efficacy of treatment and sanitation respectively; ρ_1 is the proportion of the naturally recovered ones who go to the recovered class; ρ_2 denotes the proportion of the recovered individuals due to treatment who move to a temporary immune state. For model (1), the dynamic behavior is completely determined by the basic reproduction number

$$\mathscr{R}_0 = \frac{\Lambda\beta_1 k}{\mu(\mu+d+\gamma+e_1u_1)(\delta+e_2u_2)K} + \frac{\Lambda\beta_2}{\mu(\mu+d+\gamma+e_1u_1)},$$

that is, the disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$ is globally asymptotically stable if $\mathscr{R}_0 < 1$; while for $\mathscr{R}_0 > 1$, there has a unique endemic equilibrium $E^* = (S^*, I^*, R^*, B^*)$ which is globally asymptotically stable (see [15]). The model was successful in explaining some basic effects of constant controls treatment and sanitation on disease transmission.

As an effective tool to predict and control disease outbreaks, stochastic epidemic models (including models driven by white noise [17]–[25], Markov switching [26]–[31] and Lévy jumps [32]–[35]) have been increasingly favored by many scholars in recent years. Unlike deterministic models, the random transmission dynamics of disease is usually based on probabilistic analysis, which provides a new perspective on the evolution of disease. However, due to the complexity of noise effect, there are still many problems to be studied, such as persistence and extinction of the disease under environmental noise and ergodicity of stochastic epidemic system. To this end, we introduce randomness into model (1) by assuming that stochastic perturbations are of the white noise type which are directly proportional to S, I, R and B (please refer to [36] for details), resulting the following Itô type model:

$$\begin{cases} dS = \Big[\Lambda + \gamma(1-\rho_1)I + e_1u_1(1-\rho_2)I + \alpha R \\ \qquad - \Big(\frac{\beta_1 B}{K+B} + \beta_2 I + \mu\Big)S\Big]dt + \sigma_1 S dW_1(t), \\ dI = \Big[\Big(\frac{\beta_1 B}{K+B} + \beta_2 I\Big)S - (\mu+d+\gamma+e_1u_1)I\Big]dt \\ \qquad + \sigma_2 I dW_2(t), \\ dR = [(\gamma\rho_1 + e_1u_1\rho_2)I - (\mu+\alpha)R]dt \\ \qquad + \sigma_3 R dW_3(t), \\ dB = [kI - (\delta+e_2u_2)B]dt + \sigma_4 B dW_4(t), \end{cases} \tag{2}$$

where $W_i(t)$ are mutually independent standard Brownian motions defined on this complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, $\sigma_i \geq 0$ denote the intensity of the white noise, $i = 1, 2, 3, 4$.

A natural question is: How does environmental noise affect the dynamic behavior of stochastic model (2), especially the persistence, extinction and ergodicity? To answer this question, we give some theoretical results in Section II, including the existence of a unique global positive solution for model (2), the exponential stability and persistence of the disease, and the existence of a unique ergodic stationary distribution. Strict mathematical proofs of the main results are presented in Section III. Later, some numerical simulations are carried out to confirm our theoretical analysis in Section IV. Finally, further discussions are provided in Section V to conclude our study. Our major contributions of the paper are: (i) This paper is the first attempt to consider a dysentery diarrhoea epidemic model with treatment and sanitation under environmental stochasticity. Although there are some stochastic epidemic models such as [19]–[23], [26]–[30], model (2) is essentially different due to the complexity its structure; (ii) A new stochastic tool for studying extinction of the disease is used, which is different from the existing literatures [21]–[23], [27] and is universal; (iii) A stochastic threshold $\mathscr{R}_0^S$ is given, which determines the persistence and extinction of the disease. Meanwhile, a meaningful result that environmental noise and constant treatment and sanitation may have certain inhibitory effects on disease transmission is obtained.

II. MAIN RESULTS

Some main results are presented in this section, including the existence and uniqueness of global positive solution, the exponential stability and persistence in mean, and the existence of a unique ergodic stationary distribution. Before that, let us denote

$$\mathscr{R}_0^S = \frac{\Lambda\beta_2}{(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)} + \frac{\Lambda\beta_1 k}{K(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)(\delta+e_2u_2+\frac{1}{2}\sigma_4^2)},$$

$$\Delta = \frac{\Lambda\beta_2}{(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)},$$

$$\Pi = \frac{1}{2}\Big[\Delta + \sqrt{\Delta^2 + 4\big(\mathscr{R}_0^S - \Delta\big)}\Big],$$
$$\Theta = \frac{1}{8}(\sigma_2^2+\sigma_4^2) + \frac{\Lambda(\beta_1\tau_1+\beta_2\tau_2K)\sigma_1^2}{2\mu\tau_2(\mu+\frac{1}{2}\sigma_1^2)K} + \min\Big\{\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2, \delta+e_2u_2+\frac{1}{2}\sigma_4^2\Big\} \times(\Pi-1).$$

Here,

$$\tau_1 = \frac{\upsilon_1}{\mu+\frac{1}{2}\sigma_2^2}, \quad \tau_2 = \frac{\upsilon_2}{\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2},$$

where (υ_1, υ_2) is any positive solution of the following matrix equation

$$(\upsilon_1, \upsilon_2)\Gamma = (\Pi-1)(\upsilon_1, \upsilon_2), \tag{3}$$

and

$$\Gamma = \begin{pmatrix} J & \frac{\Lambda\beta_1}{K(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)} \\ \frac{k}{\delta+e_2u_2+\frac{1}{2}\sigma_4^2} & -1 \end{pmatrix},$$

where $J = -1 + \frac{\Lambda\beta_2}{(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)}$.

Now we are in the position to state our main results.

Theorem 1: Given initial valve $(S(0), I(0), R(0), B(0)) \in \mathbb{R}_+^4$, *model* (2) *admits a unique positive solution* $(S(t), I(t), R(t), B(t))$ *for* $t \geq 0$; *furthermore, the solution will remain in* $\mathbb{R}_+^4$ *with probability one.*

Theorem 2: Let $(S(t), I(t), R(t), B(t))$ *be the solution of stochastic model* (2) *with any given initial value* $(S(0), I(0), R(0), B(0)) \in \mathbb{R}_+^4$.

(i) *If* $\mathscr{R}_0^S < 1$, *then*

$$\limsup_{t\to\infty}\frac{1}{t}\ln[\tau_1I(t)+\tau_2B(t)] \leq \Theta,$$
$$\limsup_{t\to\infty}\frac{1}{t}\ln R(t) \leq \Big[-(\mu+\alpha+\frac{\sigma_3^2}{2})\Big] \vee \Theta \ a.s.$$

Especially, if $\Theta < 0$, *then*

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} B(t) = 0,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(\tau)d\tau = \frac{\Lambda}{\mu} \ a.s.$$

(ii) *If* $\mathscr{R}_0^S > 1$, *then*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(\tau)d\tau \geq \frac{\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2}{\eta} \times(\mathscr{R}_0^S-1),$$

where η *is defined as* (13).

Remark 1: (i) *Theorem 2 gives the sufficient conditions for the infected individual to approach zero with probability one and to persist. It is worthy to note that, when* $\sigma_i = 0$ $(i = 1, 2, 4)$, $\mathscr{R}_0 = \mathscr{R}_0^S$, *which shows that* $\mathscr{R}_0^S$ *is the stochastic version of* $\mathscr{R}_0$; *when* $\sigma_i \neq 0$ $(i = 1, 2, 4)$, $\mathscr{R}_0^S < \mathscr{R}_0$, *implying the possibility that* $\mathscr{R}_0^S < 1 < \mathscr{R}_0$. *This means that environmental noise may suppress the spread of disease. Furthermore, it is interesting to find that the stochastic version of* $\mathscr{R}_0^S$ *is independent of the noise intensity* σ_3.

(ii) *We may rewrite* $\mathscr{R}_0^S$ *as* $\mathscr{R}_0^S = \mathscr{R}_1^S + \mathscr{R}_2^S$, *where* $\mathscr{R}_1^S$ *and* $\mathscr{R}_2^S$, *as shown at the bottom of the page. Biologically,* $\mathscr{R}_1^S$ *and* $\mathscr{R}_2^S$ *are the basic reproduction numbers corresponding to the ingesting shigella from a contaminated environment and through human to human, respectively.*

(iii) *After direct calculation, we easily obtain* $\frac{\partial\mathscr{R}_0^S}{\partial u_1} < 0$, $\frac{\partial\mathscr{R}_0^S}{\partial u_2} < 0$. *Thus, the basic reproduction number* $\mathscr{R}_0^S$ *decreases with implementing the controls* u_1 *and* u_2. *Similarly, the basic reproduction number increases with* β_1, β_2, k *and* Λ.

Theorem 3: If $\mathscr{R}_0^S > 1$, *then model* (2) *admits a unique stationary distribution* $\pi(\cdot)$ *and it has the ergodic property.*

Remark 2: Theorem 3 tells us that although there is no positive equilibrium point in model (2), *under the condition of* $\mathscr{R}_0^S > 1$, *there exists a unique ergodic stationary distribution, which to some extent reflects that the solution of the model is weakly stable and persistent in mean.*

It should be pointed out that condition $\Theta < 0$ in Theorem 2 means that the noise intensity cannot be too large, while $\mathscr{R}_0^S < 1$ shows that the noise intensity can not be too small, that is, the infected individuals tend to zero only at an appropriate noise intensity.

III. PROOFS OF MAIN RESULTS

In this section, we provide the detailed proofs of the main results illustrated in Section II.

A. PROOF OF THEOREM 1

Proof: Since the coefficients of model (2) do not satisfy the linear growth condition, for given initial value $(S(0), I(0), R(0), B(0)) \in \mathbb{R}_+^4$, there only exists a unique local solution on $t \in [0, \tau_e)$, where τ_e is the explosion time. Now we only show $\tau_e = \infty$ a.s., which implies the global existence of the solution. To this end, let $n_0 \geq 1$ be sufficiently large such that $S(0)$, $I(0)$, $R(0)$, and $B(0)$ all lie within the interval $[\frac{1}{n_0}, n_0]$. For each integer $n \geq n_0$, define the stopping

$$\mathscr{R}_1^S = \frac{\Lambda\beta_1k}{K(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)(\delta+e_2u_2+\frac{1}{2}\sigma_4^2)},$$
$$\mathscr{R}_2^S = \frac{\Lambda\beta_2}{(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)}.$$

$\tau_n = \inf\{t \in [0,\tau_e) : \min\{S,I,R,B\} \leq \frac{1}{n}$ or $\max\{S,I,R,B\} \geq n\}$. Clearly, τ_n is increasing in n, and for the empty set $\emptyset$ we have $\inf\emptyset = \infty$. If let $\tau_\infty = \lim_{n\to\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ a.s. Next, we prove $\tau_\infty = \infty$ by contradiction, implying $\tau_e = \infty$. If this statement is false, there exist three constants $T > 0$, $n_1 \geq n_0$ and $\epsilon \in (0,1)$ such that $\mathbb{P}\{\Omega_n\} \geq \epsilon$ for all $n \geq n_1$, where $\Omega_n = \{\tau_n \leq T\}$.

Define a function $V_1 : \mathbb{R}^4_+ \to \mathbb{R}_+$ as

$$\begin{aligned}V_1(S,I,R,B) &= S - m_1 - m_1 \ln\frac{S}{m_1} + I - 1 - \ln I\\ &\quad + R - 1 - \ln R + m_2(B - 1 - \ln B),\end{aligned}$$

where m_1, $m_2 > 0$ are determined later. Applying Itô's formula yields

$$\begin{aligned}&\mathscr{L}V_1\\ &\leq \Lambda + [\gamma(1-\rho_1) + e_1u_1(1-\rho_2)]I + m_1(\beta_1 + \beta_2 I + \mu\\ &\quad + \frac{1}{2}\sigma_1^2) - (\mu+d+\gamma+e_1u_1)I + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2 + (\gamma\rho_1\\ &\quad + e_1u_1\rho_2)I + \mu + \alpha + \frac{1}{2}\sigma_3^2 + m_2(kI + \delta + e_2u_2 + \frac{1}{2}\sigma_4^2)\\ &= [m_1\beta_2 + m_2k - (\mu+d)]I + \Lambda + m_1(\beta_1 + \mu + \frac{1}{2}\sigma_1^2) + d\\ &\quad + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2 + \mu + \alpha + \frac{1}{2}\sigma_3^2\\ &\quad + m_2(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2).\end{aligned}$$

Choose $m_1 = \frac{\mu+d}{2\beta_2}$ and $m_2 = \frac{\mu+d}{2k}$ such that $m_1\beta_2 + m_2k - (\mu+d) = 0$, then

$$\begin{aligned}\mathscr{L}V_1 &\leq \Lambda + m_1(\beta_1 + \mu + \frac{1}{2}\sigma_1^2) + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\\ &\quad + \mu + \alpha + \frac{1}{2}\sigma_3^2 + m_2(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2) := C.\end{aligned}$$

The rest of the proof is standard and can be referred to Theorem 3.1 in [17]. So we omit it here. □

B. PROOF OF THEOREM 2

Below two lemmas are very useful in proving Theorem 2, where the proof of Lemma 1 is omitted because it is standard (see Lemma 1 in [37]).

Lemma 1: The solution $(S(t), I(t), R(t), B(t))$ established in Theorem 1 satisfies

$$\begin{aligned}&\limsup_{t\to\infty}[S(t)+I(t)+R(t)+B(t)] < \infty \text{ and}\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_1 S(\tau)dW_1(\tau) = 0,\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_2 I(\tau)dW_2(\tau) = 0,\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_3 R(\tau)dW_3(\tau) = 0\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_4 B(\tau)dW_4(\tau) = 0\end{aligned}$$

Lemma 2: Let $U(I,B) = \tau_1 I + \tau_2 B$, where τ_1 and τ_2 are defined as Theorem 2. Then,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \frac{\tau_1\beta_1 B(\tau)}{KU(\tau)}\Big(S(\tau) - \frac{\Lambda}{\mu}\Big)d\tau \leq 0 \; a.s., \quad (4)$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \frac{\tau_1\beta_2 I(\tau)}{U(\tau)}\Big(S(\tau) - \frac{\Lambda}{\mu}\Big)d\tau \leq 0 \; a.s. \quad (5)$$

Proof: From model (2), $d(S+I+R) \leq [\Lambda - \mu(S+I+R)]dt + \sigma_1 S dW_1(t) + \sigma_2 I dW_2(t) + \sigma_3 R dW_3(t)$, which shows that

$$\begin{aligned}&S + I + R\\ &\leq \frac{\Lambda}{\mu} + \Big(S(0)+I(0)+R(0) - \frac{\Lambda}{\mu}\Big)e^{-\mu t} + \int_0^t e^{-\mu(t-\tau)}\\ &\quad \times \Big[\sigma_1 S(\tau)dW_1(\tau) + \sigma_2 I(\tau)dW_2(\tau) + \sigma_3 S(\tau)dW_3(\tau)\Big].\end{aligned}$$

Then,

$$\begin{aligned}&\int_0^t \frac{\tau_1\beta_1 B(\tau)}{KU(\tau)}\Big(S(\tau) - \frac{\Lambda}{\mu}\Big)d\tau \leq \int_0^t \frac{\tau_1\beta_1 B(\tau)}{KU(\tau)}\Big\{\Big(S(0)\\ &\quad + I(0) + R(0) - \frac{\Lambda}{\mu}\Big)e^{-\mu\tau} - I(\tau) - R(\tau) + \int_0^\tau e^{-\mu(\tau-s)}\\ &\quad \times \Big[\sigma_1 S(s)dW_1(s) + \sigma_2 I(s)dW_2(s) + \sigma_3 S(s)dW_3(s)\Big]\Big\}d\tau\\ &\leq \frac{\tau_1\beta_1[S(0)+I(0)+R(0)]}{\tau_2\mu K} + \int_0^t \frac{\tau_1\beta_1 B(\tau)}{KU(\tau)}\int_0^\tau e^{-\mu(\tau-s)}\\ &\quad \times \Big[\sigma_1 S(s)dW_1(s) + \sigma_2 I(s)dW_2(s) + \sigma_3 S(s)dW_3(s)\Big]d\tau\\ &= \frac{\tau_1\beta_1[S(0)+I(0)+R(0)]}{\tau_2\mu K}\\ &\quad + \frac{\tau_1\beta_1}{K}\Big\{\int_0^t \sigma_1 S(s)e^{\mu s}\Big[\int_s^t \frac{B(\tau)e^{-\mu\tau}}{U(\tau)}d\tau\Big]dW_1(s)\\ &\quad + \int_0^t \sigma_2 I(s)e^{\mu s}\Big[\int_s^t \frac{B(\tau)e^{-\mu\tau}}{U(\tau)}d\tau\Big]dW_2(s)\\ &\quad + \int_0^t \sigma_3 R(s)e^{\mu s}\Big[\int_s^t \frac{B(\tau)e^{-\mu\tau}}{U(\tau)}d\tau\Big]dW_3(s)\Big\}\\ &:= \frac{\tau_1\beta_1[S(0)+I(0)+R(0)]}{\tau_2\mu K} + \frac{\tau_1\beta_1}{K}(M_1+M_2+M_3).\end{aligned}$$

Here,

$$\begin{aligned}\langle M_1, M_1\rangle_t &= \int_0^t \sigma_1^2 S^2(s)e^{2\mu s}\Big[\int_s^t \frac{B(\tau)e^{-\mu\tau}}{U(\tau)}d\tau\Big]^2 ds\\ &\leq \frac{\sigma_1^2 \sup_{\tau\in[0,t]} S^2(\tau)}{\tau_2^2\mu^2}\int_0^t e^{2\mu s}(e^{-2\mu t} + e^{-2\mu s})ds\\ &\leq \frac{2\sigma_1^2 t \sup_{\tau\in[0,t]} S^2(\tau)}{\tau_2^2\mu^2},\end{aligned}$$

which follows from strong law of large numbers and Lemma 1 that $\lim_{t\to\infty}\frac{M_1(t)}{t} = 0$. Similarly, $\lim_{t\to\infty}\frac{M_2(t)}{t} = \lim_{t\to\infty}\frac{M_3(t)}{t} = 0$. Consequently, $\limsup_{t\to\infty}\frac{1}{t}\int_0^t \frac{\tau_1\beta_1 B(\tau)}{KU(\tau)}(S(\tau) - \frac{\Lambda}{\mu})d\tau \leq 0$. Similarly, $\limsup_{t\to\infty}\frac{1}{t}\int_0^t \frac{\tau_1\beta_2 I(\tau)}{U(\tau)}(S(\tau) - \frac{\Lambda}{\mu})d\tau \leq 0$. □

1) DISEASE EXTINCTION OF STOCHASTIC MODEL (2)

In this subsection, we prove Theorem 2(i) about the extinction of disease for model (2).

Proof: Applying Itô's formula to $U(I,B)$ defined in Lemma 2 and then integrating from 0 to t lead to

$$\frac{1}{t}\ln\frac{U(t)}{U(0)}=\frac{1}{t}\int_0^t(I_1(\tau)+I_2(\tau))d\tau+\frac{M_4(t)+M_5(t)}{t},\quad (6)$$

where

$$\begin{aligned}I_1(t)&=\frac{1}{U}\Big\{\tau_1\Big[\Big(\frac{\beta_1B}{K+B}+\beta_2I\Big)S-(\mu+d+\gamma+e_1u_1)I\Big]\\&\quad+\tau_2[kI-(\delta+e_2u_2)B]\Big\},\\I_2(t)&=-\frac{\sigma_2^2\tau_1^2I^2+\sigma_4^2\tau_2^2B^2}{2U^2},\\M_4(t)&=\int_0^t\frac{\sigma_2\tau_1I(\tau)}{U(\tau)}dW_2(\tau),\\M_5(t)&=\int_0^t\frac{\sigma_4\tau_2B(\tau)}{U(\tau)}dW_4(\tau).\end{aligned}$$

Following the strong law of large numbers yields

$$\lim_{t\to\infty}\frac{M_4(t)+M_5(t)}{t}=0\ a.s.\quad (7)$$

Notice that

$$\begin{aligned}I_1(t)&=\frac{1}{U}\Big\{\tau_1\Big[\Big(\frac{\beta_1B}{K+B}+\beta_2I\Big)S-(\mu+d+\gamma+e_1u_1)I\Big]\\&\quad+\tau_2[kI-(\delta+e_2u_2)B]\Big\}\le\frac{1}{U}\Big\{\frac{\tau_1\beta_1B}{K}\Big(S-\frac{\Lambda}{\mu}\Big)\\&\quad+\tau_1\beta_2I\Big(S-\frac{\Lambda}{\mu}\Big)+\frac{\tau_1\Lambda\beta_1}{\mu K}B-\tau_1(\mu+d+\gamma+e_1u_1)I\\&\quad+\frac{\tau_1\Lambda\beta_2}{\mu}I+\tau_2[kI-(\delta+e_2u_2)B]\Big\}\le\frac{1}{U}\Big\{\frac{1}{2}\sigma_2^2\tau_1I\\&\quad+\frac{\tau_1\beta_1B}{K}\Big(S-\frac{\Lambda}{\mu}\Big)+\tau_1\beta_2I\Big(S-\frac{\Lambda}{\mu}\Big)+\frac{1}{2}\sigma_4^2\tau_2B\Big\}\\&\quad+\frac{\Lambda(\beta_1\tau_1+\beta_2\tau_2K)\sigma_1^2}{2\mu\tau_2(\mu+\frac{1}{2}\sigma_1^2)K}+\frac{\Pi-1}{U}(\upsilon_1I+\upsilon_2B)\\&\le\frac{1}{U}\Big\{\frac{\tau_1\beta_1B}{K}\Big(S-\frac{\Lambda}{\mu}\Big)+\tau_1\beta_2I\Big(S-\frac{\Lambda}{\mu}\Big)+\frac{1}{2}\sigma_2^2\tau_1I\\&\quad+\frac{1}{2}\sigma_4^2\tau_2B\Big\}+\frac{\Lambda(\beta_1\tau_1+\beta_2\tau_2K)\sigma_1^2}{2\mu\tau_2(\mu+\frac{1}{2}\sigma_1^2)K}+\min\Big\{\mu+d\\&\quad+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2,\delta+e_2u_2+\frac{1}{2}\sigma_4^2\Big\}(\Pi-1),\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{U}\Big(\frac{1}{2}\sigma_2^2\tau_1I+\frac{1}{2}\sigma_4^2\tau_2B\Big)+I_2(t)\\&\quad=\frac{1}{2}\sigma_2^2\Big(\frac{\tau_1I}{U}-\frac{\tau_1^2I^2}{U^2}\Big)+\frac{1}{2}\sigma_4^2\Big(\frac{\tau_2B}{U}-\frac{\tau_2^2B^2}{U^2}\Big)\\&\quad=\frac{\tau_1\tau_2IB}{2U^2}(\sigma_2^2+\sigma_4^2)\le\frac{1}{8}(\sigma_2^2+\sigma_4^2).\end{aligned}\quad (8)$$

It then follows from (6) and Lemma 2 that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\ln U(t)\\&\le\min\Big\{\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2,\delta+e_2u_2+\frac{1}{2}\sigma_4^2\Big\}(\Pi-1)\\&\quad+\frac{1}{8}(\sigma_2^2+\sigma_4^2)+\frac{\Lambda(\beta_1\tau_1+\beta_2\tau_2K)\sigma_1^2}{2\mu\tau_2(\mu+\frac{1}{2}\sigma_1^2)K}:=\Theta\ a.s.,\end{aligned}\quad (9)$$

which is the desired assertion.

Next we assume diffusion process $\widetilde{R}(t)$ defined by

$$\begin{cases}d\widetilde{R}(t)=-(\mu+\alpha)\widetilde{R}(t)dt+\sigma_3\widetilde{R}(t)dW_3(t),\\ \widetilde{R}(0)=R(0).\end{cases}\quad (10)$$

Then, $d(R(t)-\widetilde{R}(t))\le[(\gamma\rho_1+e_1u_1\rho_2)I-(\mu+\alpha)(R(t)-\widetilde{R}(t))]dt+\sigma_3(R(t)-\widetilde{R}(t))dW_3(t)$, which together with stochastic comparison theorem implies that

$$\begin{aligned}&R(t)-\widetilde{R}(t)\le(\gamma\rho_1+e_1u_1\rho_2)\exp\Big\{-\Big(\mu+\alpha+\frac{\sigma_3^2}{2}\Big)t\\&+\sigma_3W_3(t)\Big\}\int_0^t\exp\Big\{\Big(\mu+\alpha+\frac{\sigma_3^2}{2}\Big)\tau-\sigma_3W_3(\tau)\Big\}I(\tau)d\tau.\end{aligned}$$

By (9), for arbitrary $\epsilon_1>0$ and $\omega\in\Omega$, there exists a $T_1=T_1(\omega)$ such that $I(t)\le\frac{1}{\tau_1}\exp((\Theta+\epsilon_1)t)$, $\forall t>T_1$. Hence, when $t>T_1$,

$$\begin{aligned}&|R(t)-\widetilde{R}(t)|\le(\gamma\rho_1+e_1u_1\rho_2)\exp\Big\{-\Big(\mu+\alpha+\frac{\sigma_3^2}{2}\Big)t\\&+\sigma_3W_3(t)\Big\}\int_0^{T_1}\exp\Big\{\Big(\mu+\alpha+\frac{\sigma_3^2}{2}\Big)\tau-\sigma_3W_3(\tau)\Big\}I(s)ds\\&+\frac{\Theta}{\tau_1}\exp\Big\{-\Big(\mu+\alpha+\frac{\sigma_3^2}{2}\Big)t+\sigma_3W_3(t)+\sigma_3\max_{\tau\le t}|W_3(\tau)|\Big\}\\&\times\int_{T_1}^t\exp\Big\{\Big(\mu+\alpha+\frac{\sigma_3^2}{2}+\Theta+\epsilon_1\Big)\tau\Big\}d\tau.\end{aligned}$$

That means that $\limsup_{t\to\infty}\frac{1}{t}\ln|R(t)-\widetilde{R}(t)|\le[-(\mu+\alpha+\frac{\sigma_3^2}{2})]\vee(\Theta+\epsilon_1)$ a.s. By the arbitrariness of ϵ_1, we know that $\limsup_{t\to\infty}\frac{1}{t}\ln|R(t)-\widetilde{R}(t)|\le[-(\mu+\alpha+\frac{\sigma_3^2}{2})]\vee\Theta$ *a.s.* Meanwhile, it can be seen from (10) that $\widetilde{R}(t)=R(0)\exp\{-(\mu+\alpha+\frac{\sigma_3^2}{2})t+\sigma_3W_3(t)\}$, which implies that $\limsup_{t\to\infty}\frac{1}{t}\ln\widetilde{R}(t)=-(\mu+\alpha+\frac{\sigma_3^2}{2})$ *a.s.* Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\ln R(t)\le\Big[-\Big(\mu+\alpha+\frac{\sigma_3^2}{2}\Big)\Big]\vee\Theta\ a.s.\quad (11)$$

Furthermore, if $\Theta<0$, it follows from (9) and (11) that

$$\max\Big\{\limsup_{t\to\infty}\frac{1}{t}\ln I(t),\limsup_{t\to\infty}\frac{1}{t}\ln R(t),\limsup_{t\to\infty}\frac{1}{t}\ln B(t)\Big\}<0,$$

which means $\lim_{t\to\infty}I(t)=\lim_{t\to\infty}R(t)=\lim_{t\to\infty}B(t)=0$ a.s. In this case, for sufficiently small $\epsilon_2>0$, there exist a set $\Omega_{\epsilon_2}\subset\Omega$ with $\mathbb{P}(\Omega_{\epsilon_2})\ge1-\epsilon_2$ and a positive constant

$T_2 = T_2(\omega)$ such that $(\Lambda - \epsilon_2 - \mu S)dt + \sigma_1 S dW_1(t) \le dS \le (\Lambda + \epsilon_2 - \mu S)dt + \sigma_1 S dW_1(t)$, $\forall \omega \in \Omega_{\epsilon_2}$, $t > T_2$. Then, by the stochastic comparison theorem and Lemma 3 in [38], we conclude that

$$\frac{\Lambda - \epsilon_2}{\mu} \le \lim_{t\to\infty} \frac{1}{t}\int_0^t S(\tau)d\tau \le \frac{\Lambda + \epsilon_2}{\mu} \; a.s.$$

Let $\epsilon_2 \to 0$, one has $\lim_{t\to\infty} \frac{1}{t}\int_0^t S(\tau)d\tau = \frac{\Lambda}{\mu}$ $a.s.$ □

2) DISEASE PERSISTENCE OF STOCHASTIC MODEL (1.2)

In this subsection, we prove Theorem 2(ii) about the persistence of disease for model (1.2).

Proof: Let

$$V_{21} = -a_1 \ln S - \ln I - a_2 \ln B + \frac{a_3}{K(\delta + e_2u_2)}B,$$

where a_1, a_2 and a_3 are positive constants to be determined later. By Itô's formula, we obtain

$$\begin{aligned}
&\mathscr{L}V_{21} \\
&= -a_1\Big[\frac{\Lambda}{S} + \frac{\gamma(1-\rho_1)I}{S} + \frac{e_1u_1(1-\rho_2)I}{S} + \frac{\alpha R}{S} \\
&\quad - \Big(\frac{\beta_1 B}{K+B} + \beta_2 I + \mu + \frac{1}{2}\sigma_1^2\Big)\Big] - \Big[\Big(\frac{\beta_1 B}{K+B} + \beta_2 I\Big)\frac{S}{I} \\
&\quad - \Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big)\Big] - a_2\Big[\frac{kI}{B} - \Big(\delta + e_2u_2 \\
&\quad + \frac{1}{2}\sigma_4^2\Big)\Big] + \frac{a_3}{K(\delta+e_2u_2)}[kI - (\delta + e_2u_2)B] \\
&\le -\frac{a_1\Lambda}{S} - \frac{\beta_1 SB}{K(1+\frac{B}{K})I} - a_3\Big(1 + \frac{B}{K}\Big) + a_1\Big(\mu + \frac{1}{2}\sigma_1^2\Big) \\
&\quad + a_2\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big) + \Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big) \\
&\quad + a_3 - \beta_2 S + \frac{a_1\beta_1}{K}B + \Big(a_1\beta_2 + \frac{a_3 k}{K(\delta+e_2u_2)}\Big)I - \frac{a_2 kI}{B} \\
&\le -4\sqrt[4]{\frac{\Lambda\beta_1\, k a_1a_2a_3}{K}} + a_3 + a_2\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big) \\
&\quad + a_1\Big(\mu + \frac{1}{2}\sigma_1^2\Big) + \Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big) - \beta_2 S \\
&\quad + \frac{a_1\beta_1}{K}B + \Big(a_1\beta_2 + \frac{a_3k}{K(\delta+e_2u_2)}\Big)I.
\end{aligned}$$

Choose a_1, a_2 and a_3 such that

$$\begin{aligned}
a_1\Big(\mu + \frac{1}{2}\sigma_1^2\Big) &= a_2\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big) \\
&= a_3 = \frac{\Lambda\beta_1 k}{K\Big(\mu + \frac{1}{2}\sigma_1^2\Big)\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big)},
\end{aligned}$$

then

$$\begin{aligned}
\mathscr{L}V_{21} \le &-\frac{\Lambda\beta_1 k}{K\Big(\mu + \frac{1}{2}\sigma_1^2\Big)\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big)} \\
&+ \Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big) \\
&- \beta_2 S + \frac{a_1\beta_1}{K}B + \Big(a_1\beta_2 + \frac{a_3k}{K(\delta+e_2u_2)}\Big)I. \quad (12)
\end{aligned}$$

Let $V_{22} = -a_4 \ln S + a_5 B$, where a_4 and a_5 are positive constants to be determined later. Then,

$$\begin{aligned}
&\mathscr{L}V_{22} \\
&= -a_4\Big[\frac{\Lambda}{S} + \frac{\gamma(1-\rho_1)I}{S} + \frac{e_1u_1(1-\rho_2)I}{S} + \frac{\alpha R}{S} \\
&\quad - \Big(\frac{\beta_1 B}{K+B} + \beta_2 I + \mu + \frac{1}{2}\sigma_1^2\Big)\Big] + a_5[kI - (\delta + e_2u_2)B] \\
&\le -\frac{a_4\Lambda}{S} + a_4\Big(\mu + \frac{1}{2}\sigma_1^2\Big) + \Big[\frac{a_4\beta_1}{K} - a_5(\delta + e_2u_2)\Big]B \\
&\quad + (a_4\beta_2 + a_5k)I,
\end{aligned}$$

which together with (12) yields that

$$\begin{aligned}
\mathscr{L}(V_{21} + V_{22}) \le &-\frac{\Lambda\beta_1 k}{K\Big(\mu + \frac{1}{2}\sigma_1^2\Big)\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big)} \\
&+ \Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big) + a_4\Big(\mu + \frac{1}{2}\sigma_1^2\Big) \\
&- \beta_2 S - \frac{a_4\Lambda}{S} + \Big[\frac{\beta_1(a_1+a_4)}{K} - a_5(\delta + e_2u_2)\Big]B \\
&+ \Big[(a_1+a_4)\beta_2 + a_5k + \frac{a_3k}{K(\delta+e_2u_2)}\Big]I \\
\le &-\frac{\Lambda\beta_1 k}{K\Big(\mu + \frac{1}{2}\sigma_1^2\Big)\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big)} \\
&+ \Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big) + a_4\Big(\mu + \frac{1}{2}\sigma_1^2\Big) \\
&- 2\sqrt{\beta_2 a_4\Lambda} + \Big[\frac{\beta_1(a_1+a_4)}{K} - a_5(\delta + e_2u_2)\Big]B \\
&+ \Big[(a_1+a_4)\beta_2 + a_5k + \frac{a_3k}{K(\delta+e_2u_2)}\Big]I.
\end{aligned}$$

Choose a_4 and a_5 such that

$$a_4\Big(\mu + \frac{1}{2}\sigma_1^2\Big) = \frac{\beta_2\Lambda}{\mu + \frac{1}{2}\sigma_1^2}, \quad a_5(\delta + e_2u_2) = \frac{\beta_1(a_1+a_4)}{K},$$

then

$$\begin{aligned}
\mathscr{L}(V_{21} + V_{22}) \le &-\frac{\Lambda\beta_1 k}{K\Big(\mu + \frac{1}{2}\sigma_1^2\Big)\Big(\delta + e_2u_2 + \frac{1}{2}\sigma_4^2\Big)} \\
&+ \Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big) - \frac{\beta_2\Lambda}{\mu + \frac{1}{2}\sigma_1^2} \\
&+ \Big[(a_1+a_4)\beta_2 + a_5k + \frac{a_3k}{K(\delta+e_2u_2)}\Big]I \\
= &-\Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S - 1) + \eta I,
\end{aligned}$$

where

$$\eta = (a_1+a_4)\beta_2 + a_5k + \frac{a_3k}{K(\delta+e_2u_2)}. \quad (13)$$

And hence

$$\begin{aligned}
&d(V_{21} + V_{22}) \\
&\le \Big[-\Big(\mu + d + \gamma + e_1u_1 + \frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S - 1) + \eta I\Big]dt \\
&\quad - \sigma_1(a_1+a_4)dW_1(t) - \sigma_2 dW_2(t) \\
&\quad - \sigma_4\Big(a_2 - a_5B - \frac{a_3B}{K(\delta+e_2u_2)}\Big)dW_4(t). \quad (14)
\end{aligned}$$

Integrating (14) from 0 to t and then dividing by t on both sides, we have

$$\begin{aligned}&\frac{1}{t}\ln\frac{V_{21}(t)+V_{22}(t)}{V_{21}(0)+V_{22}(0)}\\&\quad\le-\Big(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S-1)\\&\qquad+\frac{1}{t}\int_0^t\eta I(\tau)d\tau+\frac{\varphi(t)}{t},\end{aligned}$$

where $\varphi(t)=\int_0^t[-\sigma_1(a_1+a_4)dW_1(\tau)-\sigma_2dW_2(\tau)-\sigma_4(a_2-a_5B(\tau)-\frac{a_3B(\tau)}{K(\delta+e_2u_2)})dW_4(\tau)]$. According to Lemma 1 and the strong law of large numbers for martingale, we get $\lim\limits_{t\to\infty}\frac{\varphi(t)}{t}=0$, which further implies that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(\tau)d\tau\ge\frac{\Big(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S-1)}{\eta}.$$

This completes the proof. □

C. PROOF OF THEOREM 3

In this subsection, we provide a complete proof of Theorem 3. Before that, we give an important lemma about the existence of ergodic stationary distribution.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^l$ described by the stochastic differential equation

$$dX(t)=b(X)dt+\sum_{r=1}^{\ell}g_r(X)dB_r(t).$$

The diffusion matrix is defined as follows

$$A(x)=(a_{ij}(x)),\quad a_{ij}(x)=\sum_{r=1}^{\ell}g_r^i(x)g_r^j(x).$$

Then, one has the following lemma about the existence of stationary distribution established by Khasminskii [39].

Lemma 3 [39]: Assume there exists a bounded open set $D\subset\mathbb{R}^l$ with a smooth boundary Σ, satisfying the following conditions:

(a1) There exists a positive number $\mathscr{M}$ such that $\sum_{i,j=1}^{l}a_{ij}(x)\lambda_i\lambda_j\ge\mathscr{M}|\lambda|^2$, for $x\in D$ and $\lambda\in\mathbb{R}^l$.

(a2) There exists a non-negative C^2-function V and a positive constant L such that $\mathscr{L}V\le-L$, for any $X\in\mathbb{R}^l\setminus D$.

Then the Markov process $X(t)$ has a unique stationary distribution $\pi(\cdot)$, and for any integrable function $f(\cdot)$ with respect to the measure π we have

$$\mathbb{P}\Big\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s))ds=\int_{\mathbb{R}^l}f(x)\pi(dx)\Big\}=1.$$

With the help of Lemma 3, we now complete the proof of Theorem 3 about the existence of ergodic stationary distribution for model (2).

Proof: Let us construct a C^2 function $V_2:\mathbb{R}_+^4\to\mathbb{R}$ as

$$V_2=M(V_{21}+V_{22})+V_{23}+V_{24},$$

where V_{21}, V_{22} are defined in Theorem 2, and

$$\begin{aligned}V_{23}&=-\ln S-\ln R-\ln B,\\V_{24}&=\frac{1}{\theta+1}\Big(S+I+R+\frac{\mu+d}{2k}B\Big)^{\theta+1}.\end{aligned}$$

Here, M and θ satisfy the conditions

$$\begin{aligned}&-M\Big(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S-1)+\Phi\le-2,\\&\mu\wedge\frac{1}{2}(\mu+d)\wedge(\delta+e_2u_2)-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)>0,\end{aligned}$$

and Φ is defined as (24). Let $D_\iota=(\frac{1}{\iota},\iota)\times(\frac{1}{\iota},\iota)\times(\frac{1}{\iota},\iota)$, then $\liminf\limits_{\iota\to\infty,(S,I,R,B)\in\mathbb{R}_+^4\setminus D_\iota}V_2(S,I,R,B)=+\infty$, which implies that V_2 exists a global minimum point $(\bar{S},\bar{I},\bar{R},\bar{B})$ in the interior of $\mathbb{R}_+^4$. So we define a nonnegative function $\widetilde{V}_2$: $\mathbb{R}_+^4\to\mathbb{R}_+$ as

$$\widetilde{V}_2(S,I,R,B)=V_2(S,I,R,B)-V_2(\bar{S},\bar{I},\bar{R},\bar{B}).$$

Notice that

$$\begin{aligned}&\mathscr{L}V_{23}\\&=-\frac{\Lambda}{S}-\frac{\gamma(1-\rho_1)I}{S}-\frac{e_1u_1(1-\rho_2)I}{S}-\frac{\alpha R}{S}\\&\quad+\frac{\beta_1B}{K+B}+\beta_2I+\mu+\frac{1}{2}\sigma_1^2-\frac{(\gamma\rho_1+e_1u_1\rho_2)I}{B}+\mu\\&\quad+\alpha+\frac{1}{2}\sigma_3^2-\frac{kI}{B}+\delta+e_2u_2+\frac{1}{2}\sigma_4^2\\&\le-\frac{\Lambda}{S}-\frac{(\gamma\rho_1+e_1u_1\rho_2)I}{R}-\frac{kI}{B}+\beta_2I+\alpha\\&\quad+\beta_1+\delta+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2,\\&\mathscr{L}V_{24}\\&=(S+I+R+\frac{\mu+d}{2k}B)^\theta[\Lambda-\mu S-\frac{1}{2}(\mu+d)I-\mu R\\&\quad-\frac{(\mu+d)(\delta+e_2u_2)}{2k}B]+\frac{1}{2}\theta(S+I+R+\frac{\mu+d}{2k}B)^{\theta-1}\\&\quad\times[\sigma_1^2S^2+\sigma_2^2I^2+\sigma_3^2R^2+\sigma_4^2(\frac{\mu+d}{2k}B)^2]\\&\le(S+I+R+\frac{\mu+d}{2k}B)^\theta[\Lambda-(\mu\wedge\frac{1}{2}(\mu+d)\wedge(\delta+e_2u_2))\\&\quad\times(S+I+R+\frac{\mu+d}{2k}B)]+\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\\&\quad\times(S+I+R+\frac{\mu+d}{2k}B)^{\theta-1}[S^2+I^2+R^2+(\frac{\mu+d}{2k}B)^2]\\&\le\Lambda(S+I+R+\frac{\mu+d}{2k}B)^\theta-\xi(S+I+R+\frac{\mu+d}{2k}B)^{\theta+1}\\&\le\Psi-\frac{1}{2}\xi[S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}],\end{aligned}$$

where $\xi=\mu\wedge\frac{1}{2}(\mu+d)\wedge(\delta+e_2u_2)-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)>0$, and

$$\begin{aligned}\Psi=\sup\Big\{&\Lambda\Big(S+I+R+\frac{\mu+d}{2k}B\Big)^\theta\\&-\frac{1}{2}\xi\Big(S+I+R+\frac{\mu+d}{2k}B\Big)^{\theta+1}\Big\}<\infty.\end{aligned}\tag{15}$$

Therefore,

$$\begin{aligned}
&\mathscr{L}\widetilde{V}_2\\
&\leq -M\Big(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S-1)\\
&\quad+M\eta I-\frac{\Lambda}{S}-\frac{kI}{B}-\frac{(\gamma\rho_1+e_1u_1\rho_2)I}{R}-\frac{1}{2}\xi\Big[S^{\theta+1}\\
&\quad+I^{\theta+1}+R^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}\Big]+\beta_2 I+\Psi+\alpha\\
&\quad+\beta_1+\delta+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2. \qquad (16)
\end{aligned}$$

Suppose that D_ϵ is a bounded closed set defined as follows

$$D_\epsilon=\Big\{\epsilon\leq S\leq\frac{1}{\epsilon},\epsilon\leq I\leq\frac{1}{\epsilon},\epsilon^2\leq R\leq\frac{1}{\epsilon^2},\epsilon^2\leq B\leq\frac{1}{\epsilon^2}\Big\}.$$

Here ϵ is a sufficiently small positive constant satisfying

$$-\frac{\Lambda}{\epsilon}+\phi_1\leq-1, \qquad (17)$$

$$-M\Big(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S-1)+\Phi+M\eta\epsilon \leq -1, \qquad (18)$$

$$-\frac{\gamma\rho_1+e_1u_1\rho_2}{\epsilon}+\phi_1\leq-1, \qquad (19)$$

$$-\frac{k}{\epsilon}+\phi_1\leq-1, \qquad (20)$$

$$-\frac{1}{4}\xi\Big(\frac{1}{\epsilon}\Big)^{\theta+1}+\phi_2\leq-1, \qquad (21)$$

$$-\frac{1}{4}\xi\Big(\frac{\mu+d}{2k\epsilon^2}\Big)^{\theta+1}+\phi_2\leq-1, \qquad (22)$$

where ϕ_1, ϕ_2 are defined in (23) and (25). For simplicity, denote $\mathbb{R}_+^4\backslash D_\epsilon\equiv\bigcup_{i=1}^{8}D_\epsilon^i$, where

$$\begin{aligned}
&D_\epsilon^1=\{0<S<\epsilon\},\quad D_\epsilon^2=\{0<I<\epsilon\},\\
&D_\epsilon^3=\{I>\epsilon,0<R<\epsilon^2\},\quad D_\epsilon^4=\{I>\epsilon,0<B<\epsilon^2\},\\
&D_\epsilon^5=\Big\{S>\frac{1}{\epsilon}\Big\},\quad D_\epsilon^6=\Big\{I>\frac{1}{\epsilon}\Big\},\\
&D_\epsilon^7=\Big\{R>\frac{1}{\epsilon^2}\Big\},\quad D_\epsilon^8=\Big\{B>\frac{1}{\epsilon^2}\Big\}.
\end{aligned}$$

Next, we prove $\mathscr{L}\widetilde{V}_2\leq-1$ on $\mathbb{R}_+^4\backslash D_\epsilon$ in eight cases.

Case 1: When $(S,I,R,B)\in D_\epsilon^1$, by (17) one can see that

$$\begin{aligned}
&\mathscr{L}\widetilde{V}_2\\
&\leq-\frac{\Lambda}{S}-\frac{1}{2}\xi\Big[S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}\Big]\\
&\quad+M\eta I+\beta_2 I+\Psi+\alpha+\beta_1+\delta+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2\\
&\quad+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2\leq-\frac{\Lambda}{\epsilon}+\phi_1\leq-1,
\end{aligned}$$

where

$$\begin{aligned}
\phi_1=\sup\Big\{&-\frac{1}{2}\xi\Big[S^{\theta+1}+I^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}\\
&+R^{\theta+1}\Big]+M\eta I+\beta_2 I+\Psi+\alpha+\beta_1\\
&+\delta+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2\Big\}. \qquad (23)
\end{aligned}$$

Case 2: When $(S,I,R,B)\in D_\epsilon^2$, it follows from (18) that

$$\begin{aligned}
&\mathscr{L}\widetilde{V}\\
&\leq-M\Big(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S-1)+M\eta I\\
&\quad+\beta_2 I+\Psi+\alpha+\beta_1+\delta+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2\\
&\quad+\frac{1}{2}\sigma_4^2-\frac{1}{2}\xi\Big[S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}\Big]\\
&\leq-M\Big(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\Big)(\mathscr{R}_0^S-1)\\
&\quad+M\eta\epsilon+\Phi\leq-1,
\end{aligned}$$

where

$$\begin{aligned}
\Phi=\sup\Big\{&-\frac{1}{2}\xi\Big[S^{\theta+1}+I^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}\\
&+R^{\theta+1}\Big]+\Psi+\beta_2 I+\alpha+\beta_1+\delta+2\mu+e_2u_2\\
&+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2\Big\}. \qquad (24)
\end{aligned}$$

Case 3: When $(S,I,R,B)\in D_\epsilon^3$, by (19) we have

$$\begin{aligned}
&\mathscr{L}\widetilde{V}\\
&\leq-\frac{(\gamma\rho_1+e_1u_1\rho_2)I}{R}-\frac{1}{2}\xi\Big[S^{\theta+1}+I^{\theta+1}\\
&\quad+R^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}\Big]+M\eta I+\beta_2 I+\Psi+\alpha\\
&\quad+\beta_1+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2\\
&\leq+\delta-\frac{\gamma\rho_1+e_1u_1\rho_2}{\epsilon}+\phi_1\leq-1.
\end{aligned}$$

Case 4: When $(S,I,R,B)\in D_\epsilon^4$, by (20) we conclude that

$$\begin{aligned}
&\mathscr{L}\widetilde{V}\\
&\leq-\frac{kI}{B}-\frac{1}{2}\xi\Big[S^{\theta+1}+I^{\theta+1}+R^{\theta+1}+(\frac{\mu+d}{2k}B)^{\theta+1}\Big]\\
&\quad+M\eta I+\beta_2 I+\Psi+\alpha+\beta_1+\delta+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2\\
&\quad+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2\leq-\frac{k}{\epsilon}+\phi_1\leq-1.
\end{aligned}$$

Case 5: When $(S,I,R,B)\in D_\epsilon^5$, (21) implies that

$$\begin{aligned}
&\mathscr{L}\widetilde{V}\\
&\leq-\frac{1}{4}\xi S^{\theta+1}+M\eta I-\frac{1}{4}\xi\Big[S^{\theta+1}+I^{\theta+1}+R^{\theta+1}\\
&\quad+(\frac{\mu+d}{2k}B)^{\theta+1}\Big]+\beta_2 I+\Psi+\alpha+\beta_1+\delta+2\mu+e_2u_2\\
&\quad+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2\leq-\frac{1}{4}\xi(\frac{1}{\epsilon})^{\theta+1}+\phi_2\leq-1,
\end{aligned}$$

where

$$\begin{aligned}
\phi_2=\sup\Big\{&-\frac{1}{4}\xi\Big[S^{\theta+1}+I^{\theta+1}+R^{\theta+1}\\
&+(\frac{\mu+d}{2k}B)^{\theta+1}\Big]+\beta_2 I+M\eta I+\Psi+\alpha+\beta_1\\
&+\delta+2\mu+e_2u_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2\Big\}. \qquad (25)
\end{aligned}$$

Case 6: When $(S, I, R, B) \in D_\epsilon^6$, using (21), then

$$\begin{aligned}
&\mathscr{L}\widetilde{V} \\
&\le -\frac{1}{4}\xi I^{\theta+1} - \frac{1}{4}\xi\Big[S^{\theta+1} + I^{\theta+1} + (\frac{\mu+d}{2k}B)^{\theta+1} \\
&\quad + R^{\theta+1}\Big] + M\eta I + \beta_2 I + \Psi + \alpha + \beta_1 + \delta + 2\mu + e_2u_2 \\
&\quad + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 \le -\frac{1}{4}\xi(\frac{1}{\epsilon})^{\theta+1} + \phi_2 \le -1.
\end{aligned}$$

Case 7: When $(S, I, R, B) \in D_\epsilon^7$, by (22), we can easily deduce that

$$\begin{aligned}
&\mathscr{L}\widetilde{V} \\
&\le -\frac{1}{4}\xi R^{\theta+1} - \frac{1}{4}\xi\Big[S^{\theta+1} + I^{\theta+1} + (\frac{\mu+d}{2k}B)^{\theta+1} \\
&\quad + R^{\theta+1}\Big] + M\eta I + \beta_2 I + \Psi + \alpha + \beta_1 + \delta + 2\mu + e_2u_2 \\
&\quad + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 \le -\frac{1}{4}\xi(\frac{1}{\epsilon^2})^{\theta+1} + \phi_2 \le -1.
\end{aligned}$$

Case 8: When $(S, I, R, B) \in D_\epsilon^8$, following (22) yields

$$\begin{aligned}
&\mathscr{L}\widetilde{V} \\
&\le -\frac{1}{4}\xi\Big[S^{\theta+1} + I^{\theta+1} + R^{\theta+1} + (\frac{\mu+d}{2k}B)^{\theta+1}\Big] + M\eta I \\
&\quad - \frac{1}{4}\xi(\frac{\mu+d}{2k}B)^{\theta+1} + \beta_2 I + \Psi + \alpha + \beta_1 + \delta + 2\mu + e_2u_2 \\
&\quad + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 \le -\frac{1}{4}\xi(\frac{\mu+d}{2k\epsilon^2})^{\theta+1} + \phi_2 \le -1.
\end{aligned}$$

From the above analysis, it is shown that $\mathscr{L}\widetilde{V}_2 \le -1$ for $(S, I, R, B) \in \mathbb{R}_+^4 \backslash D_\epsilon$, which implies that the condition (a2) in Lemma 3 holds with $L = 1$.

And also, choose $\mathscr{M} = \min_{(S,I,R,B)\in D_\epsilon}\{\sigma_1^2S^2, \sigma_2^2I^2, \sigma_3^2R^2, \sigma_4^2B^2\}$ such that

$$\begin{aligned}
&\sum_{i,j=1}^{4} a_{ij}(S, I, R, B)\lambda_i\lambda_j \\
&\qquad = \sigma_1^2S^2\lambda_1^2 + \lambda_2^2I^2\eta_2^2 + \lambda_3^2R^2\eta_3^2 + \lambda_4^2B^2\eta_4^2 \ge \mathscr{M}\|\lambda\|^2
\end{aligned}$$

for all $(S, I, R, B) \in D_\epsilon$, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$, then the condition (a1) in Lemma 3 holds. So, model (2) has a unique stationary distribution $\pi(\cdot)$ and it is ergodic. This completes the proof. □

IV. NUMERICAL SIMULATIONS

In this section, we numerically simulate the solution of model (2) to check our theoretical results by use of the method in [40] and then find the effects of white noise and control on the spread of disease. In the following, we always select $(S(0), I(0), R(0), B(0)) = (520, 30, 1, 5)$ and parameters $\Lambda = 2.5$, $\gamma = 0.001$, $\rho_1 = 0.4$, $\rho_2 = 0$, $e_1 = 1$, $e_2 = 0.1$, $\beta_2 = 0.0001$, $\beta_1 = 0.0001$, $K = 400$, $\mu = 0.004$, $d = 0.0015$, $\alpha = 0.0025$, $\delta = 0.03$, $k = 0.2$ unless otherwise specified. Now let us do our numerical simulation in three cases.

Case 1: $\sigma_1 = 0.001$, $\sigma_2 = 0.18$, $\sigma_3 = 0.04$, $\sigma_4 = 0.01$, $u_1 = 0.05$, $u_2 = 0.1$. In this case, simple calculations show that $\mathscr{R}_0 = 1.12 > 1$, which shows that the system without random perturbation is uniformly persistent. This can be seen from see the blue lines in Fig. 1.

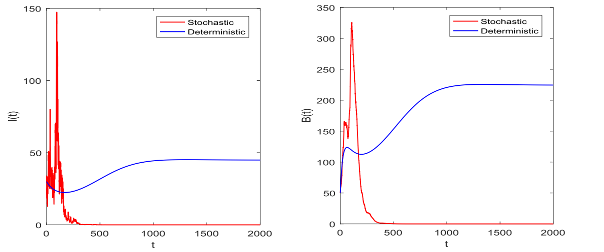

FIGURE 1. The left are time series of infected humans; the right are times series of pathogen. Here, $\sigma_1 = 0.001$, $\sigma_2 = 0.18$, $\sigma_3 = 0.04$, $\sigma_4 = 0.01$, $u_1 = 0.05$, $u_2 = 0.1$ and $\mathscr{R}_0 = 1.12 > 1$, $\mathscr{R}_0^S = 0.87 < 1$.

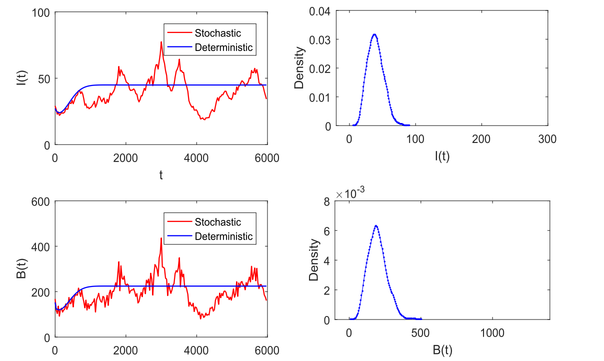

FIGURE 2. The left are time series of infected humans and pathogen, respectively; the right are its corresponding probability density functions. Here, $\sigma_1 = 0.001$, $\sigma_2 = 0.01$, $\sigma_3 = 0.04$, $\sigma_4 = 0.01$, $u_1 = 0.05$, $u_2 = 0.1$ and $\mathscr{R}_0 = 1.12 > 1$, $\mathscr{R}_0^S = 1.1 > 1$.

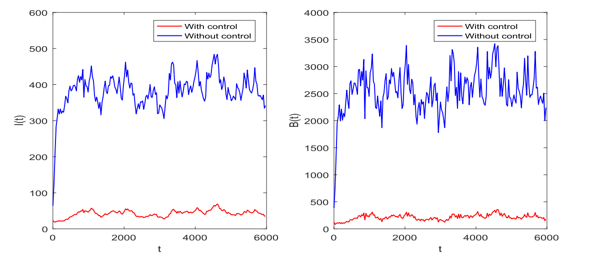

FIGURE 3. Simulations of model (2) with and without controls. The left are time series of infected humans; the right are times series of pathogen.

On the other hand, $\mathscr{R}_0^S = 0.87 < 1$, and $\varrho = -0.001 < 0$, where $v_1 = \Pi$ and $v_2 = \frac{\Lambda\beta_1}{K(\mu+\frac{1}{2}\sigma_1^2)(\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2)}$. This together with Theorem 2 yields that under random perturbation, all solution trajectories representing infected humans with dysentery diarrhoea drop to zero and the pathogen population is also eliminated from the community. Fig. 1 confirms these theoretical results, see the red lines.

Case 2: $\sigma_1 = 0.001$, $\sigma_2 = 0.01$, $\sigma_3 = 0.04$, $\sigma_4 = 0.01$, $u_1 = 0.05$, $u_2 = 0.1$. By calculating we can see that $\mathscr{R}_0^S = 1.1 > 1$. Then model (2) is persistent and has a unique ergodic stationary distribution by Theorems 2 and 3, see Fig. 2.

Comparing the above two cases, we find that large white noise may suppress the spread of disease, while small white noise can not change the original persistence of disease.

Case 3: Let $u_1 = u_2 = 0$ and noise intensities are same as Case 2. As pointed in Remark 2, treatment and sanitation may reduce the reproduction number, which implies that it is possible to decrease the infected and pathogen population using controls. Fig. 3 confirms these analysis.

V. CONCLUDING REMARKS

To better understand the spread of disease in random environment, in this paper, we proposed a stochastic dysentery diarrhoea epidemic model with controls disturbed by white noise. We first showed the existence of global positive solution of the model. Then we gave a stochastic threshold $\mathscr{R}_0^S$, which determines the extinction and persistence of the disease. Lastly, based on Khasminskii's theory, we proved that the model has a unique ergodic stationary distribution under the condition of persistence in mean. Theoretical analysis and numerical simulations show that: (i) Large white noise may suppress the spread of disease while small white noise cannot change the original persistence of disease; (ii) Constant treatment and sanitation have important positive role in controlling the spread of disease.

It is especially necessary to point out that the method proposed in this paper is universal and can be used to study other types of stochastic epidemic models, for example, stochastic HIV-1 infection model with logistic growth or Beddington-DeAngelis incidence rate and so on. Here are two examples to illustrate the universality of this method from other perspectives.

Example 1: In nature, besides the Gaussian white noise, there are other types of noise such as colored noise, which can cause the system to switch from one environmental regime to another. Usually, the switching between environmental regimes is memoryless and the waiting time for the next switching follows the exponential distribution [41]–[45]. Hence the colored noise can be modeled by a continuous time Markov chain $\{r(t)\}_{t\geq 0}$ taking values in a finite state space $\mathbb{N} = \{1, 2, \ldots, N\}$. With this in mind, model (2) can be extended to the following form:

$$\begin{cases} dS = \Big[\Lambda(r(t)) + \gamma(r(t))(1-\rho_1(r(t)))I + e_1(r(t)) \\ \quad \times u_1(r(t))(1-\rho_2(r(t)))I + \alpha(r(t))R - \Big(\dfrac{\beta_1(r(t))B}{K(r(t))+B} \\ \quad + \beta_2(r(t))I + \mu(r(t))\Big)S\Big]dt + \sigma_1(r(t))SdW_1(t), \\ dI = \Big[\Big(\dfrac{\beta_1(r(t))B}{K(r(t))+B} + \beta_2(r(t))I\Big)S - (\mu(r(t)) + d(r(t)) \\ \quad + \gamma(r(t)) + e_1(r(t))u_1(r(t)))I\Big]dt + \sigma_2(r(t))IdW_2(t), \\ dR = [(\gamma(r(t))\rho_1(r(t)) + e_1(r(t))u_1(r(t))\rho_2(r(t)))I \\ \quad - (\mu(r(t)) + \alpha(r(t)))R]dt + \sigma_3(r(t))RdW_3(t), \\ dB = [k(r(t))I - (\delta(r(t)) + e_2(r(t))u_2(r(t)))B]dt \\ \quad + \sigma_4(r(t))BdW_4(t). \end{cases} \tag{26}$$

For the sake of argument, we assume that the generator $Q = (q_{ij})_{N\times N}$ of the Markov chain is governed by

$$\mathbb{P}\{r(t+\Delta t) = j | r(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j, \\ 1 + q_{ii}\Delta t + o(\Delta t), & \text{if } i = j, \end{cases}$$

where $\Delta t > 0$ and q_{ij} is the transition rate from state i to state j and $q_{ij} \geq 0$ if $i \neq j$ while $q_{ii} = -\sum_{i\neq j} q_{ij}$. In addition, we further assume that the Markov chain $r(t)$ is irreducible and hence has a unique ergodic stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N)$ satisfying

$$\pi Q = 0, \ \sum_{i\in\mathbb{N}} \pi_i = 1 \text{ and } \pi_i > 0, \quad \forall i \in \mathbb{N}.$$

Now we denote $\hat{f} = \min_{m\in\mathbb{N}}\{f(m)\}$, $\check{f} = \max_{m\in\mathbb{N}}\{f(m)\}$, $J_1 = \sum_i^N \pi_i(\delta(i) + e_2(i)u_2(i) + \frac{1}{2}\sigma_4^2(i))(\sum_i^N \pi_i\sqrt{\Lambda(i)\beta_2(i)})^2$, $J_2 = (\sum_i^N \pi_i \sqrt[4]{\frac{\Lambda(i)\beta_1(i)k(i)}{K(i)}})^4$, $J_3 = \sum_i^N \pi_i(\mu(i) + \frac{1}{2}\sigma_1^2(i))\sum_i^N \pi_i(\mu(i) + d(i) + \gamma(i) + e_1(i)u_1(i) + \frac{1}{2}\sigma_2^2(i)) \sum_i^N \pi_i(\delta(i) + e_2(i)u_2(i) + \frac{1}{2}\sigma_4^2(i))$, and

$$\mathscr{R}_0^C = \frac{J_1 + J_2}{J_3}.$$

Using the method proposed in this paper, the persistence in mean and existence of an ergodic stationary distribution for hybrid model (26) can be investigated easily. Therefore, the following conclusions are valid.

Theorem 4: Assume $\mathscr{R}_0^C > 1$, then for any initial value $(S(0), I(0), R(0), B(0), r(0)) \in \mathbb{R}_+^4 \times \mathbb{N}$, the solution of model (26) has the following property

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(\tau)d\tau \geq \frac{\sum_i^N \pi_i(\mu(i) + d(i) + \gamma(i) + e_1(i)u_1(i) + \frac{1}{2}\sigma_2^2(i))}{\check{\eta}_1} \times (\mathscr{R}_0^C - 1) \ a.s.,$$

where

$$\eta_1 = (b_1 + b_2)\beta_2(i) + b_3k(i) + \frac{b_4k(i)}{K(i)(\delta(i) + e_2(i)u_2(i))},$$

$$b_1 = \frac{(\sum_i^N \pi_i \sqrt[4]{\frac{\Lambda(i)\beta_1(i)k(i)}{K(i)}})^4}{(\sum_i^N \pi_i(\mu(i)+\frac{1}{2}\sigma_1^2(i)))^2 \sum_i^N \pi_i(\delta(i)+e_2(i)u_2(i)+\frac{1}{2}\sigma_4^2(i))},$$

$$b_2 = \frac{(\sum_i^N \pi_i\sqrt{\Lambda(i)\beta_2(i)})^2}{(\sum_i^N \pi_i(\mu(i) + \frac{1}{2}\sigma_1^2(i)))^2},$$

$$b_3 = \frac{\check{\beta}_1}{\hat{K}(\hat{\delta}+\hat{e}_2\hat{u}_2)},$$

$$b_4 = \frac{(\sum_i^N \pi_i \sqrt[4]{\frac{\Lambda(i)\beta_1(i)k(i)}{K(i)}})^4}{\sum_i^N \pi_i(\mu(i)+\frac{1}{2}\sigma_1^2(i))\sum_i^N \pi_i(\delta(i)+e_2(i)u_2(i)+\frac{1}{2}\sigma_4^2(i))}.$$

Theorem 5: If $\mathscr{R}_0^C > 1$, then model (26) admits a unique stationary distribution and it has the ergodic property.

Example 2: Considering seasonal fluctuation and other factors, similar to [37], [46]–[48], model (2) can also be extended to the following periodic form:

$$\begin{cases} dS = \Big[\Lambda(t)+\gamma(t)(1-\rho_1(t))I+e_1(t)u_1(t)(1-\rho_2(t))I \\ \quad +\alpha(t)R-\Big(\frac{\beta_1(t)B}{K(t)+B}+\beta_2(t)I+\mu(t)\Big)S\Big]dt+\sigma_1(t)SdW_1(t), \\ dI = \Big[\Big(\frac{\beta_1(t)B}{K(t)+B}+\beta_2(t)I\Big)S-(\mu(t)+d(t)+\gamma(t) \\ \quad +e_1(t)u_1(t))I\Big]dt+\sigma_2(t)IdW_2(t), \\ dR = [(\gamma(t)\rho_1(t)+e_1(t)u_1(t)\rho_2(t))I-(\mu(t)+\alpha(t))R]dt \\ \quad +\sigma_3(t)RdW_3(t), \\ dB = [k(t)I-(\delta(t)+e_2(t)u_2(t))B]dt+\sigma_4(t)BdW_4(t), \end{cases} \quad (27)$$

where all parameters are continuous ϖ-periodic functions.

When f is an integrable function, we denote $\langle f\rangle_\varpi = \frac{1}{\varpi}\int_0^\varpi f(\tau)d\tau$; while for a continuous ϖ-periodic function, denote $f^l = \min_{t\in[0,\varpi]} f(t)$, $f^u = \max_{t\in[0,\varpi]} f(t)$. Define a parameter

$$\mathscr{R}_0^\varpi = \frac{J_4+J_5}{J_6}.$$

where $J_4 = \langle\delta+e_2u_2+\frac{1}{2}\sigma_4^2\rangle_\varpi\langle\sqrt{\Lambda\beta_2}\rangle_\varpi^2$, $J_5 = \langle\sqrt[4]{\frac{\Lambda\beta_1 k}{K}}\rangle_\varpi^4$, $J_6 = \langle\mu+\frac{1}{2}\sigma_1^2\rangle_\varpi\langle\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\rangle_\varpi\langle\delta+e_2u_2+\frac{1}{2}\sigma_4^2\rangle_\varpi$. We then get the following two results, whose proof are similar to Theorems 2(ii) and 3.

Theorem 6: Assume $\mathscr{R}_0^\varpi > 1$, then for any initial value $(S(0), I(0), R(0), B(0)) \in \mathbb{R}_+^4$, the solution $(S(t), I(t), R(t), B(t))$ of model (27) has

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(\tau)d\tau \geq \frac{\langle\mu+d+\gamma+e_1u_1+\frac{1}{2}\sigma_2^2\rangle_\varpi}{\eta_2^u} \times(\mathscr{R}_0^\varpi-1),$$

where $\eta_2 = (c_1+c_2)\beta_2(t)+c_3k(t)+\frac{c_4k(t)}{K(t)(\delta(t)+e_2(t)u_2(t))}$, $c_1 = \frac{\langle\sqrt[4]{\frac{\Lambda\beta_1 k}{K}}\rangle_\varpi^4}{\langle\mu+\frac{1}{2}\sigma_1^2\rangle_\varpi^2\langle\delta+e_2u_2+\frac{1}{2}\sigma_4^2\rangle_\varpi}$, $c_2 = \frac{\langle\sqrt{\Lambda\beta_2}\rangle_\varpi^2}{\langle\mu+\frac{1}{2}\sigma_1^2\rangle_\varpi^2}$, $c_3 = \frac{\beta_1^u}{K^l(\delta^l+e_2^l u_2^l)}$, $c_4 = \frac{\langle\sqrt[4]{\frac{\Lambda\beta_1 k}{K}}\rangle_\varpi^4}{\langle\mu+\frac{1}{2}\sigma_1^2\rangle_\varpi\langle\delta+e_2u_2+\frac{1}{2}\sigma_4^2\rangle_\varpi}$.

Theorem 7: If $\mathscr{R}_0^\varpi > 1$, then model (27) admits a positive ϖ-periodic solution.

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