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Point Cloud Registration Algorithm Based on Laplace Mixture Model

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ABSTRACT Registering point clouds quickly and accurately has always been a challenging task. A lot of research based on Gaussian mixture model is widely used in recent years. However, few people use other models for point cloud matching. Therefore, this paper proposes a point cloud registration algorithm based on the Laplace mixture model. In this paper, sampling variance is used to replace the variance of the likelihood estimation to successfully overcome the nonlinear problem. In addition, the Laplace model has strong robustness, which is very suitable for point cloud matching of 3D laser scanning. In the experiment, compared with several other algorithms, proposed method quickly and accurately registers point clouds.

INDEX TERMS Laplace model, point cloud registration, rigid, affine.

I. INTRODUCTION

Point registration technology has been highly concerned in many fields, such as computer vision, pattern recognition, mobile robotics, machine learning, medical imaging and geographic information system [1]–[8]. At present, the point cloud registration methods are mainly focused on two types: 1) the improved methods based on the Iterative Close Point (ICP) algorithm [9]; 2) the probability-based approaches. In this paper, our research mainly concentrates on probability-based methods.

In the first category, ICP algorithm is a classic example that uses directly point cloud coordinates and extracts descriptors to assist matching and registration. Due to ICP algorithm is only suitable for rigid registration, many researchers have improved ICP algorithm in order to achieve more complex point cloud registration. Ying *et al.* put forward the Scale-ICP algorithm in [10], which introduces an affine factor, for a total of 7 undetermined variables, to register point clouds on different scales. The Scale-ICP algorithm has shown fast convergence in experiments, but it can only be applied to register point clouds transformed by a single scaling factor. In [11], Ho. J *et al.* proposed an algebraic approach to affine registration of point sets, however, it is used to register point clouds that are affine transformed in three fixed-directions. Wang *et al.* [12] proposed a multidirectional affine registration (MDAR) algorithm, it solves the same problem in [11]. Du *et al.* proposed a scaling ICP algorithm [13] and improved upon scaling ICP algorithm in [14]. However, the affine ICP algorithm [14] often shows poor registration results on complex situation, due to the limitations of the convergence domains. Yang *et al.* [15] put forward a method, Go-ICP (the Globally Optimal ICP), which is based on the well-established branch-and-bound (BnB) theory. These ICP-based algorithms show good registration results on some simple point registration. Feng *et al.* proposed the PCR-GW algorithm [16], the algorithm is based on the gray wolf optimizer to register point clouds, and it improves the registration speed and registration accuracy. Shu *et al.* proposed the Whitening-ICP algorithm [17], the registration error of it was small and it reduced the computational complexity of point cloud affine registration. Nevertheless, it is difficult for them to register more complex point clouds, such as arbitrary affine deformations and complex non-rigid point clouds.

In the second category, this type of registration methods considers the alignment of two point sets as a probability density estimation problem. The feature of these methods is the widespread use of Gaussian Mixture Models (GMMs). Among these methods, the representative and widely used methods are GMMREG algorithm [18] and coherent point drift (CPD) algorithm [19]. With the development of GMM-based registration algorithms, many algorithms of this type have been gradually proposed [20]–[25]. These algorithms avoid the iteration of the ICP and can be

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applied in rigid and non-rigid point set registration, but the probability estimation algorithm usually relies on the EM (Expectation maximization algorithm), which means they consume more time. In addition, they also have exposed their limitations when there are wrong correspondences in real-world tasks, though they perform reasonably well when the spatial transformation is not complex.

In this paper, we proposed a novel point cloud registration method which is based on Laplace Model. It is known that Laplace model is not widely used in engineering but the field of economics [26]. In addition, the error metric in the proposed method strictly is minimizing the modulus of the closest point residual vector, unlike calculating the L2 norm in GMM, which makes the point cloud less prone to serious deformation in the process of registration. Extensive experiments on simulation and measurement data sets reveal the superiority of the proposed method over state-of-the-art competitors.

II. POINT CLOUD REGISTRATION METHOD BASED ON LAPLACE MIXTURE MODEL

A. LAPLACE MIXTURE MODEL

The density function of a multivariate Laplace distribution can be described as

$$f(x) = \frac{2}{(2\pi)^{D/2}|\Sigma|^{1/2}}\left(\frac{x^T\Sigma^{-1}x}{2}\right)^{\nu/2} K_\nu\left(\sqrt{2x^T\Sigma^{-1}x}\right) \tag{1}$$

where $x \in R^D$, and $\nu = (2-D)/2$; K_ν is the modified Bessel function, $\sum$ is a covariance matrix. Based on the kernel density estimation strategy, the Laplace mixture model (2) can be derived by (1).

$$f_L(x) = \sum_{k=1}^{K}\frac{2\alpha_k}{(2\pi)^{D/2}|\Sigma_k|^{1/2}}\left(\frac{\|x-\mu_k\|^2_{\Sigma_k^{-1}}}{2}\right)^{\nu/2} \times K_\nu(\sqrt{2}\,\|x-\mu_k\|_{\Sigma_k^{-1}}) \tag{2}$$

where $\|\cdot\|^2_{\Sigma_k^{-1}} = (\cdot)^T\Sigma_k^{-1}(\cdot)$ and $\sum_{k=1}^{K}\alpha_k = 1, \alpha_k > 0$, K is order in the model. However, in calculation, the Bessel function is a complex transcendental function. In addition, $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\exp(-x)$, in non-parametric statistics, low-dimensional kernel functions are often spheroidized into high-dimensional kernel functions. So, in this paper, we use a degradation model of (2) whose $\nu = \frac{1}{2}$.

$$f_L(x) = \frac{1}{C(D)}\sum_{k=1}^{K}\frac{\alpha_k}{\sigma^D}\exp\left(-\frac{\|x-\mu_k\|}{\sigma}\right) \tag{3}$$

where σ is the variance, μ_k is the mean of the k-th model and $C(D)$ is a Jacobian coefficient. The Jacobian coefficients for (3) can be summarized as shown in Table 1.

TABLE 1. Jacobi an coefficient.

D	1	2	3	4	5	7
$C(D)$	2	2π	8π	$12\pi^2$	$64\pi^2$	$768\pi^3$

B. PROBABILISTIC MODEL OF REGISTRATION

We consider that the source point cloud and the target point cloud $X_{D\times N} = (x_1, x_2, \ldots, x_N)$ $Y_{D\times M} = (y_1, y_2, \ldots, y_M)$ need to be registered. That is, M, N are the number of points in the point clouds, respectively. Let $T(Y,\theta)$ be the transform operator for Y where θ represents all parameters. If the points in the point cloud $T(Y,\theta)$ are used as the mean in the model (3), then a probability density function of the error can be described as

$$p_L(x_i) = \frac{1}{C(D)}\sum_{k=1}^{M}\frac{\alpha_k}{\sigma^D}\exp\left(-\frac{\|x_i - T(y_k,\theta)\|}{\sigma}\right) \tag{4}$$

Furthermore, a log-likelihood function of the error probability can be obtained.

$$l(\theta,\sigma) = \sum_{i=1}^{N}\left\{\ln\sum_{k=1}^{M}\frac{\alpha_{i,k}}{\sigma^D}\exp\left(-\frac{\|x_i - T(y_k,\theta)\|}{\sigma}\right)\right\} \tag{5}$$

where constant coefficients are ignored. Clearly, accurate θ maximizes $l(\theta)$. According to Jensen inequality,

$$l(\theta,\sigma) \geq -ND\ln\sigma - \sum_{i=1}^{N}\sum_{k=1}^{M}\frac{\alpha_{i,k}}{\sigma}\|x_i - T(y_k,\theta)\| \tag{6}$$

Let the right side of the inequality be $Q(\theta,\sigma)$. Clearly,

$$Q(\theta,\sigma) = -ND\ln\sigma - \sum_{i=1}^{N}\sum_{k=1}^{M}\frac{\alpha_{i,k}}{\sigma}\|x_i - T(y_k,\theta)\| \tag{7}$$

Based on the Minorize-Maximization algorithm, the maximized $Q(\theta,\sigma)$ increases $l(\theta,\sigma)$. In the nonlinear optimization process, iterative operations have to be used. Because the Laplace mixture model is essentially a stratified sampling model, then

$$\alpha_{i,k} = \frac{\exp\left(-\frac{1}{\sigma}\|x_i - T(y_k,\theta)\|\right)}{\sum_{j=1}^{M}\exp\left(-\frac{1}{\sigma}\left\|x_i - T(y_j,\theta)\right\|\right)} \tag{8}$$

Due to the lack of prior information, based on the Bayes hypothesis, the no information prior distribution of $\alpha_{i,k}$ is $\alpha_{i,k} = \frac{1}{M}$.

III. AFFINE REGISTRATION METHOD

The point cloud is inevitably affine transformed due to factors such as shape differences or thermal expansion of the object being scanned. That is,

$$T(y_k,\theta) = Ay_k + t \tag{9}$$

where A is affine matrix, t is translation vector.

Then,

$$Q(A,t,\sigma) = -ND\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{N}\sum_{k=1}^{M}\alpha_{i,k}\left\|x_i - Ay_k - t\right\| \tag{10}$$

The Q function maximization problem can be divided into two sub-problems:

$$\begin{cases} \sigma = \arg\min\limits_{\sigma} ND\ln\sigma + \frac{1}{\sigma}\sum_{i=1}^{N}\sum_{k=1}^{M}\alpha_{i,k}\left\|x_i - Ay_k - t\right\| \\ (A,t) = \arg\min\limits_{(A,t)} \sum_{i=1}^{N}\sum_{k=1}^{M}\alpha_{i,k}\left\|x_i - Ay_k - t\right\| \end{cases} \tag{11}$$

Taking partial derivative of Q with σ and equate it to zero,

$$\sigma = \frac{1}{ND}\sum_{i=1}^{N}\sum_{k=1}^{M}\alpha_{i,k}\left\|x_i - Ay_k - t\right\| \tag{12}$$

According to the above formula, the first optimization problem can be solved. However, the above formula calculation is difficult to calculate quickly. Fortunately, the sample variance can be approximated to the variance. Hence

$$\sigma^2 \approx \hat{\sigma}^2 = \frac{1}{ND}\sum_{i=1}^{N}\sum_{k=1}^{M}\alpha_{i,k}\left\|x_i - Ay_k - t\right\|^2 \tag{13}$$

On the other hand, the second optimization problem is equivalent to

$$(A,t) = \arg\min_{(A,t)} \sigma^2 \tag{14}$$

Based on the relationship between sample variance and variance, we can get

$$(A,t) = \arg\min_{(A,t)} \hat{\sigma}^2 \tag{15}$$

Taking partial derivative of sample variance $\hat{\sigma}^2$ with respect to t and equate it to zero, we can obtain:

$$t = \frac{1}{N}\left(XG^T I_M^T - AYGI_N^T\right) \tag{16}$$

where $\left(g_{i,k}\right) = \left(\alpha_{i,k}\right)$, and I_N, I_M $\left(I_N \in R^N, I_M \in R^M\right)$ are column vectors with all elements. For convenience, let $\overline{X} = \frac{1}{N}XG^T I_M^T$, $\overline{Y} = \frac{1}{N}YGI_N^T$. Then,

$$t = \overline{X} - A\overline{Y} \tag{17}$$

By (15) and (16), taking partial derivative of $\hat{\sigma}^2$ with respect to A and equate it to zero. Then,

$$A = \left(X - \overline{X}I_N^T\right)G^T\left(Y - \overline{Y}I_M^T\right)^T \times \left[\left(Y - \overline{Y}I_M^T\right)diag\left(I_N^T G\right)\left(Y - \overline{Y}I_M^T\right)^T\right]^{-1} \tag{18}$$

Summarizing the above derivation, we can generalize the entire registration algorithm to the LMM-Affine algorithm. And, the algorithm flow is shown in the following table.

LMM-Affine Algorithm

Step 1: Target point cloud X and source point cloud Y, set iter_max=100, allowable error $\varepsilon = 10^{-5}$;

Step 2: Initialize the affine matrix and the translation vector$A = I, t = 0, \alpha_{i,k} = \frac{1}{M}$ and calculate the sample variance $\widehat{\sigma}^2$ by (13);

Step 3: Update t and A by (17) and (18) respectively, calculate $\widehat{Y} = AY + t$ and update $\alpha_{i,k}$ by (8);

Step 4: Update the sample variance $\widehat{\sigma}^2$ by (13);

Step 5: If $\|\Delta(\theta,\sigma)\|^2 > \varepsilon$, set iter = iter +1; if iter>iter_max, set iter=1and go to step 3; until $\|\Delta(\theta,\sigma)\|^2 < \varepsilon$, the iteration ends.

IV. REGISTRATION TEST

To prove the effectiveness of the algorithm proposed in this paper, bunny, face and elephant 3D point cloud data provided by Stanford University were used for registration verification. The simulation was based on MATLAB 2016a, whose environment was configured for a 2.8 GHz CPU with 4 GB RAM.

A. AFFINE REGISTRATION TEST

To test the performance of LMM algorithm, the point cloud is affine transformed. Both Scale-ICP and GO-ICP are not suitable for registering point clouds that are affine transformed, so they are not compared here. We compared the following three methods Affine-ICP [14], MDAR [12], PCR-GW [16], Whitening-ICP [17] and CPD [19]. The registration result is shown in Fig. 1 and Table 2.

As can be seen from Fig.1, Affine-ICP algorithm has poor registration results, due to the limitations of the convergence domain. MDAR is just a multi-directional affine algorithm, it allows the point cloud to expand and contract in the three directions of X, Y, and Z. Therefore, its effect is not stable during arbitrary affine registration, which depends on the degree of affine deformation. From Fig.1, we also can see that CPD, Whitening-ICP and LMM can register point clouds that are affine transformed. However, it can be seen from Table 2 that CPD demonstrates almost the same accuracy effect to LMM in terms of RMSE, but it takes significantly more time to perform. This means that during the same time period, the RMSE of LMM can exceed that of CPD. As for Whitening-ICP and PCR-GW, they take less time to perform than LMM, but they perform worse in terms of RMSE. Especially in many engineering applications, registration must achieve high accuracy in a short time. Therefore, LMM algorithm provides more practical value in engineering.

B. EXPERIMENTAL TEST

In this section, some experimental data was used to test the algorithms. we use the portable laser scanner (HANDYSCAN 700TM portable laser scanner) that is shown in Fig. 2 to collect data from objects. We scanned each actual object

TABLE 2. Affine registration results on 50 trials for three point clouds.

Metric	POINT CLOUD	Affine-ICP	MDAR	PCR-GW	Whitening-ICP	CPD	LMM
	Bunny	0.8420	1.6298	0.1568	0.81	$<10^{-8}$	$<10^{-8}$
AveError (mm)	Elephant	124.6	538.5	0.0393	$<10^{-8}$	$<10^{-8}$	$<10^{-8}$
	Face	0.7425	1.1957	0.1055	$<10^{-8}$	$<10^{-8}$	$<10^{-8}$
	Bunny	35.7	13.3	6.68	20.3	163.2	**25.1**
AveTime (Sec)	Elephant	1.4	8.8	2.21	13.4	138.1	**27.1**
	Face	0.2	0.3	0.19	0.7	1.2	**0.8**

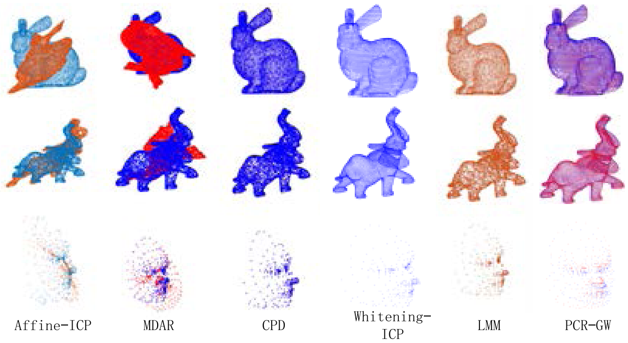

FIGURE 1. Affine registration results of several algorithms.

FIGURE 2. Scanning process and HANDYSCAN 700TM portable laser scanner.

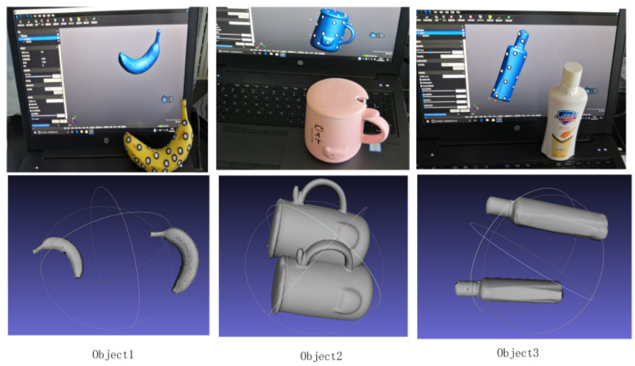

FIGURE 3. Scanned objects and software systems.

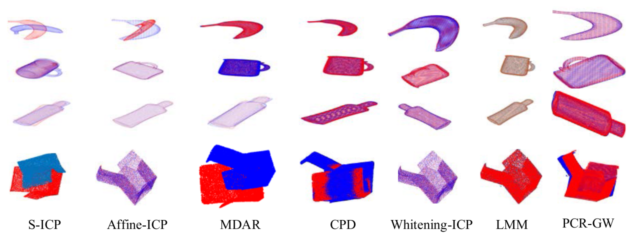

FIGURE 4. Experimental registration results of several algorithms.

twice to get two sets of point clouds. Considering the existence of the ground truth affine transformation, we first label the object and then the object can be automatically located by the scanner in the scanning process, so that the interference of the ground and the background can be ignored. The scanned point clouds are displayed on Meshlab, as shown in Fig.3.

As shown in Fig. 3, the scanned objects are in the first row and the point clouds reconstructed by software are in the second row, respectively. In fact, because the objects being scanned may be reflective, there are some areas that cannot be scanned. Therefore, on the whole, the collected point cloud distribution is relatively uniform and matching these point clouds is a great challenge to these algorithms. The registration results of several algorithms are shown in Fig.4.

Fig. 4 shows that the CPD algorithm, the Whitening-ICP algorithm and the LMM algorithm have satisfactory results; however, the results of the Scale-ICP algorithm are not satisfactory. In fact, the initial registration algorithm in the Scale-ICP algorithm always has an ambiguity in terms of direction. Affine-ICP algorithm often has poor registration results due to the limitations of the convergence domain. MDAR has good registration effect in this section because these actual objects have not undergone significant deformation during scanning. The LMM algorithm performs better than PCR-GW and Whitening-ICP in terms of RMSE, which is also shown in Table 3. The LMM algorithm produces excellent registration results and consumes less time compared to CPD algorithm. This means that during the same time period, the RMSE of LMM can exceed that of CPD. Especially in many engineering applications, registration must achieve

TABLE 3. Affine registration results on 50 trials for three point clouds.

Metric	Point cloud	Scale-ICP	Affine-ICP	PCR-GW	MDAR	CPD	White-Affine	LMM
Ave Error (mm)	banana	17.9895	18.0051	1.6	0.4394	0.3964	0.4511	**0.3078**
	cup	4.4861	0.3102	1.64	0.8858	0.3867	0.4188	**0.2863**
	bottle	4.5913	0.7682	2.18	1.3873	0.4451	0.4031	**0.3737**
	box	80	0.5993	3.95	21	11.74	0.5571	**0.794**
Ave Time (Sec)	banana	11.2	294	1.26	9.7	1303	34.32	**69.0**
	cup	28.4	68.4	2.26	26.5	1861	412.55	**592.1**
	bottle	14.5	129.1	1.51	9.8	3072	101.05	**384.3**
	box	45.44	147.86	10.1	21.43	12	147.86	**115.12**

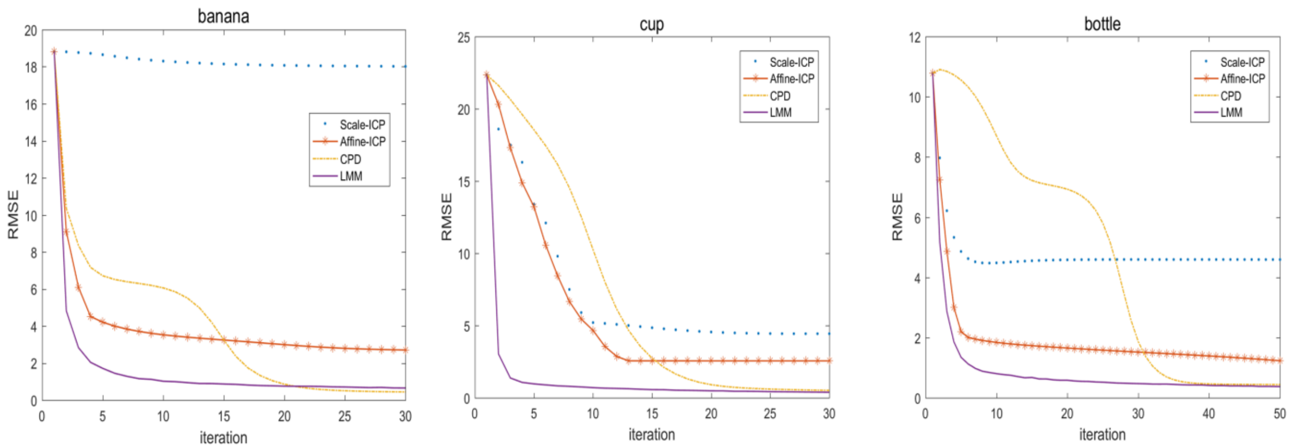

FIGURE 5. Several algorithm iterative curves.

high accuracy in a short time. Therefore, LMM algorithm provides more practical value in engineering.

Since different simulation platforms will produce different time consumption, here are the convergence curves of several iterative algorithms in Fig.5. As can be seen from Fig. 5, LMM algorithm displays excellent convergence obviously. It also shows that the LMM algorithm based on Laplace kernel is easier to converge than the CPD based on Gauss kernel.

V. CONCLUSION

In this paper, a point cloud registration algorithm based on Laplace mixture model is proposed. Generally speaking, it is more difficult to optimize this model than a Gaussian mixture model. However, the derived nonlinear equations are solved by a logarithm inequality. In addition, the calculation structure of Laplace model is simple, and the uniform sampling strategy is quite successful in reducing the computational complexity, which makes the algorithm very efficient. In the test, LMM shows better efficiency and convergence than the other algorithms. The algorithm performance of LMM proves to be more valuable for the practical application.

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