IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received September 15, 2021, accepted October 3, 2021, date of publication October 5, 2021, date of current version October 18, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3118055

Soft Rough q-Rung Orthopair m-Polar Fuzzy Sets and q-Rung Orthopair m-Polar Fuzzy Soft Rough Sets and Their Applications

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This work was supported by the Key Projects of Natural Science Research in Colleges and Universities of Anhui Province under Grant KJ2020A0642.

ABSTRACT The notion of a q-rung orthopair fuzzy soft rough set ($^{q}ROFSRS$) appeared as an extension of q-rung orthopair fuzzy set ($^{q}ROFS$) and q-rung orthopair fuzzy soft set ($^{q}ROFSS$) with the aid of rough set (RS) definition. Thus, $^{q}ROFSRS$ and m-polar fuzzy set ($_{m}PFS$) are convenient to deal with uncertain knowledge which helps us to solve many problems in decision making. In this paper, we define the soft rough q-rung orthopair m-polar fuzzy sets ($^{q}RO_{\mathrm{m}}PFS$) and q-rung orthopair m-polar fuzzy soft rough sets ($^{q}RO_{\mathrm{m}}PFSRS$) through crisp soft and q- rung orthopair (q-RO) m-polar fuzzy soft approximation space. The related characteristics of these models are also studied. Then, we construct two new algorithms for these models to solve MADM issues. The successful application and corresponding comparative analyses proves that our proposed models are rational and effective.

INDEX TERMS q-rung orthopair fuzzy soft rough set, m-polar fuzzy set, soft rough q-rung orthopair m-polar fuzzy sets, q-rung orthopair m-polar fuzzy soft rough sets, multi-attribute decision making.

I. INTRODUCTION

The rapid of research articles become very huge, especially in mathematics. Numerous suggestions were made to solve real-world problems using mathematical techniques by way of appropriate equations or formulas in helping decision makers to make their best decisions. To solve problems involving uncertainty, fuzzy sets (FS) was introduced by Zadeh [1] in 1965.

Later in 1982, Pawlak introduced the notion called Rough Sets (RS) [2], [3]. The beauty of RS is it is able to divide the area into three parts (Lower, Upper, and Boundary region). This idea comes from the meaning of the topology concept. Eight years later, Dubois and Prade [4] combine the notion of RS and FS, to form rough fuzzy sets and fuzzy rough sets. Since then, many researchers studied further on RS and FS as in the following published articles [5]–[15].

The associate editor coordinating the review of this manuscript and approving it for publication was Geng-Ming Jiang.

To reduce the uncertainty and vagueness of knowledge, Molodtsov [16] developed soft sets (SS). Feng *et al.* [17] established the soft rough sets (SRS) by merging SS and RS in 2011. Also, in 2017, Yager [18] defined a new concept called q-rung orthopair fuzzy sets ($^{q}ROFS$) as a refinement to the notion of Pythagorean fuzzy sets (PFS) [19], [20] and intuitionistic fuzzy sets (IFS) [21]. IFS and PFS are considered as special cases of $^{q}ROFS$, when $q = 1$ and $q = 2$, respectively. There are numerous research on IFS [22]–[27], PFS [28]–[36] and $^{q}ROFS$ [37]–[45].

In 1994, as an extension of FS whose membership grade range is $[-1, 1]$, bipolar fuzzy sets (BFS) was proposed by Zhang [46]. In a BFS, the membership grade 0 of a variable means that the variable is irrelevant to the corresponding property, the membership grade $(0, 1]$ of a variable points out that the variable somewhat fulfills the property, while the membership grade $[-1, 0)$ of a variable point out that the variable somewhat satisfies the implicit counter-property. The idea which lies behind such description is connected with the existence of "bipolar information" (e.g., plus information and minus information) about the given set. Plus information represents what is granted to be possible, while minus information represents what is considered to be impossible. Then to generalize the BFS to help experts to deal with uncertainty, the meaning of m-polar fuzzy sets ($_mPS$) was mooted by Chen *et al.* [47]. They proved that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical tools. In many real-life complicated problems, data sometimes comes from an employee ($n \geq 2$), that is, multipolar information (not just bipolar information, which corresponds to two-valued logic) exists. There are many applications of m-polar fuzzy sets to decision-making problems when it is compulsory to make assessments with a group of agreements. Akram *et al.* [48], [49] proposed the soft rough m-polar fuzzy and m-polar fuzzy soft rough sets. By merging the concepts of SRS, PFS, and $_mPS$, Riaz and Hashmi [50] investigated the Pythagorean m-polar fuzzy sets (P_mPFS), soft rough Pythagorean m-polar fuzzy sets (SRP_mPFS) and Pythagorean m-polar fuzzy soft rough sets (P_mPFSRS). The concept of q-rung orthopair m-polar fuzzy sets ($^qRO_mPFS$) was then defined by Riaz *et al.* [51].

Using the notions of SS, SRS, and ^qROFS, Hussain *et al.* [52] proposed the q-rung orthopair fuzzy soft sets and their application. Wang *et al.* [53] explained the ^qROF soft rough sets(^qROFSRS) with a few applications. Riaz *et al.* [54] introduced the notion of soft rough q-rung orthopair fuzzy sets and some of their properties were discussed. Thereafter, many researchers studied SRS, SS, and their applications such as [55]–[60], [64].

From these interesting studies, we intend to develop a hybrid of SRS and $^qRO_mPFS$ and put forward a new model called q-rung orthopair m-polar fuzzy soft rough sets ($^qRO_mPFSRS$) and soft rough q-rung orthopair fuzzy sets (SR^qRO_mPFS). These combinations provide us with the property of $^qRO_mPFS$ and soft rough sets together which maximize the handling of uncertain data. Thus our proposed methods are generalized extensions of Akram *et al.* [48], Riaz and Hashmi [50] and Riaz *et al.* [54]. When $q = 1$, the presented formula reduces to those methods in [48] and [49] and if $q = 2$, it reduces to those methods in [50]. Our proposed method will cater for m sets which make our studies are reliable, compared to [54] which catered for only a single set. Their relevant properties will be investigated, a few definitions and theorems will be promulgated along with illustrative examples. We will then proceed to construct two algorithms along with their applications. Finally, we will run comparative analyses on the outcomes of those two algorithms.

The structure of this paper is as follows. The preliminary of basic notions will be introduced in Section 2. Section 3 will discuss the novel concept of SR^qRO_mPFS and the related characteristics. The hybrid concept of $^qRO_mPFSRS$ will be proposed and its associated properties are discussed in Section 4. In Section 5, we give an illustrative example to show the applicability of the proposed constructed algorithms along with the comparative analyses, followed by the conclusion in Section 6.

II. PRELIMINARIES

Now, we give some basic notions on IFS, PFS and ^qROF before defining soft rough q-rung orthopair m-polar fuzzy sets SR^qRO_mPFS in the next section.

Definition 1 ([21], [22]): If Ξ is the origin set. For every $\hat{\mathscr{H}} \in \Xi$, if we have a membership grade $\vartheta_{\mathcal{E}} : \Xi \to [0, 1]$ and a non-membership grade $\varkappa_{\mathcal{E}} : \Xi \to [0, 1]$. Define the IFS $\mathcal{E}$ as indicated below.

$$\mathcal{E} = \{(\hat{\mathscr{H}}, \vartheta_{\mathcal{E}}(\hat{\mathscr{H}}), \varkappa_{\mathcal{E}}(\hat{\mathscr{H}}))\},$$

where $0 \leq \vartheta_{\mathcal{E}}(\hat{\mathscr{H}}) + \varkappa_{\mathcal{E}}(\hat{\mathscr{H}}) \leq 1$.

Also, $\hat{\mathscr{H}} = (\vartheta_{\hat{\mathscr{H}}}, \varkappa_{\hat{\mathscr{H}}})$ is said to be an intuitionistic fuzzy number (IFN), if

$$0 \leq \vartheta_{\hat{\mathscr{H}}},\ \varkappa_{\hat{\mathscr{H}}} \leq 1,\ \varrho_{\hat{\mathscr{H}}} = 1 - \vartheta_{\mathcal{E}}(\hat{\mathscr{H}}) - \varkappa_{\mathcal{E}}(\hat{\mathscr{H}}),$$

and $0 \leq \vartheta_{\hat{\mathscr{H}}} + \varkappa_{\hat{\mathscr{H}}} \leq 1$.

To treat some problem in IFS which appeared in real issues, Yager in 2014 defined Pythagorean fuzzy sets (PFSs) as follows.

Definition 2 ([19], [20]): If Ξ is the origin set. For every $\hat{\mathscr{H}} \in \Xi$, if we have a membership grade $\vartheta_{\mathcal{E}} : \Xi \to [0, 1]$ and a non-membership grade $\varkappa_{\mathcal{E}} : \Xi \to [0, 1]$. Define the PFS $\mathcal{E}$ as indicated below.

$$\mathcal{E} = \{(\hat{\mathscr{H}}, \vartheta_{\mathcal{E}}(\hat{\mathscr{H}}), \varkappa_{\mathcal{E}}(\hat{\mathscr{H}}))\},$$

where $0 \leq \vartheta^2_{\mathcal{E}}(\hat{\mathscr{H}}) + \varkappa^2_{\mathcal{E}}(\hat{\mathscr{H}}) \leq 1$.

Also, $\hat{\mathscr{H}} = (\vartheta_{\hat{\mathscr{H}}}, \varkappa_{\hat{\mathscr{H}}})$ is said to be a Pythagorean fuzzy number (PFN), if

$$\varrho_{\hat{\mathscr{H}}} = \sqrt{1 - \vartheta^2_{\mathcal{E}}(\hat{\mathscr{H}}) - \varkappa^2_{\mathcal{E}}(\hat{\mathscr{H}})}, \quad 0 \leq \vartheta^2_{\hat{\mathscr{H}}} + \varkappa^2_{\hat{\mathscr{H}}} \leq 1.$$

Generalizing further, Yager presented the notion of q-rung orthopair fuzzy sets in 2017 (q-ROFs) as follows.

Definition 3 [18]: If Ξ is the origin set. For every $\hat{\mathscr{H}} \in \Xi$, if we have a membership grade $\vartheta_{\mathcal{E}} : \Xi \to [0, 1]$ and a non-membership grade $\varkappa_{\mathcal{E}} : \Xi \to [0, 1]$. Define the q-ROFs $\mathcal{E}$ as indicated below.

$$\mathcal{E} = \{(\hat{\mathscr{H}}, \vartheta_{\mathcal{E}}(\hat{\mathscr{H}}), \varkappa_{\mathcal{E}}(\hat{\mathscr{H}}))\},$$

where $0 \leq \vartheta^{\nabla}_{\mathcal{E}}(\hat{\mathscr{H}}) + \varkappa^{\nabla}_{\mathcal{E}}(\hat{\mathscr{H}}) \leq 1$, where $\nabla \geq 1$.

Also, $\hat{\mathscr{H}} = (\vartheta_{\hat{\mathscr{H}}}, \varkappa_{\hat{\mathscr{H}}})$ is said to be a q-ROF number (q-ROFN), if

$$\varrho_{\hat{\mathscr{H}}} = \sqrt[\nabla]{1 - \vartheta^{\nabla}_{\mathcal{E}}(\hat{\mathscr{H}}) - \varkappa^{\nabla}_{\mathcal{E}}(\hat{\mathscr{H}})}, \quad 0 \leq \vartheta^{\nabla}_{\hat{\mathscr{H}}} + \varkappa^{\nabla}_{\hat{\mathscr{H}}} \leq 1.$$

Definition 4 [18]: If $\hat{\mathscr{E}}_1 = (\vartheta_{\hat{\mathscr{E}}_1}, \varkappa_{\hat{\mathscr{E}}_1})$ and $\hat{\mathscr{E}}_2 = (\vartheta_{\hat{\mathscr{E}}_2}, \varkappa_{\hat{\mathscr{E}}_2})$, for $\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2$ is q-ROFNs. Then $\forall \hat{\mathscr{H}} \in \Xi$, we have the following relation.

(1) $\hat{\mathscr{E}}^c_1 = \{(\hat{\mathscr{H}}, \varkappa_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}), \vartheta_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}))\}$.

(2) $\hat{\mathscr{E}}_1 = \hat{\mathscr{E}}_2 \iff \vartheta_{\hat{\mathscr{E}}_1} = \vartheta_{\hat{\mathscr{E}}_2}$ and $\varkappa_{\hat{\mathscr{E}}_1} = \varkappa_{\hat{\mathscr{E}}_2}$.

(3) $\hat{\mathscr{E}}_1 \leq \hat{\mathscr{E}}_2 \iff \vartheta_{\hat{\mathscr{E}}_1} \leq \vartheta_{\hat{\mathscr{E}}_2}$ and $\varkappa_{\hat{\mathscr{E}}_1} \leq \varkappa_{\hat{\mathscr{E}}_2}$.

(4) $\hat{\mathscr{E}}_1 \cap \hat{\mathscr{E}}_2 = \{(\hat{\mathscr{H}}, \vartheta_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \wedge \vartheta_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}), \varkappa_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \vee \varkappa_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}))\}$.

(5) $\hat{\mathscr{E}}_1 \cup \hat{\mathscr{E}}_2 = \{(\hat{\mathscr{H}}, \vartheta_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \vee \vartheta_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}), \varkappa_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \wedge \varkappa_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}))\}$.

(6) $\hat{\mathscr{H}}_1 - \hat{\mathscr{H}}_2 = \hat{\mathscr{H}}_1 \cap \hat{\mathscr{H}}_2^c$.

(7) $\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2 =$
$(\hat{\mathscr{H}}, \sqrt[\nabla]{(\vartheta_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla + (\vartheta_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla - (\vartheta_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}, \varkappa_{\hat{\mathcal{E}}_1}\varkappa_{\hat{\mathcal{E}}_2})$.

(8) $\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2 =$
$(\hat{\mathscr{H}}, \vartheta_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{(\varkappa_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla + (\varkappa_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla - (\varkappa_{\hat{\mathcal{E}}_1}\varkappa_{\hat{\mathcal{E}}_2})^\nabla})$.

Ali [43] gave another property on q-ROFN below.

Definition 5 [43]: If $\hat{\mathscr{E}} = (\vartheta_{\hat{\mathscr{E}}}, \varkappa_{\hat{\mathscr{E}}})$ is a q-ROFN, then we have the following.

$$\Box\hat{\mathscr{E}} = \left(\vartheta_{\hat{\mathscr{E}}}, (1 - (\vartheta_{\hat{\mathscr{E}}})^\nabla)^{\frac{1}{\nabla}}\right)$$
$$\Diamond\hat{\mathscr{E}} = \left(\varkappa_{\hat{\mathscr{E}}}, (1 - (\varkappa_{\hat{\mathscr{E}}})^\nabla)^{\frac{1}{\nabla}}\right)$$

Next, Chen *et. al.* [47] defined m-polar fuzzy sets as follows.

Definition 6 [47]: If Ξ is the origin set, where $\phi : \Xi \to [0, 1]^m$ is the set of all m-polar fuzzy sets on Ξ.

Riaz and Hashmi [50] extended it to a Pythagorean form below.

Definition 7 [50]: If Ξ is the origin set. For every $\hat{\mathscr{H}} \in \Xi$, if we have a membership grade $\vartheta^r_{\mathcal{E}} : \Xi \to [0, 1]$ and a non-membership grade $\varkappa^r_{\mathcal{E}} : \Xi \to [0, 1]$. Define the Pythagorean m-polar fuzzy sets (P_mPFS) $\mathcal{E}$ as indicated below.

$$\mathcal{E} = \{(\hat{\mathscr{H}}, \vartheta^r_{\mathcal{E}}(\hat{\mathscr{H}}), \varkappa^r_{\mathcal{E}}(\hat{\mathscr{H}}))\},$$

where $0 \leq (\vartheta^r_{\mathcal{E}}(\hat{\mathscr{H}}))^2 + (\varkappa^r_{\mathcal{E}}(\hat{\mathscr{H}}))^2 \leq 1$, where $r = 1, 2, .., m$.

Riaz *et.al.* [51] further extended m-polar fuzzy sets of Chen *et. al.* [47] to q-rung orthopair form below.

Definition 8 [51]: If Ξ is the origin set. For every $\hat{\mathscr{H}} \in \Xi$, if we have a membership grade $\vartheta^r_{\mathcal{E}} : \Xi \to [0, 1]$ and a non-membership grade $\varkappa^r_{\mathcal{E}} : \Xi \to [0, 1]$. Define the q-rung orthopair m-polar fuzzy sets (${}^qRO_mPFS$) $\mathcal{E}$ as indicated below.

$$\mathcal{E} = \{(\hat{\mathscr{H}}, \vartheta^r_{\mathcal{E}}(\hat{\mathscr{H}}), \varkappa^r_{\mathcal{E}}(\hat{\mathscr{H}}))\},$$

where $0 \leq (\vartheta^r_{\mathcal{E}}(\hat{\mathscr{H}}))^\nabla + (\varkappa^r_{\mathcal{E}}(\hat{\mathscr{H}}))^\nabla \leq 1$, where $r = 1, 2, .., m$ and $\nabla \geq 1$.

Definition 9 [51]: If $\hat{\mathscr{E}}_1 = (\vartheta^r_{\hat{\mathscr{E}}_1}, \varkappa^r_{\hat{\mathscr{E}}_1})$ and $\hat{\mathscr{E}}_2 = (\vartheta^r_{\hat{\mathscr{E}}_2}, \varkappa^r_{\hat{\mathscr{E}}_2})$, for $\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2$ is ${}^qRO_mPFN$. Then $\forall \hat{\mathscr{H}} \in \Xi$, we have the following relation.

(1) $\hat{\mathscr{E}}^c_1 = \{(\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}), \vartheta^r_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}))\}$.

(2) $\hat{\mathscr{E}}_1 = \hat{\mathscr{E}}_2 \iff \vartheta^r_{\hat{\mathscr{E}}_1} = \vartheta^r_{\hat{\mathscr{E}}_2}$ and $\varkappa^r_{\hat{\mathscr{E}}_1} = \varkappa^r_{\hat{\mathscr{E}}_2}$.

(3) $\hat{\mathscr{E}}_1 \leq \hat{\mathscr{E}}_2 \iff \vartheta^r_{\hat{\mathscr{E}}_1} \leq \vartheta^r_{\hat{\mathscr{E}}_2}$ and $\varkappa^r_{\hat{\mathscr{E}}_1} \leq \varkappa^r_{\hat{\mathscr{E}}_2}$.

(4) $\hat{\mathscr{E}}_1 \cap \hat{\mathscr{E}}_2 = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}))\}$.

(5) $\hat{\mathscr{E}}_1 \cup \hat{\mathscr{E}}_2 = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathscr{E}}_1}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathscr{E}}_2}(\hat{\mathscr{H}}))\}$.

(6) $\hat{\mathscr{H}}_1 - \hat{\mathscr{H}}_2 = \hat{\mathscr{H}}_1 \cap \hat{\mathscr{H}}_2^c$.

(7) $\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2 =$
$(\hat{\mathscr{H}}, \sqrt[\nabla]{(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2})$.

(8) $\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2 =$
$(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{(\varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla + (\varkappa^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla - (\varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2}))^\nabla})$

Molodtsov [16] defined the soft set as below.

Definition 10 [16]: If Ξ is the origin set, and let $\mathscr{E} \subseteq \Xi$. So, $\hat{\mathscr{S}} = (\mathscr{F}, \mathscr{A})$ is a soft set over Ξ, when $\mathscr{A} \subseteq \mathscr{E}$ and $\mathscr{F} : \mathscr{A} \to \mathcal{P}(\Xi)$.

Lately, the notion of q-rung orthopair fuzzy soft set (${}^qROFSS$) was investigated as follows.

Definition 11 [52]: If Ξ is the origin set. For every $\hat{\mathscr{H}} \in \Xi$, let $\mathscr{A} \subseteq \mathcal{E}$ and $\mathscr{F} : \mathscr{A} \to {}^qROFSS(\Xi)$. Then define the ${}^qROFSS$ $\mathcal{E}$ as indicated below.

$$\mathscr{F} = \{(\hat{\mathscr{H}}, \vartheta_{\mathcal{E}}(\hat{\mathscr{H}}), \varkappa_{\mathcal{E}}(\hat{\mathscr{H}}))\},$$

where $0 \leq \vartheta^\nabla_{\mathcal{E}}(\hat{\mathscr{H}}) + \varkappa^\nabla_{\mathcal{E}}(\hat{\mathscr{H}}) \leq 1$, where $\nabla \geq 1$.

Also, $\hat{\mathscr{H}} = (\vartheta_{\hat{\mathscr{H}}}, \varkappa_{\hat{\mathscr{H}}})$ is said to be a q-ROFS number (q-ROFSN), if

$$\varrho_{\hat{\mathscr{H}}} = \sqrt[\nabla]{1 - \vartheta^\nabla_{\mathcal{E}}(\hat{\mathscr{H}}) - \varkappa^\nabla_{\mathcal{E}}(\hat{\mathscr{H}})},\ 0 \leq \vartheta^\nabla_{\hat{\mathscr{H}}} + \varkappa^\nabla_{\hat{\mathscr{H}}} \leq 1.$$

Wang *et al.* [53] defined ${}^qROFSS$ from blow.

Definition 12 [53]: If Ξ is the origin set. For every $\hat{\mathscr{H}} \in \Xi$ and let $(\mathcal{F},\mathscr{A})$ be a ${}^qROFSS$. Then for $\mathscr{E} \subseteq \Xi \times \mathscr{A}$ is ${}^qROFSS$ relation is defined as follows.

$$\mathscr{E} = \{(\hat{\mathscr{H}}, \vartheta_{\mathcal{E}}(\hat{\mathscr{H}}), \varkappa_{\mathcal{E}}(\hat{\mathscr{H}}))\},$$

where $0 \leq \vartheta^\nabla_{\mathcal{E}}(\hat{\mathscr{H}}) + \varkappa^\nabla_{\mathcal{E}}(\hat{\mathscr{H}}) \leq 1$, where $\nabla \geq 1$.

Also, $\hat{\mathscr{H}} = (\vartheta_{\hat{\mathscr{H}}}, \varkappa_{\hat{\mathscr{H}}})$ is said to be a q-ROFSR number (q-ROFSRN), if

$$\varrho_{\hat{\mathscr{H}}} = \sqrt[\nabla]{1 - \vartheta^\nabla_{\mathcal{E}}(\hat{\mathscr{H}}) - \varkappa^\nabla_{\mathcal{E}}(\hat{\mathscr{H}})},\ 0 \leq \vartheta^\nabla_{\hat{\mathscr{H}}} + \varkappa^\nabla_{\hat{\mathscr{H}}} \leq 1.$$

III. SOFT ROUGH q-RUNG ORTHOPAIR m-POLAR FUZZY SETS

In this section, we will define and illustrate the notion of soft rough q-rung orthopair m-polar fuzzy sets SR^qRO_mPFS and also discuss their relevant properties.

Definition 13: If Ξ is the origin set, $\oint$ is the provisory features, and σ is the crisp soft relation, then $(\Xi, \oint, \sigma)$ is a CSAS. For any $\hat{\mathcal{E}} \in {}^qRO_mPFS(\oint)$, the soft rough ${}^qRO_mPFS$-lower and soft rough ${}^qRO_mPFS$-upper approximations (SR^qRO_mPFSLA, SR^qRO_mPFSUA), which are denoted by $\underline{\mathscr{K}}$ and $\overline{\mathscr{K}}$,respectively, are as follows.

$$\underline{\mathscr{K}}(\hat{\mathcal{E}}) = \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{H}})} (\vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{H}})} (\varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})\Big)\Big\},$$
$$\overline{\mathscr{K}}(\hat{\mathcal{E}}) = \Big\{\Big(\hat{\mathscr{H}}, \bigvee_{\vartheta \in \sigma(\hat{\mathscr{H}})} (\vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})), \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{H}})} (\varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})\Big)\Big\},$$

where $\hat{\mathscr{H}} \in \Xi$ and $\nabla = 1, 2, \ldots, n$. If $\underline{\mathscr{K}}(\hat{\mathcal{E}}) \neq \overline{\mathscr{K}}(\hat{\mathcal{E}})$, then $\hat{\mathcal{E}}$ is a soft rough q-rung orthopair m-polar fuzzy sets, otherwise, it is definable.

Example 1: If $\Xi = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2, \hat{\mathscr{H}}_3, \hat{\mathscr{H}}_4, \hat{\mathscr{H}}_5\}$ is the origin set and $\oint = \{\oint_1, \oint_2, \oint_3, \oint_4\}$ is the features set. Suppose that

$(\mathcal{S}, \oint)$ be a soft set on Ξ as

$$\mathcal{S}(\oint_1) = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2, \hat{\mathscr{H}}_3\}, \mathcal{S}(\oint_2) = \{\hat{\mathscr{H}}_2, \hat{\mathscr{H}}_4, \hat{\mathscr{H}}_5\},$$
$$\mathcal{S}(\oint_3) = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2, \hat{\mathscr{H}}_3, \hat{\mathscr{H}}_4\}, \mathcal{S}(\oint_4) = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_4, \hat{\mathscr{H}}_5\}.$$

Thus the relation is as follows

$$\sigma = \{(\hat{\mathscr{H}}_1, \oint_1), (\hat{\mathscr{H}}_1, \oint_3), (\hat{\mathscr{H}}_1, \oint_4), (\hat{\mathscr{H}}_2, \oint_1), (\hat{\mathscr{H}}_2, \oint_2), (\hat{\mathscr{H}}_2, \oint_3), (\hat{\mathscr{H}}_3, \oint_1), (\hat{\mathscr{H}}_3, \oint_3), (\hat{\mathscr{H}}_4, \oint_2), (\hat{\mathscr{H}}_4, \oint_3), (\hat{\mathscr{H}}_4, \oint_4), (\hat{\mathscr{H}}_5, \oint_2), (\hat{\mathscr{H}}_5, \oint_4)\},$$

and $\hat{\mathcal{E}} \in {}^{q}RO_{m}PFS(\oint)$ such that

$$\hat{\mathcal{E}} = \{(\oint_1, (0.73, 0.12), (0.81, 0.42)), (\oint_2, (0.39, 0.11), (0.55, 0.11)), (\oint_3, (0.91, 0.18), (0.32, 0.12)), (\oint_4, (0.87, 0.24), (0.78, 0.21))\}.$$

Then we count the values of the $SR^{q}RO_{m}PFSLA$ and $SR^{q}RO_{m}PFSUA$ as follows.

$\underline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.73, 0.24), (0.32, 0.42)), (\hat{\mathscr{H}}_2, (0.39, 0.18), (0.32, 0.42)), (\hat{\mathscr{H}}_3, (0.73, 0.18), (0.32, 0.42)), (\hat{\mathscr{H}}_4, (0.39, 0.24), (0.32, 0.21)), (\hat{\mathscr{H}}_5, (0.39, 0.24), (0.55, 0.21))\}$,

$\overline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.91, 0.12), (0.81, 0.12)), (\hat{\mathscr{H}}_2, (0.91, 0.11), (0.81, 0.11)), (\hat{\mathscr{H}}_3, (0.91, 0.12), (0.81, 0.12)), (\hat{\mathscr{H}}_4, (0.91, 0.11), (0.78, 0.11)), (\hat{\mathscr{H}}_5, (0.87, 0.11), (0.78, 0.11))\}$.

Theorem 1: Let $(\Xi, \oint, \sigma)$ be a CSAS. For every $\hat{\mathscr{H}}, \hat{\mathscr{H}}_1 \in \Xi$, then the following conditions hold.

(1) $\underline{\mathscr{K}}(\hat{\mathscr{H}}) = (\overline{\mathscr{K}}(\hat{\mathscr{H}}^c))^c$.
(1) If $\hat{\mathscr{H}} \subseteq \hat{\mathscr{H}}_1$, then $\underline{\mathscr{K}}(\hat{\mathscr{H}}) \subseteq \underline{\mathscr{K}}(\hat{\mathscr{H}}_1)$.
(3) $\underline{\mathscr{K}}(\hat{\mathscr{H}} \cap \hat{\mathscr{H}}_1) = \underline{\mathscr{K}}(\hat{\mathscr{H}}) \cap \underline{\mathscr{K}}(\hat{\mathscr{H}}_1)$.
(4) $\underline{\mathscr{K}}(\hat{\mathscr{H}} \cup \hat{\mathscr{H}}_1) \supseteq \underline{\mathscr{K}}(\hat{\mathscr{H}}) \cup \underline{\mathscr{K}}(\hat{\mathscr{H}}_1)$.
(1′) $\overline{\mathscr{K}}(\hat{\mathscr{H}}) = (\underline{\mathscr{K}}(\hat{\mathscr{H}}^c))^c$.
(2′) If $\hat{\mathscr{H}} \subseteq \hat{\mathscr{H}}_1$, then $\overline{\mathscr{K}}(\hat{\mathscr{H}}) \subseteq \overline{\mathscr{K}}(\hat{\mathscr{H}}_1)$.
(3′) $\overline{\mathscr{K}}(\hat{\mathscr{H}} \cap \hat{\mathscr{H}}_1) \subseteq \overline{\mathscr{K}}(\hat{\mathscr{H}}) \cap \overline{\mathscr{K}}(\hat{\mathscr{H}}_1)$.
(4′) $\overline{\mathscr{K}}(\hat{\mathscr{H}} \cup \hat{\mathscr{H}}_1) = \overline{\mathscr{K}}(\hat{\mathscr{H}}) \cup \overline{\mathscr{K}}(\hat{\mathscr{H}}_1)$.

Proof: (1) From Definition 13, we have the following formulas.

$$\begin{aligned}(\overline{\mathscr{K}}(\hat{\mathscr{H}}^c))^c &= \{(\hat{\mathscr{X}}, \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}^c)), \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}^c))\}^c \\ &= \{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}))\} \\ &= \underline{\mathscr{K}}(\hat{\mathscr{H}}).\end{aligned}$$

(2) Since $\hat{\mathscr{H}} \subseteq \hat{\mathscr{H}}_1$, so from Definition 13, we have

$$\begin{aligned}\underline{\mathscr{K}}(\hat{\mathscr{H}}) &= \{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}))\} \\ &\subseteq \{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}}))\} \\ &= \underline{\mathscr{K}}(\hat{\mathscr{H}}_1).\end{aligned}$$

(3) $\underline{\mathscr{K}}(\hat{\mathscr{H}} \cap \hat{\mathscr{H}}_1) =$

$$\{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}} \cap \hat{\mathscr{H}}_1}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}} \cap \hat{\mathscr{H}}_1}(\hat{\mathscr{X}}))\}$$

$$\begin{aligned}&= \underline{\mathscr{K}}(\hat{\mathscr{H}} \cap \hat{\mathscr{H}}_1) \\ &= \{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}) \wedge \vartheta^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}) \vee \varkappa^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}}))\} \\ &= \{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}})) \wedge \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}})) \vee \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}}))\} \\ &= \underline{\mathscr{K}}(\hat{\mathscr{H}}) \cap \underline{\mathscr{K}}(\hat{\mathscr{H}}_1).\end{aligned}$$

(4) $\underline{\mathscr{K}}(\hat{\mathscr{H}} \cup \hat{\mathscr{H}}_1) =$

$$\{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}} \cup \hat{\mathscr{H}}_1}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}} \cup \hat{\mathscr{H}}_1}(\hat{\mathscr{X}}))\}$$

$$\begin{aligned}&= \underline{\mathscr{K}}(\hat{\mathscr{H}} \cap \hat{\mathscr{H}}_1) = \{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}) \wedge \vartheta^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}}) \vee \varkappa^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}}))\} \\ &\supseteq \{(\hat{\mathscr{X}}, \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}})) \vee \bigwedge_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\vartheta^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}})), \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}}(\hat{\mathscr{X}})) \vee \bigvee_{\vartheta \in \sigma(\hat{\mathscr{X}})} (\varkappa^r_{\hat{\mathscr{H}}_1}(\hat{\mathscr{X}}))\} \\ &= \underline{\mathscr{K}}(\hat{\mathscr{H}}) \cup \underline{\mathscr{K}}(\hat{\mathscr{H}}_1).\end{aligned}$$

The proofs of (1’)- (4’) can be similarly proven as those proofs of (1) - (4).

IV. q-RUNG ORTHOPAIR m-POLAR FUZZY SOFT ROUGH SETS

Below, we construct the concept of q-rung orthopair m-polar fuzzy soft rough sets ${}^{q}RO_{m}PFSRS$, and will discuss their properties. Henceforth, the notions of $\mathcal{I}$, $\mathcal{J}$ and $(\mathcal{I}, \mathcal{J})$-cut sets will be proposed and their characteristics will be put forward.

Definition 14: Suppose Ξ is the origin set and $\oint$ is the provisory features for some $\hat{\mathcal{E}} \subseteq \Xi$. If we have a mapping $\mu : \hat{\mathcal{E}} \rightarrow {}^{q}RO_{m}PFS(\Xi)$, then $(\mu, \hat{\mathcal{E}})$ is called q-rung orthopair m-polar fuzzy sets (${}^{q}RO_{m}PFS$), where ${}^{q}RO_{m}PFS(\Xi)$ is the set of all q-rung orthopair m-polar fuzzy subsets of the origin set Ξ.

Definition 15: If $(\mu, \hat{\mathcal{E}})$ is a ${}^{q}RO_{m}PFSS$, then a q-rung orthopair m-polar fuzzy subset ν of $\Xi \times \oint$ is called a q-rung orthopair m-polar fuzzy soft relation as below.

$$\nu = \{((\rho, \tau), \vartheta^{\nabla}_{\nu}(\rho, \tau), \varkappa^{\nabla}_{\nu}(\rho, \tau)) : (\rho, \tau) \in \Xi \times \oint, \nabla = 1, 2, \ldots, n\},$$

where $\vartheta^{\nabla}_{\nu}(\rho, \tau), \varkappa^{\nabla}_{\nu}(\rho, \tau) \in [0, 1]$ are the membership and non-membership scale, respectively, under the term of

$$0 \leq \vartheta^{\nabla}_{\nu}(\rho, \tau) + \varkappa^{\nabla}_{\nu}(\rho, \tau) \leq 1.$$

This relation can be viewed as the following, ν, as shown at the bottom of the next page.

Definition 16: If Ξ is the origin set, $\oint$ is the provisory features, and ν is the ${}^{q}RO_{m}PFSRS$ relation, then $(\Xi, \oint, \nu)$ is a ${}^{q}RO_{m}PFS$-approximation space. For any $\hat{\mathcal{E}} \in {}^{q}RO_{m}PFS(\oint)$), the ${}^{q}RO_{m}PF$ soft rough-lower and ${}^{q}RO_{m}PF$ soft rough-upper approximations, which are denoted by $\underline{\mathscr{S}}$ and $\overline{\mathscr{S}}$, respectively, are as follows.

$$\underline{\mathscr{S}}(\hat{\mathcal{E}}) = \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge \varkappa_{\hat{\mathcal{E}}}^{r}(\tau)\big)\Big\},$$

$$\overline{\mathscr{S}}(\hat{\mathcal{E}}) = \Big\{\Big(\hat{\mathscr{H}}, \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} (\vartheta_{\nu}^{r}(\rho,\tau) \wedge \vartheta_{\hat{\mathcal{E}}}^{r}(\tau), \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \varkappa_{\hat{\mathcal{E}}}^{r}(\tau)\big)\Big)\Big\},$$

where $\hat{\mathscr{H}} \in \Xi$ and $\nabla = 1, 2, \ldots, n$. If $\underline{\mathscr{S}}(\hat{\mathcal{E}}) \neq \overline{\mathscr{S}}(\hat{\mathcal{E}})$, then $\hat{\mathcal{E}}$ is a q-rung orthopair m-polar fuzzy soft rough sets (${}^{q}RO_{m}PFSRS$), otherwise, it is definable.

Example 2: If $\Xi = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2\}$ is the origin set and $\oint = \{\oint_1, \oint_2, \oint_3\}$ is the features set. Suppose that the q-rung orthopair m-polar fuzzy soft relation $\nu : \Xi \to \oint$ as set in the following matrix, ν, as shown at the bottom of the page.

Suppose we have $\hat{\mathcal{E}} \in {}^{q}RO_{m}PFS(\oint)$ such that

$\hat{\mathcal{E}} = \big\{\big(\oint_1, (0.718, 0.318), (0.618, 0.118), (0.513, 0.213)\big),$
$\big(\oint_2, (0.813, 0.518), (0.313, 0.513), (0.418, 0.713)\big),$
$\big(\oint_3, (0.413, 0.318), (0.618, 0.412), (0.713, 0.312)\big)\big\}.$

Hence, we count the lower and upper approximations as below.

$$\underline{\mathscr{S}}(\hat{\mathcal{E}}) = \big\{\big(\hat{\mathscr{H}}_1, (0.413, 0.518), (0.381, 0.513), (0.513, 0.451)\big), \big(\hat{\mathscr{H}}_2, (0.482, 0.518), (0.487, 0.513), (0.418, 0.618)\big)\big\}$$

and

$$\overline{\mathscr{S}}(\hat{\mathcal{E}}) = \big\{\big(\hat{\mathscr{H}}_1, (0.718, 0.382), (0.519, 0.481), (0.513, 0.282)\big), \big(\hat{\mathscr{H}}_2, (0.718, 0.318), (0.618, 0.181), (0.617, 0.282)\big)\big\}.$$

Theorem 2: Let $(\Xi, \oint, \nu)$ is a ${}^{q}RO_{m}PFS$-approximation space. For every $\hat{\mathcal{E}}, \hat{\mathcal{E}}_1 \in \Xi$, then the next conditions hold.

(1) $\underline{\mathscr{S}}(\hat{\mathcal{E}}) = \big(\overline{\mathscr{S}}(\hat{\mathcal{E}}^{c})\big)^{c}$.
(2) If $\hat{\mathcal{E}} \subseteq \hat{\mathcal{E}}_1$, then $\underline{\mathscr{S}}(\hat{\mathcal{E}}) \subseteq \underline{\mathscr{S}}(\hat{\mathcal{E}}_1)$.
(3) $\underline{\mathscr{S}}(\hat{\mathcal{E}} \cap \hat{\mathcal{E}}_1) = \underline{\mathscr{S}}(\hat{\mathcal{E}}) \cap \underline{\mathscr{S}}(\hat{\mathcal{E}}_1)$.
(4) $\underline{\mathscr{S}}(\hat{\mathcal{E}} \cup \hat{\mathcal{E}}_1) \supseteq \underline{\mathscr{S}}(\hat{\mathcal{E}}) \cup \underline{\mathscr{S}}(\hat{\mathcal{E}}_1)$.
(5) $\underline{\mathscr{S}}(\hat{\mathcal{E}}) \subseteq \hat{\mathcal{E}} \subseteq \overline{\mathscr{S}}(\hat{\mathcal{E}})$.
(1′) $\overline{\mathscr{S}}(\hat{\mathcal{E}}) = \big(\underline{\mathscr{S}}(\hat{\mathcal{E}}^{c})\big)^{c}$.
(2′) If $\hat{\mathcal{E}} \subseteq \hat{\mathcal{E}}_1$, then $\overline{\mathscr{S}}(\hat{\mathcal{E}}) \subseteq \overline{\mathscr{S}}(\hat{\mathcal{E}}_1)$.
(3′) $\overline{\mathscr{S}}(\hat{\mathcal{E}} \cap \hat{\mathcal{E}}_1) \subseteq \overline{\mathscr{S}}(\hat{\mathcal{E}}) \cap \overline{\mathscr{S}}(\hat{\mathcal{E}}_1)$.
(4′) $\overline{\mathscr{S}}(\hat{\mathcal{E}} \cup \hat{\mathcal{E}}_1) = \overline{\mathscr{S}}(\hat{\mathcal{E}}) \cup \overline{\mathscr{S}}(\hat{\mathcal{E}}_1)$.

Proof: (1) From Definition 16, we have the following formulas.

$$\begin{aligned}
&\big(\overline{\mathscr{S}}(\hat{\mathcal{E}}^{c})\big)^{c} = \\
&\Big\{\Big(\hat{\mathscr{H}}, \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} (\vartheta_{\nu}^{r}(\rho,\tau) \wedge \vartheta_{\hat{\mathcal{E}}^{c}}^{r}(\tau), \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \varkappa_{\hat{\mathcal{E}}^{c}}^{r}(\tau)\big)\Big)\Big\}^{c} \\
&= \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} (1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}}^{r}(\tau), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge \varkappa_{\hat{\mathcal{E}}}^{r}(\tau)\big)\Big\} \\
&= \underline{\mathscr{S}}(\hat{\mathcal{E}}).
\end{aligned}$$

(2) Since $\hat{\mathcal{E}} \subseteq \hat{\mathcal{E}}_1$, so from Definition 16, we have

$$\begin{aligned}
&\underline{\mathscr{S}}(\hat{\mathcal{E}}) = \\
&\Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge \varkappa_{\hat{\mathcal{E}}}^{r}(\tau)\big)\Big\} \\
&\subseteq \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}_1}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge \varkappa_{\hat{\mathcal{E}}_1}^{r}(\tau)\big)\Big\} \\
&= \underline{\mathscr{S}}(\hat{\mathcal{E}}_1).
\end{aligned}$$

(3)
$$\begin{aligned}
&\underline{\mathscr{S}}(\hat{\mathcal{E}} \cap \hat{\mathcal{E}}_1) = \\
&\Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}\cap\hat{\mathcal{E}}_1}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge \varkappa_{\hat{\mathcal{E}}\cap\hat{\mathcal{E}}_1}^{r}(\tau)\big)\Big\} \\
&= \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee (\vartheta_{\hat{\mathcal{E}}}^{r}(\tau) \wedge \vartheta_{\hat{\mathcal{E}}_1}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge (\varkappa_{\hat{\mathcal{E}}}^{r}(\tau) \wedge \varkappa_{\hat{\mathcal{E}}_1}^{r}(\tau))\big)\Big\} \\
&= \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau)\varkappa_{\hat{\mathcal{E}}}^{r}(\tau)\big)\Big\} \\
&\wedge \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}_1}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge \varkappa_{\hat{\mathcal{E}}_1}^{r}(\tau)\big)\Big\} \\
&= \underline{\mathscr{S}}(\hat{\mathcal{E}}) \cap \underline{\mathscr{S}}(\hat{\mathcal{E}}_1).
\end{aligned}$$

(4)
$$\begin{aligned}
&\underline{\mathscr{S}}(\hat{\mathcal{E}} \cup \hat{\mathcal{E}}_1) = \\
&\Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee \vartheta_{\hat{\mathcal{E}}\cup\hat{\mathcal{E}}_1}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge \varkappa_{\hat{\mathcal{E}}\cup\hat{\mathcal{E}}_1}^{r}(\tau)\big)\Big\} \\
&\supseteq \Big\{\Big(\hat{\mathscr{H}}, \bigwedge_{\vartheta\in\oint(\hat{\mathscr{H}})} \big((1-\vartheta_{\nu}^{r}(\rho,\tau)) \vee (\vartheta_{\hat{\mathcal{E}}}^{r}(\tau) \vee \vartheta_{\hat{\mathcal{E}}_1}^{r}(\tau)\big), \bigvee_{\vartheta\in\oint(\hat{\mathscr{H}})} \big(\vartheta_{\nu}^{r}(\rho,\tau) \wedge (\varkappa_{\hat{\mathcal{E}}}^{r}(\tau) \wedge \varkappa_{\hat{\mathcal{E}}_1}^{r}(\tau))\big)\Big\}
\end{aligned}$$

$$\nu = \begin{pmatrix}
(\vartheta_{\nu}^{1}(\rho_1,\tau_1), \varkappa_{\nu}^{1}(\rho_1,\tau_1)) & (\vartheta_{\nu}^{2}(\rho_1,\tau_1), \varkappa_{\nu}^{2}(\rho_1,\tau_1)) & \cdots & (\vartheta_{\nu}^{m}(\rho_1,\tau_1), \varkappa_{\nu}^{m}(\rho_1,\tau_1)) \\
(\vartheta_{\nu}^{1}(\rho_2,\tau_2), \varkappa_{\nu}^{1}(\rho_2,\tau_2)) & (\vartheta_{\nu}^{2}(\rho_2,\tau_2), \varkappa_{\nu}^{2}(\rho_2,\tau_2)) & \cdots & (\vartheta_{\nu}^{m}(\rho_2,\tau_2), \varkappa_{\nu}^{m}(\rho_2,\tau_2)) \\
\vdots & \vdots & \ddots & \vdots \\
(\vartheta_{\nu}^{1}(\rho_t,\tau_t), \varkappa_{\nu}^{1}(\rho_t,\tau_t)) & (\vartheta_{\nu}^{2}(\rho_t,\tau_t), \varkappa_{\nu}^{2}(\rho_t,\tau_t)) & \cdots & (\vartheta_{\nu}^{m}(\rho_t,\tau_t), \varkappa_{\nu}^{m}(\rho_t,\tau_t))
\end{pmatrix}$$

$$\nu = \left(\begin{array}{c|c}
 & \oint_1 : \big((0.618, 0.312), (0.519, 0.418), (0.718, 0.138)\big) \\
\hat{\mathscr{H}}_1 & \oint_2 : \big((0.718, 0.318), (0.619, 0.418), (0.451, 0.512)\big) \\
 & \oint_3 : \big((0.618, 0.213), (0.418, 0.118), (0.513, 0.318)\big) \\
\hline
 & \oint_1 : \big((0.718, 0.187), (0.819, 0.113), (0.738, 0.238)\big) \\
\hat{\mathscr{H}}_2 & \oint_2 : \big((0.618, 0.313), (0.513, 0.517), (0.618, 0.418)\big) \\
 & \oint_3 : \big((0.518, 0.418), (0.413, 0.313), (0.617, 0.213)\big)
\end{array}\right)$$

$$= \{(\hat{\mathscr{H}}, \bigwedge_{\vartheta \in \oint(\hat{\mathscr{H}})} ((1-\vartheta^r_\nu(\rho,\tau)) \vee \vartheta^r_{\hat{\mathcal{E}}}(\tau)), \bigvee_{\vartheta \in \oint(\hat{\mathscr{H}})} (\vartheta^r_\nu(\rho,\tau)\varkappa^r_{\hat{\mathcal{E}}}(\tau))\}$$
$$\vee\{(\hat{\mathscr{H}}, \bigwedge_{\vartheta \in \oint(\hat{\mathscr{H}})} ((1-\vartheta^r_\nu(\rho,\tau)) \vee \vartheta^r_{\hat{\mathcal{E}}_1}(\tau)), \bigvee_{\vartheta \in \oint(\hat{\mathscr{H}})} (\vartheta^r_\nu(\rho,\tau) \wedge \varkappa^r_{\hat{\mathcal{E}}_1}(\tau))\}$$

$$= \underline{\mathscr{L}}(\hat{\mathcal{E}}) \cup \underline{\mathscr{L}}(\hat{\mathcal{E}}_1).$$

(5) It is clear from Definition 16.

The proofs of (1')- (4') can be similarly proven as those proofs of (1) - (4).

A. SOME PROPERTIES

In this segment we will propose a few definitions, propositions and illustrative examples to describe a few properties of our proposed notion on ${}^{q}RO_{m}PFSRS$.

Definition 17: If we have two ${}^{q}RO_{m}PFSRS$ $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $\hat{\mathcal{E}}_1 \leq \hat{\mathcal{E}}_2 \iff \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \leq \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})$ & $\varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \geq \varkappa^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})$.

(2) $\hat{\mathcal{E}}_1 \cup \hat{\mathcal{E}}_2 = (\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))$.

(3) $\hat{\mathcal{E}}_1 \cap \hat{\mathcal{E}}_2 = (\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))$.

(4) $\hat{\mathcal{E}}_1 \circ \hat{\mathcal{E}}_2 = (\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))$.

(5) $\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2 =$
$$(\hat{\mathscr{H}}, \sqrt[\nabla]{(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\vartheta^r_{\hat{\mathcal{E}}_2})$$

(6) $\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2 =$
$$(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{(\varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla + (\varkappa^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla - (\varkappa^r_{\hat{\mathcal{E}}_1}\vartheta^r_{\hat{\mathcal{E}}_2}))^\nabla})$$

(7) $\gamma\hat{\mathcal{E}}_1 = (\hat{\mathscr{H}}, \sqrt[\nabla]{1-(1-(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla)^\gamma}, (\varkappa^r_{\hat{\mathcal{E}}_1})^\gamma), \gamma \geq 0.$

(8) $\hat{\mathcal{E}}_1^\gamma = (\hat{\mathscr{H}}, (\vartheta^r_{\hat{\mathcal{E}}_1})^\gamma, \sqrt[\nabla]{1-(1-(\varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla)^\gamma}), \gamma \geq 0.$

(9) $\hat{\mathcal{E}}_1^c = (\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}), \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))$.

(10) $\hat{\emptyset} = (\hat{\mathscr{H}}, 0, 1)$.

(11) $\hat{\mathcal{A}} = (\hat{\mathscr{H}}, 1, 0)$.

Example 3: If we have $\hat{\mathcal{E}} = \{(\hat{\mathscr{H}}_1, (0.718, 0.521), (0.512, 0.618)), (\hat{\mathscr{H}}_2, (0.819, 0.513), (0.418, 0.213))\}$ and $\hat{\mathcal{D}} = \{(\hat{\mathscr{H}}_1, (0.716, 0.113), (0.725, 0.418)), (\hat{\mathscr{H}}_2, (0.916, 0.411), (0.311, 0.616))\}$ of ${}^{q}RO_2PFSRS$ through Ξ, then we have the below results.

(1) $\hat{\mathcal{E}} \not\leq \hat{\mathcal{D}}$ and $\hat{\mathcal{D}} \not\leq \hat{\mathcal{E}}$.

(2) $\hat{\mathcal{E}} \vee \hat{\mathcal{D}} =$
$\{(\hat{\mathscr{H}}_1, (0.718, 0.113), (0.725, 0.418)), (\hat{\mathscr{H}}_2, (0.916, 0.411), (0.418, 0.213))\}$.

(3) $\hat{\mathcal{E}} \wedge \hat{\mathcal{D}} =$
$\{(\hat{\mathscr{H}}_1, (0.716, 0.521), (0.512, 0.618)), (\hat{\mathscr{H}}_2, (0.819, 0.513), (0.311, 0.616))\}$.

(4) $\hat{\mathcal{E}} \circ \hat{\mathcal{D}} =$
$\{(\hat{\mathscr{H}}_1, (0.718, 0.521), (0.512, 0.618)), (\hat{\mathscr{H}}_2, (0.819, 0.513), (0.616, 0.213))\}$.

(5) $\hat{\mathcal{E}}^c =$
$\{(\hat{\mathscr{H}}_1, (0.521, 0.718), (0.618, 0.512)), (\hat{\mathscr{H}}_2, (0.513, 0.819), (0.213, 0.418))\}$
and
$\hat{\mathcal{D}}^c =$
$\{(\hat{\mathscr{H}}_1, (0.113, 0.716), (0.418, 0.725)), (\hat{\mathscr{H}}_2, (0.411, 0.916), (0.616, 0.311))\}$

Example 4: If we have $\hat{\mathcal{E}} = \{(\hat{\mathscr{H}}_1, (0.531, 0.222), (0.412, 0.204), (0.555, 0.301), (0.156, 0.870)), (\hat{\mathscr{H}}_2, (0.831, 0.231), (0.732, 0.444), (0.830, 0.010), (0.812, 0.110),), (\hat{\mathscr{H}}_3, (0.766, 0.244), (0.456, 0.140), (0.571, 0.473), (0.611, 0.142),)\}$ and $\hat{\mathcal{D}} = \{(\hat{\mathscr{H}}_1, (0.514, 0.345), (0.819, 0.009), (0.700, 0.227), (0.153, 0.625)), (\hat{\mathscr{H}}_2, (0.712, 0.106), (0.513, 0.300), (0.729, 0.115), (0.822, 0.200),), (\hat{\mathscr{H}}_3, (0.632, 0.301), (1, 0), (0.768, 0.072), (0, 1),)\}$ of ${}^{q}RO_4PFSRS$ through Ξ, then the following outcomes hold.

(1) $\hat{\mathcal{E}} \oplus \hat{\mathcal{D}} =$
$\{(\hat{\mathscr{H}}_1, (0.687, 0.076), (0.852, 0.002), (0.804, 0.068), (0.217, 0.544)), (\hat{\mathscr{H}}_2, (0.921, 0.025), (0.816, 0.133), (0.924, 0.001), (0.943, 0.022)), (\hat{\mathscr{H}}_3, (0.867, 0.073), (1, 0), (0.850, 0.034), (0.611, 0.142),)\}$

(2) $\hat{\mathcal{E}} \otimes \hat{\mathcal{D}} =$
$\{(\hat{\mathscr{H}}_1, (0.273, 0.372), (0.337, 0.204), (0.389, 0.338), (0.024, 0.905)), (\hat{\mathscr{H}}_2, (0.592, 0.238), (0.376, 0.482), (0.605, 0.115), (0.667, 0.210),), (\hat{\mathscr{H}}_3, (0.484, 0.346), (0.456, 0.140), (0.439, 0.474), (0, 1),)\}$

Proposition 1: If we have $\hat{\mathcal{E}}, \hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_3$ is ${}^{q}RO_{m}PFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $\hat{\emptyset} \vee \hat{\mathcal{E}} = \hat{\mathcal{E}}, \hat{\emptyset} \wedge \hat{\mathcal{E}} = \hat{\emptyset}$.
(2) $\hat{\mathcal{A}} \vee \hat{\mathcal{E}} = \hat{\mathcal{A}}, \hat{\mathcal{A}} \wedge \hat{\mathcal{E}} = \hat{\mathcal{E}}$.
(3) $\hat{\mathcal{E}} \vee \hat{\mathcal{E}} = \hat{\mathcal{E}}, \hat{\mathcal{E}} \wedge \hat{\mathcal{E}} = \hat{\mathcal{E}}$.
(4) $\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_2 \vee \hat{\mathcal{E}}_1, \hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_2 \wedge \hat{\mathcal{E}}_1$.
(5) $\hat{\mathcal{E}}_1 \vee (\hat{\mathcal{E}}_2 \vee \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) \vee \hat{\mathcal{E}}_3, \hat{\mathcal{E}}_1 \wedge (\hat{\mathcal{E}}_2 \wedge \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) \wedge \hat{\mathcal{E}}_3$.
(6) $\hat{\mathcal{E}}_1 \vee (\hat{\mathcal{E}}_2 \wedge \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) \wedge (\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_3), \hat{\mathcal{E}}_1 \wedge (\hat{\mathcal{E}}_2 \vee \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) \vee (\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_3)$.
(7) $\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2 \preceq \hat{\mathcal{E}}_1, \hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2 \preceq \hat{\mathcal{E}}_1, \hat{\mathcal{E}}_1 \preceq \hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2, \hat{\mathcal{E}}_2 \preceq \hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2$.
(8) $(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)^c = \hat{\mathcal{E}}_1^c \vee \hat{\mathcal{E}}_2^c, (\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)^c = \hat{\mathcal{E}}_1^c \wedge \hat{\mathcal{E}}_2^c$.

Proof: The proofs are trivial.

Proposition 2: If we have $\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_3$ is ${}^{q}RO_{m}PFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_2 \oplus \hat{\mathcal{E}}_1$.
(2) $\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_2 \otimes \hat{\mathcal{E}}_1$.
(3) $\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \oplus \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2) \oplus \hat{\mathcal{E}}_3$.
(4) $\hat{\mathcal{E}}_1 \otimes (\hat{\mathcal{E}}_2 \otimes \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2) \otimes \hat{\mathcal{E}}_3$.

Proof:

(1) $\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2 =$
$$(\hat{\mathscr{H}}, \sqrt[\nabla]{(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2})$$
$$= (\hat{\mathscr{H}}, \sqrt[\nabla]{(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_2}\varkappa^r_{\hat{\mathcal{E}}_1})$$
$$= \hat{\mathcal{E}}_2 \oplus \hat{\mathcal{E}}_1.$$

(2) The proof is similar to the proof of (1).

(3) $\varkappa_{\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \oplus \hat{\mathcal{E}}_3)} = \varkappa^r_{\hat{\mathcal{E}}_1}(\varkappa^r_{\hat{\mathcal{E}}_2}\varkappa^r_{\hat{\mathcal{E}}_3}) = \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2}\varkappa^r_{\hat{\mathcal{E}}_3} = (\varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2})\varkappa^r_{\hat{\mathcal{E}}_3} = \varkappa_{(\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2) \oplus \hat{\mathcal{E}}_3}, \vartheta_{\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \oplus \hat{\mathcal{E}}_3)}$, as shown at the bottom of the next page.

(4) The proof is similar to the proof of (3).

Proposition 3: If we have $\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_3$ is ${}^{q}RO_{m}PFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \vee \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2) \vee (\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_3)$.

(2) $\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \wedge \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2) \wedge (\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_3)$.
(3) $\hat{\mathcal{E}}_1 \otimes (\hat{\mathcal{E}}_2 \vee \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2) \vee (\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_3)$.
(4) $\hat{\mathcal{E}}_1 \otimes (\hat{\mathcal{E}}_2 \wedge \hat{\mathcal{E}}_3) = (\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2) \wedge (\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_3)$.

Proof:

(1) $\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \vee \hat{\mathcal{E}}_3) =$

$$\Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))\big)^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}(\varkappa^r_{\hat{\mathcal{E}}_2} \wedge \varkappa^r_{\hat{\mathcal{E}}_3})\Big)$$

$$= \Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla \vee \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla \vee \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))\big)^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2} \wedge \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_3})\Big)$$

$\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2 =$

$$\Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2}\Big)$$

$\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_3 =$

$$\Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_3}\Big)$$

$(\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2) \vee (\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_3)$

$$= \Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla \vee \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla \vee \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))\big)^\nabla},$$
$$\varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2} \wedge \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_3})\Big)$$

$$= \hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \vee \hat{\mathcal{E}}_3).$$

(2) $\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \wedge \hat{\mathcal{E}}_3) =$

$$\Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))\big)^\nabla},$$
$$\varkappa^r_{\hat{\mathcal{E}}_1}(\varkappa^r_{\hat{\mathcal{E}}_2} \vee \varkappa^r_{\hat{\mathcal{E}}_3})\Big)$$

$$= \Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla \wedge \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla \wedge \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))\big)^\nabla},$$
$$\varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2} \vee \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_3})\Big)$$

$\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2 =$

$$\Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2}\Big)$$

$\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_3 =$

$$\Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_3}\Big)$$

$(\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2) \wedge (\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_3)$

$$= \Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla \wedge \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla \wedge \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))\big)^\nabla},$$
$$\varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2} \vee \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_3})\Big)$$

$$= \hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \wedge \hat{\mathcal{E}}_3).$$

(3)-(4) The proofs are similar to the proofs of (1) and (2).

Proposition 4: If we have $\hat{\mathcal{E}}_1$, $\hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_3$ are ${}^{q}RO_{m}PFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $(\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2)^c = \hat{\mathcal{E}}_1^c \otimes \hat{\mathcal{E}}_2^c$.
(2) $(\hat{\mathcal{E}}_1 \otimes \hat{\mathcal{E}}_2)^c = \hat{\mathcal{E}}_1^c \oplus \hat{\mathcal{E}}_2^c$.

Proof:

(1) $(\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2)^c =$

$$\Big(\hat{\mathscr{H}}, \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla}, \varkappa^r_{\hat{\mathcal{E}}_1}\varkappa^r_{\hat{\mathcal{E}}_2}\Big)^c$$

$$= \Big(\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\varkappa^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_1}\vartheta^r_{\hat{\mathcal{E}}_2})^\nabla}\Big)$$
$$= \Big(\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}_1}, \vartheta^r_{\hat{\mathcal{E}}_1}\Big) \otimes \Big(\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}_2}, \vartheta^r_{\hat{\mathcal{E}}_2}\Big)$$
$$= \hat{\mathcal{E}}_1^c \otimes \hat{\mathcal{E}}_2^c.$$

(2) The proof is similar to the proof of (1).

Definition 18: If we have $\hat{\mathcal{E}} = \{(\vartheta_1, \varkappa_1), (\vartheta_2, \varkappa_2), \ldots, (\vartheta_m, \varkappa_m)\}$ is ${}^{q}RO_{m}PFN$, then we define the assort ($\mathscr{R}$) and accuracy ($\mathcal{R}$) functions of $\hat{\mathcal{E}}$ as follows.

$$\mathscr{R}(\hat{\mathcal{E}}) = \frac{1}{2m}\Big(m + \sum_{i=1}^{m}(\vartheta^\nabla - \varkappa^\nabla)\Big)$$

$$\mathcal{R}(\hat{\mathcal{E}}) = \frac{1}{m}\Big(\sum_{i=1}^{m}(\vartheta^\nabla + \varkappa^\nabla)\Big).$$

Definition 19: If we have two ${}^{q}RO_{m}PFN$ $\hat{\mathcal{E}}_1 = \{({}^1\vartheta_1, {}^1\varkappa_1), ({}^1\vartheta_2, {}^1\varkappa_2), \ldots, ({}^1\vartheta_m, {}^1\varkappa_m)\}$, $\hat{\mathcal{E}}_2 = \{({}^2\vartheta_1, {}^2\varkappa_1), ({}^2\vartheta_2, {}^2\varkappa_2), \ldots, ({}^2\vartheta_m, {}^2\varkappa_m)\}$, then the following hold.

(1) If $\mathscr{R}(\hat{\mathcal{E}}_1) > \mathscr{R}(\hat{\mathcal{E}}_2)$, then $\hat{\mathcal{E}}_1 > \hat{\mathcal{E}}_2$.
(2) If $\mathscr{R}(\hat{\mathcal{E}}_1) = \mathscr{R}(\hat{\mathcal{E}}_2)$ and $\mathcal{R}(\hat{\mathcal{E}}_1) > \mathcal{R}(\hat{\mathcal{E}}_2)$, then $\hat{\mathcal{E}}_1 > \hat{\mathcal{E}}_2$.

Definition 20: If $\hat{\mathscr{E}} = (\vartheta^r_{\hat{\mathscr{E}}}, \varkappa^r_{\hat{\mathscr{E}}})$ is a ${}^{q}RO_{m}PFSRS$, then we have the following.

$$\Box\hat{\mathscr{E}} = \Big(\vartheta^r_{\hat{\mathscr{E}}}, \sqrt[\nabla]{1 - (\vartheta^r_{\hat{\mathscr{E}}})^\nabla}\Big)$$
$$\Diamond\hat{\mathscr{E}} = \Big(\varkappa^r_{\hat{\mathscr{E}}}, \sqrt[\nabla]{1 - (\varkappa^r_{\hat{\mathscr{E}}})^\nabla}\Big)$$

Proposition 5: If we have $\hat{\mathcal{E}}$ is ${}^{q}RO_{m}PFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $\Box\Box\hat{\mathcal{E}} = \Box\hat{\mathcal{E}}$.
(2) $\Diamond\Diamond\hat{\mathcal{E}} = \Diamond\hat{\mathcal{E}}$.
(3) $\Diamond\Box\hat{\mathcal{E}} = \Box\hat{\mathcal{E}}$.
(4) $\Box\Diamond\hat{\mathcal{E}} = \Diamond\hat{\mathcal{E}}$.
(5) $(\Box\hat{\mathcal{E}}^c)^c = \Diamond\hat{\mathcal{E}}$.
(6) $(\Diamond\hat{\mathcal{E}}^c)^c = \Box\hat{\mathcal{E}}$.

$$\vartheta_{\hat{\mathcal{E}}_1 \oplus (\hat{\mathcal{E}}_2 \oplus \hat{\mathcal{E}}_3)} = \sqrt[\nabla]{\begin{array}{l}\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \Big(\sqrt[\nabla]{(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla}\Big) \\ -\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla\Big(\sqrt[\nabla]{(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla}\Big)^\nabla\end{array}}$$
$$= \sqrt[\nabla]{\begin{array}{l}\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big((\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla\big) \\ -\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla\big((\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla\big)\end{array}}$$
$$= \sqrt[\nabla]{\begin{array}{r}\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\big)^\nabla + \big((\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla + (\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla - (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}}))^\nabla\big) \\ -\big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\big)^\nabla - \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla + \big(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}})\vartheta^r_{\hat{\mathcal{E}}_3}(\hat{\mathscr{H}})\big)^\nabla\end{array}}$$
$$= \vartheta_{(\hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2) \oplus \hat{\mathcal{E}}_3}.$$

Proof:

(1) From Definition 20 we have,

$$\Box\hat{\mathcal{E}} = \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \Box\Box\hat{\mathcal{E}}.$$

(2) The proof is similar to the proof of (1).

(3) $\Box\hat{\mathcal{E}} = \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}\right)$

$$\Diamond\Box\hat{\mathcal{E}} = \left(\hat{\mathscr{H}}, \sqrt[\nabla]{1-\left(\sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}\right)^\nabla}, \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \Box\hat{\mathcal{E}}.$$

(4) The proof is similar to the proof of (3).

(5) $\Box\hat{\mathcal{E}}^c = \left(\hat{\mathscr{H}}, (\vartheta^r_{\hat{\mathcal{E}}})^c(\hat{\mathscr{H}}), \sqrt[\nabla]{1-((\vartheta^r_{\hat{\mathcal{E}}})^c(\hat{\mathscr{H}}))^\nabla}\right)$

$$= \left(\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$(\Box\hat{\mathcal{E}}^c)^c = \left(\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}\right)^c$$
$$= \left(\hat{\mathscr{H}}, \sqrt[\nabla]{1-(\varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}))^\nabla}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})\right)$$
$$= \Diamond\hat{\mathcal{E}}.$$

(6) The proof is similar to the proof of (5).

Proposition 6: If we have $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ are ${}^qRO_mPFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Box\hat{\mathcal{E}}_1 \vee \Box\hat{\mathcal{E}}_2$.
(2) $\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Box\hat{\mathcal{E}}_1 \wedge \Box\hat{\mathcal{E}}_2$.
(3) $\Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Diamond\hat{\mathcal{E}}_1 \vee \Diamond\hat{\mathcal{E}}_2$.
(4) $\Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Diamond\hat{\mathcal{E}}_1 \wedge \Diamond\hat{\mathcal{E}}_2$.

Proof:

(1) $\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) =$

$$\left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla \vee (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla}\right) \vee \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \Box\hat{\mathcal{E}}_1 \vee \Box\hat{\mathcal{E}}_2.$$

(2) $\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) =$

$$\left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla \wedge (\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_1}(\hat{\mathscr{H}}))^\nabla}\right) \wedge \left(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}), \sqrt[\nabla]{1-(\vartheta^r_{\hat{\mathcal{E}}_2}(\hat{\mathscr{H}}))^\nabla}\right)$$
$$= \Box\hat{\mathcal{E}}_1 \wedge \Box\hat{\mathcal{E}}_2.$$

(3)-(4) The proofs are similar to the proofs of (1) and (2).

Proposition 7: If we have $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ are ${}^qRO_mPFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $\Box\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(2) $\Box\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.
(3) $\Diamond\Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(4) $\Diamond\Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.
(5) $\Diamond\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(6) $\Diamond\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.
(7) $\Box\Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(8) $\Box\Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.

Proof: The proofs follow from Propositions 5 and 6.

Proposition 8: If we have $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ are ${}^qRO_mPFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $(\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)^c)^c = \Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(2) $(\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)^c)^c = \Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.
(3) $(\Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)^c)^c = \Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(4) $(\Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)^c)^c = \Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.

Proof: The proofs follow from Proposition 5.

Proposition 9: If we have $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ are ${}^qRO_mPFSRS$ through Ξ and $\hat{\mathscr{H}} \in \Xi$, then the following characteristics hold.

(1) $(\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)^c)^c = \Diamond\Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Box\Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(2) $(\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)^c)^c = \Diamond\Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Box\Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.
(3) $(\Diamond(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)^c)^c = \Box\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2) = \Diamond\Box(\hat{\mathcal{E}}_1 \vee \hat{\mathcal{E}}_2)$.
(4) $(\Diamond(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)^c)^c = \Box\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2) = \Diamond\Box(\hat{\mathcal{E}}_1 \wedge \hat{\mathcal{E}}_2)$.

Proof: The proofs follow from Propositions 5, 7 and 8.

B. $(\mathcal{I}, \mathcal{J})$-CUT SETS

Definition 21: If $\hat{\mathcal{E}} \in {}^qRO_mPFSRS$ and $\mathcal{I} \in [0, 1]$, then the $\mathcal{I}$-cut for $\hat{\mathcal{E}}$ is defined as,

$$\hat{\mathcal{E}}_{\mathcal{I}} = \{\hat{\mathscr{H}} \in \xi : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\},$$

and is called a strong (robust) $\mathcal{I}$-cut if

$$\hat{\mathcal{E}}^{\mathcal{I}} = \{\hat{\mathscr{H}} \in \xi : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) > \mathcal{I}\}.$$

Example 5: From Example 4, if $\mathcal{I} = 0.456$, we get the next values. $\hat{\mathcal{E}}_{0.456} = \{\hat{\mathscr{H}}_2, \hat{\mathscr{H}}_3\}$ and $\hat{\mathcal{E}}^{0.456} = \{\hat{\mathscr{H}}_2\}$

Proposition 10: If we have $\hat{\mathcal{E}}, \hat{\mathcal{D}}$ is ${}^qRO_mPFSRS$ through Ξ and $\mathcal{I} \in [0, 1]$, then the following characteristics hold.

(1) $\hat{\mathcal{E}}^c_{\mathcal{I}} = (\hat{\mathcal{E}}^{\mathcal{I}})^c$.
(2) $\hat{\mathcal{E}}^{\mathcal{I}} \preceq \hat{\mathcal{E}}_{\mathcal{I}}$.
(3) $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})_{\mathcal{I}} = \hat{\mathcal{E}}_{\mathcal{I}} \wedge \hat{\mathcal{D}}_{\mathcal{I}}$.
(4) $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{\mathcal{I}} = \hat{\mathcal{E}}_{\mathcal{I}} \vee \hat{\mathcal{D}}_{\mathcal{I}}$.
(5) $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})^{\mathcal{I}} = \hat{\mathcal{E}}^{\mathcal{I}} \wedge \hat{\mathcal{D}}^{\mathcal{I}}$.
(6) $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})^{\mathcal{I}} = \hat{\mathcal{E}}^{\mathcal{I}} \vee \hat{\mathcal{D}}^{\mathcal{I}}$.

Proof:

(1) Let $\hat{\mathcal{E}} = \{\hat{\mathscr{H}} \in \Xi, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})\}$. Then $\hat{\mathcal{E}}^c = \{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})\}$. Hence, $\hat{\mathcal{E}}^c_{\mathcal{I}} = \{\hat{\mathscr{H}} : \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\}$ and $(\hat{\mathcal{E}}^c)^{\mathcal{I}} = \{\hat{\mathscr{H}} : \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) > \mathcal{I}\}$. Thus $(\hat{\mathcal{E}}^{\mathcal{I}})^c = \{\hat{\mathscr{H}} : \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\} = \hat{\mathcal{E}}^c_{\mathcal{I}}$.

(2) Follows from Definition 21.

(3) Since $\hat{\mathcal{E}} \wedge \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})_{\mathcal{I}} = \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\}$

$$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\} \wedge \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\}$$
$$= \hat{\mathcal{E}}_{\mathcal{I}} \wedge \hat{\mathcal{D}}_{\mathcal{I}}.$$

(4) Since $\hat{\mathcal{E}} \vee \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{\mathcal{I}} = \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\}$
$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\} \vee \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\}$
$= \hat{\mathcal{E}}_{\mathcal{I}} \vee \hat{\mathcal{D}}_{\mathcal{I}}$.

(5) Since $\hat{\mathcal{E}} \wedge \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})^{\mathcal{I}} = \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) > \mathcal{I}\}$
$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) > \mathcal{I}\} \wedge \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) > \mathcal{I}\}$
$= \hat{\mathcal{E}}^{\mathcal{I}} \wedge \hat{\mathcal{D}}^{\mathcal{I}}$.

(6) Since $\hat{\mathcal{E}} \vee \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})^{\mathcal{I}} = \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) > \mathcal{I}\}$
$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) > \mathcal{I}\} \vee \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) > \mathcal{I}\}$
$= \hat{\mathcal{E}}^{\mathcal{I}} \vee \hat{\mathcal{D}}^{\mathcal{I}}$.

Definition 22: If $\hat{\mathcal{E}} \in {}^{q}RO_{m}PFSRS$ and $\mathcal{J} \in [0, 1]$, then the $\mathcal{J}$-cut for $\hat{\mathcal{E}}$ is defined as,

$$\hat{\mathcal{E}}_{\mathcal{J}} = \{\hat{\mathscr{H}} \in \xi : \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\},$$

and is called a strong (robust) $\mathcal{J}$-cut if

$$\hat{\mathcal{E}}^{\mathcal{J}} = \{\hat{\mathscr{H}} \in \xi : \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) < \mathcal{J}\}.$$

Example 6: From Example 4, if $\mathcal{J} = 0.140$, we get the next values. $\hat{\mathcal{E}}_{0.456} = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_3\}$ and $\hat{\mathcal{E}}^{0.456} = \{\hat{\mathscr{H}}_1\}$

Proposition 11: If we have $\hat{\mathcal{E}}, \hat{\mathcal{D}}$ is ${}^{q}RO_{m}PFSRS$ through Ξ and $\mathcal{J} \in [0, 1]$, then the following characteristics hold.
(1) $\hat{\mathcal{E}}^c_{\mathcal{J}} = (\hat{\mathcal{E}}^{\mathcal{J}})^c$.
(2) $\hat{\mathcal{E}}^{\mathcal{J}} \preceq \hat{\mathcal{E}}_{\mathcal{J}}$.
(3) $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})_{\mathcal{J}} = \hat{\mathcal{E}}_{\mathcal{J}} \wedge \hat{\mathcal{D}}_{\mathcal{J}}$.
(4) $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{\mathcal{J}} = \hat{\mathcal{E}}_{\mathcal{J}} \vee \hat{\mathcal{D}}_{\mathcal{J}}$.
(5) $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})^{\mathcal{J}} = \hat{\mathcal{E}}^{\mathcal{J}} \wedge \hat{\mathcal{D}}^{\mathcal{J}}$.
(6) $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})^{\mathcal{J}} = \hat{\mathcal{E}}^{\mathcal{J}} \vee \hat{\mathcal{D}}^{\mathcal{J}}$.

Proof: (1) Let $\hat{\mathcal{E}} = \{\hat{\mathscr{H}} \in \Xi, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})\}$. Then $\hat{\mathcal{E}}^c = \{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}), \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}})\}$. Hence, $\hat{\mathcal{E}}^c_{\mathcal{J}} = \{\hat{\mathscr{H}} : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}$ and $(\hat{\mathcal{E}}^c)^{\mathcal{J}} = \{\hat{\mathscr{H}} : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) < \mathcal{J}\}$. Thus $(\hat{\mathcal{E}}^{\mathcal{J}})^c = \{\hat{\mathscr{H}} : \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\} = \hat{\mathcal{E}}^c_{\mathcal{J}}$.

(2) Follows from Definition 22.

(3) As $\hat{\mathcal{E}} \wedge \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})_{\mathcal{J}} = \{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}$
$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\} \wedge \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}$
$= \hat{\mathcal{E}}_{\mathcal{J}} \wedge \hat{\mathcal{D}}_{\mathcal{J}}$.

(4) As $\hat{\mathcal{E}} \vee \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{\mathcal{J}} = \{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}$
$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\} \vee \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}$
$= \hat{\mathcal{E}}_{\mathcal{J}} \vee \hat{\mathcal{D}}_{\mathcal{J}}$.

(5) As $\hat{\mathcal{E}} \wedge \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})^{\mathcal{J}} = \{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) < \mathcal{J}\}$
$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) < \mathcal{J}\} \wedge \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) < \mathcal{J}\}$
$= \hat{\mathcal{E}}^{\mathcal{J}} \wedge \hat{\mathcal{D}}^{\mathcal{J}}$.

(6) As $\hat{\mathcal{E}} \vee \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})^{\mathcal{J}} = \{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) < \mathcal{J}\}$
$= \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) < \mathcal{J}\} \vee \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) < \mathcal{J}\}$
$= \hat{\mathcal{E}}^{\mathcal{J}} \vee \hat{\mathcal{D}}^{\mathcal{J}}$.

Definition 23: If $\hat{\mathcal{E}} \in {}^{q}RO_{m}PFSRS$ and $(\mathcal{I}, \mathcal{J}) \in [0, 1]$ and $\mathcal{I}+\mathcal{J} \in [0, 1]$, then four types of cuts, that is, $(\mathcal{I}, \mathcal{J})$-cut, $(\mathcal{I}_{\mathcal{S}}, \mathcal{J})$-cut, $(\mathcal{I}, \mathcal{J}_{\mathcal{S}})$-cut and $(\mathcal{I}, \mathcal{J})_{\mathcal{S}}$-cut for $\hat{\mathcal{E}}$ are defined, respectively, are as follows.

$$\hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} = \{\hat{\mathscr{H}} \in \xi : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}.$$
$$\hat{\mathcal{E}}^{\mathcal{I}}_{\mathcal{J}} = \{\hat{\mathscr{H}} \in \xi : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) > \mathcal{I}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}.$$
$$\hat{\mathcal{E}}^{\mathcal{J}}_{\mathcal{I}} = \{\hat{\mathscr{H}} \in \xi : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) < \mathcal{J}\}.$$
$$\hat{\mathcal{E}}^{(\mathcal{I},\mathcal{J})} = \{\hat{\mathscr{H}} \in \xi : \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) > \mathcal{I}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) < \mathcal{J}\}.$$

Proposition 12: If we have $\hat{\mathcal{E}}, \hat{\mathcal{D}}$ is ${}^{q}RO_{m}PFSRS$ through Ξ and $\mathcal{I}, \mathcal{J} \in [0, 1]$, then the following characteristics hold.
(1) $\hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} = \hat{\mathcal{E}}_{\mathcal{I}} \wedge \hat{\mathcal{E}}_{\mathcal{J}}$
(2) $\hat{\mathcal{E}} \preceq \hat{\mathcal{D}} \iff \hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} \preceq \hat{\mathcal{D}}_{(\mathcal{I},\mathcal{J})}$
(3) $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})_{(\mathcal{I},\mathcal{J})} = \hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} \wedge \hat{\mathcal{D}}_{(\mathcal{I},\mathcal{J})}$.
(4) $(\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{(\mathcal{I},\mathcal{J})} \succeq \hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} \vee \hat{\mathcal{D}}_{(\mathcal{I},\mathcal{J})}$.
(5) If $\mathcal{I}_1 \geq \mathcal{I}_2$ and $\mathcal{J}_1 \leq \mathcal{J}_2$, then $\hat{\mathcal{E}}_{\mathcal{I}_1} \preceq \hat{\mathcal{E}}_{\mathcal{I}_2}, \hat{\mathcal{E}}_{\mathcal{J}_1} \preceq \hat{\mathcal{E}}_{\mathcal{J}_2}$ and $\hat{\mathcal{E}}_{(\mathcal{I}_1,\mathcal{J}_1)} \preceq \hat{\mathcal{E}}_{(\mathcal{I}_2,\mathcal{J}_2)}$

Proof: (1)-(2) Straightforward using Definition 23.

(3) Since $\hat{\mathcal{E}} \wedge \hat{\mathcal{D}} = \{(\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}), \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}))\}$.

So, $(\hat{\mathcal{E}} \wedge \hat{\mathcal{D}})_{(\mathcal{I},\mathcal{J})} = \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \wedge \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \geq \mathcal{I}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \vee \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\}$
$= (\{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\} \wedge \{\hat{\mathscr{H}}, \vartheta^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \geq \mathcal{I}\}) \wedge (\{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{E}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\} \wedge \{\hat{\mathscr{H}}, \varkappa^r_{\hat{\mathcal{D}}}(\hat{\mathscr{H}}) \leq \mathcal{J}\})$
$= (\hat{\mathcal{E}}_{\mathcal{I}} \wedge \hat{\mathcal{D}}_{\mathcal{I}}) \wedge (\hat{\mathcal{E}}_{\mathcal{J}} \wedge \hat{\mathcal{D}}_{\mathcal{J}})$
$= (\hat{\mathcal{E}}_{\mathcal{I}} \wedge \hat{\mathcal{E}}_{\mathcal{J}}) \wedge (\hat{\mathcal{D}}_{\mathcal{I}} \wedge \hat{\mathcal{D}}_{\mathcal{J}})$
$= \hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} \wedge \hat{\mathcal{D}}_{(\mathcal{I},\mathcal{J})}$.

(4) As $\hat{\mathcal{E}} \preceq \hat{\mathcal{E}} \vee \hat{\mathcal{D}}$ and $\hat{\mathcal{D}} \preceq \hat{\mathcal{E}} \vee \hat{\mathcal{D}}$, then from (2), we have $\hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} \preceq (\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{(\mathcal{I},\mathcal{J})}$ and $\hat{\mathcal{D}}_{(\mathcal{I},\mathcal{J})} \preceq (\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{(\mathcal{I},\mathcal{J})}$. Therefore $\hat{\mathcal{E}}_{(\mathcal{I},\mathcal{J})} \vee \hat{\mathcal{D}}_{(\mathcal{I},\mathcal{J})} \preceq (\hat{\mathcal{E}} \vee \hat{\mathcal{D}})_{(\mathcal{I},\mathcal{J})}$.

(5) Follows from Definitions 21, 22 and 23, and the property (1) of Proposition 4.28.

V. APPLICATIONS

Here, we construct two algorithms to solve MCDM issues via soft rough q-rung orthopair m-polar fuzzy sets ($SR^{q}RO_{m}PFS$) and q-RO m-polar fuzzy soft rough sets (${}^{q}RO_{m}PFSRS$).

These algorithms will aid managers to make decisions using our proposed models via the lower and upper approximations.

A. DESCRIPTION

Let $\Xi = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2, \ldots, \hat{\mathscr{H}}_t\}$ be t number of computer programmers and $\oint = \{\oint_1, \oint_2, \ldots, \oint_r\}$ be r features required of these programmers by the institution which placed the advertisement. The institution establishes several criteria to best choose desirable candidates with the following features: Communication Skill $\oint_1$, Personality $\oint_2$, Experience $\oint_3$, Self-Dependability $\oint_4$. We will build a crisp soft relation for the first method σ over $\Xi \times \oint$ and q-RO m-polar fuzzy soft relation for the second method $\nu : \Xi \to \oint$. Therefore, through the proposed methods $SR^{q}RO_{m}PFS$ and ${}^{q}RO_{m}PFSRS$, we introduce the following two subsections to aid with the managerial decision..

B. $SR^{q}RO_{m}PFS$ APPROACH

The following steps in Algorithm 1 establishes our new approach using the q-ROF m-polar fuzzy sets and crisp soft approximation space.

Algorithm 1 Algorithm for $SR^{q}RO_{m}PFS$

Input: Ξ is the origin set and $\oint$ is the provisory features.
Output: Decision Making.
1: Investigate the crisp soft relation σ based on the data provided.
2: Establish $\hat{\mathcal{E}} \in {}^{q}RO_{m}PFS(\oint)$.
3: Compute $\underline{\mathscr{K}}(\hat{\mathcal{E}})$ ($SR^{q}RO_{m}PFSLA$) and $\overline{\mathscr{K}}(\hat{\mathcal{E}})$ ($SR^{q}RO_{m}PFSUA$).
4: Compute $\underline{\mathscr{K}}(\hat{\mathcal{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathcal{E}})$ from Definition 17.
5: Compute the consequence of all features in $\underline{\mathscr{K}}(\hat{\mathcal{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathcal{E}})$ from Definition 18.
6: Assort the features by Definition 19.
7: Obtain the decision.

Now, we give the following illustrated example of the proposed approach.

Suppose $\Xi = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2, \hat{\mathscr{H}}_3, \hat{\mathscr{H}}_4, \hat{\mathscr{H}}_5\}$ is the origin set of candidates and $\oint = \{\oint_1, \oint_2, \oint_3, \oint_4\}$ is the features set. Thus the relation is as follows

$\sigma = \{(\hat{\mathscr{H}}_1, \oint_2), (\hat{\mathscr{H}}_1, \oint_3), (\hat{\mathscr{H}}_2, \oint_1), (\hat{\mathscr{H}}_2, \oint_2), (\hat{\mathscr{H}}_2, \oint_4), (\hat{\mathscr{H}}_3, \oint_1), (\hat{\mathscr{H}}_3, \oint_4), (\hat{\mathscr{H}}_4, \oint_4), (\hat{\mathscr{H}}_5, \oint_2), (\hat{\mathscr{H}}_5, \oint_4)\}$.

Hence, we have the following results.

$\mathcal{S}(\hat{\mathscr{H}}_1) = \{\oint_2, \oint_3\}, \mathcal{S}(\hat{\mathscr{H}}_2) = \{\oint_1, \oint_2, \oint_4\}, \mathcal{S}(\hat{\mathscr{H}}_3) = \{\oint_1, \oint_4\}, \mathcal{S}(\hat{\mathscr{H}}_4) = \{\oint_4\}, \mathcal{S}(\hat{\mathscr{H}}_5) = \{\oint_2, \oint_4\}$.

Then we set the q-RO 3-polar fuzzy subsets of Ξ as follows.

$\hat{\mathcal{E}} = \{(\oint_1, (0.67, 0.21), (0.71, 0.28), (0.78, 0.31)), (\oint_2, (0.81, 0.21), (0.73, 0.31), (0.69, 0.18)), (\oint_3, (0.89, 0.12), (0.78, 0.31), (0.74, 0.44)), (\oint_4, (0.81, 0.38), (0.67, 0.17), (0.65, 0.16))\}$.

Through these data, we can now compute the lower and upper approximations of $\hat{\mathcal{E}}$ as follows.

$\underline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.81, 0.21), (0.73, 0.31), (0.69, 0.44)), (\hat{\mathscr{H}}_2, (0.67, 0.38), (0.67, 0.31), (0.65, 0.31)), (\hat{\mathscr{H}}_3, (0.67, 0.38), (0.67, 0.28), (0.65, 0.31)), (\hat{\mathscr{H}}_4, (0.81, 0.38), (0.67, 0.17), (0.65, 0.16)), (\hat{\mathscr{H}}_5, (0.81, 0.38), (0.67, 0.31), (0.65, 0.18))\}$,

$\overline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.89, 0.12), (0.78, 0.31), (0.74, 0.18)), (\hat{\mathscr{H}}_2, (0.81, 0.21), (0.73, 0.17), (0.78, 0.16),), (\hat{\mathscr{H}}_3, (0.81, 0.21), (0.71, 0.17), (0.78, 0.16)), (\hat{\mathscr{H}}_4, (0.81, 0.38), (0.67, 0.17), (0.65, 0.16)), (\hat{\mathscr{H}}_5, (0.81, 0.21), (0.73, 0.17), (0.69, 0.16))\}$.

Henceforth, we count the ring sum for these information as below.

If $\nabla = 1$.

$\underline{\mathscr{K}}(\hat{\mathcal{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.979, 0.025), (0.941, 0.096), (0.919, 0.079)), (\hat{\mathscr{H}}_2, (0.937, 0.080), (0.911, 0.053), (0.923, 0.050),), (\hat{\mathscr{H}}_3, (0.937, 0.080), (0.904, 0.048), (0.923, 0.050)), (\hat{\mathscr{H}}_4, (0.964, 0.144), (0.891, 0.030), (0.878, 0.027)), (\hat{\mathscr{H}}_5, (0.964, 0.080), (0.911, 0.053), (0.892, 0.029))\}$

If $\nabla = 2$.

$\underline{\mathscr{K}}(\hat{\mathcal{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.964, 0.025), (0.904, 0.096), (0.873, 0.079)), (\hat{\mathscr{H}}_2, (0.900, 0.080), (0.862, 0.053), (0.879, 0.050),), (\hat{\mathscr{H}}_3, (0.900, 0.080), (0.852, 0.048), (0.879, 0.050)), (\hat{\mathscr{H}}_4, (0.939, 0.144), (0.834, 0.030), (0.816, 0.027)), (\hat{\mathscr{H}}_5, (0.939, 0.080), (0.862, 0.053), (0.835, 0.029))\}$

If $\nabla = 3$.

$\underline{\mathscr{K}}(\hat{\mathcal{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.952, 0.025), (0.880, 0.096), (0.844, 0.079)), (\hat{\mathscr{H}}_2, (0.876, 0.080), (0.830, 0.053), (0.852, 0.050),), (\hat{\mathscr{H}}_3, (0.876, 0.080), (0.820, 0.048), (0.852, 0.050)), (\hat{\mathscr{H}}_4, (0.921, 0.144), (0.800, 0.030), (0.780, 0.027)), (\hat{\mathscr{H}}_5, (0.921, 0.080), (0.830, 0.053), (0.800, 0.029))\}$

If $\nabla = 5$.

$\underline{\mathscr{K}}(\hat{\mathcal{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathcal{E}}) = \{(\hat{\mathscr{H}}_1, (0.934, 0.025), (0.847, 0.096), (0.808, 0.079)), (\hat{\mathscr{H}}_2, (0.847, 0.080), (0.793, 0.053), (0.820, 0.050),), (\hat{\mathscr{H}}_3, (0.847, 0.080), (0.781, 0.048), (0.820, 0.050)), (\hat{\mathscr{H}}_4, (0.895, 0.144), (0.759, 0.030), (0.738, 0.027)), (\hat{\mathscr{H}}_5, (0.895, 0.080), (0.793, 0.053), (0.760, 0.029))\}$

Next, we compute the assort of each variable.

If $\nabla = 1$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.9398,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.9313,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.931,\ \mathscr{R}(\hat{\mathscr{H}}_4) = 0.922,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.9342.$$

If $\nabla = 2$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.9235,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.9097,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.9088,\ \mathscr{R}(\hat{\mathscr{H}}_4) = 0.898,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.9123.$$

If $\nabla = 3$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.9127,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.8958,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.895,\ \mathscr{R}(\hat{\mathscr{H}}_4) = 0.8833,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.8982.$$

If $\nabla = 5$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.8982,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.8795,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.8783,\ \mathscr{R}(\hat{\mathscr{H}}_4) = 0.8652,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.881.$$

Finally, we rank the alternatives as follows.

If $\nabla = 1$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4.$$

If $\nabla = 2$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4.$$

If $\nabla = 3$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4.$$

If $\nabla = 5$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4.$$

C. $^{q}RO_{m}PFSRS$ APPROACH

The following steps in Algorithm 2 establishes our new approach using the q-ROF m-polar fuzzy soft rough sets and crisp soft approximation space.

Now, we give the following illustrated example using the proposed approach.

Presume that $\Xi = \{\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2, \hat{\mathscr{H}}_3, \hat{\mathscr{H}}_4, \hat{\mathscr{H}}_5\}$ is the origin set and $\oint = \{\oint_1, \oint_2, \oint_3, \oint_4\}$ is the features set.

Hence, we have the q-RO 3-polar fuzzy soft relation as in the following matrix.

$$\nu = \left(\begin{array}{c|l}
 & \oint_1 : \big((0.68, 0.43), (0.73, 0.35), (0.81, 0.31)\big) \\
\hat{\mathscr{H}}_1 & \oint_2 : \big((0.86, 0.18), (0.75, 0.41), (0.73, 0.21)\big) \\
 & \oint_3 : \big((0.91, 0.15), (0.83, 0.41), (0.81, 0.51)\big) \\
 & \oint_4 : \big((0.85, 0.41), (0.73, 0.35), (0.69, 0.23)\big) \\ \hline
 & \oint_1 : \big((0.73, 0.41), (0.71, 0.37), (0.83, 0.41)\big) \\
\hat{\mathscr{H}}_2 & \oint_2 : \big((0.82, 0.43), (0.78, 0.31), (0.68, 0.23)\big) \\
 & \oint_3 : \big((0.81, 0.31), (0.78, 0.41), (0.72, 0.18)\big) \\
 & \oint_4 : \big((0.79, 0.53), (0.68, 0.46), (0.67, 0.51)\big) \\ \hline
 & \oint_1 : \big((0.71, 0.51), (0.69, 0.41), (0.76, 0.51)\big) \\
\hat{\mathscr{H}}_3 & \oint_2 : \big((0.82, 0.52), (0.76, 0.36), (0.63, 0.28)\big) \\
 & \oint_3 : \big((0.85, 0.41), (0.71, 0.51), (0.73, 0.11)\big) \\
 & \oint_4 : \big((0.75, 0.18), (0.67, 0.41), (0.63, 0.43)\big) \\ \hline
 & \oint_1 : \big((0.73, 0.31), (0.75, 0.13), (0.78, 0.32)\big) \\
\hat{\mathscr{H}}_4 & \oint_2 : \big((0.85, 0.13), (0.71, 0.11), (0.68, 0.28)\big) \\
 & \oint_3 : \big((0.86, 0.23), (0.68, 0.51), (0.69, 0.19)\big) \\
 & \oint_4 : \big((0.78, 0.17), (0.63, 0.31), (0.61, 0.38)\big) \\ \hline
 & \oint_1 : \big((0.73, 0.13), (0.81, 0.21), (0.85, 0.16)\big) \\
\hat{\mathscr{H}}_5 & \oint_2 : \big((0.89, 0.11), (0.81, 0.31), (0.78, 0.21)\big) \\
 & \oint_3 : \big((0.96, 0.12), (0.86, 0.21), (0.83, 0.31)\big) \\
 & \oint_4 : \big((0.87, 0.36), (0.76, 0.26), (0.74, 0.14)\big)
\end{array}\right)$$

Then we set the q-RO 3-polar fuzzy subsets of Ξ as follows.

$\hat{\mathscr{E}} = \big\{\big(\oint_1, (0.67, 0.21), (0.71, 0.28), (0.78, 0.31)\big),$
$\big(\oint_2, (0.81, 0.21), (0.73, 0.31), (0.69, 0.18)\big),$
$\big(\oint_3, (0.89, 0.12), (0.78, 0.31), (0.74, 0.44)\big),$
$\big(\oint_4, (0.81, 0.38), (0.67, 0.17), (0.65, 0.16)\big)\big\}.$

Using these information, we can now compute the lower and upper approximations of $\hat{\mathscr{E}}$ as follows.

$\underline{\mathscr{L}}(\hat{\mathscr{E}}) =$
$\big\{\big(\hat{\mathscr{H}}_1, (0.81, 0.38), (0.73, 0.31), (0.69, 0.44)\big),$
$\big(\hat{\mathscr{H}}_2, (0.67, 0.38), (0.67, 0.31), (0.65, 0.44)\big),$
$\big(\hat{\mathscr{H}}_3, (0.67, 0.38), (0.67, 0.31), (0.65, 0.44)\big),$
$\big(\hat{\mathscr{H}}_4, (0.81, 0.38), (0.67, 0.31), (0.65, 0.44)\big),$
$\big(\hat{\mathscr{H}}_5, (0.81, 0.38), (0.67, 0.31), (0.65, 0.44)\big)\big\},$

$\overline{\mathscr{S}}(\hat{\mathscr{E}}) =$
$\big\{\big(\hat{\mathscr{H}}_1, (0.89, 0.12), (0.78, 0.27), (0.74, 0.27)\big),$
$\big(\hat{\mathscr{H}}_2, (0.81, 0.19), (0.73, 0.29), (0.78, 0.31), \big),$
$\big(\hat{\mathscr{H}}_3, (0.81, 0.15), (0.71, 0.31), (0.78, 0.31)\big),$
$\big(\hat{\mathscr{H}}_4, (0.81, 0.14), (0.67, 0.28), (0.65, 0.31)\big),$
$\big(\hat{\mathscr{H}}_5, (0.81, 0.12), (0.73, 0.24), (0.69, 0.22)\big)\big\}.$

Henceforth, we count the ring sum for these information as below.

If $\nabla = 1$.
$\underline{\mathscr{K}}(\hat{\mathscr{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathscr{E}}) =$
$\big\{\big(\hat{\mathscr{H}}_1, (0.979, 0.046), (0.941, 0.084), (0.919, 0.119)\big),$
$\big(\hat{\mathscr{H}}_2, (0.937, 0.072), (0.911, 0.0899), (0.923, 0.136), \big),$
$\big(\hat{\mathscr{H}}_3, (0.937, 0.057), (0.904, 0.096), (0.923, 0.136)\big),$
$\big(\hat{\mathscr{H}}_4, (0.964, 0.053), (0.891, 0.087), (0.878, 0.136)\big),$
$\big(\hat{\mathscr{H}}_5, (0.964, 0.046), (0.911, 0.074), (0.892, 0.097)\big)\big\}$

If $\nabla = 2$.
$\underline{\mathscr{K}}(\hat{\mathscr{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathscr{E}}) =$
$\big\{\big(\hat{\mathscr{H}}_1, (0.964, 0.046), (0.904, 0.084), (0.874, 0.119)\big),$
$\big(\hat{\mathscr{H}}_2, (0.900, 0.072), (0.862, 0.0899), (0.888, 0.136), \big),$
$\big(\hat{\mathscr{H}}_3, (0.900, 0.057), (0.853, 0.096), (0.888, 0.136)\big),$
$\big(\hat{\mathscr{H}}_4, (0.939, 0.053), (0.834, 0.087), (0.816, 0.136)\big),$
$\big(\hat{\mathscr{H}}_5, (0.939, 0.046), (0.862, 0.074), (0.835, 0.097)\big)\big\}$

If $\nabla = 3$.
$\underline{\mathscr{K}}(\hat{\mathscr{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathscr{E}}) =$
$\big\{\big(\hat{\mathscr{H}}_1, (0.952, 0.046), (0.879, 0.084), (0.844, 0.119)\big),$
$\big(\hat{\mathscr{H}}_2, (0.876, 0.072), (0.831, 0.0899), (0.852, 0.136), \big),$
$\big(\hat{\mathscr{H}}_3, (0.876, 0.057), (0.828, 0.096), (0.852, 0.136)\big),$
$\big(\hat{\mathscr{H}}_4, (0.921, 0.053), (0.800, 0.087), (0.800, 0.136)\big),$
$\big(\hat{\mathscr{H}}_5, (0.921, 0.046), (0.831, 0.074), (0.801, 0.097)\big)\big\}$

If $\nabla = 5$.
$\underline{\mathscr{K}}(\hat{\mathscr{E}}) \oplus \overline{\mathscr{K}}(\hat{\mathscr{E}}) =$
$\big\{\big(\hat{\mathscr{H}}_1, (0.934, 0.046), (0.847, 0.084), (0.808, 0.119)\big),$
$\big(\hat{\mathscr{H}}_2, (0.847, 0.072), (0.793, 0.0899), (0.820, 0.136), \big),$
$\big(\hat{\mathscr{H}}_3, (0.847, 0.057), (0.781, 0.096), (0.820, 0.136)\big),$

Algorithm 2 Algorithm for $^{q}RO_{m}PFSRS$

Input: Ξ is the origin set and $\oint$ is the provisory features.
Output: Decision Making.
1: Investigate the crisp soft relation σ based on the data provided.
2: Establish $\hat{\mathscr{E}} \in {}^{q}RO_{m}PFS(\oint)$.
3: Compute $\underline{\mathscr{L}}(\hat{\mathscr{E}})$ and $\overline{\mathscr{S}}(\hat{\mathscr{E}})$ from Definition 16.
4: Compute $\underline{\mathscr{L}}(\hat{\mathscr{E}}) \oplus \overline{\mathscr{S}}(\hat{\mathscr{E}})$ from Definition 17.
5: Compute the consequence of all features in $\underline{\mathscr{L}}(\hat{\mathscr{E}}) \oplus \overline{\mathscr{S}}(\hat{\mathscr{E}})$ from Definition 18.
6: Assort the features by Definition 19.
7: Obtain the decision.

TABLE 1. Table for scores using different ∇ for $SR^{q}RO_{m}PFS$.

Different approaches	Obtain a decision					
	$\hat{\mathscr{H}}_1$	$\hat{\mathscr{H}}_2$	$\hat{\mathscr{H}}_3$	$\hat{\mathscr{H}}_4$	$\hat{\mathscr{H}}_5$	
$\nabla = 1$	0.9398	0.9313	0.931	0.922	0.9342	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4$
$\nabla = 2$	0.9235	0.9097	0.9088	0.898	0.9123	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4$
$\nabla = 3$	0.9127	0.8958	0.895	0.8833	0.8982	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4$
$\nabla = 5$	0.8982	0.8795	0.8783	0.8652	0.881	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4$

$$\big(\hat{\mathscr{H}}_4, (0.896, 0.053), (0.759, 0.087), (0.738, 0.136)\big),$$
$$\big(\hat{\mathscr{H}}_5, (0.896, 0.046), (0.793, 0.074), (0.760, 0.097)\big)\}$$

Then, we compute the assort of each variable as next.

If $\nabla = 1$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.9317,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.9122,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.9125,$$
$$\mathscr{R}(\hat{\mathscr{H}}_4) = 0.9095,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.925.$$

If $\nabla = 2$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.9155,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.891,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.892,$$
$$\mathscr{R}(\hat{\mathscr{H}}_4) = 0.8855,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.9032.$$

If $\nabla = 3$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.9043,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.8769,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.8778,$$
$$\mathscr{R}(\hat{\mathscr{H}}_4) = 0.8742,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.8893.$$

If $\nabla = 5$.

$$\mathscr{R}(\hat{\mathscr{H}}_1) = 0.89,\ \mathscr{R}(\hat{\mathscr{H}}_2) = 0.8604,\ \mathscr{R}(\hat{\mathscr{H}}_3) = 0.8599,$$
$$\mathscr{R}(\hat{\mathscr{H}}_4) = 0.8528,\ \mathscr{R}(\hat{\mathscr{H}}_5) = 0.872.$$

Finally, we rank the alternatives as follows.

If $\nabla = 1$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_4.$$

If $\nabla = 2$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_4.$$

If $\nabla = 3$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_4.$$

If $\nabla = 5$.

$$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4.$$

D. COMPARATIVE ANALYSES

In this section, we will explain the merits of the proposed methods by comparisons between ours, that is, $SR^{q}RO_{m}PFS$ and $^{q}RO_{m}PFSRS$, and the previous methods, that is, soft rough m-polar fuzzy sets and m-polar fuzzy soft rough sets by Akram *et al.* [48], soft rough Pythagorean fuzzy set and Pythagorean fuzzy soft rough set by Riaz and Hashmi [50] and soft rough q-rung orthopair fuzzy sets and q-rung orthopair fuzzy soft rough sets by Riaz *et al.* [54]. The novel approaches to solve MADM issues can be seen as illustrated in Tables 1 and 2.

Table 1 shows the ordering outcomes for different ∇ (i.e., Akram *et al.* [48], Riaz and Hashmi [50] and our proposed methods) for $SR^{q}RO_{m}PFS$. The best selection of the proposed different approaches is by hiring programmer $\hat{\mathscr{H}}_1$. This means that our model is reliable and rational.

TABLE 2. Table for scores using different ∇ for $^{q}RO_{m}PFSRS$.

Different approaches	Obtain a decision					
	$\hat{\mathscr{H}}_1$	$\hat{\mathscr{H}}_2$	$\hat{\mathscr{H}}_3$	$\hat{\mathscr{H}}_4$	$\hat{\mathscr{H}}_5$	
$\nabla = 1$	0.9317	0.9122	0.9125	0.9095	0.925	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_4$
$\nabla = 2$	0.9155	0.891	0.892	0.8855	0.9032	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_4$
$\nabla = 3$	0.9043	0.8769	0.8778	0.8742	0.8893	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_4$
$\nabla = 5$	0.89	0.8604	0.8599	0.8528	0.872	$\hat{\mathscr{H}}_1 > \hat{\mathscr{H}}_5 > \hat{\mathscr{H}}_2 > \hat{\mathscr{H}}_3 > \hat{\mathscr{H}}_4$

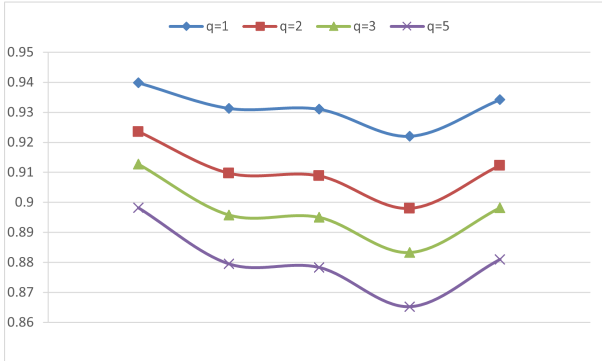

FIGURE 1. The representation of the $SR^{q}RO_{m}PFS$ for different ∇.

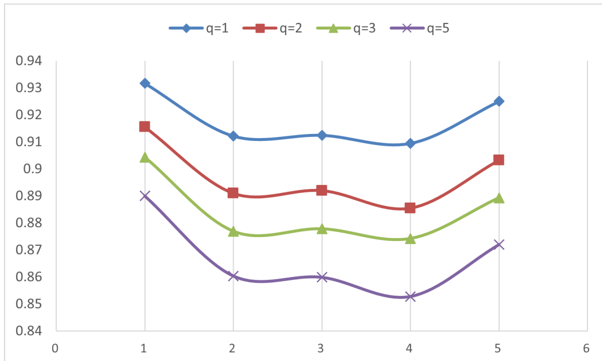

FIGURE 2. The representation of the $^{q}RO_{m}PFSRS$ for different ∇.

Table 2 shows the ordering outcomes for different ∇ (i.e., Akram *et al.* [48], Riaz and Hashmi [50] and our proposed methods) for $^{q}RO_{m}PFSRS$. The best selection of the proposed different approaches is by hiring programmer $\hat{\mathscr{H}}_1$. This means that our model is reasonable and effective.

We can also show the differences between different ∇ (i.e., Akram *et al.* [48], Riaz and Hashmi [50] and our proposed methods) using the following two figures, Figure 1 and Figure 2.

Figure 1 illustrates the comparisons on the outcomes for $\nabla = 1, 2, 3, 5$ for $SR^{q}RO_{m}PFS$, which means that the $\hat{\mathscr{H}}_1$ alternative is the best choice for this institution under the given requirements.

Figure 2 illustrates the comparisons on the outcomes for $\nabla = 1, 2, 3, 5$ for $^{q}RO_{m}PFSRS$, which means that the $\hat{\mathscr{H}}_1$

alternative is the best choice for this institution under the given requirements.

Figure 2 illustrates the comparisons on the outcomes for $\nabla = 1, 2, 3, 5$ (i.e., Akram *et al.* [48], Riaz and Hashmi [50] and our proposed methods) for ${}^{q}RO_{m}PFSRS$, which means that the $\hat{\mathscr{H}_1}$ alternative is the best choice for this institution under the given requirements. Note that the data used here cannot be processed by the methods of Riaz *et al.* [54] which can only handle a single set. Hence, our proposed methods have overcome the hurdle of set limitations of the previous existing methods of Akram *et al.* [48], Riaz and Hashmi [50] and Riaz *et al.* [54].

VI. CONCLUSION

We have constructed new algorithms using soft rough q-RO m-polar fuzzy sets ($SR^{q}RO_{m}PFS$) and q-RO m-polar fuzzy soft rough sets (${}^{q}RO_{m}PFSRS$) to provide us with novel approaches to help make a decision on managerial problems. These new models proved their effectiveness and reliability, as can be seen in Tables 1 and 2, and displayed on Figures 1 and 2. The characteristics related to these models have also been discussed. We have established two different groups of steps for these new models according to the crisp soft and q-RO m-polar fuzzy soft approximation space to solve MADM problems. The comparative analyses indicated that the proposed approaches yield consistent results. In future, we shall extend the proposed methods to a variety of other environments such as the T-spherical power Muirhead operators [62], multi-objective programming [64], neurogenetics [65] and polynomial zeros [66]–[68].

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