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A New Multi Attribute Decision Making Method Based on the T-Spherical Hesitant Fuzzy Sets

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ABSTRACT In this paper, we first introduce the concept of T-spherical hesitant fuzzy set (T-SHFS) on the basis of the combination of T-spherical fuzzy sets (T-SFSs) and hesitant fuzzy sets (HFSs). This model can provide more accuracy in expressing fuzzy and indeterminate data. Then, we develop the basic operational laws of T-SHFSs and study their properties. Also, we propose the T-spherical hesitant fuzzy weighted averaging (T-SHFWA) operator and the T-spherical hesitant fuzzy weighted geometric (T-SHFWG) operator and investigate their properties. Furthermore, two newly approaches to multi attribute decision making within the framework of T-SHFS are developed on the basis of the proposed aggregation operators. An illustrative example is given to demonstrate the effectiveness of the developed approaches in dealing with uncertain and indeterminate decision making problems. Finally, the proposed models' performance is evaluated to the existing models. Results from this comparison analysis reveal a great similarity and consistency while using other models.

INDEX TERMS Aggregation operators, hesitant fuzzy set, MADM, spherical fuzzy set, T-spherical fuzzy set.

I. INTRODUCTION

Decision theory has been a chief field of investigation in many scientific disciplines were the area of choice under uncertainty represents the heart of decision theory. The process of decision-making, in the majority of cases, consists of the evaluation of alternatives and the choice of the most preferable from them [1]–[3]. Multiple attribute decision making (MADM) - firstly proposed by Churchman *et al.* [4] - refers to rank alternatives or select the best choice based on multiple attribute evaluation values of the different alternatives. In traditional MADM problems, decision-makers usually provide deterministic measurements to express their preference. On account of the time limitations, decision-makers' capability, and the increasing uncertainty of problems and the complexity of human's cognitive information, decision-makers frequently find it difficult to provide deterministic measurements when attempting to solve MADM problems. To tackle this issue, Zadeh [5] originated fuzzy set (FS) to process the fuzzy information, and then it was used to easily characterized the attribute values in uncertain MADM problems. However, in some cases, fuzzy set alone cannot precisely describe vague, ambiguous, incomplete and indeterminate information in MADM problems. To effectively deal with these cases, many generalizations and variations of fuzzy sets have been generated. Among them, we underline, for their relevance in this paper, hesitant fuzzy sets (HFS) [6], intutionistic fuzzy set (IFS) [7], Pythagorean fuzzy set (PYFS) [8] and picture fuzzy set (PFS) [9].

PFS is a generalization of FS and IFS which is characterized by a truth membership function (T), indeterminacy membership function (I), and falsity membership function(F), each of which lies in the standard interval of [0, 1] and their summation is greater than or equal 0 and less than or equal 1, i.e., $0 \leq T + I + F \leq 1$. Basically, PFS has been originated to express the indeterminacy, which cannot be accurately expressed in the traditional FS and IFS. PFS has further been developed to picture hesitant fuzzy set(PHFS) [10] and spherical fuzzy set (SFS) [11]. In SFS memberships grades are gratifying the condition $0 \leq T^2 + I^2 + F^2 \leq 1$ instead of $0 \leq T + I + F \leq 1$ as is in PFS. SFS has further been extended to spherical hesitant fuzzy set (SHFS) [12] which is a generalization of PHFS. Comparing with PHFS to model uncertain information, SHFS has wider feasible region, which can handle more room of uncertainty.

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In addition, IFS and PYFS has been extended to q-rung orthopair fuzzy set (q-ROFS) [13] which is more flexible than IFS and PYFS since the sum of the qth power of the truth membership and falsity membership less than 1. In other words, q-ROFSs relax the constraint of IFS and PYFSs.

In order to enhance the quality of q-ROFS, so it can deal with vagueness and impreciseness, Li *et al.* [14] proposed q-rung picture fuzzy set (q-RPFS), which takes the advantages of both q-ROFS and PFS. To effectively aggregate q-RPF data, some aggregation operators have been developed [14]–[19]. In [14], the authors proposed the q-rung picture linguistic weighted Heronian mean (q-RPLWHM) and the q-rung picture linguistic geometric Heronian mean (q-RPLGHM) operators. He *et al.*, developed some q-RPF Dombi Hamy mean (q-RPFDHM) operators in [15]. Akram *et al.* [16], proposed Einstein operational laws for q-rung picture fuzzy numbers and introduced some q-rung picture fuzzy Einstein weighted averaging operators. Specific types of q-RPF Yager average and geometric aggregation operators have been studied in [17]. Yang *et al.*, [18] presented the interval q-RPF Heronian mean (IVq-RPtFHM) operators based on the new operational laws of the IVq-RPtF numbers. Pinar and Boran [19] introduced a novel distance measure for q-RPFS. The proposed distance measure is used in q-RPF ELECTRE integrated with TOPSIS method.

Meanwhile, to consider human's hesitance, q-ROFS has been extended to q-rung orthopair hesitant fuzzy set (q-ROHFS) by Liu *et al.* [20]. Authors proposed the distance measures between q-ROHFSs and developed TOPSIS method into the proposed measures. Hussain *et al.* [21] also presented a new concept of hesitant q-rung orthopair fuzzy set (H_qROFS) and proposed the H_qROF weighted averaging (H_qROFWA) and H_qROF weighted geometric (H_qROFWG)operators. Wang *et al.* [22] defined the distance, similarity measures and the entropy of q-ROHFSs. Based on these concepts, they constructed a TOPSIS model under the q-ROHF environment. Wang *et al.* [23] presented the dual hesitant q-rung orthopair fuzzy set (DHq-ROFS) for handling real MADM problems. To aggregate the information in DHq-ROFSs some Muirhead mean operators are defined. Afterward, the defined operators are used to solve the MADM with DHq-ROF numbers. In the same year Wang *et al.* [24] proposed some power Heronian mean operators under the q-ROHF environment to deal with green supplier selection in supply chain management. Yang and Pang [25] developed a new MADM based on the q-ROHFS, where the linear programming technique for multidimensional analysis of preference (LINMAP) and TOPSIS have been extended to q-ROHF environment in order to handle MADM problems.

Mahmood *et al.* [11] proposed T-spherical fuzzy set (T-SFS) as a generalization of q-ROFS and q-RPFS. Some aggregation operators have been developed on T-SFS. [11], [26]–[31]. In [11], the authors proposed the T- spherical fuzzy weighted geometric operator. Garg *et al.*, developed some weighted ordered weighted and hybrid geometric aggregation operators in [26]. Garg *et al.* [27], define several weighted averaging and geometric power aggregation operators. The stated operators named as T-spherical fuzzy weighted, ordered weighted, hybrid averaging and geometric operators for the collection of the T-SFSs. Zeng *et al.* [28] proposed some new Einstein operations for TSF numbers on which they define some T-spherical Einstein interactive averaging and geometric operators. In [29], a generalized parameter is defined for TSFSs and based on the proposed parameter, a group generalized TSF geometric operators are proposed. In the complex space, Ali *et al.* [30] proposed two aggregation operators to the complex TSF numbers, including weighted geometric and weighted averaging operators. In addition, Karaaslan *et al.* [31], introduced some complex TSF Dombi weighted aggregation operators on the basis of the Dombi t-norm and t-conorm.

Summarizing above discussion, we combine the advantages of both T-SFS and HFS by proposing the notion of T-spherical hesitant fuzzy set(T-SHFS) which is a more comprehensive model to cope with the most real complicated situations that can not be handled by the existing models. In the T-SHFS, for each element of the reference set, the grades of truth, indeterminacy and falsity memberships consist of the set of several values in the interval [0, 1] rather than a single value. Furthermore, the sum of the qth power of the truth membership, the qth power of the indeterminacy membership and the qth power of the falsity membership is not more than 1, which makes it more influential and resilient than other current models. Thus, the contributions of this paper are stated as follows: (1) the novel notion of T-SHFS with its operations are proposed; (2) to effectively aggregate T-SHF data, we propose the aggregation operators, called T-SHFWA and T-SHFWG operators; (3) MADM problem is addressed based on T-SHFNs by utilizing the T-SHFWA and T-SHFWG operators. To examine the efficiency and validity of the proposed models, we solve an example by using the proposed operators;(4) the proposed operators are compared with existing operators with an MADM example.

The remaining portions of the paper is set out as follows. In section 2, we present some essential concepts related to PHFS, SHFS, q-ROFS, T-SFS and q-ROHFS. In Section 3, we provide the formal concept of the T-SHFS and define the score and accuracy function of the T-SHFNs. The operational rules of the T-SHFNs along with their properties are also introduced in this section. We propose the T-SHFWA operator and the T-SHFWG operator in Section 4. Properties of these operators are also investigated in this section. On the basis of the T-SHFWA operator and the T-SHFWG operator, two new methods for MADM are proposed in Sction 5. To verify the application of the suggested models, Section 6 provides real example about the ranking of mobile phone products. The comparison analysis with other existing models has been conducted in Section 7. Finally, Section 8, outlines the main results in this research.

II. BASIC CONCEPTS

We shortly review the basic concepts of PHFS, SHFS, q-ROFS, T-SFS and q-ROHFS which be utilized in the subsequent research.

A. PHFS

Definition 2 [10]: Let M be a finite set, $M \neq \phi$. A PHFS F on M is represented as:

$$F = \{< m, \alpha(m), \beta(m), \gamma(m) > | m \in M\},$$

where $\alpha(m) = \{\mu : \mu \in \alpha(m)\}$, $\beta(m) = \{\nu : \nu \in \beta(m)\}$, $\gamma(m) = \{\xi : \xi \in \gamma(m)\}$ are three sets of values in [0, 1], representing the truth, neutral, and falsity membership degrees, where $0 \leq \mu^+ + \nu^+ + \xi^+ \leq 1$, such that $\mu^+ = \cup_{\mu\in\alpha(m)} \max\{\mu\}$, $\nu^+ = \cup_{\nu\in\beta(m)} \max\{\nu\}$, and $\xi^+ = \cup_{\xi\in\gamma(m)} \max\{\xi\}$.

In order to rank the picture hesitant fuzzy numbers, Wang and Li [10] developed the score and accuracy function as:

Definition 3 [10]: Suppose $N = \{\alpha, \beta, \gamma\}$ is a picture hesitant fuzzy number, the numbers of values in α, β, γ are l, m, n, respectively. The score function is stated as

$$\Gamma(N) = \frac{1 + \frac{1}{l}\sum_{i=1}^{l}\mu_i - \frac{1}{m}\sum_{i=1}^{m}\nu_i - \frac{1}{n}\sum_{i=1}^{n}\xi_i}{2}, \Gamma(N) \in [0, 1]$$

The accuracy function is stated as

$$\lambda(N) = \frac{\frac{1}{l}\sum_{i=1}^{l}\mu_i + \frac{1}{p}\sum_{i=1}^{p}\nu_i + \frac{1}{q}\sum_{i=1}^{q}\xi_i}{2}, \lambda(N) \in [0, 1].$$

As an extension of PHFS, Khan *et al.* [12] proposed the SHFS which is more powerful and flexible since the sum of the square power of the positive membership, the square power of the neutral membership and the square power of the negative membership is not exceed 1. The formal definition of the SHFS is as follows.

B. SPHERICAL HESTANT FUZZY SET (SHFS)

Definition 2 [12]: Consider the ground set $M \neq \phi$. A SHFS F on M can be defined as follows.

$$F = \{< m, \alpha(m), \beta(m), \gamma(m) > | m \in M\},$$

where $\alpha(m) = \{\mu : \mu \in [0, 1]\}$, $\beta(m) = \{\nu : \nu \in [0, 1]\}$, $\gamma(m) = \{\xi : \xi \in [0, 1]\}$ are three sets of values in [0, 1], representing the positive, neutral, and negative membership degrees, with the condition $0 \leq (\mu^+)^2 + (\nu^+)^2 + (\xi^+)^2 \leq 1$, such that $\mu^+ = \cup_{\mu\in\alpha(m)} \max\{\mu\}$, $\nu^+ = \cup_{\nu\in\beta(m)} \max\{\nu\}$, and $\xi^+ = \cup_{\xi\in\gamma(m)} \max\{\xi\}$.

C. Q-ROFS

Definition 4 [13]: If M is a finite set. Then the q-ROFS H on M is defined as

$$H = \{(\mu, T_H(\mu), F_H(\mu)) : \mu \in M\},$$

where $T_H(\mu)$ and $F_H(\mu)$ represent degree of positive membership and the degree of negative membership respectively, where $T_H(\mu)$ and $F_H(\mu)$ belong to the unit interval [0, 1] and $0 \leq (T_H(\mu))^q + (F_H(\mu))^q \leq 1$ $(q \geq 1), \forall \mu \in M$. The indeterminacy degree of μ in M is

$$\pi_H(\mu) = \left(T_H^q(\mu) + F_H^q(\mu) - T_H^q(\mu)F_H^q(\mu)\right)^{1/q}.$$

Mahmood *et al.* [10] define the T-SFS by taking the neutral membership degree into account in q-ROFSs.

D. T-SPHERICAL FUZZY SET (T-SFS)

Definition 5 [10]: If M is a fixed set. A T-SFS H on M is defined as follows.

$$H = \{(\mu, T_H(\mu), I_H(\mu), F_H(\mu)) : \mu \in M\},$$

where $T_H(\mu)$, $I_H(\mu)$ and $F_H(\mu)$ belong to the unit interval [0, 1] and $0 \leq (T_H(\mu))^q + (I_H(\mu))^q + (F_H(\mu))^q \leq 1$ $(q \geq 1)$, for all $\mu \in M$. The degree of refusal membership of μ to M is defined as

$$\pi_H(\mu) = \left(1 - \left[(T_H(\mu))^q + (I_H(\mu))^q + (F_H(\mu))^q\right]\right)^{1/q}$$

E. Q-ROHFS

Based on the q-ROFS and HFS, Liu *et al.* [20] define the q-ROHFS as follows.

Definition 6 [20]: Let $M = \{m_1, m_2, \ldots, m_n\}$ be a fixed set. The q-ROHFS on M is defined as

$$H = \{< m_i, D(m_i), E(m_i) >_q : m_i \in M\} (q \geq 1),$$

where $D(m_i)$ and $E(m_i)$ are two sets of values in [0, 1], denoting all the possible qth rung memberships and the qth rung nonmemberships of the elements $m_i \in M$, and they satisfy:

$0 \leq \alpha, \beta \leq 1$, $0 \leq (\alpha^+)^q + (\beta^+)^q \leq 1$, where $\alpha \in D(m_i)$, $\beta \in E(m_i)$, $\alpha^+ = \max_{\alpha\in D(m_i)}\{\alpha\}$, $\beta^+ = \max_{\beta\in E(m_i)}\{\beta\}$ for all $m_i \in M$.

The qth rung orthopair hesitancy membership degree ξ can be defined as $\xi = (1 - \alpha^q - \beta^q)^{1/q}$.

The basic operational rules of q-ROHFNs are defined as follows.

Definition 7 [20]: Let $\tilde{Q} = \langle D_{\tilde{Q}}, E_{\tilde{Q}}\rangle_q$, $\tilde{Q}_1 = \langle D_{\tilde{Q}_1}, E_{\tilde{Q}_1}\rangle_q$ and $\tilde{Q}_2 = \langle D_{\tilde{Q}_2}, E_{\tilde{Q}_2}\rangle_q$ be three q-ROHFNs. $\lambda > 0, q \geq 1$. Then the operational laws of q-ROHFNs are represented as:

1) $\tilde{Q}_1 \oplus \tilde{Q}_2 = \bigcup_{\alpha_{\tilde{Q}_i}\in D_{\tilde{Q}_i}, \beta_{\tilde{Q}_i}\in E_{\tilde{Q}_i}} \left\langle \{(\alpha^q_{\tilde{Q}_1} + \alpha^q_{\tilde{Q}_2} - \alpha^q_{\tilde{Q}_1}\alpha^q_{\tilde{Q}_2})^{\frac{1}{q}}\}, \{\beta_{\tilde{Q}_1}\beta_{\tilde{Q}_2}\}\right\rangle_q, i = 1, 2.$

2) $\tilde{Q}_1 \otimes \tilde{Q}_2 = \bigcup_{\alpha_{\tilde{Q}_i}\in D_{\tilde{Q}_i}, \beta_{\tilde{Q}_i}\in E_{\tilde{Q}_i},} \left\langle \{\{\alpha_{\tilde{Q}_1}\alpha_{\tilde{Q}_2}\}, \{(\beta^q_{\tilde{Q}_1} + \beta^q_{\tilde{Q}_2} - \beta^q_{\tilde{Q}_1}\beta^q_{\tilde{Q}_2})^{\frac{1}{q}}\}\right\rangle_q, i = 1, 2.$

3) $\lambda\tilde{Q} = \bigcup_{\alpha_{\tilde{Q}}\in D_{\tilde{Q}}, \beta_{\tilde{Q}}\in E_{\tilde{Q}}} \left\langle \{(1 - (1 - \alpha^q_{\tilde{Q}})^{\lambda})^{\frac{1}{q}}\}, \{\beta^q_{\tilde{Q}}\}\right\rangle_q$

4) $\tilde{Q}^{\lambda} = \bigcup_{\alpha_{\tilde{Q}}\in D_{\tilde{Q}},\beta_{\tilde{Q}}\in E_{\tilde{Q}}} \left\langle \{\alpha_{\tilde{Q}}^{q}\}, \{(1-(1-\beta_{\tilde{Q}}^{q})^{\lambda})^{\frac{1}{q}}\}\right\rangle_{q}$

5) $\tilde{Q}^{c} = \bigcup_{\alpha_{\tilde{Q}}\in D_{\tilde{Q}},\beta_{\tilde{Q}}\in E_{\tilde{Q}}} \left\langle \{\beta_{\tilde{Q}}\}, \{\alpha_{\tilde{Q}}\}\right\rangle_{q}$, where $\tilde{Q}^{c}$ is the inverse of $\tilde{Q}$.

The following is the definition of the score and accuracy functions of the q-ROHFN.

Definition 8 [20]: If $\tilde{Q} = \langle D_{\tilde{Q}}, E_{\tilde{Q}}\rangle_q$ is a q-ROHFN, such that $\gamma(D_{\tilde{Q}})$ and $\gamma(E_{\tilde{Q}})$ equal the number of objects in sets $D_{\tilde{Q}}$ and $E_{\tilde{Q}}$, respectively, then a score function $\Gamma(\tilde{Q})$ is:

$$\Gamma(\tilde{Q}) = \frac{1}{\gamma(D_{\tilde{Q}})}\sum_{\alpha_{\tilde{Q}_j}\in D_{\tilde{Q}}}\alpha_{\tilde{Q}_j}^{q} - \frac{1}{\gamma(E_{\tilde{Q}})}\sum_{\beta_{\tilde{Q}_j}\in E_{\tilde{Q}}}\beta_{\tilde{Q}_j}^{q}.$$

Accuracy function of q-ROHFN $\lambda(\tilde{Q})$ can be stated as

$$\lambda(\tilde{Q}) = \frac{1}{\gamma(D_{\tilde{Q}})}\sum_{\alpha_{\tilde{Q}_k}\in D_{\tilde{Q}}}\alpha_{\tilde{Q}_k}^{q} + \frac{1}{\gamma(E_{\tilde{Q}})}\sum_{\beta_{\tilde{Q}_k}\in E_{\tilde{Q}}}\beta_{\tilde{Q}_k}^{q}.$$

Hussain *et al.* [21] provide the basic notions of related aggregation operators of the q-ROHFN as follows.

Definition 9 [21]: Let $\tilde{Q}_k = \langle D_{\tilde{Q}_k}, E_{\tilde{Q}_k}\rangle_q (k = 1, 2, \ldots, n)$ be a collection of q-ROHFNs. The q-ROHFWA operator is a mapping $q-ROHFWA : q-ROHFN^{n} \longrightarrow q-ROHFN$, defined by

$q-ROHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_k) = w_1\tilde{Q}_1 \oplus w_2\tilde{Q}_2 \oplus \ldots \oplus w_n\tilde{Q}_n$, where $w = (w_1, w_2, \ldots, w_n)$ represents the weight vector of$\tilde{Q}_k (k = 1, 2, \ldots, n)$, $0 \leqslant w_k \leqslant 1$ and $\sum_{k=1}^{n} w_k = 1$.

Based on the operational rules of the q-ROHFNs, the aggregation result for the q-ROHFNs is given as in Theorem 1.

Theorem 1 [21]: If $\tilde{Q}_k = \langle D_{\tilde{Q}_k}, E_{\tilde{Q}_k}\rangle_q (k = 1, 2, \ldots, n)$ is a set of q-ROHFNs, then the aggregated value by using the q-ROHFWA operator remains q-ROHFN and

$$q-ROHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \bigcup_{\alpha_{\tilde{Q}_k}\in D_{\tilde{Q}_k},\beta_{\tilde{Q}_k}\in E_{\tilde{Q}_k}} \left\langle \left(1-\prod_{k=1}^{n}(1-\alpha_{\tilde{Q}_k}^{q})^{w_k}\right)^{\frac{1}{q}}, \prod_{k=1}^{n}\beta_{\tilde{Q}_k}^{w_k}\right\rangle_q,$$
$k = 1, 2.$

q-ROHFWG operator is also defined in [21], as follows.

Definition 10 [21]: Consider the collection $\tilde{Q}_j = \langle D_{\tilde{Q}_j}, E_{\tilde{Q}_j}\rangle_q (j = 1, 2, \ldots, k)$ of q-ROHFNs. The q-ROHFWG operator is a mapping $q-ROHFWG : q-ROHFN^{K} \longrightarrow q-ROHFN$, defined by

$q-ROHFWG(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_k) = \tilde{Q}_1^{w_1} \otimes \tilde{Q}_2^{w_2} \otimes \ldots \otimes \tilde{Q}_k^{w_k}$, where $w = (w_1, w_2, \ldots, w_k)$ is the weight vector of$\tilde{Q}_j (j = 1, 2, \ldots, k)$, $0 \leqslant w_j \leqslant 1$ and $\sum_{j=1}^{k} w_j = 1$.

The aggregation result for q-ROHFNs through operation rule is given as in Theorem 2.

Theorem 2 [21]: If $\tilde{Q}_k = \langle D_{\tilde{Q}_k}, E_{\tilde{Q}_k}\rangle_q (k = 1, 2, \ldots, n)$ is a set of q-ROHFNs, then the aggregated value by utilizing the q-ROHFWG operator is still q-ROHFN and

$$q-ROHFWG(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \bigcup_{\alpha_{\tilde{Q}_k}\in D_{\tilde{Q}_k},\beta_{\tilde{Q}_k}\in E_{\tilde{Q}_k}} \left\langle \prod_{k=1}^{n}\beta_{\tilde{Q}_k}^{w_k}, \left(1-\prod_{k=1}^{n}(1-\alpha_{\tilde{Q}_k}^{q})^{w_k}\right)^{\frac{1}{q}}\right\rangle_q,$$
$k = 1, 2.$

In the following, we propose the notion of T-SHFS as an extension of q-ROHFS by adding the hesitant neutral membership function to the structure of the q-ROHFS.

III. T-SPHERICAL HESITANT FUZZY SET

We will propose the definition of T-SHFS as follows:

Definition 11: Let M be a fixed set such that $M = \{m_1, \ldots, m_n\}$. Then a T-SHFS on M can be defined as $Q = \{\langle m_i, T(m_i), I(m_i), F(m_i)\rangle_q : m_i \in M\}$. Where $T(m_i), I(m_i)$, and $F(m_i)$ are three sets of values in $[0, 1]$, denoting, respectively all the possible truth membership degrees, indeterminacy membership degrees and falsity membership degrees of the elements $m_i \in M$ to the set Q and they satisfy the following conditions:

$x, y, z \in [0, 1]$, $0 \leq (x^{+})^{q} + (y^{+})^{q} + (z^{+})^{q} \leq 1$ $(q \geq 1)$, for all $x \in T(m_i)$, $y \in I(m_i)$, $z \in F(m_i)$, where $x^{+} = \max_{x\in T(m_i)}\{x\}$, $y^{+} = \max_{y\in I(m_i)}\{y\}$, $z^{+} = \max_{z\in F(m_i)}\{z\}$, for each $m_i \in M$.

The degree of refusal membership can be defined as $\pi = [1-(x^{q}+y^{q}+z^{q})]^{\frac{1}{q}}$.

If the set M has only one element, i.e. $M = \{m\}$, then the T-SHFS Q is reduced to $\langle T(m), I(m), F(m)\rangle_q$. For convenience, we call $\langle T(m), I(m), F(m)\rangle_q$ a T-spherical hesitant fuzzy number (T-SHFN), denoted by $\tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\rangle_q$.

Remark 1: If each of T(m), I(m) and F(m) has only one value in $[0, 1]$, then the T-SHFN $\langle T(m), I(m), F(m)\rangle_q$ is reduced to a T-SFN.

Remark 2: If $q = 1$, the T-SHFN $\langle T(m), I(m), F(m)\rangle_q$ is reduced to SHFN.

Definition 12: Let $\tilde{Q} = \langle T(m), I(m), F(m)\rangle_q$ be a T-SHFN, where J, K and L represent the number of values in the sets $T(m), I(m)$ and $F(m)$ respectively. Then a score function $\Gamma(\tilde{Q})$ is defined as:

$$\Gamma(\tilde{Q}) = \frac{1+\frac{1}{J^{q}}(\sum_{x\in T(m)}x)^{q} - \frac{1}{K^{q}}(\sum_{y\in I(m)}y)^{q} - \frac{1}{L^{q}}(\sum_{z\in F(m)}z)^{q}}{2},$$
$$\Gamma(\tilde{Q}) \in [0, 1]$$

The accuracy function is defined as:

$$\lambda(\tilde{Q}) = \frac{\frac{1}{J^{q}}(\sum_{x\in T(m)}x)^{q} + \frac{1}{K^{q}}(\sum_{y\in I(m)}y)^{q} + \frac{1}{L^{q}}(\sum_{z\in F(m)}z)^{q}}{2}$$

According to the definition of score and accuracy functions above, the method for comparing two T-SHFNs can be defined as follows.

Definition 13: If $\tilde{Q}_1$ and $\tilde{Q}_2$ are two T-SHFNs. Then we can define the comparison method as follows.

1) If $\Gamma(\tilde{Q}_1) > \Gamma(\tilde{Q}_2)$. Then $\tilde{Q}_1$ is greater than $\tilde{Q}_2$, denoted by $\tilde{Q}_1 > \tilde{Q}_2$.
2) If $\Gamma(\tilde{Q}_1) = \Gamma(\tilde{Q}_2)$ and $\lambda(\tilde{Q}_1) > \lambda(\tilde{Q}_2)$. Then $\tilde{Q}_1$ is greater than $\tilde{Q}_2$, denoted by $\tilde{Q}_1 > \tilde{Q}_2$.
3) If $\Gamma(\tilde{Q}_1) = \Gamma(\tilde{Q}_2)$ and $\lambda(\tilde{Q}_1) = \lambda(\tilde{Q}_2)$. Then $\tilde{Q}_1$ equals $\tilde{Q}_2$, that is, $\tilde{Q}_1$ is indifferent to $\tilde{Q}_2$ denoted by $\tilde{Q}_1 = \tilde{Q}_2$.

In the following, we propose the operational laws of T-SHFNs, namely, addition, multiplication, scalar multiplication, and power operations. The operations for T-SHFNs are defined based on the Archimedean t-conorm, and t-norm and are as follows.

Definition 14: Let $\tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\rangle_q$, $\tilde{Q}_1 = \langle T_{\tilde{Q}_1}, I_{\tilde{Q}_1}, F_{\tilde{Q}_1}\rangle_q$ and $\tilde{Q}_2 = \langle T_{\tilde{Q}_2}, I_{\tilde{Q}_2}, F_{\tilde{Q}_2}\rangle_q$ be three T-SHFNs. $\lambda > 0, q \geq 1$. Then the operational laws of T-SHFNs are represented as:

1) $\tilde{Q}_1 \oplus \tilde{Q}_2 = \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ (x^q_{\tilde{Q}_1} + x^q_{\tilde{Q}_2} - x^q_{\tilde{Q}_1} x^q_{\tilde{Q}_2})^{\frac{1}{q}} \right\}, \left\{ y_{\tilde{Q}_1} y_{\tilde{Q}_2} \right\}, \left\{ z_{\tilde{Q}_1} z_{\tilde{Q}_2} \right\} \right\rangle_q, i = 1, 2.$
2) $\tilde{Q}_1 \otimes \tilde{Q}_2 = \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ \{ x_{\tilde{Q}_1} x_{\tilde{Q}_2} \right\}, \left\{ (y^q_{\tilde{Q}_1} + y^q_{\tilde{Q}_2} - y^q_{\tilde{Q}_1} y^q_{\tilde{Q}_2})^{\frac{1}{q}} \right\}, \left\{ (z^q_{\tilde{Q}_1} + z^q_{\tilde{Q}_2} - z^q_{\tilde{Q}_1} z^q_{\tilde{Q}_2})^{\frac{1}{q}} \right\} \right\rangle_q, i = 1, 2.$
3) $\lambda\tilde{Q} = \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle \left\{ (1 - (1 - x^q_{\tilde{Q}})^{\lambda})^{\frac{1}{q}} \right\}, \left\{ y^q_{\tilde{Q}} \right\}, \left\{ z^q_{\tilde{Q}} \right\} \right\rangle_q$
4) $\tilde{Q}^{\lambda} = \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle \left\{ x^q_{\tilde{Q}} \right\}, \left\{ (1 - (1 - y^q_{\tilde{Q}})^{\lambda})^{\frac{1}{q}} \right\}, \left\{ (1 - (1 - z^q_{\tilde{Q}})^{\lambda})^{\frac{1}{q}} \right\} \right\rangle_q$
5) $\tilde{Q}^c = \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle \left\{ z_{\tilde{Q}} \right\}, \left\{ y_{\tilde{Q}} \right\}, \left\{ x_{\tilde{Q}} \right\} \right\rangle_q$, where $\tilde{Q}^c$ is the inverse of $\tilde{Q}$.

In view of the above definition, we prove the following results.

Theorem 3: If $\tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\rangle_q$, $\tilde{Q}_1 = \langle T_{\tilde{Q}_1}, I_{\tilde{Q}_1}, F_{\tilde{Q}_1}\rangle_q$ and $\tilde{Q}_2 = \langle T_{\tilde{Q}_2}, I_{\tilde{Q}_2}, F_{\tilde{Q}_2}\rangle_q$ are three T-SHFNs. $\lambda > 0, q \geq 1$. Then,

1) $\tilde{Q}_1 \oplus \tilde{Q}_2 = \tilde{Q}_2 \oplus \tilde{Q}_1$
2) $\tilde{Q}_1 \otimes \tilde{Q}_2 = \tilde{Q}_2 \otimes \tilde{Q}_1$
3) $\lambda(\tilde{Q}_1 \oplus \tilde{Q}_2) = \lambda\tilde{Q}_1 \oplus \lambda\tilde{Q}_2$
4) $(\tilde{Q}_1 \otimes \tilde{Q}_2)^{\lambda} = \tilde{Q}_1{}^{\lambda} \otimes \tilde{Q}_2{}^{\lambda}$

Proof: We give the proofs of (1) and (3).

1) Based on Definition 14,
$$\tilde{Q}_1 \oplus \tilde{Q}_2 = \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ (x^q_{\tilde{Q}_1} + x^q_{\tilde{Q}_2} - x^q_{\tilde{Q}_1} x^q_{\tilde{Q}_2})^{\frac{1}{q}} \right\}, \left\{ y_{\tilde{Q}_1} y_{\tilde{Q}_2} \right\}, \left\{ z_{\tilde{Q}_1} z_{\tilde{Q}_2} \right\} \right\rangle_q, i = 1, 2.$$
$$= \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ (x^q_{\tilde{Q}_2} + x^q_{\tilde{Q}_1} - x^q_{\tilde{Q}_2} x^q_{\tilde{Q}_1})^{\frac{1}{q}} \right\}, \left\{ y_{\tilde{Q}_2} y_{\tilde{Q}_1} \right\}, \left\{ z_{\tilde{Q}_2} z_{\tilde{Q}_1} \right\} \right\rangle_q, i = 1, 2.$$
$$= \tilde{Q}_2 \oplus \tilde{Q}_1$$
2) The proof is similar to that of (1).
3) According to (1) and (3) in Definition 14, we can get for the left side of the Equation
$$\lambda(\tilde{Q}_1 \oplus \tilde{Q}_2) = \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \lambda\left\langle \left\{ (x^q_{\tilde{Q}_1} + x^q_{\tilde{Q}_2} - x^q_{\tilde{Q}_1} x^q_{\tilde{Q}_2})^{\frac{1}{q}} \right\}, \left\{ y_{\tilde{Q}_1} y_{\tilde{Q}_2} \right\}, \left\{ z_{\tilde{Q}_1} z_{\tilde{Q}_2} \right\} \right\rangle_q, i = 1, 2.$$
$$= \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ \left[1 - (1 - (x^q_{\tilde{Q}_1} + x^q_{\tilde{Q}_2} - x^q_{\tilde{Q}_1} x^q_{\tilde{Q}_2}))^{\lambda} \right]^{\frac{1}{q}} \right\}, \left\{ (y_{\tilde{Q}_1} y_{\tilde{Q}_2})^{\lambda} \right\}, \left\{ (z_{\tilde{Q}_1} z_{\tilde{Q}_2})^{\lambda} \right\} \right\rangle_q, i = 1, 2.$$
$$= \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ (1 - (1 - x^q_{\tilde{Q}_1})^{\lambda} (1 - x^q_{\tilde{Q}_2})^{\lambda})^{\frac{1}{q}} \right\}, \left\{ (y_{\tilde{Q}_1})^{\lambda} (y_{\tilde{Q}_2})^{\lambda} \right\}, \left\{ (z_{\tilde{Q}_1})^{\lambda} (z_{\tilde{Q}_2})^{\lambda} \right\} \right\rangle_q, i = 1, 2.$$
For the right side of the Equation, we can get:
$$\lambda\tilde{Q}_1 = \bigcup_{x_{\tilde{Q}_1}\in T_{\tilde{Q}_1}, y_{\tilde{Q}_1}\in I_{\tilde{Q}_1}, z_{\tilde{Q}_1}\in F_{\tilde{Q}_1}} \left\langle \left\{ (1 - (1 - x^q_{\tilde{Q}_1})^{\lambda})^{\frac{1}{q}} \right\}, \left\{ (y_{\tilde{Q}_1})^{\lambda} \right\}, \left\{ (z_{\tilde{Q}_1})^{\lambda} \right\} \right\rangle_q,$$
$$\lambda\tilde{Q}_2 = \bigcup_{x_{\tilde{Q}_2}\in T_{\tilde{Q}_2}, y_{\tilde{Q}_2}\in I_{\tilde{Q}_2}, z_{\tilde{Q}_2}\in F_{\tilde{Q}_2}} \left\langle \left\{ (1 - (1 - x^q_{\tilde{Q}_2})^{\lambda})^{\frac{1}{q}} \right\}, \left\{ (y_{\tilde{Q}_2})^{\lambda} \right\}, \left\{ (z_{\tilde{Q}_2})^{\lambda} \right\} \right\rangle_q,$$
moreover, since
$$\lambda\tilde{Q}_1 + \lambda\tilde{Q}_2 = \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ \left[1 - (1 - x^q_{\tilde{Q}_1})^{\lambda} + (1 - (1 - x^q_{\tilde{Q}_2})^{\lambda} - (1 - (1 - x^q_{\tilde{Q}_1})^{\lambda} (1 - (1 - x^q_{\tilde{Q}_2})^{\lambda} \right]^{\frac{1}{q}} \right\}, \left\{ (y_{\tilde{Q}_1})^{\lambda} (y_{\tilde{Q}_2})^{\lambda} \right\}, \left\{ (z_{\tilde{Q}_1})^{\lambda} (z_{\tilde{Q}_2})^{\lambda} \right\} \right\rangle_q,$$
$$= \bigcup_{x_{\tilde{Q}_i}\in T_{\tilde{Q}_i}, y_{\tilde{Q}_i}\in I_{\tilde{Q}_i}, z_{\tilde{Q}_i}\in F_{\tilde{Q}_i}} \left\langle \left\{ (1 - (1 - x^q_{\tilde{Q}_1})^{\lambda} (1 - x^q_{\tilde{Q}_2})^{\lambda})^{\frac{1}{q}} \right\}, \left\{ (y_{\tilde{Q}_1})^{\lambda} (y_{\tilde{Q}_2})^{\lambda} \right\}, \left\{ (z_{\tilde{Q}_1})^{\lambda} (z_{\tilde{Q}_2})^{\lambda} \right\} \right\rangle_q, i = 1, 2.$$
Thus, we get $\lambda(\tilde{Q}_1 \oplus \tilde{Q}_2) = \lambda\tilde{Q}_1 \oplus \lambda\tilde{Q}_2$.
4) The proof is similar to that of (3).

Example 1: Let $\tilde{Q}_1 = \langle\{0.3, 0.2\}, 0.7, \{0.5, 0.6\}\rangle_3$, and $\tilde{Q}_2 = \langle 0.8, \{0.5, 0.1\}, \{0.4, 0.6\}\rangle_3$ be two 3-SHFNs. If $\lambda = 4$, then

1) $\tilde{Q}_1 \oplus \tilde{Q}_2 = \langle\{0.8068, 0.802\}, \{0.35, 0.07\}, \{0.2, 0.3, 0.24, 0.36\}\rangle_3$
2) $\tilde{Q}_1 \otimes \tilde{Q}_2 = \langle\{0.24, 0.16\}, \{0.7519, 0.7004\}, \{0.5656, 0.6796, 0.6432, 0.7276\}\rangle_3$
3) $\lambda\tilde{Q}_1 = \langle\{0.4698, 0.3162\}, 0.2401, \{0.0625, 0.1296\}\rangle_3$
4) $\lambda\tilde{Q}_2 = \langle 0.9807, \{0.0625, 0.0001\}, \{0.0256, 0.1296\}\rangle_3$
5) $\lambda(\tilde{Q}_1 \oplus \tilde{Q}_2) = \langle\{0.9827, 0.9813\}, \{0.015, 0\}, \{0.0016, 0.0081, 0.0033, 0.0167\}\rangle_3$
6) $\lambda\tilde{Q}_1 \oplus \lambda\tilde{Q}_2 = \langle\{0.9827, 0.9813\}, \{0.015, 0\}, \{0.0016, 0.0081, 0.0033, 0.0167\}\rangle_3$
Obviously, $\lambda(\tilde{Q}_1 \oplus \tilde{Q}_2) = \lambda\tilde{Q}_1 \oplus \lambda\tilde{Q}_2$
7) $\tilde{Q}_1{}^{\lambda} = \langle\{0.0081, 0.0016\}, 0.9335, \{0.7451, 0.8537\}\rangle_3$
8) $\tilde{Q}_2{}^{\lambda} = \langle 0.4096, \{0.7451, 0.1586\}, \{0.6148, 0.8537\}\rangle_3$

9) $\tilde{Q_1}^{\lambda} \otimes \tilde{Q_2}^{\lambda} = \langle\{0.0033, 0.0006\}, \{0.9621, 0.9337\}, \{0.8192, 0.9199, 0.8921, 0.9499\}\rangle_3$
10) $(\tilde{Q}_1 \otimes \tilde{Q}_2)^{\lambda} = \langle\{0.0033, 0.0006\}, \{0.9621, 0.9337\}, \{0.8192, 0.9199, 0.8921, 0.9499\}\rangle_3$
That is $\tilde{Q_1}^{\lambda} \otimes \tilde{Q_2}^{\lambda} = (\tilde{Q}_1 \otimes \tilde{Q}_2)^{\lambda}$.

IV. T-SHF AGGREGATION OPERATORS

On the basis of the operational rules of T-SHFNs in Section 3, we will propose the T-SHFWA operator and the T-SHFWG operator and investigate their properties.

A. THE T-SHFWA OPERATOR

Definition 15: Suppose $\tilde{Q}_k = \langle T_{\tilde{Q}_k}, I_{\tilde{Q}_k}, F_{\tilde{Q}_k}\rangle_q (k = 1, 2, \ldots, n)$ is a collection of T-SHFNs. The T-SHFWA operator is a mapping $T-SHFWA : T-SHFN^n \longrightarrow T-SHFN$, defined by

$T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = w_1\tilde{Q}_1 \oplus w_2\tilde{Q}_2 \oplus \ldots \oplus w_n\tilde{Q}_n$, where $w = (w_1, w_2, \ldots, w_n)$ is the weight vector of $\tilde{Q}_k (k = 1, 2, \ldots, n)$, $0 \leqslant w_k \leqslant 1$ and $\sum_{k=1}^{k} w_k = 1$.

On the basis of the operators rules of the T-SHFNs, we derive the following theorem.

Theorem 4: If $\tilde{Q}_k = \langle T_{\tilde{Q}_k}, I_{\tilde{Q}_k}, F_{\tilde{Q}_k}\rangle_q (k = 1, 2, \ldots, n)$ is a collection of T-SHFNs, then the accumulated value by using the T-SHFWA operator is still T-SHFN and

$$T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left(1 - \prod_{k=1}^{n} (1 - x_{\tilde{Q}_k}^q)^{w_k}\right)^{\frac{1}{q}}, \prod_{k=1}^{n} y_{\tilde{Q}_k}^{w_k}, \prod_{k=1}^{n} z_{\tilde{Q}_k}^{w_k} \right\rangle_q \tag{1}$$

Proof: We will use the method of mathematical induction to prove this theorem.

1) For $n = 2$, since $w_1\tilde{Q}_1 = \bigcup_{x_{\tilde{Q}_1} \in T_{\tilde{Q}_1}, y_{\tilde{Q}_1} \in I_{\tilde{Q}_1}, z_{\tilde{Q}_1} \in F_{\tilde{Q}_1}} \left\langle \left\{(1 - (1 - x_{\tilde{Q}_1}^q)^{w_1})^{\frac{1}{q}}\right\}, \{y_{\tilde{Q}_1}^{w_1}\}, \{z_{\tilde{Q}_1}^{w_1}\} \right\rangle_q$
$w_2\tilde{Q}_2 = \bigcup_{x_{\tilde{Q}_2} \in T_{\tilde{Q}_2}, y_{\tilde{Q}_2} \in I_{\tilde{Q}_2}, z_{\tilde{Q}_2} \in F_{\tilde{Q}_2}} \left\langle \left\{(1 - (1 - x_{\tilde{Q}_2}^q)^{w_2})^{\frac{1}{q}}\right\}, \{y_{\tilde{Q}_2}^{w_2}\}, \{z_{\tilde{Q}_2}^{w_2}\} \right\rangle_q$, then $T-SHFWA(\tilde{Q}_1, \tilde{Q}_2) = w_1\tilde{Q}_2 + w_2\tilde{Q}_2$

$$= \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left\{\left[1 - (1 - x_{\tilde{Q}_1}^q)^{w_1} + (1 - (1 - x_{\tilde{Q}_2}^q)^{w_2}) - (1 - (1 - x_{\tilde{Q}_1}^q)^{w_1})(1 - (1 - x_{\tilde{Q}_1}^q)^{w_2})\right]^{\frac{1}{q}}\right\}, \left\{(y_{\tilde{Q}_1})^{w_1}(y_{\tilde{Q}_2})^{w_2}\right\}, \left\{(z_{\tilde{Q}_1})^{w_1}(z_{\tilde{Q}_2})^{w_2}\right\} \right\rangle_q, \text{for } k = 1, 2$$

$$= \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \{(1 - (1 - x_{\tilde{Q}_1}^q)^{w_1} (1 - x_{\tilde{Q}_2}^q)^{w_2})^{\frac{1}{q}}\}, \{(y_{\tilde{Q}_1})^{w_1}(y_{\tilde{Q}_2})^{w_2}\}, \{(z_{\tilde{Q}_1})^{w_1}(z_{\tilde{Q}_2})^{w_2}\} \right\rangle_q, k = 1, 2.$$

$$= \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left\{(1 - \prod_{k=1}^{2} (1 - x_{\tilde{Q}_k}^q)^{w_k})^{\frac{1}{q}}\right\}, \left\{\prod_{k=1}^{2} y_{\tilde{Q}_k}^{w_k}\right\}, \left\{\prod_{k=1}^{2} z_{\tilde{Q}_k}^{w_k}\right\} \right\rangle_q.$$

Obviously Equation (1) holds for $n = 2$.

2) If Equation (1) holds for $k = n$, then

$$T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left\{(1 - \prod_{k=1}^{n} (1 - x_{\tilde{Q}_k}^q)^{w_k})^{\frac{1}{q}}\right\}, \left\{\prod_{k=1}^{n} y_{\tilde{Q}_k}^{w_k}\right\}, \left\{\prod_{k=1}^{n} z_{\tilde{Q}_k}^{w_k}\right\} \right\rangle_q.$$

When $k = n + 1$, based on the operational rules of the T-SHFNs, we have

$$T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n, \tilde{Q_{n+1}}) = T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) \oplus w_{n+1}Q_{n+1}$$

$$= \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left\{(1 - \prod_{k=1}^{n} (1 - x_{\tilde{Q}_k}^q)^{w_k})^{\frac{1}{q}}\right\}, \left\{\prod_{k=1}^{n} y_{\tilde{Q}_k}^{w_k}\right\}, \left\{\prod_{k=1}^{n} z_{\tilde{Q}_k}^{w_k}\right\} \right\rangle_q \oplus \bigcup_{x_{\tilde{Q}_{n+1}} \in T_{\tilde{Q}_{n+1}}, y_{\tilde{Q}_{n+1}} \in I_{\tilde{Q}_{n+1}}, z_{\tilde{Q}_{n+1}} \in F_{\tilde{Q}_{n+1}}} \left\langle \left\{(1 - (1 - x_{\tilde{Q}_{n+1}}^q)^{w_{n+1}})^{\frac{1}{q}}\right\}, \{y_{\tilde{Q}_{n+1}}^{w_{n+1}}\}, \{z_{\tilde{Q}_{n+1}}^{w_{n+1}}\} \right\rangle_q$$

$$= \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left\{\left[1 - \prod_{k=1}^{n}(1 - x_{\tilde{Q}_k}^q)^{w_k} + (1 - (1 - x_{\tilde{Q}_{n+1}}^q)^{w_{n+1}}) - (1 - \prod_{k=1}^{n}(1 - x_{\tilde{Q}_k}^q)^{w_k})(1 - (1 - x_{\tilde{Q}_{n+1}}^q)^{w_{n+1}})\right]^{\frac{1}{q}}\right\}, \left\{\prod_{k=1}^{n}(y_{\tilde{Q}_k})^{w_k}(y_{\tilde{Q}_{n+1}})^{w_{n+1}}\right\}, \left\{\prod_{k=1}^{n}(z_{\tilde{Q}_k})^{w_k}(z_{\tilde{Q}_{n+1}})^{w_{n+1}}\right\} \right\rangle_q, \text{for } k = 1, 2, ., n$$

$$= \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left\{\left(1 - \prod_{k=1}^{n} (1 - x_{\tilde{Q}_k}^q)^{w_k} (1 - x_{Q_{n+1}}^q)^{w_{n+1}}\right)^{\frac{1}{q}}\right\}, \left\{\prod_{k=1}^{n+1} y_{\tilde{Q}_k}^{w_k}\right\}, \left\{\prod_{k=1}^{n+1} z_{\tilde{Q}_k}^{w_k}\right\} \right\rangle_q.$$

$$= \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \left\{\left(1 - \prod_{k=1}^{n+1} (1 - x_{\tilde{Q}_k}^q)^{w_k}\right)^{\frac{1}{q}}\right\}, \left\{\prod_{k=1}^{n+1} y_{\tilde{Q}_k}^{w_k}\right\}, \left\{\prod_{k=1}^{n+1} z_{\tilde{Q}_k}^{w_k}\right\} \right\rangle_q,$$

$k = 1, 2, \ldots, n + 1$. That is Equation (1) holds for $k = n + 1$. According to steps (1)and (2), we can get Equation (1) holds for any k.

In the following, we present the properties of the T-SHFWA operator along with their proofs.

Theorem 5 (Idempotency): Suppose $\tilde{Q}_k = \langle T_{\tilde{Q}_k}, I_{\tilde{Q}_k}, F_{\tilde{Q}_k}\rangle_q$ $(k = 1, 2, \ldots, n)$ is a collection of T-SHFNs. If $\tilde{Q}_k = \tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\rangle_q, k = 1, 2, \ldots, n$, then $T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\rangle_q$.

Proof: According to Theorem 4, since $\tilde{Q}_k = \tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\rangle_q$, we have

$$\begin{aligned}
&T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n)\\
&= \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle \left(1-\prod_{k=1}^{n}(1-x_{\tilde{Q}}^{q})^{w_k}\right)^{\frac{1}{q}}, \prod_{k=1}^{n} y_{\tilde{Q}}^{w_k}, \prod_{k=1}^{n} z_{\tilde{Q}}^{w_k}\right\rangle_q\\
&= \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle \left(1-(1-x_{\tilde{Q}}^{q})^{\sum_{k=1}^{n} w_k}\right)^{\frac{1}{q}}, y_{\tilde{Q}}^{\sum_{k=1}^{n} w_k}, z_{\tilde{Q}}^{\sum_{k=1}^{n} w_k}\right\rangle_q\\
&= \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle \left(1-(1-x_{\tilde{Q}}^{q})\right)^{\frac{1}{q}}, y_{\tilde{Q}}, z_{\tilde{Q}}\right\rangle_q\\
&= \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle x_{\tilde{Q}}, y_{\tilde{Q}}, z_{\tilde{Q}}\right\rangle_q\\
&= \bigcup_{x_{\tilde{Q}}\in T_{\tilde{Q}}, y_{\tilde{Q}}\in I_{\tilde{Q}}, z_{\tilde{Q}}\in F_{\tilde{Q}}} \left\langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\right\rangle_q = \tilde{Q}.
\end{aligned}$$

Theorem 6 (Boundedness): Let $\tilde{Q}_k = \langle T_{\tilde{Q}_k}, I_{\tilde{Q}_k}, F_{\tilde{Q}_k}\rangle_q (k = 1, 2, \ldots, n)$ be T-SHFNs. If $\tilde{Q}^- = \left\langle \{x^-\}, \{y^+\}, \{z^+\}\right\rangle_q$ and $\tilde{Q}^+ = \left\langle \{x^+\}, \{y^-\}, \{z^-\}\right\rangle_q$,
where,

$$\begin{aligned}
x^- &= \min_{x_{\tilde{Q}_k}\in T_{\tilde{Q}_k}}\{x_{\tilde{Q}_k}\}, & y^- &= \min_{y_{\tilde{Q}_k}\in I_{\tilde{Q}_k}}\{y_{\tilde{Q}_k}\},\\
z^- &= \min_{z_{\tilde{Q}_k}\in F_{\tilde{Q}_k}}\{z_{\tilde{Q}_k}\}, & x^+ &= \max_{x_{\tilde{Q}_k}\in T_{\tilde{Q}_k}}\{x_{\tilde{Q}_k}\},\\
y^+ &= \min_{y_{\tilde{Q}_k}\in I_{\tilde{Q}_k}}\{y_{\tilde{Q}_k}\}, & z^+ &= \max_{z_{\tilde{Q}_k}\in F_{\tilde{Q}_k}}\{z_{\tilde{Q}_k}\},
\end{aligned}$$

then,

$$\tilde{Q}^- \leq T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) \leq \tilde{Q}^+.$$

Proof: For $q \geq 1$, since $x^- \leq x_{\tilde{Q}_k} \leq x^+$, then $(x^-)^q \leq (x_{\tilde{Q}_k})^q$,

$$\begin{aligned}
&1-(x^-)^q \geq 1-(x_{\tilde{Q}_k})^q, \left(1-(x^-)^q\right)^{w_k} \geq \left(1-(x_{\tilde{Q}_k})^q\right)^{w_k},\\
&\prod_{k=1}^{n}\left(1-(x^-)^q\right)^{w_k} \geq \prod_{k=1}^{n}\left(1-(x_{\tilde{Q}_k})^q\right)^{w_k},\\
&1-\prod_{k=1}^{n}\left(1-(x^-)^q\right)^{w_k} \leq 1-\prod_{k=1}^{n}\left(1-(x_{\tilde{Q}_k})^q\right)^{w_k},\\
&\left(1-\prod_{k=1}^{n}\left(1-(x^-)^q\right)^{w_k}\right)^{\frac{1}{q}} \leq \left(1-\prod_{k=1}^{n}\left(1-(x_{\tilde{Q}_k})^q\right)^{w_k}\right)^{\frac{1}{q}}\\
&= x^-,
\end{aligned}$$

similarly, we have

$$\left(1-\prod_{k=1}^{n}\left(1-x_{\tilde{Q}_k})^q\right)^{w_k}\right)^{\frac{1}{q}} \leq \left(1-\prod_{k=1}^{n}\left(1-(x^+)^q\right)^{w_k}\right)^{\frac{1}{q}} = x^+.$$

As $y^- \leq y_{\tilde{Q}_k} \leq y^+$, then

$$\begin{aligned}
&(y^-)^{w_k} \leq (y_{\tilde{Q}_k})^{w_k} \leq (y^+)^{w_k},\\
&\prod_{k=1}^{n}(y^-)^{w_k} \leq \prod_{k=1}^{n}(y_{\tilde{Q}_k})^{w_k} \leq \prod_{k=1}^{n}(y^+)^{w_k},\\
&y^- \leq \prod_{k=1}^{n}(y_{\tilde{Q}_k})^{w_k} \leq y^+.
\end{aligned}$$

Similarly, as $z^- \leq z_{\tilde{Q}_k} \leq z^+$, we have $z^- \leq \prod_{k=1}^{n}(z_{\tilde{Q}_k})^{w_k} \leq z^+$.

Let $T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \tilde{Q} = \left\langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}}\right\rangle_q$, then

$$\begin{aligned}
\Gamma(\tilde{Q}) &= \left[1+\frac{1}{J^q}\Big(\sum_{x_{\tilde{Q}}\in T_{\tilde{Q}}} x_{\tilde{Q}}\Big)^q - \frac{1}{K^q}\Big(\sum_{y_{\tilde{Q}}\in I_{\tilde{Q}}} y_{\tilde{Q}}\Big)^q\right.\\
&\left.-\frac{1}{L^q}\Big(\sum_{z_{\tilde{Q}}\in F_{\tilde{Q}}} z_{\tilde{Q}}\Big)^q\right]\Big/2 \geq \left[1+\frac{1}{(J^-)^q}\Big(\sum_{x^-_{\tilde{Q}}\in T_{\tilde{Q}}} x^-_{\tilde{Q}}\Big)^q\right.\\
&\left.-\frac{1}{(k^-)^q}\Big(\sum_{y^+_{\tilde{Q}}\in I_{\tilde{Q}}} y^+_{\tilde{Q}}\Big)^q - \frac{1}{(L^-)^q}\Big(\sum_{z^+_{\tilde{Q}}\in F_{\tilde{Q}}} z^+_{\tilde{Q}}\Big)^q\right]\Big/2 = \Gamma((\tilde{Q})^-)
\end{aligned}$$

and,

$$\begin{aligned}
\Gamma(\tilde{Q}) &= \left[1+\frac{1}{J^q}\Big(\sum_{x_{\tilde{Q}}\in T_{\tilde{Q}}} x_{\tilde{Q}}\Big)^q - \frac{1}{K^q}\Big(\sum_{y_{\tilde{Q}}\in I_{\tilde{Q}}} y_{\tilde{Q}}\Big)^q\right.\\
&\left.-\frac{1}{L^q}\Big(\sum_{z_{\tilde{Q}}\in F_{\tilde{Q}}} z_{\tilde{Q}}\Big)^q\right]\Big/2 \leq \left[1+\frac{1}{(J^+)^q}\Big(\sum_{x^+_{\tilde{Q}}\in T_{\tilde{Q}}} x^+_{\tilde{Q}}\Big)^q\right.\\
&\left.-\frac{1}{(k^+)^q}\Big(\sum_{y^-_{\tilde{Q}}\in I_{\tilde{Q}}} y^-_{\tilde{Q}}\Big)^q - \frac{1}{(L^+)^q}\Big(\sum_{z^-_{\tilde{Q}}\in F_{\tilde{Q}}} z^-_{\tilde{Q}}\Big)^q\right]\Big/2 = \Gamma((\tilde{Q})^+),
\end{aligned}$$

where J^-, K^-, and L^- are the number of values in the sets $T_{\tilde{Q}}, I_{\tilde{Q}}$ and $F_{\tilde{Q}}$, respectively, such that $T_{\tilde{Q}}, I_{\tilde{Q}}$ and $F_{\tilde{Q}}$ belong to $(\tilde{Q})^-$. J^+, K^+, and L^+ are the number of values in the sets $T_{\tilde{Q}}, I_{\tilde{Q}}$ and $F_{\tilde{Q}}$, respectively, which belong to $(\tilde{Q})^+$.

Note that $J^- = K^- = L^- = J^+ = K^+ = L^+ = n$, where n represents the number of the T-SHFNs.

We obtain $\tilde{Q}^- \leq T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) \leq \tilde{Q}^+$.

Theorem 7 (Monotonicity): Suppose $\tilde{Q}_k (k = 1, 2, \ldots, n)$ and $\tilde{Q}^*_k (k = 1, 2, \ldots, n)$ are two collections of T-SHFNs. If for all k, $\tilde{Q}_k \leq \tilde{Q}^*_k$. Then, $T-SHFWA(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) \leq T-SHFWA(\tilde{Q}^*_1,$
$\tilde{Q}^*_2, \ldots, \tilde{Q}^*_n)$.

Proof: The proof can be gained from Theorem 6.

B. THE T-SHFWG OPERATOR

The definition of T-SHFWG operator can be stated as follows.

Definition 15: Consider the collection $\tilde{Q}_k = \langle T_{\tilde{Q}_k}, I_{\tilde{Q}_k}, F_{\tilde{Q}_k}\rangle_q (k = 1, 2, \ldots, n)$ of T-SHFNs. Moreover, if $w = (w_1, w_2, \ldots, w_n)$ is the weight vector of $\tilde{Q}_k(k = 1, 2, \ldots, n), 0 \leqslant w_k \leqslant 1$ and $\sum_{k=1}^{n} w_k = 1$. Then the mapping for T-SHFWG operator is defined as:

$T - SHFWG : T - SHFN^n \longrightarrow T - SHFN$, such that $T - SHFWG(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \tilde{Q}_1^{w_1} \otimes \tilde{Q}_2^{w_2} \otimes \ldots \otimes \tilde{Q}_n^{w_n}$.

The aggregation result for T-SHFNs through operation rules is given as in the following theorem.

Theorem 8: Suppose that $\tilde{Q}_k = \langle T_{\tilde{Q}_k}, I_{\tilde{Q}_k}, F_{\tilde{Q}_k} \rangle_q$, $(k = 1, 2, \ldots, n)$ be the collection of T-SHFNs. Let us consider the weight vector $w = (w_1, w_2, \ldots, w_n)$ of $\tilde{Q}_k$. Then,

$$T - SHFWG(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \bigcup_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}, y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}, z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \left\langle \prod_{k=1}^{n} x_{\tilde{Q}_k}^{w_k}, \left(1 - \prod_{k=1}^{n} (1 - y_{\tilde{Q}_k}^q)^{w_k}\right)^{\frac{1}{q}}, \left(1 - \prod_{k=1}^{n} (1 - z_{\tilde{Q}_k}^q)^{w_k}\right)^{\frac{1}{q}} \right\rangle_q \tag{2}$$

Proof: The proof is like to that of Theorem 4.

Correspondingly, T-SHFWG operator has the coming properties.

Theorem 9 (Idempotency): Consider the collection $\tilde{Q}_k = \langle T_{\tilde{Q}_k}, I_{\tilde{Q}_k}, F_{\tilde{Q}_k} \rangle_q (k = 1, 2, \ldots, n)$ of T-SHFNs. If $\tilde{Q}_k(k = 1, 2, \ldots, n)$ are equal, i.e., $\tilde{Q}_k = \tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}} \rangle_q, k = 1, 2, \ldots, n$, then $T - SHFWG(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) = \tilde{Q} = \langle T_{\tilde{Q}}, I_{\tilde{Q}}, F_{\tilde{Q}} \rangle_q$.

Proof: Proof follows from Theorem 5.

Theorem 10 (Boundedness): If $\tilde{Q}_k$ $(k = 1, 2, \ldots, n)$ are the set of T-SHFNs. Consider $w = (w_1, w_2, \ldots, w_n)$ is the weight vector of $\tilde{Q}_k$ such that $\tilde{Q}_k \geq 0(k = 1, 2, \ldots, n)$, where $w_k \in [0, 1]$ with $\sum_{k=1}^{n} w_k = 1$. Then,

$\tilde{Q}^- \leq T - SHFWG(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) \leq \tilde{Q}^+$., where

$$\tilde{Q}^- = \left\langle T_{(\tilde{Q})^-}, I_{(\tilde{Q})^-}, F_{(\tilde{Q})^-} \right\rangle = \left\langle \{x^-\}, \{y^+\}, \{z^+\} \right\rangle_q,$$

and,

$$\tilde{Q}^+ = \left\langle T_{(\tilde{Q})^+}, I_{(\tilde{Q})^+}, F_{(\tilde{Q})^+} \right\rangle = \left\langle \{x^+\}, \{y^-\}, \{z^-\} \right\rangle_q,$$

where,

$$\begin{aligned} x^- &= \min_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}} \{x_{\tilde{Q}_k}\}, & y^- &= \min_{y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}} \{y_{\tilde{Q}_k}\}, \\ z^- &= \min_{z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \{z_{\tilde{Q}_k}\}, & x^+ &= \max_{x_{\tilde{Q}_k} \in T_{\tilde{Q}_k}} \{x_{\tilde{Q}_k}\}, \\ y^+ &= \min_{y_{\tilde{Q}_k} \in I_{\tilde{Q}_k}} \{y_{\tilde{Q}_k}\}, & z^+ &= \max_{z_{\tilde{Q}_k} \in F_{\tilde{Q}_k}} \{z_{\tilde{Q}_k}\}. \end{aligned}$$

Proof: Proof directly follows from Theorem 6.

Theorem 11 (Monotonicity): Consider $\tilde{Q}_k(k = 1, 2, \ldots, n)$ and $\tilde{Q}_k^*(k = 1, 2, \ldots, n)$ be the collection of two families of T-SHFNs. If for all k, $\tilde{Q}_k \leq \tilde{Q}_k^*$. Then, $T - SHFWG(\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_n) \leq T - SHFWG(\tilde{Q}_1^*, \tilde{Q}_2^*, \ldots, \tilde{Q}_n^*)$.

Proof: Proof is straightforward as Theorem 7.

V. DECISION-MAKING METHOD BASED ON THE T-SHFWA AND T-SHFWG OPERATORS

In this section, based on the T-SHFWA and T-SHFWG operators, we will come up with two new approaches for MADM with T-SHF information. Suppose $M = \{M_1, M_2, \ldots, M_n\}$ is a collection of n alternatives which are assessed according to k attributes $F = \{F_1, F_2, \ldots, F_k\}$. If the weight vector of the attributes is $w = \{w_1, w_2, \ldots, w_k\}$, where $\sum_{j=1}^{k} w_j = 1 \quad w_j \geq 0$. Suppose that $D = \left[\tilde{Q}_{ij}\right]_{n \times k}$ represents the decision matrix, where $\tilde{Q}_{ij} = \left\langle T_{ij}, I_{ij}, F_{ij} \right\rangle_q$ is the assessment value of alternative M_i, for attribute F_j, which is represented by T-SHFN. Then the algorithm for solving the MADM problem based on the T-SHFWA operator or T-SHFWG operator is explained below.

Algorithm 1:

Step 1: Normalize the decision matrix D by converting all cost attributes to benefit attributes. The converted values can be calculated by the following formula

$$\tilde{Q}_{ij} = \begin{cases} \left\langle T_{ij}, I_{ij}, F_{ij} \right\rangle_q & \text{for benefit attributes} \\ \left\langle F_{ij}, I_{ij}, T_{ij} \right\rangle_q & \text{for cost attributes} \end{cases} \tag{3}$$

Step 2: Aggregate all attributes values $\tilde{Q}_{ij}$ $(j = 1, 2, \ldots, k)$ to obtain the overall value $\tilde{S}_i$ by T-SHFWA operator such that $\tilde{S}_i = T - SHFWA\left(\tilde{Q}_{i1}, \tilde{Q}_{i2}, \ldots, \tilde{Q}_{iK}\right)$ or by T-SHFWG operator where $\tilde{S}_i = T - SHFWG\left(\tilde{Q}_{i1}, \tilde{Q}_{i2}, \ldots, \tilde{Q}_{iK}\right)$.

Step 3: Rank $\tilde{S}_i$ $(i = 1, 2, \ldots, n)$ based on the score function $\Gamma(\tilde{S}_i)$ and accuracy function $\lambda(\tilde{S}_i)$ by Definition 12.

Step 4: Rank the corresponding alternatives according to their scores to get the best one.

VI. ILLUSTRATIVE EXAMPLE

To reveal the application of the suggested models, the following example about the ranking of mobile phone products is adapted from [32].

Example 2: Suppose that there are four mobile phones which are recorded as follows. M_1: iPhone 8, M_2: iPhone 8 plus, M_3: Samsung Galaxy Note 8, and M_4: Huawei Mate 10 Pro.

The proposed methods are utilized to rank these phones according to the following features. F_1: Battery life, F_2: Camera, F_3: Screen, F_4: Sound (Audio), F_5: Battery draining. Assume the weights of the attributes are $w = (0.3, 0.3, 0.2, 0.1, 0.1)$, under the T-SHF environment. The evaluation information are described as T-SHFNs with $q = 3$. Subsequently the T-SHF evaluation matrix D is attained as shown in Table 1.

Then we can rank the alternative mobile phones using Algorithm 1.

Step 1: Normalize $D = \left[\tilde{Q}_{ij}\right]_{n \times k}$: Because the only cost attribute in this example is battery draining F_5, we convert it to benefit attribute using Equation 3, and then we obtain D^* as in Table 2.

Step 2: Utilizing the T-SHFWA operator, the overall evaluation values of the alternatives is obtained as:

TABLE 1. The T-SHF decision matrix *D*.

$Feature$	M_1	M_2	M_3	M_4
F_1	$\langle\{0.1,0.3,0.2\},\{0.5,0.7\},\{0.8,0.7\}\rangle_q$	$\langle\{0.3\},\{0.4,0.1\},\{0.5,0.6\}\rangle_q$	$\langle\{0.5\},\{0.1,0.3\},\{0.2,0.3,0.4\}\rangle_q$	$\langle\{0.8,0.7\},\{0.1\},\{0.2,0.3\}\rangle_q$
F_2	$\langle\{0.4\},\{0.5,0.6\},\{0.8,0.3,0.6\}\rangle_q$	$\langle\{0.5,0.4\},\{0.2\},\{0.7,0.6\}\rangle_q$	$\langle\{0.5,0.6\},\{0.5\},\{0.5,0.6\}\rangle_q$	$\langle\{0.8\},\{0.2,0.4\},\{0,0.1\}\rangle_q$
F_3	$\langle\{0.5,0.4\},\{0.6\},\{0.8\}\rangle_q$	$\langle\{0.5,0.6\},\{0.4,0.5\},\{0.5\}\rangle_q$	$\langle\{0.6,0.7,0.8\},\{0.2\},\{0.4\}\rangle_q$	$\langle\{0.7,0.6\},\{0.1\},\{0.2\}\rangle_q$
F_4	$\langle\{0.6\},\{0.9\},\{0.4,0.5\}\rangle_q$	$\langle\{0.7\},\{0.4,0.5\},\{0.6\}\rangle$	$\langle\{0.7\},\{0.5,0.3\},\{0.5\}\rangle_q$	$\langle\{0.8\},\{0.5\},\{0,0.1\}\rangle_q$
F_5	$\langle\{0.8\},\{0.2,0.5\},\{0.1\}\rangle_q$	$\langle\{0.7\},\{0.7\},\{0.3,0.2\}\rangle_q$	$\langle\{0.5\},\{0.7,0.3\},\{0.8\}\rangle_q$	$\langle\{0.2\},\{0.1,0.4\},\{0.9\}\rangle_q$

TABLE 2. The normalized decision matrix *D**.

$Feature$	M_1	M_2	M_3	M_4
F_1	$\langle\{0.1,0.3,0.2\},\{0.5,0.7\},\{0.8,0.7\}\rangle_q$	$\langle\{0.3\},\{0.4,0.1\},\{0.5,0.6\}\rangle_q$	$\langle\{0.5\},\{0.1,0.3\},\{0.2,0.3,0.4\}\rangle_q$	$\langle\{0.8,0.7\},\{0.1\},\{0.2,0.3\}\rangle_q$
F_2	$\langle\{0.4\},\{0.5,0.6\},\{0.8,0.3,0.6\}\rangle_q$	$\langle\{0.5,0.4\},\{0.2\},\{0.7,0.6\}\rangle_q$	$\langle\{0.5,0.6\},\{0.5\},\{0.5,0.6\}\rangle_q$	$\langle\{0.8\},\{0.2,0.4\},\{0,0.1\}\rangle_q$
F_3	$\langle\{0.5,0.4\},\{0.6\},\{0.8\}\rangle_q$	$\langle\{0.5,0.6\},\{0.4,0.5\},\{0.5\}\rangle_q$	$\langle\{0.6,0.7,0.8\},\{0.2\},\{0.4\}\rangle_q$	$\langle\{0.7,0.6\},\{0.1\},\{0.2\}\rangle_q$
F_4	$\langle\{0.6\},\{0.9\},\{0.4,0.5\}\rangle_q$	$\langle\{0.7\},\{0.4,0.5\},\{0.6\}\rangle$	$\langle\{0.7\},\{0.5,0.3\},\{0.5\}\rangle_q$	$\langle\{0.8\},\{0.5\},\{0,0.1\}\rangle_q$
F_5	$\langle\{0.1\},\{0.2,0.5\},\{0.8\}\rangle_q$	$\langle\{0.3,0.2\},\{0.7\},\{0.7\}\rangle_q$	$\langle\{0.8\},\{0.7,0.3\},\{0.5\}\rangle_q$	$\langle\{0.9\},\{0.1,0.4\},\{0.2\}\rangle_q$

$\tilde{S}_1 = \Big\{\{0.4104, 0.4245, 0.4143, 0.3836, 0.3998, 0.388\}, \{0.5015, 0.5496, 0.5298, 0.5807, 0.5549, 0.6081, 0.5863, 0.6425\}, \{0.7463, 0.5561, 0.6846, 0.7632, 0.5686, 0.7001, 0.717, 0.5343, 0.6578, 0.7332, 0.5463, 0.6726\}\Big\}$

$\tilde{S}_2 = \Big\{\{0.4833, 0.4808, 0.5093, 0.5071, 0.4558, 0.453, 0.4853, 0.4892\}, \{0.3436, 0.3514, 0.3593, 0.3674, 0.2267, 0.2318, 0.237, 0.2424\}, \{0.5824, 0.5561, 0.6153, 0.5875\}\Big\}$

$\tilde{S}_3 = \Big\{\{0.5994, 0.6237, 0.6591, 0.6221, 0.644, 0.6764\}, \{0.2656, 0.244, 0.2524, 0.2319, 0.3692, 0.3392, 0.3509, 0.3224\}, \{0.3631, 0.3836, 0.4101, 0.4333, 0.4471, 0.4724\}\Big\}$

$\tilde{S}_4 = \Big\{\{0.7998, 0.7905, 0.7753, 0.7645\}, \{0.1446, 0.1661, 0.1781, 0.2045\}, \{0, 0, 0, 0.1516, 0, 0, 0, 0.1712\}\Big\}$

Step 3: Calculate the score values $\Gamma(\tilde{S}_i)$ of each alternative by Definition 12. We obtain $\Gamma(\tilde{S}_1) = 0.2991, \Gamma(\tilde{S}_2) = 0.443, \Gamma(\tilde{S}_3) = 0.5798, \Gamma(\tilde{S}_4) = 0.737$.

Step 4: We get the rank of the four mobile phones as.

$M_4 > M_3 > M_2 > M_1$. Therefore, Huawei mobile phone is the optimal choice.

In step 2, if we use the T-SHFWG operator to aggregate attribute values, then:

$\tilde{S}_1 = \Big\{\{0.2505, 0.3478, 0.308, 0.2393, 0.3326, 0.2946\}, \{0.8442, 0.6204, 0.6334, 0.641, 0.6639, 0.6705, 0.6808, 0.6869\}, \{0.7825, 0.7352, 0.7365, 0.7844, 0.6363, 0.739, 0.7551, 0.6691, 0.7004, 0.7573, 0.6726, 0.7034\}\Big\}$

$\tilde{S}_2 = \Big\{\{0.4214, 0.4047, 0.4371, 0.4197, 0.3943, 0.3786, 0.4089, 0.3926\}, \{0.4319, 0.4427, 0.453, 0.4627, 0.3967, 0.4097, 0.4219, 0.4332\}, \{0.6611, 0.5714, 0.6329, 0.5971\}\Big\}$

$\tilde{S}_3 = \Big\{\{0.5618, 0.5791, 0.5951, 0.5935, 0.6121, 0.6287\}, \{0.4527, 0.3836, 0.4362, 0.3591, 0.4641, 0.3998, 0.4486, 0.4247\}, \{0.4294, 0.4776, 0.4388, 0.4851, 0.4563, 0.4991\}\Big\}$

$\tilde{S}_4 = \Big\{\{0.788, 0.7641, 0.7571, 0.7341\}, \{0.2533, 0.283, 0.3215, 0.3406\}, \{0.1686, 0.1698, 172, 0.1731, 0.2195, 0.2202, 0.2216, 0.2222\}\Big\}$.

Then we obtain the score values of each alternative as $\Gamma(\tilde{S}_1) = 0.1669, \Gamma(\tilde{S}_2) = 0.377, \Gamma(\tilde{S}_3) = 0.518, \Gamma(\tilde{S}_4) = 0.703$.

Based on the above score values, the order of the alternative mobile phones is $M_4 > M_3 > M_2 > M_1$.

Obviously, these two approaches have the same ranking result.

VII. THE COMPARISON ANALYSIS WITH THE OTHER METHODS

Apparently, T-SHFS is improved from SHFS which is the basis of the suggested approach. Also, the models H_qROFS and PHFS are the closest in structure to T-SHFS. Hence, to further ensure the validity of the proposed methods and explore their advantages, we compare the suggested operators with some SHF operators, some H_qROF operators and some PHF operators, including SHF weighted averaging (SHFWA) operator [12], SHF weighted geometric (SHFWG) operator [12], H_qROF weighted averaging (H_qROFWA) operator [21], H_qROF weighted geometric (H_qROFWG)operator [21], PHF weighted averaging (PHFWA) operator [10] and PHF weighted geometric(PHFWG) operator [10].

TABLE 3. Ranking results from different methods for example 2.

Methods	Score values	Ranking results
SHFWA operator ($q=2$)	$\Gamma(\tilde{S}_1)=-2.7128, \Gamma(\tilde{S}_2)=0.3413, \Gamma(\tilde{S}_3)=0.2786, \Gamma(\tilde{S}_4)=1.3684$	$M_4>M_2>M_3>M_1$
SHFWG operator ($q=2$)	$\Gamma(\tilde{S}_1)=-3.2904, \Gamma(\tilde{S}_2)=-0.1163, \Gamma(\tilde{S}_3)=-0.0955, \Gamma(\tilde{S}_4)=0.8384$	$M_4>M_3>M_2>M_1$
H_qROFWA operator ($T_Q, F_Q \in [0,1], I_Q=0, q=3$)	$\Gamma(\tilde{S}_1)=-0.2474, \Gamma(\tilde{S}_2)=-0.0887, \Gamma(\tilde{S}_3)=0.1854, \Gamma(\tilde{S}_4)=0.4789$	$M_4>M_3>M_2>M_1$
H_qROFWG operator ($T_Q, F_Q \in [0,1], I_Q=0, q=3$)	$\Gamma(\tilde{S}_1)=-0.3545, \Gamma(\tilde{S}_2)=-0.1676, \Gamma(\tilde{S}_3)=0.1105, \Gamma(\tilde{S}_4)=0.4334$	$M_4>M_3>M_2>M_1$
PHFWA operator ($q=1$)	$\Gamma(\tilde{S}_1)=0.0663, \Gamma(\tilde{S}_2)=0.301, \Gamma(\tilde{S}_3)=0.4543, \Gamma(\tilde{S}_4)=0.7823$	$M_4>M_3>M_2>M_1$
PHFWG operator ($q=1$)	$\Gamma(\tilde{S}_1)=0, \Gamma(\tilde{S}_2)=0.2213, \Gamma(\tilde{S}_3)=0.3932, \Gamma(\tilde{S}_4)=0.6851$	$M_4>M_3>M_2>M_1$
q-ROSNHFWA operator in this paper ($q=3$)	$\Gamma(\tilde{S}_1)=0.2991, \Gamma(\tilde{S}_2)=0.443, \Gamma(\tilde{S}_3)=0.5798, \Gamma(\tilde{S}_4)=0.737$	$M_4>M_3>M_2>M_1$
q-ROSNHFWG operator in this paper ($q=3$)	$\Gamma(\tilde{S}_1)=0.1669, \Gamma(\tilde{S}_2)=0.377, \Gamma(\tilde{S}_3)=0.518, \Gamma(\tilde{S}_4)=0.703$	$M_4>M_3>M_2>M_1$

In order to conduct the comparison, we solve the same decision making problem above using these operators. It is worth mentioning that the H_qROFSs only have two membership degrees, a truth and a falsity membership degrees, while the T-SHFS is constructed by three membership degrees, which are truth, indeterminacy and falsity membership degrees. Thus, the H_qROFS is a special case of T-SHFS and can be easily represented in the form of T-SHFS. In other words, the H_qROFS is a T-SHFS with indeterminacy term equal zero, so it cannot handle the T-SHFNs. Thus, to compare the suggested method with these in [22], we put the degree of indeterminacy to 0 in the suggested operator. For PHFS and SHFS they are also considered as special cases of T-SHFS with $q = 1$ and $q = 2$, respectively. In particular, we set $q = 1$ in the proposed operators when we compare the PHF operators with the proposed operators and $q = 2$, while comparing SHF operators with the proposed operators.

The comparison results are shown as Table 3. From Table 3, it is evident that the ranking results produced by the proposed methods are almost the same as those produced by other methods. However, their optimal selections are the same, which can prove the effectiveness of our methods very well. As mentioned above, H_qROFNs do not have the membership function which represents the neutrality, which will lead to the lack of some information, while the proposed T-SHFNs include truth, neutral and false membership degrees and give decision makers a more flexible environment to avoid information loss in the decision making process.

Wang and Li's [10] method is based on PHFW operators, which need simple calculations, but its area of applications very limited. It can only deal with the problem that parameters are represented as PHFNs, and PHFNs cannot fully represent the actual decision information, as the summation of its potential positive, neutral, and negative membership degrees must be between 0 and 1, i.e, ($0 \leq x^+ + y^+ + Z^+ \leq 1$). Thus, it will easily cause the bias of the data.

Khan *et al.*'s [12] method is based on the SHF operators which use the SHFNs. This makes the SHF operators more flexible than PHFW operators as the structure of SHFNs is with a constraint that the sum of the characteristic functions may exceeded from the unit interval but their square must belong to the unit interval, i.e, $0 \leq (x^+)^2 + (y^+)^2 + (Z^+)^2 \leq 1$.

However, the proposed method is based on the T-SHFW operators, which is expressed based on the T-SHFN. The T-SHFN is more flexible than the PHFN and SHFN because it needs the summation of potential truth, neutral and falsity membership degrees must meet $0 \leq (x^+)^q + (y^+)^q + (Z^+)^q \leq 1$ ($q \geq 1$). It can represent the attributes more comprehensively. For the proposed T-SHFWA and T-SHFWG operators, we can find that they are best ways to express the neutral data by the T-SHFNs because they make the decision making process easier and changeable by a parameter q. By increasing q, the scope of the evaluated information will be wider, thus avoiding the bias of information.

VIII. CONCLUSION

This paper proposed the T-SHFSs, provided the operational laws of the T-SHFNs and discussed their properties. The proposed T-SHFSs inherent the advantages and superiorities of the SHFS, q-ROHFS and q-RPFS so that it is more flexible and powerful. Based on the operational laws of the T-SHFNs, the T-SHFWA operator and T-SHFWG opertor are defined. The properties of the proposed operators such as the idempotency, boundedness and monotonicity are investigated. Afterwards, we proposed two new methods for MADM, where the attributes values is in the form of T-SHFNs. We applied the newly developed methods to rank mobile phone products based on online reviews. The comparative study is also provided and the results show a great similarity and consistency while using other ranking methods such as SHFWA operator, SHFWG opertor, H_qROFWA operator, H_qROFWG operator, GPHFWA operator and GPHFWG operator.

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