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Iterative LMI Approach to Robust Hierarchical Control of Homogenous Linear Multi-Agent Systems Subject to Polytopic Uncertainty and External Disturbance

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ABSTRACT This paper investigates design of robust hierarchical control for linear multi-agent systems (MAS) subject to polytopic uncertainty and external disturbance. Each agent of MAS is described by a homogeneous linear time-invariant dynamic model. The control structure has two layers, namely, an upper layer and a lower layer. Local actions are executed in the lower layer. Each agent shares information through an undirected graph with neighboring agents in the upper layer to achieve the global stabilization and disturbance attenuation. We employ a parameter-dependent Lyapunov function to formulate the robust control design of the local and global feedback in terms of bilinear matrix inequalities and incorporate constraints on disturbance attenuation. We propose the sufficient condition, where system matrices and Lyapunov variables are separated. The parameterization of the controller depends on a common slack variable instead of the Lyapunov matrices. We develop an iterative approach based on the coordinated optimization to solve sub-problems over linear matrix inequalities. Numerical examples are provided to demonstrate the effectiveness of the proposed robust control designs. It is shown that our proposed robust control designs outperform other robust control designs in terms of achievable disturbance attenuation and maximum admissible uncertainty bound.

INDEX TERMS Multi-agent systems, polytopic uncertainty, disturbance attenuation, hierarchical control, parameter-dependent bilinear matrix inequality.

I. INTRODUCTION

In recent years, there have been researches investigating the hierarchical control of multi-agent systems (MAS) because of its wide applications in various areas containing biological networks, energy management systems, transportation networks, etc. The key concept of hierarchical control framework is to attain certain global objectives by using local measurements of subsystems (i.e., agents) and inter-agent cooperation via information exchange. Various classes of network systems were effectively coped with by this framework [1].

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In general control systems, uncertainties and external disturbances are known sources of stability and performance degradation [2]. It is thus worth investigating the problem of disturbance attenuation for uncertain MAS. The robustness of MAS against disturbance and uncertainties can be handled in terms of \mathcal{H}_∞ and \mathcal{H}_2 norms [3]. One common approach for robust analysis and synthesis is Lyapunov theory. Finding a Lyapunov function which guarantees robust stability for all admissible systems is the main task. There are two main types of Lyapunov functions. The first type, referred to as quadratic function, is a common parameter-independent Lyapunov function (PILF) which explicitly depends on time. A sufficient condition to guarantee stability of MAS is derived in terms of linear matrix inequalities and

then is called parameter-independent linear matrix inequality (PILMI) approach. This approach gives conservative results due to the following causes [4]. First, it utilizes only a fixed Lyapunov function for stability and performance for the entire uncertain domain. Second, design specifications use a common Lyapunov matrix and it is difficult to search the solution due to the product between the Lyapunov matrix variable and system matrices. The second type is a parameter-dependent Lyapunov function (PDLF) whose matrix variable depends on uncertain parameters. PDLF can reduce the conservatism of robust stability and performance [5], [6]. Therefore, this approach has gained notable attractions recently. When utilizing this approach, robust analysis and control conditions are formulated in terms of parameter-dependent bilinear matrix inequalities (PDBMIs) involving uncertain parameters [7].

There has been a number of studies dealing with single uncertain and perturbed systems using PDLMI, e.g., [8]–[12]. However, only a few works have been existed for uncertain and perturbed MAS. For example, [13] investigated the leader-follower consensus of higher order MAS with directed and switching topology. A PDLF sufficient condition for obtaining consensus was formulated as a PDLMI. [14] addressed the stabilization and disturbance attenuation for homogeneous MAS with the norm-bounded uncertainty. The formulation incorporates \mathcal{H}_∞ and \mathcal{H}_2 design criteria on disturbance attenuation. A quadratic Lyapunov function is employed, and robust design of feedback controllers is given in terms of LMI. The limitation of this approach is a common Lyapunov function for entire uncertainty domain. In addition, [15] addressed the design of robust consensus controllers for homogeneous MAS in which agents contain polytopic uncertainty and external disturbance and inter-agent communications are subject to uncertainty. Then, a robust \mathcal{H}_∞ stabilization was formulated by a PDLF and a sufficient condition was derived as a PDLF. Nevertheless, the controller parameterization depends on the Lyapunov matrix variable and the obtained result may be still conservative. To obtain a robust controller independent of uncertain parameters, it is possible to find a parameter-independent Lyapunov matrix variable utilizing the PILMI approach. An effective way to overcome this conservativeness is to separate the Lyapunov function with system matrices by introducing additional slack variables. This is called the dilated approach.

To our best knowledge, there has not been any work using the dilated approach for the robust hierarchical control of perturbed and uncertain MAS. This motivates us to investigate the dilated BMI for attaining specific disturbance attenuation of uncertain MAS by using an affine PDLF with respect to uncertain parameters. The derived controller is independent of non-common Lyapunov matrix variables but is dependent of a common slack variable [16]. Moreover, robust hierarchical controllers can be designed using the new dilated BMI conditions to improve system performance specified by disturbance attenuation. More specifically, we propose

a systematic design approach of hierarchical \mathcal{H}_∞ and \mathcal{H}_2 controllers for linear MAS subject to polytopic uncertainty using the dilated BMI conditions. Our contributions are as follows.

- New robust design conditions are derived using PDLF, which includes quadratic PILF as a special case. The BMI design criteria are then determined and effectively solved by an iterative approach based on the coordinated optimization of sub-problems over linear matrix inequalities.
- We compare the proposed BMI method with the PILMI method and the previous robust control method [14] to show the performance improvement in our proposed design. In particular, our proposed robust controllers give the upper bound of disturbance attenuation less than that obtained by other approaches. In addition, the proposed robust controllers yield stabilization of MAS with a larger maximum admissible bound of uncertainty.

The organization of this paper is as follows. We introduce problem formulation in Section II. Robust hierarchical \mathcal{H}_∞ and \mathcal{H}_2 state feedback control laws are proposed in Section III. Section IV illustrates numerical examples. Finally, conclusions are given in Section V.

II. PROBLEM FORMULATION

A. MODEL OF AGENTS

The dynamics of agents considered in the current work consists of polytopic uncertainty and external disturbance as follows.

$$\begin{cases} \dot{x}_i(t) = A(\delta)x_i(t) + B(\delta)u_i(t) + B_d(\delta)d_i(t), \\ y_i(t) = C(\delta)x_i(t) + D(\delta)u_i(t), \end{cases} \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^q$, $y_i(t) \in \mathbb{R}^m$, and $d_i(t) \in \mathcal{L}_2^l[0, \infty)$ represent the state vector, control input vector, measured output, and external disturbance input of the i -th agent, respectively. Here, $\delta = [\delta_1, \dots, \delta_M]^T$ is the time-invariant polytopic uncertain vector which belongs to the unit simplex Θ given by

$$\Theta \triangleq \left\{ \delta \in \mathbb{R}^M : \sum_{k=1}^M \delta_k = 1, \delta_k \geq 0, k = 1, \dots, M \right\}, \quad (2)$$

where M is the number of vertices. The matrices $A(\delta) \in \mathbb{R}^{n \times n}$, $B(\delta) \in \mathbb{R}^{n \times q}$, $C(\delta) \in \mathbb{R}^{m \times n}$, $B_d(\delta) \in \mathbb{R}^{n \times l}$, and $D(\delta) \in \mathbb{R}^{m \times q}$, $0 < q \leq n$ denote uncertain system matrix, input matrix, output matrix, disturbance input matrix, and direct transition matrix, respectively. Those uncertain matrices are of polytopic type, which are represented in term of the uncertainty vector δ as follows,

$$\begin{aligned} A(\delta) &= \sum_{k=1}^M \delta_k A_k, & B(\delta) &= \sum_{k=1}^M \delta_k B_k, & B_d(\delta) &= \sum_{k=1}^M \delta_k B_{dk}, \\ C(\delta) &= \sum_{k=1}^M \delta_k C_k, & D(\delta) &= \sum_{k=1}^M \delta_k D_k, \end{aligned} \quad (3)$$

where $A_k, B_k, B_{dk}, C_k,$ and D_k are known constant matrices. Then the overall homogeneous MAS composing of N identical dynamical agents is represented by

$$\begin{cases} \dot{x}(t) = \tilde{A}(\delta)x(t) + \tilde{B}(\delta)u(t) + \tilde{B}_d(\delta)d(t), \\ y(t) = \tilde{C}(\delta)x(t) + \tilde{D}(\delta)u(t), \end{cases} \quad (4)$$

where

$$\begin{aligned} \tilde{A}(\delta) &= I_N \otimes A(\delta), & \tilde{B}(\delta) &= I_N \otimes B(\delta), \\ \tilde{B}_d(\delta) &= I_N \otimes B_d(\delta), \\ \tilde{C}(\delta) &= I_N \otimes C(\delta), & \tilde{D}(\delta) &= I_N \otimes D(\delta), \\ x(t) &\triangleq [x_1(t)^T \ \dots \ x_N(t)^T]^T \in \mathbb{R}^{nN}, \\ u(t) &\triangleq [u_1(t)^T \ \dots \ u_N(t)^T]^T \in \mathbb{R}^{mN}, \\ y(t) &\triangleq [y_1(t)^T \ \dots \ y_N(t)^T]^T \in \mathbb{R}^{mN}, \\ d(t) &\triangleq [d_1(t)^T \ \dots \ d_N(t)^T]^T \in \mathbb{R}^{lN}. \end{aligned} \quad (5)$$

The notation \otimes denotes the Kronecker product.

The following reviews some basic concepts of graph theory. An undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ stands for communication topology of MAS where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ and $\mathcal{E} = \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\} \subseteq \mathcal{V} \times \mathcal{V}$ are the sets of vertex and edge, respectively. Next, $\mathcal{N}_i \triangleq \{j : (v_i, v_j) \in \mathcal{E}\}$ represents the set of neighbors of the i -th agent. The graph \mathcal{G} is connected if there exists a path between any two vertices. $\mathcal{A} = [a_{ij}]$ denotes an adjacency matrix of \mathcal{G} where $a_{ij} = 0$ if $(v_i, v_j) \notin \mathcal{E}$ and $a_{ij} = a_{ji} > 0$ if $(v_i, v_j) \in \mathcal{E}$. In addition, $\mathcal{D} = \text{diag}\{\text{deg}_i\}_{i=1, \dots, N}$ indicates degree matrix of \mathcal{G} in which $\text{deg}_i \triangleq \sum_{j \in \mathcal{N}_i} a_{ij}$ is degree of i -th vertex. Laplacian matrix \mathcal{L} of \mathcal{G} is derived by $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Note that \mathcal{L} is symmetric positive semi-definite, i.e., $\mathcal{L} = \mathcal{L}^T \geq 0$, and $\mathcal{L}1_N = 0$.

The following assumptions will be employed.

Assumption 1: $(A_k, B_k), \forall k = 1, \dots, M$ is stabilizable.

Assumption 2: Graph \mathcal{G} is fixed, undirected, and connected.

The \mathcal{H}_∞ and \mathcal{H}_2 norms of the closed-loop transfer function $G_{yd}(s, \delta)$ from $d(t)$ to $y(t)$ of system (4) [3] are utilized to assess the effect of parametric uncertainty and exogenous disturbance.

$$\begin{aligned} \|G_{yd}(s, \delta)\|_\infty &= \sup_{\nu \in \mathbb{R}, \delta \in \Theta} \bar{\sigma}(G_{yd}(j\nu, \delta)) \\ &= \sup_{\delta \in \Theta, 0 \neq d(t) \in \mathcal{L}_2^N[0, +\infty)} \frac{\|y(t)\|_2}{\|d(t)\|_2}, \\ \|G_{yd}(s, \delta)\|_2^2 &= \sup_{\omega \in \mathbb{R}, \delta \in \Theta} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace}(G_{yd}(j\omega, \delta)^T G_{yd}(j\omega, \delta)) d\omega, \end{aligned} \quad (6)$$

where $\bar{\sigma}(\cdot)$ indicates the largest singular value of a matrix.

The objective of this work is to design a hierarchical feedback controller for the perturbed uncertain MAS (4) such that a given constraint on inter-agent information exchange

depicted by graph \mathcal{G} is satisfied. The control input of network system consists of two terms, i.e., the local control input $u_l(t)$ and the global control input $u_g(t)$,

$$u(t) = u_l(t) + u_g(t) = -F_c x(t), \quad (8)$$

where

$$u_l(t) = -(I_N \otimes F_l)x(t), \quad \text{and} \quad u_g(t) = -(\mathcal{L} \otimes F_u)x(t).$$

$F_l \in \mathbb{R}^{q \times n}$ and $F_u \in \mathbb{R}^{q \times n}$ indicate the local and global feedback controller gains, respectively. It is proven from [1], [17] that the information exchange among agents can be assured by using the hierarchical feedback gain F_c which belongs to the following class

$$\mathcal{F} \triangleq \{F_c \in \mathbb{R}^{(Nq) \times (Nn)} | F_c = I_N \otimes F_l + \mathcal{L} \otimes F_u\}. \quad (9)$$

Subsequently the controller structure modified from [1], [17], [18] is exhibited in Figure 1.

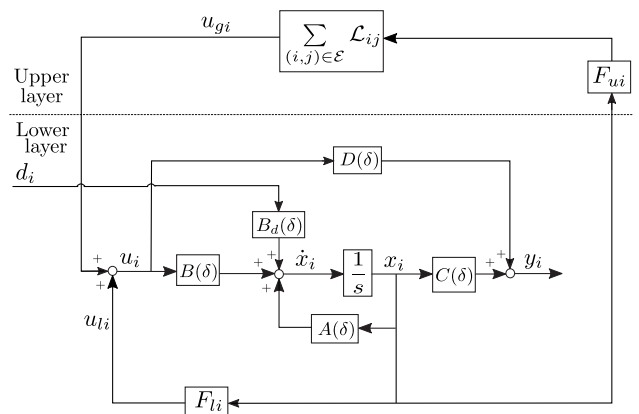


FIGURE 1. Hierarchical control structure of the i -th agent.

In this paper, we employ the design approach of the nominal MAS in [1], [17], [18] and propose the robust design to handle the model uncertainty of MAS and the influence of input external disturbances to output performance. In addition, we improve the robust control in [14] which is developed for homogeneous MAS with the norm-bounded uncertainty. The new robust design ensures the robust stabilization and reduces disturbance attenuation for homogeneous MAS with the polytopic uncertainty. The proposed approach constitutes a significant improvement over the previous works [1], [14], [17], [18].

Let $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_N\}$ with diagonal elements being eigenvalues of \mathcal{L} . Since the Laplacian matrix \mathcal{L} is positive semi-definite [19], it can be diagonalized with an orthogonal matrix $U \in \mathbb{R}^{N \times N}$, i.e., $U^T \mathcal{L} U = \Lambda$. The following changes of variables are made,

$$\bar{x} = (U^T \otimes I_n)x, \quad \bar{y} = (U^T \otimes I_m)y, \quad \bar{d} = (U^T \otimes I_l)d. \quad (10)$$

Next, substituting controller (8) into MAS model (4) and then multiplying the term $(U^T \otimes I_n)$ to both

sides of (4) yields

$$\begin{cases} \dot{\bar{x}}(t) = \bar{A}(\delta)\bar{x}(t) + \bar{B}_d(\delta)\bar{d}(t) \\ \bar{y}(t) = \bar{C}(\delta)\bar{x}(t), \end{cases} \quad (11)$$

where $\bar{A}(\delta) \triangleq I_N \otimes (A(\theta) - B(\delta)F_l) - \Lambda \otimes (B(\delta)F_u)$, $\bar{C}(\delta) \triangleq I_N \otimes (C(\delta) - D(\delta)F_l) - \Lambda \otimes (D(\delta)F_u)$. It is obvious that the system (11) is decomposed into N independent subsystems as follows,

$$\begin{cases} \dot{\bar{x}}_i(t) = \bar{A}_i(\delta)\bar{x}_i(t) + B_d(\delta)\bar{d}_i(t), \\ \bar{y}_i(t) = \bar{C}_i(\delta)\bar{x}_i(t), \end{cases} \quad (12)$$

where $\bar{A}_i(\delta) \triangleq A(\delta) - B(\delta)(F_l + \lambda_i F_u)$, $\bar{C}_i(\delta) \triangleq C(\delta) - D(\delta)(F_l + \lambda_i F_u)$. It is obvious that performance of (11) can be simultaneously synthesized through those N subsystems (12). Let $G_{\bar{y}\bar{d}}(s, \delta) \triangleq \bar{C}(\delta)(sI_{N \times n} - \bar{A}(\delta))^{-1}\bar{B}_d(\delta)$ and $G_{\bar{y}_i\bar{d}_i}(s, \delta) \triangleq \bar{C}_i(\delta)(sI_n - \bar{A}_i(\delta))^{-1}B_d(\delta)$ be the transfer function from $\bar{d}(t)$ to $\bar{y}(t)$ of system (11) and from $\bar{d}_i(t)$ to $\bar{y}_i(t)$ of system (12), respectively. Similarly to [19], [20], it is followed from (10), (11), and (12) that

$$\begin{aligned} G_{\bar{y}\bar{d}}(s, \delta) &= \text{diag}\{G_{\bar{y}_i\bar{d}_i}(s, \delta)\}_{i=1, \dots, N} \\ &= (U^T \otimes I_l)(G_{yd}(s, \delta))(U \otimes I_m). \end{aligned} \quad (13)$$

B. DESIGN PROBLEMS

In this research, we investigate two control design problems. First, the robust \mathcal{H}_∞ control design problem is described in the following.

Problem 1 (Robust \mathcal{H}_∞ Control Design): Design a controller (8) such that the closed-loop system (1) satisfies the performance criterion given by $\|G_{yd}(s, \delta)\|_\infty < \gamma_\infty, \forall \delta \in \Theta$ when $d(t) \neq 0, \forall d(t) \in \mathcal{L}_2^{IN}[0, \infty)$, where $\gamma_\infty > 0$ is a given disturbance attenuation level.

Second, the robust \mathcal{H}_2 control design problem is presented below.

Problem 2 (Robust \mathcal{H}_2 Control Design): Let $\gamma_2 > 0$ is a given disturbance suppression index, synthesize a controller (8) to assure the closed-loop system (1) performance indicated by $\|G_{yd}(s, \delta)\|_2 < \gamma_2, \forall \delta \in \Theta$ for any disturbance with infinite energy $d(t) \neq 0, d(t) \in \mathbb{R}^{IN}$.

III. MAIN RESULTS

This section first provides sufficient conditions for robust \mathcal{H}_∞ control design and robust \mathcal{H}_2 control design and then presents their design procedures.

A. ROBUST \mathcal{H}_∞ CONTROL DESIGN

Theorem 1: Given $\gamma_\infty > 0$, the robust \mathcal{H}_∞ control design problem is solved if there exist symmetric positive definite matrices $X_{ik} \in \mathbb{R}^{n \times n}, i = 1, \dots, N, k = 1, \dots, M, G_1, G_2 \in \mathbb{R}^{q \times n}, G \in \mathbb{R}^{n \times n}$, and a scalar number $\alpha_\infty > 0$ such that

$$\begin{bmatrix} \text{sym}(A_{\infty i}) & \Xi_{12} & B_{dk} & C_{\infty i}^T \\ \Xi_{21} & -\alpha_\infty(G + G^T) & 0_{n \times l} & \alpha_\infty C_{\infty} \\ B_{dk}^T & 0_{l \times n} & -\gamma_\infty^2 I_l & 0_{l \times m} \\ C_{\infty i} & \alpha_\infty C_{\infty i} & 0_{m \times l} & -I_{m \times m} \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M, \quad (14)$$

where $A_{\infty i} = A_k G - B_k(G_1 + \lambda_i G_2)$, $C_{\infty i} = C_k G - D_k(G_1 + \lambda_i G_2)$, $\Xi_{12} = X_{ik} - G^T + \alpha_\infty A_{\infty i}$, and $\Xi_{21} = X_{ik} - G + \alpha_\infty A_{\infty i}^T$. The local and global controller gains are then computed by $F_l = G_1 G^{-1}$ and $F_u = G_2 G^{-1}$.

Proof: See Appendix A for the proof. ■

Finally, let $\bar{\gamma}_\infty = \gamma_\infty^2$. The upper bound for robust \mathcal{H}_∞ control can be found by solving the following optimization problem,

$$\begin{aligned} \min_{\bar{\gamma}_\infty, \alpha_\infty, G_1, G_2, G, X_{ik}} \quad & \bar{\gamma}_\infty \\ \text{s.t.} \quad & \bar{\gamma}_\infty > 0, \quad \alpha_\infty > 0, \quad X_{ik} > 0, \end{aligned} \quad (14) \text{ is satisfied.} \quad (15)$$

It is worth noting that because there exist product terms between decision variables α_∞ and (G_1, G_2, G) , the conditions (14) are BMIs. Thus, global minimization $\bar{\gamma}_\infty$ in (15) cannot be found generally by utilizing convex optimization algorithms. However, it can be observed that the conditions (14) become LMIs in (X_{ik}, G_1, G_2, G) if α_∞ is fixed. Furthermore, if (X_{ik}, G_1, G_2, G) are fixed, then conditions (14) become LMIs in α_∞ . On the other hand, a possible way to solve this kind of optimization is to use the coordinate optimization [21], [22]. The main idea of this technique is solving LMIs optimization problems in each coordinate alternatively to achieve suboptimal controller. Let us explain how to apply the iterative approach for (15) as follows. We first start with a given α_∞ and a corresponding $\bar{\gamma}_\infty$. Then by fixing decision variables either α_∞ or (G_1, G_2, G) , we alternatively solve minimization problems over LMI constraints. When the number of iterations increases, the value of $\bar{\gamma}_\infty$ is monotonically decreasing. The stopping criteria of this process is either the improvement in the value of $\bar{\gamma}_\infty$ is less than the desired absolute tolerance or the LMIs minimization problems are infeasible.

Algorithm 1: Iterative LMI of Robust Parameter-Dependent \mathcal{H}_∞ Control (RPHIC)

Input: the system matrices

$(A_k, B_k, B_{dk}, C_k, D_k), k = 1, \dots, M$, Laplacian matrix \mathcal{L} , and chosen absolute tolerance ϵ_∞ .

1. Let $j = 0$ where j is the iteration index.

2. Initialize $\alpha_{\infty,0} > 0$.

3. Fix α_∞ at $\alpha_{\infty,0}$.

Solve (15) to find (X_{ik}, G_1, G_2, G) and $\bar{\gamma}_\infty$. Then set $\bar{\gamma}_{\infty,0} = \bar{\gamma}_\infty$.

for $j := 1$ to $iter_{max}$ **do**

 (i) set $j = j + 1$;

 (ii) solve (15) to compute (α_∞, X_{ik}) and γ_∞ where (G_1, G_2, G) are fixed at previous most value. Then set $\alpha_{\infty,j} = \alpha_\infty$;

 (iii) solve (15) to derive (X_{ik}, G_1, G_2, G) and $\bar{\gamma}_\infty$ where α_∞ is fixed at $\alpha_{\infty,j}$. Then set $\bar{\gamma}_{\infty,j} = \bar{\gamma}_\infty$; Until either $|\bar{\gamma}_{\infty,j} - \bar{\gamma}_{\infty,j-1}| \leq \epsilon_\infty$ or LMIs minimization problem in (ii) or (iii) is infeasible.

end

Next, we propose the procedure of the robust \mathcal{H}_∞ control design composing of three main steps.

- 1) Derive Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N \times N}$ of graph \mathcal{G} .
- 2) Solve the optimization problem (15) by utilizing Algorithm 1 to obtain solution in terms of $(\bar{\gamma}_\infty, G_1, G_2, G, \alpha_\infty, X_{ik})$ in robust \mathcal{H}_∞ control design.
- 3) Compute hierarchical state feedback controller (8).

Remark 1: We will compare the proposed RPHIC with a robust quadratic \mathcal{H}_∞ control (RQHIC) that guarantees the \mathcal{H}_∞ performance of the uncertain MAS (12). Design of RQHIC is based on a common quadratic Lyapunov function. Denote $\gamma_{\infty,q}$ be the disturbance attenuation of RQHIC. Let $\bar{\gamma}_{\infty,q} = \gamma_{\infty,q}^2$. We find the Lyapunov function candidate

$$V_i(x_i(t)) = x_i(t)^T \bar{P} x_i(t), \quad i = 1, \dots, N$$

where $\bar{P} = \bar{P}^T \in \mathbb{R}^{n \times n}$, $\bar{P} \succ 0$. Let $X = \bar{P}^{-1}$. We adopt similar steps in the proposed robust \mathcal{H}_∞ control design to determine minimization of $\gamma_{\infty,q}$ which can be found by solving the following optimization problem. Let $\bar{\gamma}_{\infty,q} = \gamma_{\infty,q}^2$.

$$\begin{aligned} \min_{\bar{\gamma}_{\infty,q}, X, G_1, G_2} \quad & \bar{\gamma}_{\infty,q} \\ \text{s.t.} \quad & \bar{\gamma}_{\infty,q} > 0, \quad X \succ 0, \end{aligned} \quad (16a)$$

$$\begin{aligned} \begin{bmatrix} \text{sym}(A_{\infty i,q}) & C_{\infty i,q}^T & B_d \\ C_{\infty i,q} & -I_m & 0_{m \times l} \\ B_d^T & 0_{l \times m} & -\bar{\gamma}_{\infty,q} I_l \end{bmatrix} < 0, \\ \forall i = 1, \dots, N. \end{aligned} \quad (16b)$$

where $A_{\infty i,q} = A_k X - B_k(G_1 + \lambda_i G_2)$, $C_{\infty i,q} = C_k X - D(G_1 + \lambda_i G_2)$. The state-feedback gain of RQHIC has the form

$$F_{\infty,q} = I_N \otimes F_{l\infty,q} + \mathcal{L} \otimes F_{u\infty,q}, \quad (17)$$

where $F_{l\infty,q} = G_1 X^{-1}$ and $F_{u\infty,q} = G_2 X^{-1}$.

It is noted that our design conditions using the parameter-dependent Lyapunov function is less conservative than that using a common quadratic Lyapunov function for the entire uncertain domain.

B. ROBUST \mathcal{H}_2 CONTROL DESIGN

The following theorem provides a sufficient condition to design of robust \mathcal{H}_2 control design.

Theorem 2: Let $\gamma_2 > 0$ be given. Denote $A_{2i} = A_k G - B_k(G_1 + \lambda_i G_2)$ and $C_{2i} = C_k G - D_k(G_1 + \lambda_i G_2)$. The robust \mathcal{H}_2 control design problem is solved if there exist $X_{ik} = X_{ik}^T \in \mathbb{R}^{n \times n}$, $X_{ik} \succ 0$, $G_1, G_2 \in \mathbb{R}^{q \times n}$, $E_{ik} = E_{ik}^T \in \mathbb{R}^{l \times l}$, $E_{ik} \succ 0$, $i = 1, \dots, N$, $k = 1, \dots, M$, and $\alpha_2 > 0$ satisfying the following LMIs

$$\begin{bmatrix} \text{sym}(A_{2i}) & X_{ik} - G^T + \alpha_2 A_{2i} & C_{2i}^T \\ X_{ik} - G + \alpha_2 A_{2i}^T & -\alpha_2(G + G^T) & \alpha_2 C_{2i}^T \\ C_{2i} & \alpha_2 C_{2i} & -\gamma_2^2 I_m \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M, \quad (18a)$$

$$\begin{bmatrix} X_{ik} & B_{dk} \\ B_{dk}^T & E_{ik} \end{bmatrix} \succ 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M, \quad (18b)$$

$$\sum_{i=1}^N \text{trace}(E_{ik}) < 1, \quad \forall k = 1, \dots, M. \quad (18c)$$

The hierarchical feedback gains are calculated by $F_l = G_1 G^{-1}$, and $F_u = G_2 G^{-1}$.

Proof: See Appendix B for the proof. ■

Lastly, denote $\bar{\gamma}_2 \triangleq \gamma_2^2$, then the smallest disturbance attenuation level can be found by solving the following minimization problem,

$$\begin{aligned} \min_{\bar{\gamma}_2, \alpha_2, G_1, G_2, G, X_{ik}, E_{ik}} \quad & \bar{\gamma}_2 \\ \text{s.t.} \quad & \bar{\gamma}_2 > 0, \quad \alpha_2 > 0, \quad X_{ik} \succ 0, \quad E_{ik} \succ 0, \\ & (18) \text{ is satisfied.} \end{aligned} \quad (19)$$

Note that the conditions (18a) are BMI constraints due to the existence of product terms including decision variables α_2 and (X_{ik}, G_1, G_2, G) . Nevertheless, it can be seen that the conditions (18) become LMI conditions either in $(X_{ik}, G_1, G_2, E_{ik})$ if α_2 is fixed or in (α_2, E_{ik}) if (X_{ik}, G_1, G_2, G) are fixed. It then allows us to apply iterative approach [21], [22] for minimization problems (19) with LMI constraints to attain desired sub-optimal controller. Let us provide the following algorithm.

Algorithm 2: Iterative LMI of Robust Parameter-Dependent \mathcal{H}_2 Control (RPH2C)

Input: the system matrices

$(A_k, B_k, B_{dk}, C_k, D_k)$, $k = 1, \dots, M$, Laplacian matrix \mathcal{L} , and chosen absolute tolerance ϵ_2 .

1. Let $j = 0$ where j is the iteration index.

2. Initialize $\alpha_{2,0} > 0$.

3. Fix α_2 at $\alpha_{2,0}$.

Solve (19) to compute $(X_{ik}, G_1, G_2, G, E_{ik})$ and $\bar{\gamma}_2$. Then set $\bar{\gamma}_{2,0} = \bar{\gamma}_2$.

for $j := 1$ **to** $iter_{max}$ **do**

(i) set $j = j + 1$;

(ii) solve (19) to compute $(\alpha_2, X_{ik}, E_{ik})$ and $\bar{\gamma}_2$ where (G_1, G_2, G) are fixed at previous most value. Then set $\alpha_{2,j} = \alpha_2$;

(iii) solve (19) to determine $(X_{ik}, G_1, G_2, G, E_{ik})$ and $\bar{\gamma}_2$ where α_2 is fixed at $\alpha_{2,j}$. Then set $\bar{\gamma}_{2,j} = \bar{\gamma}_2$;

Until either $|\bar{\gamma}_{2,j} - \bar{\gamma}_{2,j-1}| \leq \epsilon_2$ or LMIs

minimization problem in (ii) or (iii) is infeasible.

end

Next, a procedure of robust \mathcal{H}_2 control design is given, which contains three main steps.

1) Determine Laplacian matrix \mathcal{L} .

2) Solve the optimization problem (19) using Algorithm 2 to attain solution in terms of $(\bar{\gamma}_2, \alpha_2, G_1, G_2, G, X_{ik}, E_{ik}, i = 1, \dots, N, k = 1, \dots, M)$ in robust \mathcal{H}_2 control design.

3) Calculate hierarchical state feedback controller (8).

Remark 2: We will compare the proposed RPH2C with a robust quadratic \mathcal{H}_2 control (RQH2C). Design of RQH2C

is based on a common matrix variable that guarantees the \mathcal{H}_2 performance of the uncertain MAS (12). We find the Lyapunov function candidate

$$V_i(x_i(t)) = x_i(t)^T \bar{P} x_i(t), \quad i = 1, \dots, N$$

where $\bar{P} = \bar{P}^T \in \mathbb{R}^{n \times n}$, $\bar{X} \succ 0$. Let $\bar{X} = \bar{P}^{-1}$. The following optimization problem is proposed to find the minimum disturbance attenuation obtained by RQH2C. A comparison in terms of minimum disturbance attenuation γ_2 to illustrate the improvement of our proposed design is given in the numerical example. Denote $\bar{\gamma}_{2,q} = \gamma_{2,q}^2$.

$$\begin{aligned} \min_{\bar{\gamma}_{2,q}, G_1, G_2, \bar{X}, E_{ik}} \quad & \bar{\gamma}_{2,q} \\ \text{s.t.} \quad & \bar{\gamma}_{2,q} > 0, \quad \bar{X} \succ 0, \quad E_{ik} \succ 0, \end{aligned} \quad (20a)$$

$$\begin{aligned} \begin{bmatrix} \text{sym}(A_{2i,q}) & B_{dk} \\ B_{dk}^T & -I_l \end{bmatrix} < 0, \\ \forall i = 1, \dots, N, \quad k = 1, \dots, M, \end{aligned} \quad (20b)$$

$$\begin{aligned} \begin{bmatrix} E_{ik} & C_{2i,q}^T \\ C_{2i,q} & \bar{X} \end{bmatrix} \succ 0, \\ \forall i = 1, \dots, N, \quad k = 1, \dots, M, \end{aligned} \quad (20c)$$

$$\begin{aligned} \sum_{i=1}^N \text{trace}(E_{ik}) < \bar{\gamma}_{2,q}, \\ \forall k = 1, \dots, M, \end{aligned} \quad (20d)$$

where $A_{2i,q} = A_k \bar{X} - B_k(G_1 + \lambda_i G_2)$, $C_{2i,q} = C_k \bar{X} - D_k(G_1 + \lambda_i G_2)$, Then the nominal hierarchical state-feedback gain of RQH2C is in the form

$$F_{c2,q} = I_N \otimes F_{l2,q} + \mathcal{L} \otimes F_{u2,q}, \quad (21)$$

where $F_{l2,q} = G_1 \bar{X}^{-1}$, and $F_{u2,q} = G_2 \bar{X}^{-1}$.

It is noted that our proposed synthesis conditions use the parameter-dependent Lyapunov function which leads to PDBMI conditions. It allows us to use multiple Lyapunov matrix variables for different vertices of uncertain agents. Hence, PDBMI design approach is less conservative than that using a common quadratic Lyapunov function for entire uncertain domain.

IV. NUMERICAL EXAMPLES

In this section, numerical examples are given to show effectiveness of the proposed robust control designs. We make comparison between results obtained from our proposed designs with those attained from other robust control designs.

For reference in the numerical results, we shall refer the robust \mathcal{H}_∞ control of polytopic MAS as robust parameter-dependent \mathcal{H}_∞ control (RPHIC) and robust \mathcal{H}_2 control of polytopic MAS as robust parameter-dependent \mathcal{H}_2 control (RPH2C). Let us refer the previous robust \mathcal{H}_∞ control and

robust \mathcal{H}_2 control of MAS subject to norm-bounded uncertainty in [14] as RHHIC and RHH2C, respectively.

We shall compare the design results between RPHIC and RHHIC and between RPH2C and RHH2C in the first and second examples, respectively. We first describe the dynamic model of MAS subject to norm-bounded uncertainty and show the equivalent MAS subject to polytopic uncertainty. Then we apply the proposed RPHIC and RPH2C for the polytopic models and compare the computed upper bound of disturbance attenuation and maximum admissible bound of uncertainty. Subsequently, we will compare the design results between RPHIC and RQHIC and between RPH2C and RQH2C in the third and fourth examples.

We utilize a notebook computer with Intel Core i7 2.6 GHz processor, 16 GB of RAM, and 64-bit operating system to solve Example 1-3. The design in Example 4 employs a desktop computer with Intel Core i7 3.20GHz processor and 32GB of RAM.

Example 1: This example aims to compare maximum uncertainty bound of two design methods, RPHIC and RHHIC. Three homogeneous dynamical agents having norm-bounded uncertainty are given [14].

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & -0.25 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0.03 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \\ C &= [0.1 \quad 0.15], \quad D = 0.05, \quad V = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \\ W &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \Delta_i = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \end{aligned}$$

where $\delta_k, k = 1, 2$ satisfy $|\delta_k| \leq \theta$, in which θ is norm-bounded uncertain parameter. An equivalent polytopic uncertainty form of this dynamic model is as follows.

$$\begin{aligned} A(\delta) &= \begin{bmatrix} 0.8 & -0.25 \\ 1 & 0 \end{bmatrix} + \delta_1 \begin{bmatrix} 0 & 0 \\ 0.04\theta & 0 \end{bmatrix} - \delta_2 \begin{bmatrix} 0 & 0 \\ 0.04\theta & 0 \end{bmatrix} \\ &+ \delta_3 \begin{bmatrix} 0 & 0 \\ 0 & 0.05\theta \end{bmatrix} - \delta_4 \begin{bmatrix} 0 & 0 \\ 0 & 0.05\theta \end{bmatrix} \\ B &= \begin{bmatrix} 0.1 \\ 0.03 \end{bmatrix}, \quad C = [0.1 \quad 0.15], \quad B_d = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \\ D &= 0.05, \quad \sum_{k=1}^4 \delta_k = 1, \quad \delta_k \geq 0. \end{aligned}$$

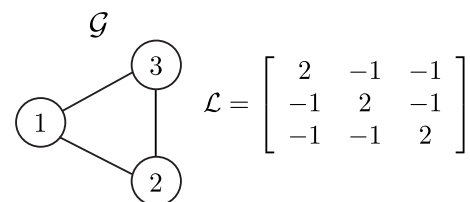


FIGURE 2. Undirected topology and Laplacian matrix of MAS.

The inter-agent communication described by triangle graph \mathcal{G} and its Laplacian matrix \mathcal{L} are depicted in Figure 2. The external disturbances $d_i(t)$ are any finite-energy signals,

e.g., band-limited signals. We will compare the disturbance attenuation between RPHIC and RHHIC.

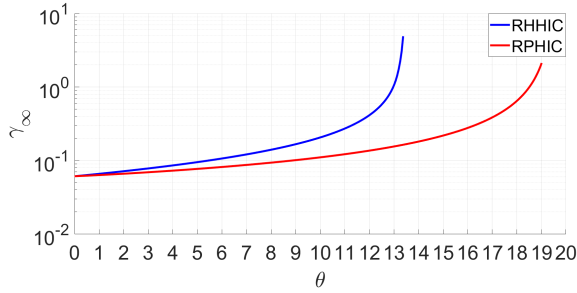


FIGURE 3. Upper bound of \mathcal{H}_∞ disturbance attenuation versus uncertainty bound θ .

Figure 3 displays the computed γ_∞ when varying θ on the interval $(0, 19.05]$ for both RPHIC and RHHIC. It can be seen that when θ is larger, γ_∞ increases. The smaller value of γ_∞ means better performance. Obviously, our proposed RPHIC attains significantly lower disturbance attenuation than that of RHHIC with the same value of θ . In addition, our proposed design conditions of RPHIC are still feasible with $\theta = 19.05$ whereas the design condition of RHHIC is feasible up to the maximum $\theta = 13.436$.

Table 1 summarizes the upper bound of γ_∞ from various design conditions. It is shown in Figure 3 and Table 1 that the proposed design condition RPHIC in Theorem 1 allows for a wider range of maximum admissible uncertainty bound θ_{max} and provides smaller disturbance attenuation γ_∞ than that of RHHIC [14]. Especially, θ_{max} is increased by 41.78%, and γ_∞ is decreased by 89.85% when $\theta = 13$. Therefore, design condition of RPHIC gives less conservative design results than the RHHIC in [14].

TABLE 1. Upper bound of \mathcal{H}_∞ disturbance attenuation versus θ .

Control method	γ_∞			
	$\theta = 0.1$	$\theta = 5$	$\theta = 13$	$\theta = 19$
RPHIC: Theorem 1	0.0615	0.0772	0.1556	2.0610
RHHIC [14]	0.0617	0.0953	1.0541	Infeasible

Note that the optimization problem (15) whose constraints are BMIs has the higher computational complexity when solved by Algorithm 1. This means that the achievement of less conservativeness is attained at the higher computational complexity.

Example 2: This example aims to compare the disturbance attenuation and the maximum achievable uncertainty bound between RPH2C and RHH2C. The MAS composes of three homogeneous dynamical agents having the following modified dynamic model [14] in which $\delta_k, k = 1, 2$ satisfy $|\delta_k| \leq \theta$ where θ denotes norm-bounded uncertain parameter.

$$A = \begin{bmatrix} -0.3 & -0.4 \\ 0.2 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$$

$$C = [0.2 \quad 0.25], \quad D = 0.2, \quad V = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.3 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.1 & 0 \\ 0.3 & 0.2 \end{bmatrix}, \quad \Delta_i = \begin{bmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{bmatrix}.$$

The model of MAS can be expressed by MAS subject to polytopic uncertainty as follows.

$$A(\delta) = \begin{bmatrix} -0.3 & -0.4 \\ 0.2 & 0.5 \end{bmatrix} + \delta_1 \begin{bmatrix} 0 & 0 \\ 0.06\theta & 0.04\theta \end{bmatrix}$$

$$- \delta_2 \begin{bmatrix} 0 & 0 \\ 0.06\theta & 0.04\theta \end{bmatrix} + \delta_3 \begin{bmatrix} 0 & 0 \\ 0.03\theta & 0 \end{bmatrix}$$

$$- \delta_4 \begin{bmatrix} 0 & 0 \\ 0.03\theta & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix},$$

$$C = [0.2 \quad 0.25], \quad B_d = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad D = 0.2,$$

$$\sum_{k=1}^4 \delta_k = 1, \quad \delta_k \geq 0.$$

TABLE 2. Upper bound of \mathcal{H}_2 disturbance attenuation versus θ .

Control method	γ_2			
	$\theta = 0.2$	$\theta = 4$	$\theta = 8$	$\theta = 17$
RPH2C: Theorem 2	0.4072	0.5953	0.9087	14.8489
RHH2C [14]	0.4148	0.8290	5.0965	Infeasible

Figure 2 depicts the inter-agent communication \mathcal{G} and its associated Laplacian matrix \mathcal{L} . The external disturbance $d_i(t)$ can be persistent signal such as white noise. We compare disturbance attenuation γ_2 obtained from RPH2C and RHH2C. The computed γ_2 as the upper bound of uncertainty θ is varied on the interval $(0, 17.2]$ is illustrated in Figure 4. It is observed that γ_2 of RPH2C is significantly smaller than that of RHH2C when θ is gradually enlarged. With a fixed θ , our proposed RPH2C yields a much lower disturbance attenuation than that of the RHH2C. Furthermore, with $\theta = 17.2$, our proposed RPH2C design still find robust state feedback, whereas the synthesis condition of RHH2C becomes infeasible. In fact, RHH2C can work with maximum uncertainty bound $\theta = 9.773$. The upper bound of γ_2 from different design conditions are given in Table 2. Figure 4 and Table 2 indicate that the proposed design condition RPH2C in Theorem 2 allows for a larger maximum admissible uncertainty bound θ_{max} and provides a lower disturbance attenuation γ_∞ than those of RHH2C in [14]. In particular, θ_{max} is improved by 75.995%, and γ_2 is decreased by 28.19% as $\theta = 4$. RPH2C effectively finds γ_2 for a wider parametric uncertainty range than the existing design condition and hence makes improvement over RHH2C.

Since the optimization problem (19) solved by using Algorithm 2 is subjected to BMI constraints, it has the higher computational complexity than RHH2C. Thus, at the higher cost of computational complexity, the less conservative result is achieved.

Example 3: This example aims to compare γ_∞ obtained by RPHIC and RQHIC and γ_2 obtained by RPH2C and RQH2C. Consider three homogeneous dynamical agents whose

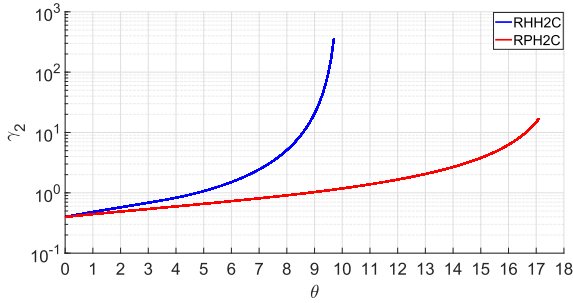


FIGURE 4. Upper bound of \mathcal{H}_2 disturbance attenuation versus uncertainty bound θ .

system matrices are given below.

$$\begin{aligned}
 A(\delta) &= \delta_1 \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix} + \delta_2 \begin{bmatrix} 0.5 & 8 & 1 \\ -1.2 & 1.5 & 0 \\ 0 & 3 & -4 \end{bmatrix} \\
 &+ \delta_3 \begin{bmatrix} 0 & 5 & 4 \\ -1.5 & 0.3 & 0.1 \\ 0.5 & 1.5 & -3 \end{bmatrix} + \delta_4 \begin{bmatrix} 0.3 & 2 & 2.5 \\ -0.5 & 0.4 & 0.2 \\ 0.7 & 1.3 & -2.4 \end{bmatrix} \\
 &+ \delta_5 \begin{bmatrix} 0.5 & 2.3 & 4.5 \\ -0.3 & 0.5 & 0.15 \\ 0.65 & 1.4 & -2.2 \end{bmatrix} \\
 &+ \delta_6 \begin{bmatrix} 0.35 & 2.2 & 4.4 \\ -0.25 & 0.45 & 0.1 \\ 0.35 & 1.3 & -1.7 \end{bmatrix} \\
 B(\delta) &= \delta_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \delta_2 \begin{bmatrix} 0.2 \\ 1.5 \\ 0.3 \end{bmatrix} + \delta_3 \begin{bmatrix} 0.2 \\ 0.6 \\ 0.8 \end{bmatrix} \\
 &+ \delta_4 \begin{bmatrix} 0.25 \\ 0.35 \\ 0.65 \end{bmatrix} + \delta_5 \begin{bmatrix} 0.3 \\ 0.45 \\ 0.75 \end{bmatrix} + \delta_6 \begin{bmatrix} 0.12 \\ 0.35 \\ 0.65 \end{bmatrix} \\
 C(\delta) &= \delta_1 [1 \ 0 \ 0] + \delta_2 [0.3 \ 0 \ 0.2] \\
 &+ \delta_3 [0 \ 0.5 \ 0.3] + \delta_4 [0.3 \ 0.25 \ 0.5] \\
 &+ \delta_5 [0.2 \ 0.45 \ 0.65] + \delta_6 [0.15 \ 0.25 \ 0.4] \\
 B_d(\delta) &= \delta_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \delta_2 \begin{bmatrix} 1.2 \\ 0.1 \\ 0.5 \end{bmatrix} + \delta_3 \begin{bmatrix} 1.1 \\ 0.2 \\ 0.5 \end{bmatrix} \\
 &+ \delta_4 \begin{bmatrix} 1.2 \\ 0.3 \\ 0.45 \end{bmatrix} + \delta_5 \begin{bmatrix} 0.9 \\ 0.2 \\ 0.55 \end{bmatrix} + \delta_6 \begin{bmatrix} 0.5 \\ 0.1 \\ 0.25 \end{bmatrix} \\
 D(\delta) &= \delta_1 + 0.5\delta_2 + 0.6\delta_3 + 0.7\delta_4 + 0.75\delta_5 + 0.55\delta_6, \\
 \sum_{i=1}^6 \delta_k &= 1, \quad \delta_k \geq 0.
 \end{aligned}$$

We will compare RPHIC with robust quadratic \mathcal{H}_∞ control (RQHIC) using a common Lyapunov variable. Figure 2 demonstrates the inter-agent communication described by triangle graph \mathcal{G} and its associated Laplacian matrix \mathcal{L} .

State-feedback gains of RPHIC are computed by Theorem 1 using Algorithm 1. The initial parameters $\alpha_{\infty,0}$ is set to be 0.01. The desired absolute tolerance ϵ_∞ is selected to

be 10^{-6} . Then, we yield feasible solutions after 18 iterations. Table 3 gives state feedback gains and obtained disturbance attenuation of both aforementioned controllers. Obviously, comparing the computed γ_∞ , our proposed RPHIC provides significantly lower disturbance attenuation than that of RQHIC. However, RPHIC spends more computational time than RHHIC.

Subsequently, we design RPH2C and compare with RQH2C. By Theorem 2, state-feedback gains of RPH2C are computed using Algorithm 2. The initial parameters $\alpha_{2,0}$ is set to be 0.02. The tolerance ϵ_2 is selected to be 10^{-6} . Then, feasible solutions are found after 38 iterations. These results are summarized in Table 4. It is observed that our proposed RPH2C yields a much lower disturbance attenuation than that of the RQH2C. RPH2C spends more computational time than RQH2C.

Example 4 (Vehicle Suspension Systems): The objective of this example is to illustrate effectiveness of proposed robust control design for a homogeneous MAS consisting of four active vehicle suspension systems.

Consider a vehicle suspension composing of one-fourth of the vehicle body mass, suspension components, and one wheel. This model with two degrees of freedom (2DOF) involving only the vertical motion of sprung and unsprung mass caused by the vertical ground displacement z_r arising from road irregularities [23], [24] captures many essential features of a real suspension system. Note that the sprung mass m_s represents the chassis at one corner of the vehicle. Unsprung mass m_u corresponds to the wheel and axle assembly at one corner of the vehicle. The suspension stiffness k_s and damping rate of suspension c_s stand for passive spring and shock absorber. k_t is tyre stiffness. z_s and z_u are vertical displacement of the sprung mass and unsprung mass. The actuator force f_s serves as a control input to eliminate the vibration of the vehicle chassis. The dynamic equation on vertical motion of a vehicle suspension model is obtained by Newton’s second law as follows.

$$\begin{aligned}
 m_s \ddot{z}_s(t) + c_s [\dot{z}_s(t) - \dot{z}_u(t)] + k_s [z_s(t) - z_u(t)] &= f_s(t), \\
 m_u \ddot{z}_u(t) + c_s [\dot{z}_u(t) - \dot{z}_s(t)] + k_s [z_u(t) - z_s(t)] \\
 + k_t [z_u(t) - z_r(t)] &= -f_s(t).
 \end{aligned} \tag{22}$$

The following assumptions are used.

- Damping of tyres and dynamic of the hydraulic actuator can be ignored.
- States $z_s(t)$ and $z_u(t)$ are measurable.
- Tyres of vehicle continuously contact with road.

Define $x_1(t) = z_s(t) - z_u(t)$, $x_2(t) = z_u(t) - z_r(t)$, $x_3(t) = \dot{z}_s(t)$, $x_4(t) = \dot{z}_u(t)$, $u(t) = f_s(t)$, where $x_1(t)$ stands for the suspension deflection, $x_2(t)$ represents tyre deflection, $x_3(t)$ and $x_4(t)$ denote sprung mass speed and unsprung mass speed, respectively, and $u(t)$ is control input. The ride comfort relates to the vertical acceleration $\ddot{z}_s(t)$ encountered by the vehicle body. To reduce the vertical acceleration of the vehicle chassis, the vehicle’s structural characteristics impose stiff limit on suspension deflection $x_1(t)$. Hitting the

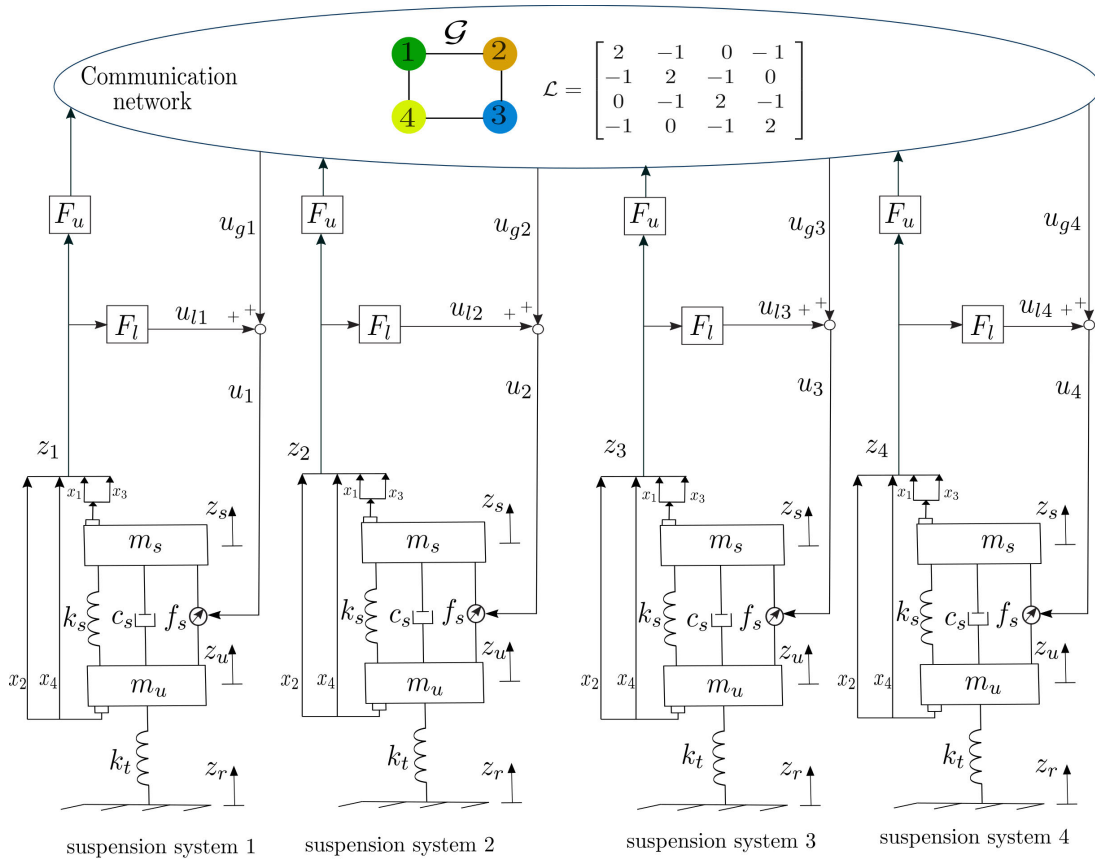


FIGURE 5. Vehicle suspension systems.

TABLE 3. Robust \$\mathcal{H}_\infty\$ feedback gains and corresponding \$\gamma_\infty\$.

Design	RPHIC	RQHIC
\$\gamma_\infty\$	1.74996	1.94604
\$F_l\$	[13.8972 13.4502 5.5436]	\$10^{10} \times [6.4235 5.8476 3.9609]\$
\$F_u\$	\$10^{-10} \times [-0.0664 -0.0277 0.1368]\$	\$[-0.3295 -0.3000 -0.2032]\$
CPU time (Sec.)	38.9796	2.125

TABLE 4. Robust \$\mathcal{H}_2\$ feedback gains and corresponding \$\gamma_2\$.

Design	RPH2C	RQH2C
\$\gamma_2\$	3.9084	25.9301
\$F_l\$	[2.5400 3.5283 3.1280]	[2.9274 4.2512 1.9155]
\$F_u\$	\$10^{-7} \times [-0.0014 -0.0471 -0.1212]\$	\$[-0.3709 -0.2346 0.2195]\$
CPU time (Sec.)	97.4531	3.0469

TABLE 5. Parametric uncertainty and \$M\$ of vehicle suspension systems.

Case	Parametric uncertainty	\$M\$
1	\$m_s\$	2
2	\$m_s\$, and \$k_t\$	4
3	\$m_s\$, \$k_t\$, and \$k_s\$	8
4	\$m_s\$, \$k_t\$, \$k_s\$, and \$m_u\$	16
5	\$m_s\$, \$k_t\$, \$k_s\$, \$m_u\$, and \$c_s\$	32

TABLE 6. Parameters of vehicle suspension system and uncertainty bound.

Parameter	Nominal value	Uncertainty bound (%)	Unit
\$m_s\$	100	\$\pm 20\$	Kg
\$k_s\$	40	\$\pm 5\$	N/m
\$c_s\$	440	\$\pm 8\$	Nm/s
\$m_u\$	20	\$\pm 5\$	Kg
\$k_t\$	45	\$\pm 5\$	N/m

deflection limit causes uncomfortable for passengers and increases vehicle wear [25], [26]. To guarantee driving safety, the vehicle wheels need to contact continuously with road.

A good road holding has good correction with tire deflection \$z_2(t)\$. Thus, the output performance which satisfies the objectives can be defined by \$y(t) = [\dot{x}_3(t) \ x_1(t) \ x_2(t)]\$.

TABLE 7. State feedback gains and γ_∞ of robust \mathcal{H}_∞ control when varying M .

M	Feedback gain		γ_∞
	F_l	F_u	
2	$10^3 \times [0.3135 \ -0.0066 \ 1.2508 \ 0.6243]$	$10^{-5} \times [0.0615 \ 0.0137 \ 0.2839 \ 0.0428]$	5.7185
4	$10^3 \times [1.8364 \ 0.3756 \ 8.2073 \ 1.6866]$	$[-1.3050 \ -0.2593 \ -5.9942 \ -0.8839]$	5.7430
8	$10^4 \times [0.3603 \ -0.0579 \ 1.7109 \ 0.2184]$	$10^{-4} \times [-0.0216 \ 0.0030 \ -0.1038 \ -0.0107]$	5.7500
16	$10^3 \times [1.4682 \ -0.7105 \ 7.3480 \ 1.0850]$	$10^{-6} [0.0504 \ -0.0264 \ 0.2608 \ 0.0206]$	6.0384
32	$10^3 \times [1.3249 \ -1.2137 \ 6.8874 \ 0.7006]$	$[-0.0070 \ 0.0067 \ -0.0376 \ -0.0011]$	6.0461

TABLE 8. Computational time of robust \mathcal{H}_∞ control design versus M .

M	$\alpha_{\infty,0}$	α_∞	# iter.	CPU time (Sec.)
2	0.5	0.49987548	2	4.3750
4	0.01	0.17408271	378	4.1063×10^3
8	0.5	0.50000181	2	7.9688
16	0.5	0.50000014	2	13.3906
32	1	0.74932245	132	6.9150×10^4

Let $z(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]^T$. Denote $d(t) = \dot{z}_r(t)$ the external disturbance. The state-space model of each vehicle suspension system can be written by

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_s} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & -\frac{c_s}{m_u} \end{bmatrix} z(t) \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -\frac{1}{m_u} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} d(t) \\ y(t) &= \begin{bmatrix} -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} z(t). \end{aligned} \tag{23}$$

In this example, we consider 4 active suspension systems. Each active suspension system is cast as an agent connecting to its neighbors by a square graph. When carrying out experiments, we vary the number of uncertainty parameters in affine model in 5 cases as given in Table 5. The uncertainty bound is caused by discrepancy around the nominal value and specified by the percentage of the nominal value. The parameters of active suspension system are assumed in Table 6.

Theorem 1 is employed to find state-feedback controller stabilization gains using Algorithm 1. These controller gains are synthesized with varying the number of vertices M and initial $\alpha_{\infty,0}$. We choose $\epsilon_\infty = 10^{-5}$. To this end, we select the best obtained γ_∞ . Table 7 and Table 8 summarize the results.

Figure 6 illustrates the γ_∞ obtained by the proposed RPHIC and the RQHIC when varying the number of vertices M from 2 to 32. Overall, γ_∞ is increased as M increases. Obviously, RQHIC gives larger disturbance attenuation γ_∞

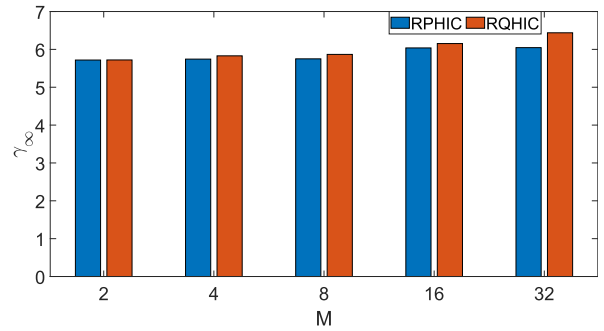


FIGURE 6. Upper bound of \mathcal{H}_∞ disturbance attenuation versus M .

than that of RPHIC. The smaller value of γ_∞ is, the better performance will be. Therefore, RPHIC outperforms RQHIC.

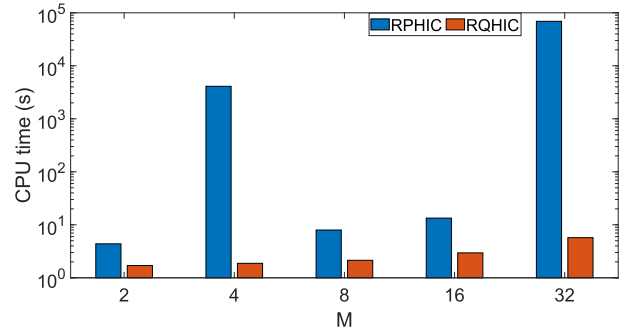


FIGURE 7. CPU time of robust \mathcal{H}_∞ control design versus M .

Figure 7 shows CPU time of RPHIC and RQHIC designs with varying number of vertices M . It is seen that RPHIC spends more time than those of RQHIC. At $M = 4$, CPU time of RPHIC is higher than that of RQHIC since it requires many iterations. Nevertheless, the proposed RPHIC gives lower γ_∞ as illustrated by Figure 6.

Next, we design RPH2C for the active suspension systems. By Theorem 2, state-feedback gains of RPH2C can be computed using Algorithm 2. The initial parameter $\alpha_{2,0}$ is varied and ϵ_2 is selected to be 10^{-5} . The design results are summarized in Table 4. It is observed that our proposed RPH2C yields much lower disturbance attenuation gains than that of RQH2C. However, RPH2C requires more computational time than RHH2C.

We employ Theorem 2 to design state-feedback controller by Algorithm 2. These controller gains and γ_2 are computed

TABLE 9. State feedback gains and γ_2 of robust \mathcal{H}_2 control when varying M .

M	Feedback gain		γ_2
	F_l	F_u	
2	$10^3 \times [0.4955 \ -0.8441 \ 3.9500 \ -3.6743]$	$[-17.8399 \ 29.9111 \ -148.8568 \ 139.5583]$	4.7707
4	$10^3 \times [0.5072 \ -0.7357 \ 3.4456 \ -3.1509]$	$[-14.2089 \ 19.5784 \ -98.4337 \ 90.6545]$	4.7800
8	$[42.8177 \ -129.5914 \ 370.1388 \ -328.3100]$	$10^{-3} \times [0.0750 \ -0.1458 \ 0.6955 \ -0.6609]$	4.7800
16	$[15.3442 \ -81.8154 \ 137.2419 \ -111.8021]$	$[0.0002 \ -0.0002 \ 0.0011 \ -0.0010]$	4.9259
32	$10^3 \times [0.1996 \ -0.3000 \ 1.3171 \ -1.1978]$	$10^{-5} \times [0.0319 \ -0.0474 \ 0.2525 \ -0.2357]$	4.9726

for various M by varying $\alpha_{2,0}$. The results are given in Table 9 and Table 10.

TABLE 10. Computational time of robust \mathcal{H}_2 control design versus M .

M	$\alpha_{2,0}$	α_2	# iter.	CPU time (Sec.)
2	2	1.99999253	3	14.8906
4	2	2.00000035	3	18.5156
8	1.25	0.55165220	55	1.2971×10^{-3}
16	1.75	0.53370718	43	3.4663×10^{-3}
32	1.75	1.74999994	2	42.8281

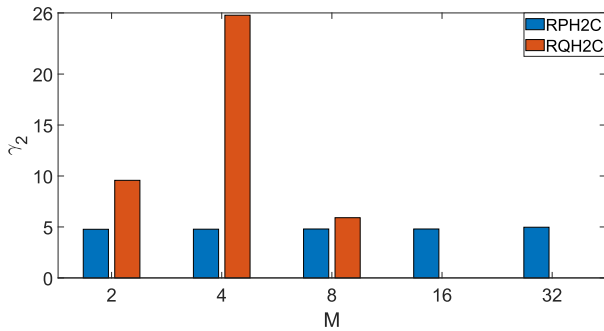


FIGURE 8. Upper bound of \mathcal{H}_2 disturbance attenuation versus M .

Figure 8 shows the value of γ_2 obtained by the proposed RPH2C and the RQH2C when varying the number of vertices M . In general, RPH2C gives a smaller value of γ_2 than RQH2C. The condition of RPH2C design is capable of finding γ_2 for all various value of M , whereas RQH2C design is infeasible at the value of M of 16 and 32. When M increases, γ_2 of RPH2C is increased. In contrast, γ_2 of RQH2C is increased for $M = 4$ and then decreased for $M = 8$. The RQH2C has a drawback due to a common Lyapunov matrix when designing robust control for the entire uncertainty domain. RPH2C gives a smaller value of γ_2 than RQH2C. Thus, it can be concluded that the RPH2C is less conservative than the RQH2C.

CPU time used by both the proposed RPH2C and RQH2C when varying the number of vertices M is shown in Figure 9. It is observed that RPH2C requires more time than those of RQH2C. There is no data of CPU time utilized by RQH2C at M of 16 and 32 since the RQH2C condition is infeasible. RPH2C is still capable of finding γ_2 for $M = 16, 32$. Note that the proposed RPH2C gives significantly lower γ_2 as depicted by Figure 8.

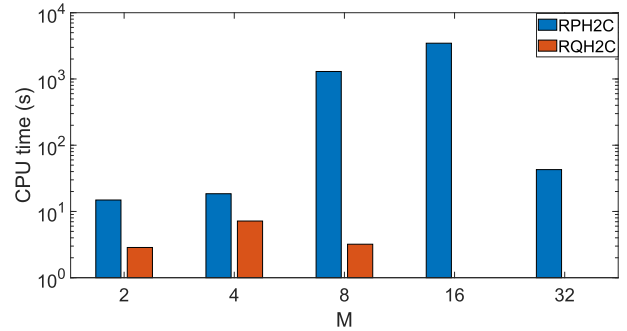


FIGURE 9. CPU time of robust \mathcal{H}_2 control design versus M .

V. CONCLUSION

We present the designs of robust hierarchical \mathcal{H}_∞ and \mathcal{H}_2 control for multi-agent systems subject to polytopic uncertainty and external disturbance. The design criteria are based on the novel sufficient conditions in terms of BMI with specified upper bound of disturbance attenuation. We develop the iterative method of the coordinated optimization to solve sub-problems over LMIs to determine the robust state feedback gains. The numerical results reveal that the novel robust control designs significantly give lower disturbance attenuation than that of the other robust controllers. In addition, the proposed designs guarantee the robust stability of MAS for a larger maximum admissible bound of uncertainty. The ongoing work is to extend the robust design of hierarchical control for heterogeneous polytopic uncertain MAS.

APPENDIX A

Note that $G + G^T > 0$ in (14) guarantees the existence of G^{-1} .

Consider the following parameter-dependent Lyapunov function candidates

$$V_i(x_i(t)) = x_i(t)^T P_i(\delta)x_i(t), \quad (24)$$

where $P_i(\delta) = \sum_{k=1}^M \delta_k P_{ik} > 0, i = 1, \dots, N, k = 1, \dots, M, \delta \in \Theta$. It can prove that $\|G_{\bar{y}_i \bar{d}_i}(s, \delta)\|_\infty < \gamma_\infty, \forall \delta \in \Theta$ of system (12) if the following inequalities hold

$$P_i(\delta)\bar{A}_i(\delta) + \bar{A}_i(\delta)^T P_i(\delta) + \frac{1}{\gamma_\infty^2} P_{ik}(\delta)B_d(\delta)B_d(\delta)^T P_i(\delta) + \bar{C}_i(\delta)^T \bar{C}_i(\delta) < 0, \quad i = 1, \dots, N, k = 1, \dots, M. \quad (25)$$

Therefore, it makes

$$\dot{V}_i(x_i(t)) + \bar{y}_i^T \bar{y}_i - \gamma_\infty^2 \bar{d}_i^T \bar{d}_i < 0, \quad i = 1, \dots, N, \quad k = 1, \dots, M. \quad (26)$$

In fact, the condition (26) can be verified as follows. It follows from (12) that

$$\dot{V}_i(x_i(t)) + \bar{y}_i^T \bar{y}_i - \gamma_\infty^2 \bar{d}_i^T \bar{d}_i = \begin{bmatrix} \bar{x}_i \\ \bar{d}_i \end{bmatrix}^T \Upsilon_\infty \begin{bmatrix} \bar{x}_i \\ \bar{d}_i \end{bmatrix}, \quad i = 1, \dots, N, \quad k = 1, \dots, M,$$

where $\Upsilon_\infty = \begin{bmatrix} \Upsilon_{\infty,11} & P_i(\delta)B_d(\delta) \\ B_d(\delta)^T P_i(\delta) & -\gamma_\infty^2 I \end{bmatrix}$, $\Upsilon_{\infty,11} = \bar{A}_i(\delta)^T P_i(\delta) + P_i(\delta)\bar{A}_i(\delta) + \bar{C}_i(\delta)^T \bar{C}_i(\delta)$. Applying Schur's complement for the above equations, the condition (26) holds if (25) is satisfied. By definition (6), it obtains $\|G_{yd}(s, \delta)\|_\infty < \gamma_\infty^2, \forall \delta \in \Theta$. Subsequently, pre- and post-multiplying $P_i(\delta)^{-1}$ and $P_i(\delta)^{-1}$ to (25), respectively, yields

$$\begin{aligned} &\bar{A}_i(\delta)P_i(\delta)^{-1} + P_i(\delta)^{-1}\bar{A}_i(\delta)^T + \frac{1}{\gamma_\infty^2}B_d(\delta)B_d(\delta)^T \\ &+ P_i(\delta)^{-1}\bar{C}_i(\delta)^T \bar{C}_i(\delta)P_i(\delta)^{-1} < 0, \\ &i = 1, \dots, N, \quad k = 1, \dots, M. \end{aligned} \quad (27)$$

Let $X_i(\delta) = P_i(\delta)^{-1}$, the inequalities (27) are equivalent to

$$\begin{aligned} &\bar{A}_i(\delta)X_i(\delta) + X_i(\delta)\bar{A}_i(\delta)^T + \frac{1}{\gamma_\infty^2}B_d(\delta)B_d(\delta)^T \\ &+ X_i(\delta)\bar{C}_i(\delta)^T \bar{C}_i(\delta)X_i(\delta) < 0, \\ &i = 1, \dots, N, \quad k = 1, \dots, M. \end{aligned} \quad (28)$$

Applying Schur's complement [27] for (28), the following conditions are yielded

$$\begin{bmatrix} \text{sym}(\bar{A}_i(\delta)X_i(\delta)) & (\bar{C}_i(\delta)X_i(\delta))^T & B_d(\delta) \\ \bar{C}_i(\delta)X_i(\delta) & -I_m & 0 \\ B_d(\delta)^T & 0 & -\gamma_\infty^2 I_l \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M. \quad (29)$$

Next, we will show that (29) obtains once the following condition holds

$$\begin{bmatrix} \text{sym}(\bar{A}_{\infty i}(\delta)) & \Xi_{\infty,12} & B_d(\delta) & \bar{C}_{\infty i}(\delta)^T \\ \Xi_{\infty,21} & -\alpha_\infty(G + G^T) & 0_{n \times l} & \alpha_\infty \bar{C}_{\infty i}(\delta) \\ B_d(\delta)^T & 0_{l \times n} & -\gamma_\infty^2 I_l & 0_{l \times m} \\ \bar{C}_{\infty i}(\delta) & \alpha_\infty \bar{C}_{\infty i}(\delta) & 0_{m \times l} & -I_m \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M, \quad (30)$$

where $\bar{A}_{\infty i}(\delta) = A_i(\delta)G$ and $\bar{C}_{\infty i}(\delta) = C_i(\delta)G$, $\Xi_{\infty,12} = X_i(\delta) - G^T + \alpha_\infty \bar{A}_{\infty i}(\delta)$, $\Xi_{\infty,21} = X_i(\delta) - G + \alpha_\infty \bar{A}_{\infty i}(\delta)^T$, and α_∞ is an arbitrarily prescribed number. The condition (30) is PDBMI.

Let $T_\infty = \begin{bmatrix} I & \bar{A}_i(\delta) & 0 & 0 \\ 0 & \bar{C}_i(\delta) & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}$. Pre- and post-multiplying (30) by T_∞ and T_∞^T , respectively, yields (29). On the other

hand, multiplying δ_k to both sides of (14) obtains

$$\begin{bmatrix} \delta_k \text{sym}(A_{\infty i}) & \delta_k \Xi_{12} & \delta_k B_{dk} & \delta_k C_{\infty i}^T \\ \delta_k \Xi_{21} & -\delta_k \alpha_\infty(G + G^T) & 0_{n \times l} & \delta_k \alpha_\infty C_\infty \\ \delta_k B_{dk}^T & 0_{l \times n} & -\gamma_\infty^2 \delta_k I_l & 0_{l \times m} \\ \delta_k C_{\infty i} & \delta_k \alpha_\infty C_{\infty i} & 0_{m \times l} & -\delta_k I_{m \times m} \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M, \quad (31)$$

where $A_{\infty i} = A_k G - B_k(G_1 + \lambda_i G_2)$, $C_{\infty i} = C_k G - D_k(G_1 + \lambda_i G_2)$, $\Xi_{12} = X_{ik} - G^T + \alpha_\infty A_{\infty i}$, and $\Xi_{21} = X_{ik} - G + \alpha_\infty A_{\infty i}^T$. Then summing up side by side of (31) yields (30) according to (2) and (3).

APPENDIX B

Note $G + G^T > 0$ in (18a) ensures the existence of G^{-1} . Recall that $\bar{A}_i(\delta) = A(\delta) - B(\delta)(F_l + \lambda_i F_u)$, $\bar{C}_i(\delta) = C(\delta) - D(\delta)(F_l + \lambda_i F_u)$. As shown in [3], the system (12) guarantee $\|G_{\bar{y}_i \bar{d}_i}(s, \delta)\|_2 < \gamma^2, \forall \delta \in \Theta$ if following inequalities hold

$$\begin{aligned} &\bar{A}_i(\delta)^T P_{ik} + P_i(\delta)\bar{A}_i(\delta) + \bar{C}_i(\delta)^T \bar{C}_i(\delta) < 0, \\ &\forall i = 1, \dots, N, \quad k = 1, \dots, M. \end{aligned} \quad (32a)$$

$$\text{trace}(B_d(\delta)^T P_i(\delta)B_d(\delta)) < \gamma_2^2, \quad (32b)$$

where $P_i(\delta) = P_i(\delta)^T = \sum_{k=1}^M \delta_k P_{ik} > 0, i = 1, \dots, N, \delta \in \Theta$ given in (2) is the matrix of parameter-dependent Lyapunov function candidates

$$V_i(x_i(t)) = x_i(t)^T P_i(\delta)x_i(t). \quad (33)$$

Pre- and post-multiplication of (32a) with $P_i(\delta)^{-1}$ gets

$$\begin{aligned} &P_i(\delta)^{-1}\bar{A}_i(\delta)^T + P_i(\delta)^{-1}\bar{C}_i(\delta)^T \bar{C}_i(\delta)P_i(\delta)^{-1} \\ &+ \bar{A}_i(\delta)P_i(\delta)^{-1} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M. \end{aligned} \quad (34)$$

Let $X_i(\delta) = \gamma_2^2 P_i(\delta)^{-1}$. Then (34) is equivalent to

$$\begin{aligned} &\frac{1}{\gamma_2^2}X_i(\delta)\bar{A}_i(\delta)^T + \frac{1}{\gamma_2^2}X_i(\delta)\bar{C}_i(\delta)^T \bar{C}_i(\delta)\frac{1}{\gamma_2^2}X_i(\delta) \\ &+ \bar{A}_i(\delta)\frac{1}{\gamma_2^2}X_i(\delta) < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M. \end{aligned} \quad (35)$$

Applying Schur's complement [27] for (35) obtains

$$\begin{bmatrix} \bar{A}_i(\delta)X_i(\delta) + X_i(\delta)\bar{A}_i(\delta)^T & (\bar{C}_i(\delta)X_i(\delta))^T \\ \bar{C}_i(\delta)X_i(\delta) & -\gamma_2^2 I_m \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M. \quad (36)$$

Subsequently, we show the condition (36) holds if the following inequalities are satisfied.

$$\begin{bmatrix} \text{sym}(A_{2i}(\delta)) & \Psi_{12} & C_{2i}(\delta)^T \\ \Psi_{21} & -\alpha_2(G + G^T) & \alpha_2 C_{2i}(\delta)^T \\ C_{2i}(\delta) & \alpha_2 C_{2i}(\delta) & -\gamma_2^2 I_m \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, \quad k = 1, \dots, M, \quad (37)$$

where $A_{2i}(\delta) = A_i(\delta)G$, $C_{2i}(\delta) = C_i(\delta)G$, $\Psi_{12} = X_i(\delta) - G^T + \alpha_2 A_{2i}(\delta)$, $\Psi_{21} X_i(\delta) - G + \alpha_2 A_{2i}(\delta)^T$, and α_2 is an arbitrarily prescribed number. The condition (37) is called PDBMI.

Let $T_2 = \begin{bmatrix} I & \tilde{A}_i(\delta) & 0 \\ 0 & \tilde{C}_i(\delta) & I \end{bmatrix}$. Pre- and post- multiplying (37) by T_2 and T_2^T , respectively, we obtain (36).

Multiplying (18a) by δ_k obtains

$$\begin{bmatrix} \text{sym}(A_{2i\delta}) & \delta_k(X_{ik} - G^T + \alpha_2 A_{2i}) & \delta_k C_{2i}^T \\ \delta_k(X_{ik} - G + \alpha_2 A_{2i}^T) & -\delta_k \alpha_2(G + G^T) & \delta_k \alpha_2 C_{2i}^T \\ \delta_k C_{2i} & \delta_k \alpha_2 C_{2i} & -\delta_k \gamma_2^2 I_m \end{bmatrix} < 0, \quad \forall i = 1, \dots, N, k = 1, \dots, M, \quad (38)$$

where $A_{2i\delta} = \delta_k(A_k G - B_k(G_1 + \lambda_i G_2))$ and $C_{2i} = C_k G - D_k(G_1 + \lambda_i G_2)$. Then summing up side by side of (38) gets (37) because of the property given in (2) and (3).

Suppose there exist $E_i(\delta) = \sum_{i=1}^N \delta_k E_{ik} \succ 0$, $\forall i = 1, \dots, N$, $k = 1, \dots, M$, such that

$$B_d(\delta)^T P_i(\delta) B_d(\delta) < E_i(\delta), \quad (39)$$

Substituting $P_i(\delta) = \gamma_2^2 X_i(\delta)^{-1}$ into (39) obtains

$$\gamma_2^2 B_d(\delta)^T X_i(\delta)^{-1}(\delta) B_d(\delta) < E_i(\delta). \quad (40)$$

Applying Schur's complement [27] for (40) gets

$$\begin{bmatrix} X_i(\delta) & B_d(\delta) \\ B_d(\delta)^T & E_i(\delta) \end{bmatrix} \succ 0, \quad \forall i = 1, \dots, N, k = 1, \dots, M, \quad (41)$$

which are satisfied if (18b) hold from numerical view point.

A necessary and sufficient condition to ensure $\|G_{\bar{y}\bar{d}}(s, \delta)\|_2^2 \leq \gamma_2^2$ is that $\sum_{i=1}^N \|G_{\bar{y}_i \bar{d}_i}(s, \delta)\|_2^2 \leq \gamma_2^2$ holds. It follows from (32) that this condition is held if (36), (41) hold, and the following condition is satisfied,

$$\sum_{i=1}^N \text{trace}(E_i(\delta)) < 1, \quad \forall k = 1, \dots, M. \quad (42)$$

Substituting $\sum_{k=1}^M \delta_k = 1$ into the right side of (42) yields

$$\sum_{k=1}^M \delta_k \left(\sum_{i=1}^N \text{trace}(E_{ik}) \right) < \sum_{k=1}^M \delta_k. \quad (43)$$

Then isolate all term involving δ_k in (43). It is sufficient to guarantee (43) by checking the sign of the finite number of coefficients $\delta_k \left(\sum_{i=1}^N \text{trace}(E_{ik}) - 1 \right) < 0$, $k = 1, \dots, M$. We eventually assure (43) if (18c) is satisfied.

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REFERENCES

- [1] D. H. Nguyen and S. Hara, "Hierarchical decentralized stabilization for networked dynamical systems by LQR selective pole shift," in *Proc. 19th World Congr. (IFAC)*, Cape Town, South Africa, Aug. 2014, pp. 5778–5783.
- [2] W. Chen, J. Yang, L. Guo, and S. Li, "Disturbance-observer-based control and related methods—An overview," *IEEE Trans. Ind. Electron.*, vol. 63, no. 2, pp. 1083–1095, Feb. 2016.
- [3] Y. Ebihara, D. Peaucelle, and D. Arzelier, *S-Variable Approach to LMI-Based Robust Control*. London, U.K.: Springer, 2015.
- [4] Y. Ebihara and T. Hagiwara, "A dilated LMI approach to continuous-time gain-scheduled controller synthesis with parameter-dependent Lyapunov variables," *Trans. Soc. Instrum. Control Eng.*, vol. 39, no. 8, pp. 734–740, 2003.
- [5] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Trans. Autom. Control*, vol. 41, no. 4, pp. 436–442, Mar. 1996.
- [6] G. I. Bara, "Robust analysis and control of parameter-dependent uncertain descriptor systems," *Syst. Control Lett.*, vol. 60, no. 5, pp. 356–364, May 2011.
- [7] M. Farhood, "LPV control of nonstationary systems: A parameter-dependent Lyapunov approach," *IEEE Trans. Autom. Control*, vol. 57, no. 1, pp. 209–215, Jan. 2012.
- [8] D. Peaucelle and D. Arzelier, "Robust performance analysis with LMI-based methods for real parametric uncertainty via parameter-dependent Lyapunov functions," *IEEE Trans. Autom. Control*, vol. 46, no. 4, pp. 624–630, Apr. 2001.
- [9] V. F. Montagner and P. L. D. Peres, "Robust stability and performance of linear time-varying systems in polytopic domains," *Int. J. Control*, vol. 77, no. 15, pp. 1343–1352, Oct. 2004.
- [10] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "Polynomially parameter-dependent Lyapunov functions for robust stability of polytopic systems: An LMI approach," *IEEE Trans. Autom. Control*, vol. 50, no. 3, pp. 365–370, Mar. 2005.
- [11] R. C. L. F. Oliveira and P. L. D. Peres, "Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1334–1340, Jul. 2007.
- [12] P. Seiler, A. Topcu, A. Packard, and G. Balas, "Parameter-dependent Lyapunov functions for linear systems with constant uncertainties," *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2410–2416, Oct. 2009.
- [13] L. Gao, X. Zhu, W. Chen, and H. Zhang, "Leader-following consensus of linear multiagent systems with state observer under switching topologies," *Math. Problems Eng.*, vol. 2013, pp. 1–12, Jan. 2013.
- [14] T. Van Pham, D. H. Nguyen, and D. Banjerdpongchai, "Design of robust hierarchical control for homogeneous linear multi-agent systems with parametric uncertainty and external disturbance," *Int. J. Control*, pp. 1–13, Nov. 2021, doi: 10.1080/00207179.2021.1992671.
- [15] W. Huang, J. Zeng, and H. Sun, "Robust consensus for linear multi-agent systems with mixed uncertainties," *Syst. Control Lett.*, vol. 76, pp. 56–65, Feb. 2015.
- [16] T. Jennawasin, M. Kawanishi, and T. Narikiyo, "Iterative LMI approach to robust state-feedback control of polynomial systems with bounded actuators," *Int. J. Control, Automat. Syst.*, vol. 17, no. 4, pp. 847–856, 2019.
- [17] D. H. Nguyen, "A sub-optimal consensus design for multi-agent systems based on hierarchical LQR," *Automatica*, vol. 55, pp. 88–94, May 2015.
- [18] D. H. Nguyen, T. Narikiyo, M. Kawanishi, and S. Hara, "Hierarchical decentralized robust optimal design for homogeneous linear multi-agent systems," 2016, *arXiv:1607.01848*.
- [19] Z. Li, Z. Duan, L. Xie, and X. Liu, "Distributed robust control of linear multi-agent systems with parameter uncertainties," *Int. J. Control*, vol. 85, no. 8, pp. 1039–1050, 2012.
- [20] Z. Li, Z. Duan, and G. Chen, "On H_∞ and H_2 performance regions of multi-agent systems," *Automatica*, vol. 47, no. 4, pp. 797–803, 2011.
- [21] L. E. Ghaoui and V. Balakrishnan, "Synthesis of fixed-structure controllers via numerical optimization," in *Proc. IEEE 33rd Conf. Decis. Control*, Lake Buena Vista, FL, USA, Dec. 1994, pp. 2678–2683.

- [22] T. Nampradit and D. Banjerdpongchai, "Iterative LMI approach to design robust state-feedback controllers for Lur'e systems with time-invariant delays," *Int. J. Control, Autom. Syst.*, vol. 13, no. 5, pp. 1086–1096, Oct. 2015.
- [23] D. L. Guo, H. Y. Hu, and J. Q. Yi, "Neural network control for a semi-active vehicle suspension with a magnetorheological damper," *J. Vibrat. Control*, vol. 10, no. 3, pp. 461–471, Mar. 2004.
- [24] J. Lei, Y.-L. Li, and X.-K. Guo, "Neural network-based supervisory control for quarter-car suspension with nonlinearity and uncertainty," *Integr. Ferroelectr.*, vol. 160, no. 1, pp. 106–126, Feb. 2015.
- [25] I. J. Fialho and G. J. Balas, "Design of nonlinear controllers for active vehicle suspensions using parameter-varying control synthesis," *Vehicle Syst. Dyn.*, vol. 33, no. 5, pp. 351–370, May 2000.
- [26] H. Du, J. Lam, and K. Y. Sze, "Design of non-fragile H_∞ controller for active vehicle suspensions," *J. Vibrat. Control*, vol. 11, no. 2, pp. 225–243, Feb. 2005.
- [27] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 1994.



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