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Finite-Time Sliding Mode Control for Uncertain Neutral Systems With Time Delays

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ABSTRACT The finite-time stability (FTS) problem of uncertain neutral time-delay systems via a sliding mode control (SMC) approach is discussed in this paper. First, we construct a suitable sliding mode surface and an SMC law, which can guarantee the system states can reach the sliding mode surface in a finite time and maintain the sliding mode. Then, through the Lyapunov stability theory and the inequality techniques, the finite time stability of the closed-loop system during reaching phase and sliding mode phase is studied, a set of sufficient conditions which ensure the system to be finite-time stability is developed. Finally, a numerical simulation example is given to illustrate the effectiveness of the results.

INDEX TERMS Finite-time stability, sliding mode control, uncertain, neutral time-delay systems.

I. INTRODUCTION

SMC is an effective nonlinear robust control method, which has strong robustness to resist parameter uncertainty and external disturbance of dynamical systems [1], [2]. At the same time, it has the advantage of excellent transient response [3], [4]. Due to these characteristics, SMC is widely used in missile guidance systems [5], motor control systems [6] and other industrial automation fields, and it becomes a hot spots which attracted many scholars in interest and importance. For example, for a second-order nonlinear dynamic system, a new nonlinear sliding mode controller is presented to resist parameter uncertainty and external disturbance [7]. The observer-based SMC problem of phase-type semi-Markovian jump systems was discussed in [4]. In [8], the authors investigated the issue of SMC with adaptive neural networks for a class of nonlinear uncertain systems. A novel asynchronous sliding mode control scheme is proposed in [9], which guarantees the desired finite-time boundedness of Markovian jump systems with sensor and actuator faulty signals. In fact, SMC is used on a variety of systems, such as Markovian jump systems [4], [10], time-delay systems [11], [12], and stochastic systems [13], [14].

For a long time, the investigators focused on Lyapunov asymptotic stability(LAS); however, this theory also has its limitations. The most important point is that there is generally no constraint on the time required for the system to reach steady state, which is not allowed in some time-critical systems, such as robot dynamic stability and communication networks. Based on this issue, Dorato proposed the finite time theory in 1961, since then, many scholars have begun to study this theory [15], [16]. FTS means that the weighted state of a system does not exceed a predetermined threshold during a finite time interval when the initial states of the system are norm-bounded [17], [18]. When there are external disturbances in the system, finite-time boundedness(FTB) will replace FTS. FTS focuses on the transient performance of the system, so it has special application scenarios and has attracted remarkable attention of researchers. Recently, two new finite-time convergence criterions are proposed based on the property of the second order differential equation [19]. For a class of discrete-time nonlinear systems, new control procedures are proposed, which can satisfy the finite-time stabilization [20]. In [21], the authors put forward a finite-time SMC law to satisfy FTB and H_∞ performance requirements. In addition, FTS is also used in fractional order systems [22], [23] and neural networks [24], [25].

Different from the common time-delay systems, the neutral system is a special kind of dynamic system, which is often used in the research of aircraft engine control, immune response, electrodynamics processes and so on [26]–[28]. Neutral systems contain delays in its state and in the derivatives of its state [29]. The above characteristics lead to more complex dynamic behavior of this kind of system, and make this kind of system have better universality than the general time-delay systems [30], [31]. In [32], for a class of

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uncertain neutral delay systems with mismatched uncertainties, the author proposed a sliding mode control law to guarantee the asymptotic stability of closed-loop systems. The problem of exponential H_∞ output tracking control for a class of switched neutral system with time-varying delay was addressed in [33]. The results of robust stability for a class of uncertain neutral system with time-varying delay can be found in [34].

To date, there have been many studies on finite-time sliding mode control, and many scholars have extended this method to various control systems [9], [21], [35]. However, to our best knowledge, there are still few results on FTS/FTB of uncertain neutral systems. Similar uncertain neutral time delay systems with SMC were studied in [32], [36], but these papers focused on the asymptotic stability of the system rather than FTS. FTS for uncertain systems over reaching phase and sliding motion phase were discussed in [21] and [38]; nevertheless, the system in these papers is relatively simple. Different from these literatures, the neutral time-delay systems investigated in this paper have better universality, and the results can better reflect the quantitative relationship between the FTS conditions and the system parameters. As mentioned above, neutral systems have a wide range of applications, FTS based on sliding mode control can guarantee the transient performance of such systems. In addition, the results of neutral time-delay systems can be easily extended to standard time-delay systems. Therefore, the study of this problem is of practical significance, which is our motivation for this paper. We summarize the main contributions of this thesis:

(a) Considering the effect of time delay and uncertainty, which makes our model have better universality;

(b) The design method of sliding mode controller is given, which has good robustness to resist parameter uncertainty.

(c) Some sufficient conditions for FTS of the SMC system in the finite time interval $[0, T]$ are given.

(d) The obtained results can describe the relationship between FTS and system parameters, and can be easily solved.

A. NOTATIONS

The following notations will be used in this paper: R^n and $R^{n\times m}$ represent the n-dimensional Euclidean space and the set of $n \times m$ real matrices, respectively. Superscript 'T' denotes the transpose of matrix. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ stand for the minimum and maximum eigenvalue of matrices. Asterisk($*$) means the term of symmetry. $\|\cdot\|$ denotes the Euclidean norm operator.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the uncertain neutral system with time-delay as follows:

$$\begin{cases} \dot{x}(t) - G\dot{x}(t-h) = (A_1 + \Delta A_1)x(t) + (A_2 + \Delta A_2) \\ \qquad \times x(t-d) + B(u(t) + f(x(t))), \\ x(\theta) = \phi(\theta), \theta \in [-\tau, 0], \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the system state, $u(t) \in R^m$ is the control input, $G \in R^{n\times n}$, $A_1 \in R^{n\times n}$, $A_2 \in R^{n\times n}$, $B \in R^{n\times m}$ are known constant matrices. $h > 0$ and $d > 0$ are time delay constants. τ takes the maximum value of h and d, $f(x(t))$ denotes non-linear known function, which satisfies the Lipchitz constraint $\|f(x(t))\| \le \beta\|x(t)\|$, here β is a positive scalar. ΔA_1 and ΔA_2 are uncertain matrices that satisfy the following condition:

$$[\Delta A_1 \ \Delta A_2] = MF(t)[N_1 \ N_2], \tag{2}$$

where, M, N_1, N_2 are constant matrices of appropriate dimensions, $F(t)$ is an unknown time-varying matrix that satisfies $F(t)^{\mathrm{T}}F(t) \le I$, $\phi(\theta) \in R^{n\times n}$ is a continue initial function vector.

Firstly we give a necessary definition and some lemmas.

Definition 1 (FTS): For given c_1, c_2, T and $R > 0$, where $c_2 > c_1 > 0$, $T > 0$, the system (1) is said to be finite-time stabilizable with respect to (c_1, c_2, T, R), if

$$\sup_{-\tau\le\theta\le0} x^{\mathrm{T}}(\theta)Rx(\theta) \le c_1 \Rightarrow x^{\mathrm{T}}(t)Rx(t) \le c_2, \quad \forall t \in [0, T]. \tag{3}$$

Lemma 1 [37]: For some real matrices of appropriate dimensions $S_1, S_2, E(t)$, here, $E^{\mathrm{T}}(t)E(t) \le I$ and a scalar υ, the following inequality holds:

$$S_1E(t)S_2 + S_2^{\mathrm{T}}E^{\mathrm{T}}(t)S_1^{\mathrm{T}} \le \upsilon S_1S_1^{\mathrm{T}} + \upsilon^{-1}S_2^{\mathrm{T}}S_2.$$

Lemma 2 (Partitioning Strategy [38]): For given c_1, c_2, T and $R > 0$, the system (1) is FTS about (c_1, c_2, T, R), if and only if there exists an auxiliary scalar satisfying $c_1 < c^* < c_2$ such that the system is FTS about (c_1, c^*, τ^*, R) during reaching phase and FTS about (c^*, c_2, T, R) during sliding motion phase, here, τ^* is the time that the system state reaches the sliding surface.

III. MAIN RESULTS

We design the following integral sliding variable

$$s(t) = H(x(t) - Gx(t-h)) - \int_0^t H(A_1 + BK_1)x(s)\mathrm{d}s - \int_0^t H(A_2 + BK_2)x(s-d)\mathrm{d}s, \tag{4}$$

where H is chosen such that HB is nonsingular, which can be attained by $H = B^{\mathrm{T}}L$ with $L > 0$. K_1, K_2 are controller gains to be designed later.

The derivative of $s(t)$ gives

$$\begin{aligned} \dot{s}(t) &= H(\dot{x}(t) - G\dot{x}(t-h)) - H(A_1 + BK_1)x(t) \\ &\quad - H(A_2 + BK_2)x(t-d) \\ &= H((\Delta A_1 - BK_1)x(t) + (\Delta A_2 - BK_2)x(t-d) \\ &\quad + B(u(t) + f(x(t)))). \end{aligned} \tag{5}$$

A. REACHABILITY ANALYSIS

In this subsection, an appropriate SMC law is designed, which can ensure the system state reaches the sliding surface in a finte time τ^* and maintain sliding motion for all subsequent time.

Theorem 1: The state trajectories of uncertain neutral system (1) can reach the sliding surface in finite time $[0, \tau^*]$ with the SMC as

$$u(t) = K_1x(t) + K_2x(t-d) - \rho(t)sign(s(t)), \tag{6}$$

where $\rho(t) = \eta + \beta||x(t)|| + ||H_M||(||N_1x(t)|| + ||N_2x(t-d)||)$, $H_M = (HB)^{-1}HM$, $\tau^* \leq \frac{1}{\hat{\eta}}\sqrt{V_1(x(0), 0)}$, $\hat{\eta} = \eta\sqrt{\frac{1}{2\lambda_{\max}((HB)^{-1})}}$. Here, $\rho > 0$, $sign(\cdot)$ stands for sign function.

Proof: Take the Lyapunov function as

$$V_1(t) = \frac{1}{2}s^{\mathrm{T}}(t)(HB)^{-1}s(t).$$

Time derivative of $V_1(t)$ equals to

$$\begin{aligned}
\dot{V}_1(x(t), t) &= s^{\mathrm{T}}(t)(HB)^{-1}\dot{s}(t)\\
&= s^{\mathrm{T}}(t)(HB)^{-1}H(MF(t)N_1x(t) - BK_1x(t)\\
&\quad +MF(t)N_2x(t-d) - BK_2x(t-d)\\
&\quad +B(u(t) + f(x(t)))\\
&= s^{\mathrm{T}}(t)(H_MF(t)N_1x(t) - K_1x(t)\\
&\quad +H_MF(t)N_2x(t-d)\\
&\quad -K_2x(t-d) + u(t) + f(x(t)))\\
&= s^{\mathrm{T}}(t)(-\rho(t)sign(s(t)) + H_MF(t)N_1x(t)\\
&\quad +H_MF(t)N_2x(t-d) + f(x(t)))\\
&\leq s^{\mathrm{T}}(t)(-(\eta + ||H_M||\,||N_1x(t)||\\
&\quad +||H_M||\,||N_2x(t-d)||)sign(s(t))\\
&\quad -\beta||x(t)||sign(s(t)) + H_MF(t)N_1x(t)\\
&\quad +H_MF(t)N_2x(t-d) + f(x(t)))\\
&\leq -\eta||s(t)|| < 0.
\end{aligned} \tag{7}$$

We get

$$2V_1(x(t), t) = s^{\mathrm{T}}(t)(HB)^{-1}s(t) \leq \lambda_{\max}((HB)^{-1})||s(t)||^2.$$

Thus

$$||s(t)|| \geq \sqrt{\frac{2V_1(x(t), t)}{\lambda_{\max}((HB)^{-1})}}.$$

Then

$$\begin{aligned}
\dot{V}_1(x(t), t) &\leq -\eta\sqrt{\frac{2V_1(x(t), t)}{\lambda_{\max}((HB)^{-1})}}\\
&= -2\hat{\eta}\sqrt{V_1(x(t), t)}.
\end{aligned} \tag{8}$$

Integrating (8) on the interval $[0, \tau^*]$, where $\tau^* > 0$ is the time to reach the sliding surface, we get

$$\sqrt{V_1(x(\tau^*), \tau^*)} - \sqrt{V_1(x(0), 0)} \leq -\hat{\eta}\tau^*,$$

which means that the system state trajectories can reach sliding surface in a finite time $\tau^*(\tau^* \leq \frac{1}{\hat{\eta}}\sqrt{V_1(x(0), 0)})$, the proof is completed.

B. FTS OVER REACHING PHASE WITHIN $[0, \tau^*]$

In this subsection, we will analyze FTB for the system (1) during the reaching phase. By substituting (6) into (1), we get

$$\dot{x}(t) - G\dot{x}(t-h) = \tilde{A}_1x(t) + \tilde{A}_2x(t-d) + B\tilde{f}(x(t)), \tag{9}$$

where $\tilde{A}_1 = A_1 + \Delta A_1 + BK_1$, $\tilde{A}_2 = A_2 + \Delta A_2 + BK_2$, $\tilde{f}(x(t)) = -\rho(t)sign(s(t)) + f(x(t))$.

Theorem 2: For given positive scalars c_1, τ^* and symmetric matrix $R > 0$, the sliding mode dynamics system (9) is FTS about (c_1, c^*, τ^*, R), if there exist a scalar $\sigma \geq 0$, $c^* > 0 (c^* > c_1)$ and symmetric matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$ such that the following conditions hold (10) and (11), as shown at the bottom of the next page: where $\tilde{P} = R^{-\frac{1}{2}}PR^{-\frac{1}{2}}$, $\tilde{Q}_1 = R^{-\frac{1}{2}}Q_1R^{-\frac{1}{2}}$, $\tilde{Q}_2 = R^{-\frac{1}{2}}Q_2R^{-\frac{1}{2}}$.

Proof: Choose the Lyapunov function as

$$\begin{aligned}
V_2(x(t), t) &= x^{\mathrm{T}}(t)\tilde{P}^{-1}x(t)\\
&\quad + \int_{t-d}^{t} \mathrm{e}^{-\sigma(t-s)}x^{\mathrm{T}}(s)\tilde{Q}_1^{-1}x(s)\mathrm{d}s\\
&\quad + \int_{t-h}^{t} \mathrm{e}^{-\sigma(t-s)}\dot{x}^{\mathrm{T}}(s)\tilde{Q}_2^{-1}\dot{x}(s)\mathrm{d}s.
\end{aligned}$$

Along the solution of system (9), we obtain

$$\begin{aligned}
\dot{V}_2(x(t), t) &= 2x^{\mathrm{T}}(t)\tilde{P}^{-1}\dot{x}(t) + x^{\mathrm{T}}(t)\tilde{Q}_1^{-1}x(t)\\
&\quad -\mathrm{e}^{-\sigma d}x^{\mathrm{T}}(t-d)\tilde{Q}_1^{-1}x(t-d)\\
&\quad -\sigma\int_{t-d}^{t} \mathrm{e}^{-\sigma(t-s)}x^{\mathrm{T}}(s)\tilde{Q}_1^{-1}x(s)\mathrm{d}s\\
&\quad +\dot{x}^{\mathrm{T}}(t)\tilde{Q}_2^{-1}\dot{x}(t) - \mathrm{e}^{-\sigma h}\dot{x}^{\mathrm{T}}(t-h)\tilde{Q}_2^{-1}\dot{x}(t-h)\\
&\quad -\sigma\int_{t-h}^{t} \mathrm{e}^{-\sigma(t-s)}\dot{x}^{\mathrm{T}}(s)\tilde{Q}_2^{-1}\dot{x}(s)ds\\
&\quad -\sigma x^{\mathrm{T}}(t)\tilde{P}^{-1}x(t) + \sigma x^{\mathrm{T}}(t)\tilde{P}^{-1}x(t).
\end{aligned}$$

Note that

$$\begin{aligned}
&\dot{V}_2(x(t), t) + \sigma V_2(x(t), t) - \pi\tilde{f}^{\mathrm{T}}(x(t))\tilde{f}(x(t))\\
&= 2x^{\mathrm{T}}(t)\tilde{P}^{-1}\dot{x}(t) + \sigma x^{\mathrm{T}}(t)\tilde{P}^{-1}x(t) + x^{\mathrm{T}}(t)\tilde{Q}_1^{-1}x(t)\\
&\quad -\mathrm{e}^{-\sigma d}x^{\mathrm{T}}(t-d)\tilde{Q}_1^{-1}x(t-d) + \dot{x}^{\mathrm{T}}(t)\tilde{Q}_2^{-1}\dot{x}(t)\\
&\quad -\mathrm{e}^{-\sigma h}\dot{x}^{\mathrm{T}}(t-h)\tilde{Q}_2^{-1}\dot{x}(t-h) - \pi\tilde{f}^{\mathrm{T}}(x(t))\tilde{f}(x(t))\\
&= 2x^{\mathrm{T}}(t)\tilde{P}^{-1}(G\dot{x}(t-h) + \tilde{A}_1x(t) + \tilde{A}_2x(t-d) + B\tilde{f}(x(t)))\\
&\quad +x^{\mathrm{T}}(t)(\sigma\tilde{P}^{-1} + \tilde{Q}_1^{-1})x(t) - \mathrm{e}^{-\sigma d}x^{\mathrm{T}}(t-d)\tilde{Q}_1^{-1}x(t-d)\\
&\quad -\mathrm{e}^{-\sigma h}\dot{x}^{\mathrm{T}}(t-h)\tilde{Q}_2^{-1}\dot{x}(t-h) + (G\dot{x}(t-h) + \tilde{A}_1x(t)\\
&\quad +\tilde{A}_2x(t-d) + B\tilde{f}(x(t)))^{\mathrm{T}}\tilde{Q}_2^{-1}(G\dot{x}(t-h) + \tilde{A}_1x(t)\\
&\quad +\tilde{A}_2x(t-d) + B\tilde{f}(x(t))) - \pi\tilde{f}^{\mathrm{T}}(x(t))\tilde{f}(x(t))\\
&= \kappa^{\mathrm{T}}(t)\Pi\kappa(t),
\end{aligned}$$

where $\kappa(t) = (x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-d), \dot{x}^{\mathrm{T}}(t-h), \tilde{f}^{\mathrm{T}}(x(t)))^{\mathrm{T}}$ the equation can be derived, as shown at the bottom of next page,

According to (10) and the Schur complement, $\Xi < 0$ means $\Pi < 0$, thus

$$\dot{V}_2(x(t), t) + \sigma V_2(x(t), t) - \pi\tilde{f}^{\mathrm{T}}(x(t))\tilde{f}(x(t)) < 0. \tag{12}$$

Multiplying both sides of (12) by $e^{\sigma t}$ and integrating from 0 to t ($t \in [0, \tau^*]$), we get

$$V_2(x(t), t) < \mathrm{e}^{-\sigma t}V_2(x(0), 0) + \int_0^t \pi\tilde{f}^{\mathrm{T}}(x(t))\tilde{f}(x(t))\,\mathrm{d}s.$$

When $t \in [0, \tau^*)$, it is obvious that $\int_0^t \pi\tilde{f}^{\mathrm{T}}(x(t))\tilde{f}(x(t))\,\mathrm{d}s$ is a bounded integral and its upper bound is assumed to be ϖ.

Note that

$$V_2(x(t),t) \geq x^{\mathrm{T}}(t)\tilde{P}^{-1}x(t) \geq \lambda_{\min}(P^{-1})x^{\mathrm{T}}(t)Rx(t) \geq \frac{1}{\lambda_{\max}(P)}x^{\mathrm{T}}(t)Rx(t),$$

and

$$\begin{aligned} & \mathrm{e}^{-\sigma t}V_2(x(0),0) \\ & \leq x^{\mathrm{T}}(0)\tilde{P}^{-1}x(0) + \int_{-d}^{0}\mathrm{e}^{\sigma s}x^{\mathrm{T}}(s)\tilde{Q}_1^{-1}x(s)\mathrm{d}s \\ & \quad + \int_{-h}^{0}\mathrm{e}^{\sigma s}\dot{x}^{T}(s)\tilde{Q}_2^{-1}\dot{x}(s)\mathrm{d}s \\ & \leq c_1(\lambda_{\max}(P^{-1}) + \lambda_{\max}(Q_1{}^{-1})\int_{-d}^{0}\mathrm{e}^{\sigma s}\mathrm{d}s \\ & \quad + \lambda_{max}(Q_2{}^{-1})\int_{-h}^{0}\mathrm{e}^{\sigma s}\mathrm{d}s) \\ & \leq c_1(\frac{1}{\lambda_{\min}(P)} + \left(\frac{1}{\lambda_{\min}(Q_1)} + \frac{1}{\lambda_{\min}(Q_2)}\right)\frac{1}{\sigma}(1-\mathrm{e}^{-\sigma\tau})), \end{aligned}$$

Thus

$$\begin{aligned} & \frac{1}{\lambda_{\max}(P)}x^{\mathrm{T}}(t)Rx(t) \\ & \leq \left(\frac{1}{\lambda_{\min}(P)} + \left(\frac{1}{\lambda_{\min}(Q_1)} + \frac{1}{\lambda_{\min}(Q_2)}\right)\frac{1}{\sigma}(1-\mathrm{e}^{-\sigma\tau})\right)c_1 \\ & \quad + \varpi. \end{aligned}$$

According to (10), we have $x^{\mathrm{T}}(t)Rx(t) < c^*, t \in [0, \tau^*]$, which means that the system (9) is FTS with respect to (c_1, c^*, τ^*, R), the proof is completed.

Remark 1: We note that the conditions of Theorem 2 contain plenty of nonlinear terms, which makes it impossible for us to solve it through LMI toolbox. To address this problem, we need to linearize the nonlinear terms in Theorem 3.

Theorem 3: For given positive scalars c_1, τ^*, ϖ and symmetric matrix $R > 0$, the sliding mode dynamics system (9) is FTS about (c_1, c^*, τ^*, R), if there exist scalars $\sigma > 0$, $\nu > 0$, $c^* > 0 (c^* > c_1)$, $\lambda_i > 0 (i = 1, 2, 3, 4)$ and symmetric matrices $X > 0, \tilde{Q}_1 > 0, \tilde{Q}_2 > 0$, real matrices W_1, W_2 satisfying the following LMIs (13)–(17), as shown at the bottom of the next page:

where $\Theta_{11} = A_1X + XA_1{}^{\mathrm{T}} + BW_1 + W_1{}^{\mathrm{T}}B^{\mathrm{T}} + \sigma X + \nu MM^{\mathrm{T}}$, and the parameters K_1 and K_2 are given by $K_1 = W_1X^{-1}$, $K_2 = W_2Q_1^{-1}$.

Proof: Substituting $\tilde{A}_1 = A_1 + \Delta A_1 + BK_1, \tilde{A}_2 = A_2 + \Delta A_2 + BK_2$ into (10), we get $\Xi = \Gamma + \Delta\Gamma$, where the equation can be derived, as shown at the bottom of next page

Let

$$T_M = \begin{bmatrix} M^{\mathrm{T}}\tilde{P}^{-1} & 0 & 0 & 0 & M^{\mathrm{T}} \end{bmatrix},$$
$$T_N = \begin{bmatrix} N_1 & N_2 & 0 & 0 & 0 \end{bmatrix}.$$

then $\Delta\Gamma = T_M^{\mathrm{T}}F(t)T_N + T_N^{\mathrm{T}}F^{\mathrm{T}}(t)T_M$.

From Lemma 1, there exists a scalar $\nu > 0$, such that inequality (18) holds (18) and (19), as shown at the bottom of the next page.

where

$$\begin{aligned} \Psi_{11} &= \tilde{P}^{-1}A_1 + A_1{}^{\mathrm{T}}\tilde{P}^{-1} + \tilde{P}^{-1}BK_1 + K_1{}^{\mathrm{T}}B^{\mathrm{T}}\tilde{P}^{-1} \\ &\quad + \sigma\tilde{P}^{-1} + \tilde{Q}_1^{-1} + \nu\tilde{P}^{-1}MM^{\mathrm{T}}\tilde{P}^{-1}, \\ \Psi_{15} &= A_1^{\mathrm{T}} + K_1^{\mathrm{T}}B^{\mathrm{T}} + \nu\tilde{P}^{-1}MM^{\mathrm{T}}. \end{aligned}$$

By Schur Lemma, it is known that $\Psi < 0$ implies (19). Multiplying both sides of (19) by $\mathrm{diag}\{\tilde{P}, \tilde{Q}_1, \tilde{Q}_2, I, I, I\}$ and letting $X = \tilde{P}, W_1 = K_1X, W_2 = K_2\tilde{Q}_1$. By Schur complement, we can know that (13) is equivalent to (19).

On the other hand, from the definition of $\tilde{P}, \tilde{Q}_1, \tilde{Q}_2$ and (14)-(17), it follows that

$$\begin{aligned} & \frac{c_1}{\lambda_{\min}(R^{\frac{1}{2}}XR^{\frac{1}{2}})} + \left(\frac{1}{\lambda_{\min}(R^{\frac{1}{2}}\tilde{Q}_1R^{\frac{1}{2}})} + \frac{1}{\lambda_{\min}(R^{\frac{1}{2}}\tilde{Q}_2R^{\frac{1}{2}})}\right) \\ & \quad \times\frac{1}{\sigma}(1-\mathrm{e}^{-\sigma\tau})c_1 + \varpi < \frac{c^*}{\lambda_{\max}(R^{\frac{1}{2}}XR^{\frac{1}{2}})}. \end{aligned} \tag{20}$$

Then, according to Theorem 2, the system (9) is FTS about (c_1, c^*, τ^*, R), the proof is completed.

C. FTS OVER SLIDING MOTION PHASE WITHIN $[\tau^*, T]$

In this subsection, we will analyze FTB for the system (1) during the sliding motion phase. When the system state move along the sliding surface, the equivalent control u_{eq} can be

$$\Xi = \begin{bmatrix} \tilde{P}^{-1}\tilde{A}_1 + \tilde{A}_1^T\tilde{P}^{-1} + \sigma\tilde{P}^{-1} + \tilde{Q}_1^{-1} & \tilde{P}^{-1}\tilde{A}_2 & \tilde{P}^{-1}G & \tilde{P}^{-1}B & \tilde{A}_1^T \\ * & -\mathrm{e}^{-\sigma d}\tilde{Q}_1^{-1} & 0 & 0 & \tilde{A}_2^T \\ * & * & -\mathrm{e}^{-\sigma h}\tilde{Q}_2^{-1} & 0 & G^T \\ * & * & * & -\pi I & B^T \\ * & * & * & * & -\tilde{Q}_2 \end{bmatrix} < 0, \tag{10}$$

$$\frac{c_1}{\lambda_{\min}(P)} + \left(\frac{1}{\lambda_{\min}(Q_1)} + \frac{1}{\lambda_{\min}(Q_2)}\right)\frac{1}{\sigma}(1-\mathrm{e}^{-\sigma\tau})c_1 + \varpi < \frac{c^*}{\lambda_{\max}(P)}, \tag{11}$$

$$\Pi = \begin{bmatrix} \tilde{P}^{-1}\tilde{A}_1 + \tilde{A}_1^{\mathrm{T}}\tilde{P}^{-1} + \sigma\tilde{P}^{-1} + \tilde{Q}_1^{-1} & \tilde{P}^{-1}\tilde{A}_2 & \tilde{P}^{-1}G & \tilde{P}^{-1}B \\ * & -\mathrm{e}^{-\sigma d}\tilde{Q}_1{}^{-1} & 0 & 0 \\ * & * & -\mathrm{e}^{-\sigma h}\tilde{Q}_2{}^{-1} & 0 \\ * & * & * & -\pi I \end{bmatrix} + \begin{bmatrix} \tilde{A}_1^{\mathrm{T}} \\ \tilde{A}_2^{\mathrm{T}} \\ G^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}\tilde{Q}_2^{-1}\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 & G & B \end{bmatrix}.$$

solved by $\dot{s}(t) = 0$.

$$u_{eq} = -(HB)^{-1}H[(\Delta A_1 - BK_1)x(t) + (\Delta A_2 - BK_2)x(t-d)] - f(x(t)). \quad (21)$$

By substituting (21) into (1), we get

$$\dot{x}(t) - G\dot{x}(t-h) = \bar{A}_1 x(t) + \bar{A}_2 x(t-d), \quad (22)$$

where

$$\bar{A}_1 = A_1 + \Delta A_1 - B(HB)^{-1}H\Delta A_1 + BK_1,$$
$$\bar{A}_2 = A_2 + \Delta A_2 - B(HB)^{-1}H\Delta A_2 + BK_2.$$

Theorem 4: For given positive scalars c_2, T and symmetric matrix $R > 0$, the closed-loop system (22) is FTS about (c^*, c_2, T, R), if there exist scalars $\alpha \geq 0, c^* > 0 (c^* < c_2)$, and symmetric matrices $\check{P} > 0, \check{Q}_1 > 0, \check{Q}_2 > 0$ such that the following conditions hold (23) and (24), as shown at the bottom of the next page:

$$\Theta = \begin{bmatrix} \Theta_{11} & A_2\tilde{Q}_1 + BW_2 & G\tilde{Q}_2 & B & XA_1^{\mathrm{T}} + W_1^{\mathrm{T}}B^{\mathrm{T}} + \nu MM^{\mathrm{T}} & XN_1^{\mathrm{T}} & X \\ * & -\mathrm{e}^{-\sigma d}\tilde{Q}_1 & 0 & 0 & \tilde{Q}_1A_2^{\mathrm{T}} + W_2{}^{\mathrm{T}}B^{\mathrm{T}} & \tilde{Q}_1N_2^{\mathrm{T}} & 0 \\ * & * & -\mathrm{e}^{-\sigma h}\tilde{Q}_2 & 0 & \tilde{Q}_2G^{\mathrm{T}} & 0 & 0 \\ * & * & * & -\pi I & B^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & -\tilde{Q}_2 + \nu MM^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & * & -\nu I & 0 \\ * & * & * & * & * & * & -\tilde{Q}_1 \end{bmatrix} < 0, \quad (13)$$

$$\lambda_1 R^{-1} < X < \lambda_2 R^{-1}, \quad (14)$$
$$\lambda_3 R^{-1} < \tilde{Q}_1, \quad (15)$$
$$\lambda_4 R^{-1} < \tilde{Q}_2, \quad (16)$$
$$\frac{c_1}{\lambda_1} + \left(\frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right)\frac{1}{\sigma}(1 - \mathrm{e}^{-\sigma\tau})c_1 + \varpi < \frac{c^*}{\lambda_2}, \quad (17)$$

$$\Delta\Gamma = \begin{bmatrix} \tilde{P}^{-1}\Delta A_1 + \Delta A_1{}^{\mathrm{T}}\tilde{P}^{-1} & \tilde{P}^{-1}\Delta A_2 & 0 & 0 & \Delta A_1^{\mathrm{T}} \\ * & 0 & 0 & 0 & \Delta A_2^{\mathrm{T}} \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \tilde{P}^{-1}A_2 + \tilde{P}^{-1}BK_2 & \tilde{P}^{-1}G & \tilde{P}^{-1}B & A_1^{\mathrm{T}} + K_1^{\mathrm{T}}B^{\mathrm{T}} \\ * & -\mathrm{e}^{-\sigma d}\tilde{Q}_1^{-1} & 0 & 0 & A_2^{\mathrm{T}} + K_2^{\mathrm{T}}B^{\mathrm{T}} \\ * & * & -\mathrm{e}^{-\sigma h}\tilde{Q}_2^{-1} & 0 & G^{\mathrm{T}} \\ * & * & * & -\pi I & B^{\mathrm{T}} \\ * & * & * & * & -\tilde{Q}_2 \end{bmatrix},$$

$$\Gamma_{11} = \tilde{P}^{-1}A_1 + A_1{}^{\mathrm{T}}\tilde{P}^{-1} + \tilde{P}^{-1}BK_1 + K_1{}^{\mathrm{T}}B^{\mathrm{T}}\tilde{P}^{-1} + \sigma\tilde{P}^{-1} + \tilde{Q}_1^{-1}.$$

$$\Gamma + \Delta\Gamma \leq \begin{bmatrix} \Psi_{11} & \tilde{P}^{-1}A_2 + \tilde{P}^{-1}BK_2 & \tilde{P}^{-1}G & \tilde{P}^{-1}B & \Psi_{15} \\ * & -\mathrm{e}^{-\sigma d}\tilde{Q}_1^{-1} & 0 & 0 & A_2^{\mathrm{T}} + K_2^{\mathrm{T}}B^{\mathrm{T}} \\ * & * & -\mathrm{e}^{-\sigma h}\tilde{Q}_2^{-1} & 0 & G^{\mathrm{T}} \\ * & * & * & -\pi I & B^{\mathrm{T}} \\ * & * & * & * & -\tilde{Q}_2 + \nu MM^{\mathrm{T}} \end{bmatrix} + \nu^{-1}T_N^{\mathrm{T}}T_N \triangleq \Psi, \quad (18)$$

$$\begin{bmatrix} \Psi_{11} & \tilde{P}^{-1}A_2 + \tilde{P}^{-1}BK_2 & \tilde{P}^{-1}G & \tilde{P}^{-1}B & \Psi_{15} & N_1^{\mathrm{T}} \\ * & -\mathrm{e}^{-\sigma d}\tilde{Q}_1^{-1} & 0 & 0 & A_2^{\mathrm{T}} + K_2^{\mathrm{T}}B^{\mathrm{T}} & N_2^{\mathrm{T}} \\ * & * & -\mathrm{e}^{-\sigma h}\tilde{Q}_2^{-1} & 0 & G^{\mathrm{T}} & 0 \\ * & * & * & -\pi I & B^{\mathrm{T}} & 0 \\ * & * & * & * & -\tilde{Q}_2 + \nu MM^{\mathrm{T}} & 0 \\ * & * & * & * & * & -\nu I \end{bmatrix} < 0, \quad (19)$$

where $\bar{P} = R^{-\frac{1}{2}}\breve{P}R^{-\frac{1}{2}}, \bar{Q}_1 = R^{-\frac{1}{2}}\breve{Q}_1R^{-\frac{1}{2}}, \bar{Q}_2 = R^{-\frac{1}{2}}\breve{Q}_2R^{-\frac{1}{2}}$.

Proof: Choose the Lyapunov function as

$$\begin{aligned}V_3(x(t),t) &= x^{\mathrm{T}}(t)\bar{P}^{-1}x(t)\\&\quad+\int_{t-d}^{t}\mathrm{e}^{-\alpha(t-s)}x^{\mathrm{T}}(s)\bar{Q}_1^{-1}x(s)\mathrm{d}s\\&\quad+\int_{t-h}^{t}\mathrm{e}^{-\alpha(t-s)}\dot{x}^{\mathrm{T}}(s)\bar{Q}_2^{-1}\dot{x}(s)\mathrm{d}s.\end{aligned}$$

Along the solution of system (22), we obtain

$$\begin{aligned}\dot{V}_3(x(t),t) &= 2x^{\mathrm{T}}(t)\bar{P}^{-1}\dot{x}(t)+x^{\mathrm{T}}(t)\bar{Q}_1^{-1}x(t)\\&\quad-\mathrm{e}^{-\alpha d}x^{\mathrm{T}}(t-d)\bar{Q}_1^{-1}x(t-d)\\&\quad-\alpha\int_{t-d}^{t}\mathrm{e}^{-\alpha(t-s)}x^{\mathrm{T}}(s)\bar{Q}_1^{-1}x(s)\mathrm{d}s\\&\quad+\dot{x}^{\mathrm{T}}(t)\bar{Q}_2^{-1}\dot{x}(t)\\&\quad-\mathrm{e}^{-\alpha h}\dot{x}^{\mathrm{T}}(t-h)\bar{Q}_2^{-1}\dot{x}(t-h)\\&\quad-\int_{t-h}^{t}\mathrm{e}^{-\alpha(t-s)}\dot{x}^{\mathrm{T}}(s)\bar{Q}_2^{-1}\dot{x}(s)\mathrm{d}s\\&\quad-\alpha x^{\mathrm{T}}(t)\bar{P}^{-1}x(t)+\alpha x^{\mathrm{T}}(t)\bar{P}^{-1}x(t).\end{aligned}$$

Note that

$$\begin{aligned}&\dot{V}_3(x(t),t)+\alpha V_3(x(t),t)\\&= 2x^{\mathrm{T}}(t)\bar{P}^{-1}\dot{x}(t)+\alpha x^{\mathrm{T}}(t)\bar{P}^{-1}x(t)+x^{\mathrm{T}}(t)\bar{Q}_1^{-1}x(t)\\&\quad-\mathrm{e}^{-\alpha d}x^{\mathrm{T}}(t-d)\bar{Q}_1^{-1}x(t-d)+\dot{x}^{\mathrm{T}}(t)\bar{Q}_2^{-1}\dot{x}(t)\\&\quad-\mathrm{e}^{-\alpha h}\dot{x}^{\mathrm{T}}(t-h)\bar{Q}_2^{-1}\dot{x}(t-h)\\&= 2x^{\mathrm{T}}(t)\bar{P}^{-1}\left(G\dot{x}(t-h)+\bar{A}_1x(t)+\bar{A}_2x(t-d)\right)\\&\quad+x^{\mathrm{T}}(t)(\alpha\bar{P}^{-1}+\bar{Q}_1^{-1})x(t)-\mathrm{e}^{-\alpha d}x^{\mathrm{T}}(t-d)\bar{Q}_1^{-1}x(t-d)\\&\quad-\mathrm{e}^{-\alpha h}\dot{x}^{\mathrm{T}}(t-h)\bar{Q}_2^{-1}\dot{x}(t-h)+(G\dot{x}(t-h)+\bar{A}_1x(t)\\&\quad+\bar{A}_2x(t-d))^{\mathrm{T}}\bar{Q}_2^{-1}(G\dot{x}(t-h)+\bar{A}_1x(t)+\bar{A}_2x(t-d))\\&= \xi^{\mathrm{T}}(t)\Phi\xi(t),\end{aligned}$$

where the equation can be derived, as shown at the bottom of next page

$\xi(t) = (x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-d), \dot{x}^{\mathrm{T}}(t-h))^{\mathrm{T}}$. By Schur complement, we can know that $\Omega < 0$ is equivalent to $\Phi < 0$, then

$$\dot{V}_3(x(t),t)+\alpha V_3(x(t),t) < 0, \tag{25}$$

Multiplying both sides of (25) by $e^{\alpha t}$ and integrating from τ^* to t ($t \in [\tau^*, T]$), we get

$$V_3(x(t),t) < \mathrm{e}^{-\alpha t}V_3(x(\tau^*),\tau^*), \tag{26}$$

Note that

$$\begin{aligned}&V_3(x(t),t)\\&\geq x^{\mathrm{T}}(t)\bar{P}^{-1}x(t) \geq \lambda_{\min}(\breve{P}^{-1})x^{\mathrm{T}}(t)Rx(t)\\&= \frac{1}{\lambda_{\max}(\breve{P})}x^{\mathrm{T}}(t)Rx(t),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{e}^{-\alpha t}V_3(x(\tau^*),\tau^*)\\&\leq x^{\mathrm{T}}(\tau^*)\bar{P}^{-1}x(\tau^*)+\int_{\tau^*-d}^{\tau^*}\mathrm{e}^{-\alpha(\tau^*-s)}x^{\mathrm{T}}(s)\bar{Q}_1^{-1}x(s)ds\\&\quad+\int_{\tau^*-h}^{\tau^*}\mathrm{e}^{-\alpha(\tau^*-s)}\dot{x}^{\mathrm{T}}(s)\bar{Q}_2^{-1}\dot{x}(s)\mathrm{d}s\\&\leq c^*(\lambda_{\max}(\breve{P}^{-1})+\lambda_{\max}(\breve{Q}_1^{-1})\int_{\tau^*-\tau}^{\tau^*}\mathrm{e}^{-\alpha(\tau^*-s)}\mathrm{d}s\\&\quad+\lambda_{\max}(\breve{Q}_2^{-1})\int_{\tau^*-\tau}^{\tau^*}\mathrm{e}^{-\alpha(\tau^*-s)}\mathrm{d}s)\\&\leq c^*\left(\frac{1}{\lambda_{\min}(\breve{P})}+\left(\frac{1}{\lambda_{\min}(\breve{Q}_1)}+\frac{1}{\lambda_{\min}(\breve{Q}_2)}\right)\frac{1}{\alpha}(1-\mathrm{e}^{-\alpha\tau})\right),\end{aligned}$$

thus,

$$\begin{aligned}&\frac{1}{\lambda_{\max}(\breve{P})}x^{\mathrm{T}}(t)Rx(t)\\&\quad\leq\left(\frac{1}{\lambda_{\min}(\breve{P})}+\left(\frac{1}{\lambda_{\min}(\breve{Q}_1)}+\frac{1}{\lambda_{\min}(\breve{Q}_2)}\right)\frac{1}{\alpha}(1-\mathrm{e}^{-\alpha\tau})\right)c^*.\end{aligned}$$

From (24), we have $x^{\mathrm{T}}(t)Rx(t) < c_2, t \in [\tau^*, T]$, which means that the system (22) is FTS with respect to $((c^*, c_2, T, R))$, the proof is completed.

Remark 2: In fact, according to lemma 2 the reaching instant τ^* is not only the end of reaching phase but also the start of sliding motion phase, so $x(\tau^*)Rx(\tau^*) < c^*$ can be used as the initial condition for Theorem 4, which connects the two segments of SMC.

Theorem 5: For given positive scalars c_2, T and symmetric matrix $R > 0$, the closed-loop system (22) is FTS about (c^*, c_2, T, R), if there exist scalars $\alpha > 0, \gamma > 0, c^* > 0 (c^* < c_2), \tilde{\lambda}_i > 0 (i = 1, 2, 3, 4)$ and symmetric matrices $\bar{X} > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0$, real matrices $\bar{W}_1, \bar{W}_2$ satisfying the following LMIs (27) and (31), as shown at the bottom of the next page:

where

$$\begin{aligned}\bar{\Theta}_{11} &= A_1\bar{X}+\bar{X}A_1^{\mathrm{T}}+B\bar{W}_1+\bar{W}_1^{\mathrm{T}}B^{\mathrm{T}}+\alpha\bar{X}\\&\quad+\gamma(M-B(HB)^{-1}HM)(M^{\mathrm{T}}-M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}}),\\\bar{\Theta}_{14} &= \bar{X}A_1^{\mathrm{T}}+\bar{W}_1^{\mathrm{T}}B^{\mathrm{T}}+\gamma(M-B(HB)^{-1}HM)(M^{\mathrm{T}}\\&\quad-M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}}),\\\bar{\Theta}_{44} &= -\bar{Q}_2+\gamma(M-B(HB)^{-1}HM)(M^{\mathrm{T}}\\&\quad-M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}}).\end{aligned}$$

$$\begin{bmatrix}\bar{P}^{-1}\bar{A}_1+\bar{A}_1^{\mathrm{T}}\bar{P}^{-1}+\alpha\bar{P}^{-1}+\bar{Q}_1^{-1} & \bar{P}^{-1}\bar{A}_2 & \bar{P}^{-1}G & \bar{A}_1^{\mathrm{T}}\\ * & -e^{-\alpha d}\bar{Q}_1^{-1} & 0 & \bar{A}_2^{\mathrm{T}}\\ * & * & -e^{-\alpha h}\bar{Q}_2^{-1} & G^{\mathrm{T}}\\ * & * & * & -\bar{Q}_2\end{bmatrix} = \Omega < 0, \tag{23}$$

$$\frac{c^*}{\lambda_{\min}(\breve{P})}+\left(\frac{1}{\lambda_{\min}(\breve{Q}_1)}+\frac{1}{\lambda_{\min}(\breve{Q}_2)}\right)\frac{1}{\alpha}(1-\mathrm{e}^{-\alpha\tau})c^* < \frac{c_2}{\lambda_{\max}(\breve{P})}, \tag{24}$$

and the parameters $\bar{K}_1$ and $\bar{K}_2$ are given by $\bar{K}_1 = \bar{W}_1\bar{X}^{-1}, \bar{K}_2 = \bar{W}_2\bar{Q}_1^{-1}$.

Proof: Substituting $\bar{A}_1 = A_1 + \Delta A_1 - B(HB)^{-1}H\Delta A_1 - BK_1$, $\bar{A}_2 = A_2 + \Delta A_2 - B(HB)^{-1}H\Delta A_2 - BK_2$ into (23), we get $\Omega = \bar{\Gamma} + \Delta\bar{\Gamma}$, where the equation can be derived, as shown at the bottom of next page

here

$$\bar{\Gamma}_{11} = \bar{P}^{-1}A_1 + A_1^{\mathrm{T}}\bar{P}^{-1} + \bar{P}^{-1}B\bar{K}_1 + \bar{K}_1^{\mathrm{T}}B^{\mathrm{T}}\bar{P}^{-1} + \alpha\bar{P}^{-1} + \bar{Q}_1^{-1},$$

$$\Delta\bar{\Gamma}_{11} = \bar{P}^{-1}\Delta A_1 + \Delta A_1^{\mathrm{T}}\bar{P}^{-1} - \bar{P}^{-1}B(HB)^{-1}H\Delta A_1 - (\bar{P}^{-1}B(HB)^{-1}H\Delta A_1)^{\mathrm{T}}.$$

Denote

$$K_M = [M^{\mathrm{T}}\bar{P}^{-1} - M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}}\bar{P}^{-1} \quad 0 \quad 0 \quad \times M^{\mathrm{T}} - M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}}],$$

$$K_N = [N_1 \quad N_2 \quad 0 \quad 0],$$

then $\Delta\bar{\Gamma} = K_M^{\mathrm{T}}F(t)K_N + K_N^{\mathrm{T}}F^{\mathrm{T}}(t)\mathrm{K}_M$.

From Lemma 1, there exists a scalar $\gamma > 0$, such that the following inequality holds:

$$\bar{\Gamma} + \Delta\bar{\Gamma} \leq \begin{bmatrix} \bar{\Psi}_{11} & \bar{P}^{-1}A_2 + \bar{P}^{-1}B\bar{K}_2 & \bar{P}^{-1}G & \bar{\Psi}_{14} \\ * & -e^{-\alpha d}\bar{Q}_1^{-1} & 0 & A_2{}^{\mathrm{T}} + \bar{K}_2^{\mathrm{T}}B^{\mathrm{T}} \\ * & * & -e^{-\alpha h}\bar{Q}_2^{-1} & G^T \\ * & * & * & \bar{\Psi}_{44} \end{bmatrix} + \gamma^{-1}K_N^{\mathrm{T}}K_N \triangleq \bar{\Psi}, \tag{32}$$

where

$$\bar{\Psi}_{11} = \bar{P}^{-1}A_1 + A_1^{\mathrm{T}}\bar{P}^{-1} + \bar{P}^{-1}B\bar{K}_1 + \bar{K}_1^{\mathrm{T}}B^{\mathrm{T}}\bar{P}^{-1} + \alpha\bar{P}^{-1} + \bar{Q}_1^{-1} + \gamma\bar{P}^{-1} \times(M - B(HB)^{-1}HM)(M^{\mathrm{T}} - M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}})\bar{P}^{-1},$$

$$\bar{\Psi}_{14} = A_1^{\mathrm{T}} + \bar{K}_1^{\mathrm{T}}B^{\mathrm{T}} + \gamma\bar{P}^{-1}(M - B(HB)^{-1}HM)(M^{\mathrm{T}} - M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}}),$$

$$\bar{\Psi}_{44} = -\bar{Q}_2 + \gamma(M - B(HB)^{-1}HM) \times(M^{\mathrm{T}} - M^{\mathrm{T}}(B(HB)^{-1}H)^{\mathrm{T}}).$$

By Schur Lemma, it is known that $\bar{\Psi} < 0$ implies

$$\begin{bmatrix} \bar{\Psi}_{11} & \bar{P}^{-1}A_2 - \bar{P}^{-1}B\bar{K}_2 & \bar{P}^{-1}G & \bar{\Psi}_{14} & N_1^{\mathrm{T}} \\ * & -e^{-\alpha d}\bar{Q}_1^{-1} & 0 & A_2^{\mathrm{T}} - \bar{K}_2^{\mathrm{T}}B^{\mathrm{T}} & N_2^{\mathrm{T}} \\ * & * & -e^{-\alpha h}\bar{Q}_2^{-1} & G^{\mathrm{T}} & 0 \\ * & * & * & \bar{\Psi}_{44} & 0 \\ * & * & * & * & -\gamma I \end{bmatrix} < 0. \tag{33}$$

Multiplying both sides of (33) by diag$\{\bar{P}, \bar{Q}_1, \bar{Q}_2, I, I\}$ and letting $\bar{X} = \bar{P}, \bar{W}_1 = \bar{K}_1\bar{X}, \bar{W}_2 = \bar{K}_2\bar{Q}_1$. By Schur complement, we can know that (27) is equivalent to (33).

On the other hand, from the definition of $\bar{P}, \bar{Q}_1, \bar{Q}_2$ and (28)-(31), it follows that

$$\frac{c^*}{\lambda_{\min}(R^{\frac{1}{2}}\bar{X}R^{\frac{1}{2}})} + \left(\frac{1}{\lambda_{\min}(R^{\frac{1}{2}}\bar{Q}_1R^{\frac{1}{2}})} + \frac{1}{\lambda_{\min}(R^{\frac{1}{2}}\bar{Q}_2R^{\frac{1}{2}})}\right) \times\frac{1}{\alpha}(1 - e^{-\alpha\tau})c^* < \frac{c_2}{\lambda_{\max}(R^{\frac{1}{2}}\bar{X}R^{\frac{1}{2}})}. \tag{34}$$

Then, according to Theorem 4, the system (22) is FTS about (c^*, c_2, T, R), the proof is completed.

Remark 3. Theorem 3 and Theorem 5 give the FTS conditions of the system state over reaching phase and sliding motion phase, respectively. If the parameters in Theorem 5 are substituted as follows: $\alpha \to \sigma, \gamma \to \nu, \tilde{\lambda}_i \to \lambda_i (i = 1, 2, 3, 4), \bar{Q}_1 \to \tilde{Q}_1, \bar{Q}_2 \to \tilde{Q}_2, \bar{X} \to X, \bar{W}_1 \to W_1, \bar{W}_2 \to W_2, \bar{K}_1 \to K_1, \bar{K}_2 \to K_2$, and solve inequalities (13)-(17) and (27)-(31), then according to lemma 2, the conditions that the SMC system state satisfies FTS about (c_1, c_2, T, R) in whole finite-time interval $[0, T]$ can be obtained.

$$\Phi = \begin{bmatrix} \bar{P}^{-1}\bar{A}_1 + \bar{A}_1^{\mathrm{T}}\bar{P}^{-1} + \alpha\bar{P}^{-1} + \bar{Q}_1^{-1} & \bar{P}^{-1}\bar{A}_2 & \bar{P}^{-1}G \\ * & -e^{-\alpha d}\bar{Q}_1^{-1} & 0 \\ * & * & -e^{-\alpha h}\bar{Q}_2^{-1} \end{bmatrix} + \begin{bmatrix} \bar{A}_1^{\mathrm{T}} \\ \bar{A}_2^{\mathrm{T}} \\ G^{\mathrm{T}} \end{bmatrix} \bar{Q}_2^{-1} [\bar{A}_1 \quad \bar{A}_2 \quad G],$$

$$\begin{bmatrix} \bar{\Theta}_{11} & A_2\bar{Q}_1 + B\bar{W}_2 & G\bar{Q}_2 & \bar{\Theta}_{14} & \bar{X}N_1{}^{\mathrm{T}} & \bar{X} \\ * & -e^{-ad}\bar{Q}_1 & 0 & \bar{Q}_1A_2{}^{\mathrm{T}} + \bar{W}_2^{\mathrm{T}}B^{\mathrm{T}} & \bar{Q}_1N_2{}^{\mathrm{T}} & 0 \\ * & * & -e^{-\alpha h}\bar{Q}_2 & \bar{Q}_2G^{\mathrm{T}} & 0 & 0 \\ * & * & * & \bar{\Theta}_{44} & 0 & 0 \\ * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & -\bar{Q}_1 \end{bmatrix} = \bar{\Theta} < 0, \tag{27}$$

$$\tilde{\lambda}_1R^{-1} < \bar{X} < \tilde{\lambda}_2R^{-1}, \tag{28}$$

$$\tilde{\lambda}_3R^{-1} < \bar{Q}_1, \tag{29}$$

$$\tilde{\lambda}_4R^{-1} < \bar{Q}_2, \tag{30}$$

$$\frac{c^*}{\tilde{\lambda}_1} + \left(\frac{1}{\tilde{\lambda}_3} + \frac{1}{\tilde{\lambda}_4}\right)\frac{1}{\alpha}(1 - e^{-\alpha\tau})c^* < \frac{c_2}{\tilde{\lambda}_2}, \tag{31}$$

Remark 4: It is now widely recognized that the time delays are often unavoidable in various practical systems and can affect system performance. Over the past years, the mixed time delays has attracted much attention and variety of results are presented in [39] and [40]. In this paper, the time delay we consider is constant. By constructing different Lyapunov-Krasovskii function, we can extend our results to mixed time-delay systems to reduce the conservativeness.

IV. NUMERICAL SIMULATION

Consider the neutral system (1) with the following matrices:

$$G = \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.1 & 0.2 & -0.6 \\ 0.2 & 0 & 0.3 \end{bmatrix}, \quad R = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -8 & 2 & 1 \\ 0 & -5 & -1 \\ 5 & -1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & -2 & 0 \\ 2 & 0 & 4 \\ 5 & 2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.1 \\ 0.1 \\ 0.3 \end{bmatrix}, M = \begin{bmatrix} -0.3 \\ 0.7 \\ 0.5 \end{bmatrix}, N_1 = \begin{bmatrix} 0.3 & -0.3 & 0.1 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0.2 & -0.4 & 0.5 \end{bmatrix}, H = \begin{bmatrix} 0.3 & 0.6 & 0.3 \end{bmatrix}.$$

Let $c_1 = 1, c^* = 2, c_2 = 5, T = 2.5, \sigma = 1, d = 0.05, h = 0.08, \eta = 0.5, \beta = 0.6$, and choose $F(t) = \sin(t), f(x(t)) = 0.6\sin(x_1(t) - x_3(t))$.

Using the method in Remark 3 to solve the LMIs in Theorem 3 and Theorem 5, we can obtain

$$X = \begin{bmatrix} 22.5147 & -0.5357 & -2.4198 \\ -0.5357 & 16.6342 & 2.8563 \\ -2.4198 & 2.8563 & 1.7731 \end{bmatrix},$$

$$\tilde{Q}_1 = \begin{bmatrix} 11.5908 & -6.3525 & -1.8589 \\ -6.3525 & 9.3303 & 2.0135 \\ -1.8589 & 2.0135 & 0.6342 \end{bmatrix},$$

$$\tilde{Q}_2 = \begin{bmatrix} 167.9825 & -21.0453 & -26.7386 \\ -21.0453 & 173.6349 & 26.1708 \\ -26.7386 & 26.1708 & 40.3957 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -326.9790 & 39.3361 & -84.2151 \end{bmatrix},$$
$$W_2 = \begin{bmatrix} -148.5988 & 44.1303 & 18.0651 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} -26.8605 & 22.0499 & -119.6755 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -16.0833 & -6.9699 & 3.4727 \end{bmatrix}.$$
$$\nu = 18.6985, \lambda_1 = 0.4526, \lambda_2 = 35.3490,$$
$$\lambda_3 = 0.0755, \lambda_4 = 18.5351.$$

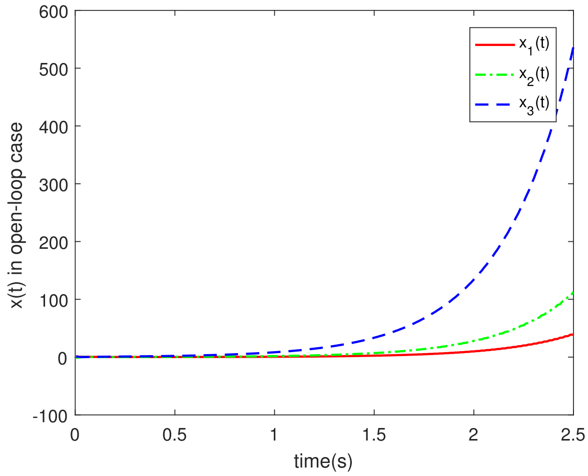

FIGURE 1. System state $x(t)$ in open-loop case.

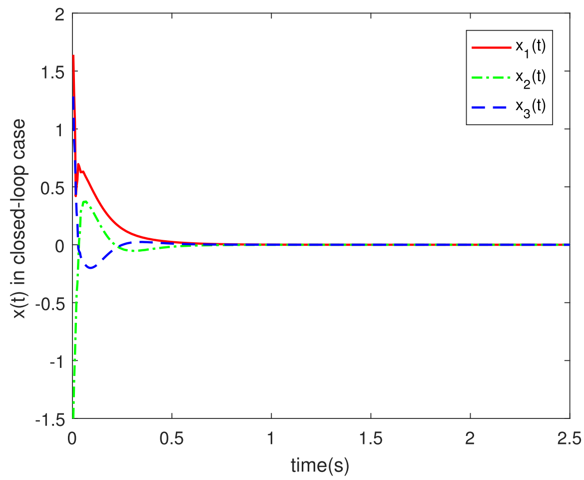

FIGURE 2. System state $x(t)$ in closed-loop case.

We note that the control law (6) contains a sign function $sign(s(t))$, which can cause severe chattering. High-frequency chattering will affect the actual control accuracy and may cause system oscillation to destroy controller components. Therefore, we replace the traditional sign function with a smooth approximate sigmoid function (35).

$$sigm(s(t)) = \frac{2}{1 - e^{-s(t)}} - 1, \tag{35}$$

Simulation results with the initial condition $x(0) = \begin{bmatrix} 0.42 & -0.7 & 0.48 \end{bmatrix}^{\mathrm{T}}$ are shown in Fig.1 - Fig.6. Fig.1 and

$$\bar{\Gamma} = \begin{bmatrix} \bar{\Gamma}_{11} & \bar{P}^{-1}A_2 + \bar{P}^{-1}B\bar{K}_2 & \bar{P}^{-1}G & A_1^{\mathrm{T}} + \bar{K}_1^{\mathrm{T}}B^{\mathrm{T}} \\ * & -\mathrm{e}^{-\alpha d}\bar{Q}_1^{-1} & 0 & A_2^{\mathrm{T}} + \bar{K}_2^{\mathrm{T}}B^{\mathrm{T}} \\ * & * & -\mathrm{e}^{-\alpha h}\bar{Q}_2^{-1} & G^{\mathrm{T}} \\ * & * & * & -\bar{Q}_2 \end{bmatrix},$$

$$\Delta\bar{\Gamma} = \begin{bmatrix} \Delta\bar{\Gamma}_{11} & \bar{P}^{-1}\Delta A_2 - \bar{P}^{-1}B(HB)^{-1}H\Delta A_2 & 0 & \Delta A_1^{\mathrm{T}} - (B(HB)^{-1}H\Delta A_1)^{\mathrm{T}} \\ * & 0 & 0 & \Delta A_2^{\mathrm{T}} - (B(HB)^{-1}H\Delta A_2)^{\mathrm{T}} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix},$$

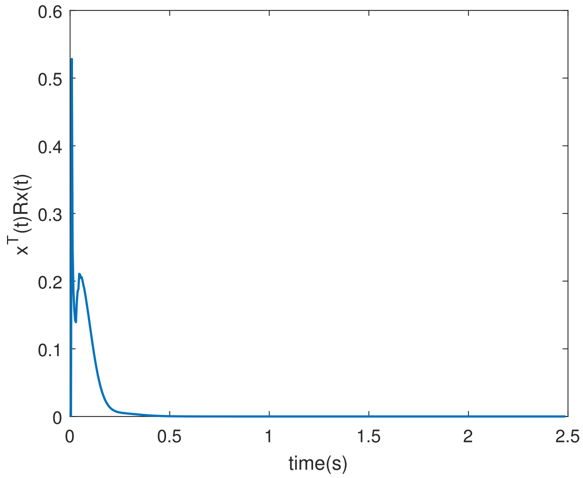

FIGURE 3. Evolution of $x^{\mathrm{T}}(t)Rx(t)$ in closed-loop case.

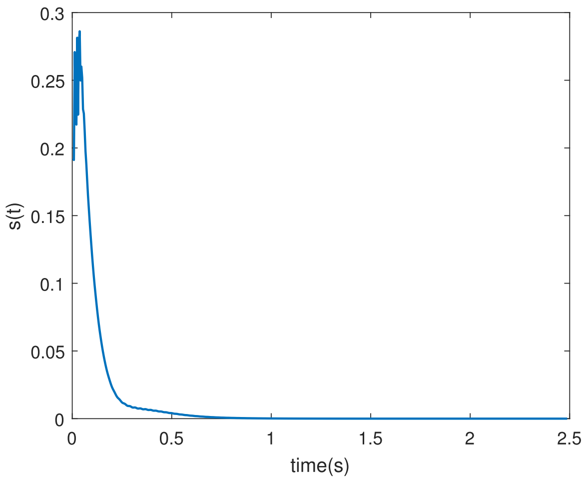

FIGURE 4. Sliding variable $s(t)$.

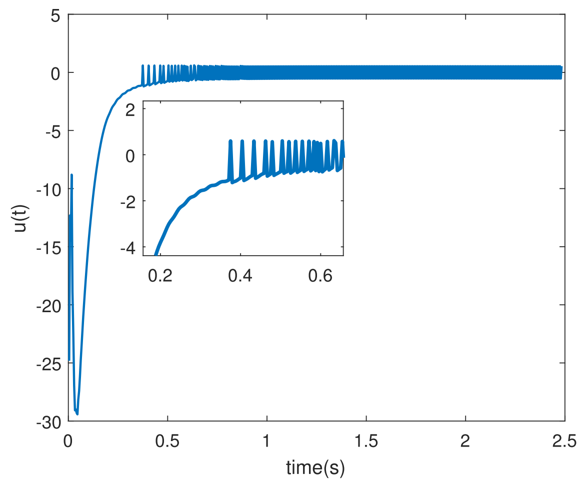

FIGURE 5. Control input $u(t)$ with sign function.

Fig.2 show the curves of the system state in open-loop and closed-loop cases respectively, which can demonstrate the effect of SMC law.

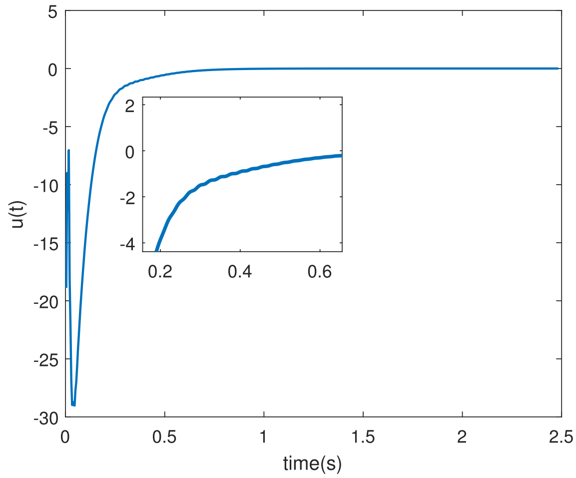

FIGURE 6. Control input $u(t)$ with sigmoid function.

From the given initial state vector $x(0)=\begin{bmatrix}0.42 & -0.7 & 0.48\end{bmatrix}^{\mathrm{T}}$ and know matrix R, we can calculate $x^T(0)Rx(0) = 0.53$, which satisfies that $x^T(0)Rx(0) \leq 1$, and from Fig.3, it is easy to see that the weighted state trajectory $x^{\mathrm{T}}(t)Rx(t)$ does not exceed $c_2 = 5$ during the time interval $[0, T]$, which indicates that the system satisfies finite-time stability about (c_1, c_2, T, R) by the SMC law we designed. Fig.4 shows the curve of sliding variable $s(t)$.

Fig.5 depicts the evolution of control input $u(t)$ with the switching function $sign(s(t))$, we can see that the control input begins to jitter at high frequency when the system state reaches the sliding surface and then remains in this state, which we do not want to see. Fig.6 depicts the evolution of control input $u(t)$ with the switching function $sigm(s(t))$, compared to Fig.5, we can find that the addition of sigmoid function can effectively reduce the chattering phenomenon.

V. CONCLUSION

Based on sliding mode control, we have investigated the finite-time stability problem of a class of uncertain neutral systems. A suitable integral-type sliding surface and an SMC law are designed, which can ensure the system satisfies FTS in time interval given. The obtained conditions are given in the form of LMIs for easy solution, a numerical simulation shows the practicability of the proposed method. The neutral time-delay systems studied in this paper have better universality than general time-delay systems, such as wireless transmission system, standard time-delay system and so on, can be converted into this kind of neutral system, FTS can ensure the transient performance requirements of such systems. It is noteworthy that the results we obtained contain a large number of variables and parameters, which increases the computational complexity, so how to simplify the results and reduce the conservativeness, deserves further research. In the future, we will use adaptive sliding mode control to study uncertain neutral systems with mixed time delays. In addition, we shall extend the proposed method to Markovian jump systems, and investigate the asynchronous sliding mode control problem.

REFERENCES

[1] U. Kotta, S. Z. Sarpturk, and Y. Istefanopulos, "On the stability of discrete time sliding mode control systems," *IEEE Trans. Autom. Control*, vol. 34, no. 9, pp. 1021–1022, Sep. 1989.

[2] C. Du, F. Li, and C. Yang, "An improved homogeneous polynomial approach for adaptive sliding-mode control of Markov jump systems with actuator faults," *IEEE Trans. Autom. Control*, vol. 65, no. 3, pp. 955–969, Mar. 2020.

[3] J. Zhang, D. Shi, and Y. Xia, "Design of sliding mode output feedback controllers via dynamic sliding surface," *Automatica*, vol. 124, Feb. 2021, Art. no. 109310.

[4] F. Li, L. Wu, P. Shi, and C.-C. Lim, "State estimation and sliding mode control for semi-Markovian jump systems with mismatched uncertainties," *Automatica*, vol. 51, pp. 385–393, Jan. 2015.

[5] S. R. Kumar, S. Rao, and D. Ghose, "Sliding-mode guidance and control for all-aspect interceptors with terminal angle constraints," *J. Guid. Control Dyn.*, vol. 35, no. 4, pp. 1230–1244, 2012.

[6] H. Mekki, O. Benzineb, D. Boukhetala, M. Tadjine, and M. Benbouzid, "Sliding mode based fault detection, reconstruction and fault tolerant control scheme for motor systems," *ISA Trans.*, vol. 57, pp. 340–351, Jul. 2015.

[7] G. Jianguo, L. Yuchao, Z. Jun, and W. Guoqing, "A new nonlinear sliding mode control system design," in *Proc. 27th Chin. Control Decis. Conf.*, Qingdao, China, May 2015, pp. 4490–4493.

[8] N. N. Wang and F. Hao, "Event-triggered sliding mode control with adaptive neural networks for uncertain nonlinear systems," *Neurocomputing*, vol. 436, pp. 184–197, May 2021.

[9] F. Li, C. Du, C. Yang, L. Wu, and W. Gui, "Finite-time asynchronous sliding mode control for Markovian jump systems," *Automatica*, vol. 109, Nov. 2019, Art. no. 108503.

[10] R. Sakthivel, R. Sakthivel, V. Nithya, P. Selvaraj, and O. M. Kwon, "Fuzzy sliding mode control design of Markovian jump systems with time-varying delay," *J. Franklin Inst.*, vol. 355, no. 14, pp. 6353–6370, Sep. 2018.

[11] L. G. Wu, X. J. Su, and P. Shi, "Sliding mode control with bounded $\mathscr{L}_2$ gain performance of Markovian jump singular time-delay systems," *Automatica*, vol. 48, no. 8, pp. 1929–1933, Aug. 2012.

[12] M. Yousefi and T. Binazadeh, "Delay-independent sliding mode control of time-delay linear fractional order systems," *Trans. Inst. Meas. Control*, vol. 40, no. 4, pp. 1212–1222, 2018.

[13] Q. Zhang, L. Li, X.-G. Yan, and S. K. Spurgeon, "Sliding mode control for singular stochastic Markovian jump systems with uncertainties," *Automatica*, vol. 79, no. 1, pp. 27–34, May 2017.

[14] L. Wu, W. X. Zheng, and H. Gao, "Dissipativity-based sliding mode control of switched stochastic systems," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 785–791, Mar. 2013.

[15] P. Dorato, "Short time stability in linear time-varying systems," *Proc. IRE Int. Conv. Rec.*, pp. 83–87, May 1961.

[16] L. Weiss and E. Infante, "Finite time stability under perturbing forces and on product spaces," *IEEE Trans. Autom. Control*, vol. AC-12, no. 1, pp. 54–59, Feb. 1967.

[17] S. Zhang, Y. Guo, S. Wang, Z. Liu, and X. Hu, "Finite–time output stability of impulse switching system with norm–bounded state constraint," *IEEE Access*, vol. 7, pp. 82927–82938, 2019.

[18] O. Mofid and S. Mobayen, "Adaptive sliding mode control for finite-time stability of quad-rotor UAVs with parametric uncertainties," *ISA Trans.*, vol. 72, pp. 1–14, Jan. 2018.

[19] P. Miao, L. Fan, and D. Zhang, "Two new finite-time convergence criterions and application to solve time varying Sylvester equation and pseudo-inverse of a matrix," *Int. J. Control Autom. Syst.*, vol. 19, no. 4, pp. 1570–1577, 2021.

[20] S. Ibrir, "Finite-time stabilization procedures for discrete-time nonlinear systems in feedback form," *IEEE Access*, vol. 9, pp. 27226–27238, 2021.

[21] L. H. Zhang, Y. Y. Zhou, W. H. Qi, J. D. Cao, J. Cheng, Y. L. Wei, X. X. Yan, and S. W. Li, "Non-fragile observer-based H_∞ finite-time sliding mode control," *Appl. Math. Comput.*, vol. 375, Jun. 2020, Art. no. 125069.

[22] O. Naifar, A. M. Nagy, A. B. Makhlouf, M. Kharrat, and M. A. Hammami, "Finite-time stability of linear fractional-order time-delay systems," *Int. J. Robust Nonlinear Control*, vol. 29, no. 1, pp. 180–187, Jan. 2019.

[23] Z. Yang, J. Zhang, J. H. Hu, and J. Mei, "New results on finite-time stability for fractional-order neural networks with proportional delay," *Neurocomputing*, vol. 442, no. 1, pp. 327–336, 2021.

[24] L. Van Hien and D. T. Son, "Finite-time stability of a class of non-autonomous neural networks with heterogeneous proportional delays," *Appl. Math. Comput.*, vol. 251, pp. 14–23, Jan. 2015.

[25] S. Tyagi and S. C. Martha, "Finite-time stability for a class of fractional-order fuzzy neural networks with proportional delay," *Fuzzy Sets Syst.*, vol. 381, pp. 68–77, Feb. 2020.

[26] Z. H. Shao, "Robust stability of uncertain neutral delay systems," *IEEE Trans. Ind. Electron.*, vol. 51, no. 3, pp. 730–733, Jun. 2004.

[27] H. Mayer, K. S. Zaenker, and U. A. D. Heiden, "A basic mathematical model of the immune response," *Chaos*, vol. 5, no. 1, pp. 155–161, Mar. 1995.

[28] R. D. Driver, "A neutral system with state-dependent delay," *J. Differ. Equ.*, vol. 54, no. 1, pp. 73–86, Aug. 1984.

[29] B. Wang and S. M. Zhong, "On novel stability criterion of neutral system with time-varying delay," *Appl. Math. Comput.*, vol. 189, no. 2, pp. 1124–1130, Jun. 2007.

[30] F. Mazenc, "Stability analysis of time-varying neutral time-delay systems," *IEEE Trans. Autom. Control*, vol. 60, no. 2, pp. 540–546, Feb. 2015.

[31] R. Rabah, G. M. Sklyar, and A. V. Rezounenko, "Stability analysis of neutral type systems in Hilbert space," *J. Differ. Equ.*, vol. 214, no. 2, p. 391–428, Jul. 2005.

[32] N. Yugang and W. C. H. Daniel, "Sliding mode control for neutral delay systems with unmatched uncertainties," in *Proc. 27th Chin. Control Conf.*, Kunming, China, Jul. 2008, pp. 571–575.

[33] S. Liu and Z. Xiang, "Exponential H_∞ output tracking control for switched neutral system with time-varying delay and nonlinear perturbations," *Circuits Syst. Signal Process.*, vol. 32, no. 1, pp. 103–121, 2012.

[34] J. Zhang, P. Shi, and J. Q. Qiu, "Robust stability criteria for uncertain neutral system with time delay and nonlinear uncertainties," *Chaos Solitons Fractals*, vol. 38, no. 1, pp. 160–167, Oct. 2008.

[35] K. Mathiyalagan and G. Sangeetha, "Finite-time stabilization of nonlinear time delay systems using LQR based sliding mode control," *J. Franklin Inst.*, vol. 356, no. 7, pp. 3948–3964, May 2019.

[36] Y. He, J. Yi, Y. Liu, and Y. Jing, "Delay-dependent sliding mode control for uncertain neutral delay systems," in *Proc. Chin. Control Decis. Conf.*, Yantai, China, Jul. 2008, pp. 5001–5005.

[37] Y. G. He, L. Zhao, and M. Y. Zhang, "Sliding mode control for a class of uncertain neutral delay systems," *Proc. Eng.*, vol. 15, no. 1, pp. 1181–1185, 2011.

[38] J. Song, Y. Niu, and Y. Zou, "Finite-time stabilization via sliding mode control," *IEEE Trans. Autom. Control*, vol. 62, no. 3, pp. 1478–1483, Mar. 2017.

[39] Y. Liu, Z. Wang, L. Ma, and F. E. Alsaadi, "Robust H_∞ control for a class of uncertain nonlinear systems with mixed time-delays," *J. Franklin Inst.*, vol. 355, no. 14, pp. 6339–6352, Sep. 2018.

[40] R. Saravanakumar, M. S. Ali, H. Huang, J. Cao, and Y. H. Joo, "Robust H_∞ state-feedback control for nonlinear uncertain systems with mixed time-varying delays," *Int. J. Control Autom. Syst.*, vol. 16, no. 1, pp. 225–233, Feb. 2018.

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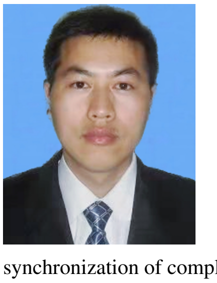

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