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Hamming Distance of Constacyclic Codes of Length p^s Over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$

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ABSTRACT Let p be any prime, s and m be positive integers. In this paper, we completely determine the Hamming distance of all constacyclic codes of length p^s over the finite commutative chain ring $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$ $(u^3=0)$. As applications, we identify all maximum distance saparable codes (i.e., optimal codes with respect to the Singleton bound) among them.

INDEX TERMS Hamming distance, constacyclic codes, optimal codes, MDS codes.

I. INTRODUCTION

Constacyclic codes form one of the most important class of codes, due to their easiness in encoding and decoding via simple shift registers, and their many practical applications. This class of codes can be seen as a generalization of cyclic codes, that have been extensively studied since the late 1950s (cf. [25]–[29]).

Let $\mathbb{F}_{p^m}$ be a finite field of p^m elements, where p is a prime, and let $\ell \geq 2$ be an integer. Then the ring $R = \mathbb{F}_{p^m}[u]/\langle u^\ell\rangle = \mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+\ldots+u^{\ell-1}\mathbb{F}_{p^m}$ $(u^\ell=0)$ is a finite commutative chain ring. Many new and good codes have been constructed by using this type of commutative chain rings (see, for instance, ([18], [31], [32]). Finite commutative chain rings also have practical applications in connections between modular lattices and linear codes over $\mathbb{F}_p+u\mathbb{F}_p$ [3].

When $\ell=2$, there are a lot of literatures on constacyclic codes over rings $\mathbb{F}_{p^m}[u]/\langle u^2\rangle = \mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ for various prime p and positive integers m (see, e.g., [1], [2], [4], [8], [10]–[13], [16], [17], [19], [30].) In particular, structure of and Hamming distance distibution of all constacyclic codes of length p^s over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ were completely determined in [8], [14], [21].

When $\ell = 3$, in 2015, [34] determined the structure of $(\delta+\alpha u^2)$-constacyclic codes of length p^s over $\mathbb{F}_{p^m}[u]/\langle u^3\rangle = \mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$. Recently, [22] obtained the structure of all constacyclic codes of length p^s over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$ by classifying them into 8 types. [33] studies the structure of repeated-root constacyclic codes of any length over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$ and provided the Hamming distnace of some of them. However, the complete Hamming distance distribution of all constacyclic codes of length p^s over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$ was still left open. That motivates us to complete that task in this paper. As an application, we use this Hamming distance distribution to identify all MDS codes among them. These MDS codes are optimal in the sense that among codes of the same length and dimension, they have the best error-correcting capacities.

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II. SOME PRELIMINARIES

For a fintie ring R, consider the set R^n of n-tuples of elements from R as a module over R in the usual way. A subest $C \subseteq R^n$ is called a linear code of length n over R if C is an R-submodule of R^n.

For a unit λ of R, the λ-constacyclic (λ-twisted) shift τ_λ on R^n is the shif

$$\tau_\lambda((x_0, x_1, \ldots, x_{n-1})) = (\lambda x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and a code C is said to be λ-constacyclic if $\tau_\lambda(C) = C$, i.e., if C is closed under the λ-constacyclic shift τ_λ. In case $\lambda = 1$, those λ-constacyclic codes are called cyclic codes, and when

$\lambda = -1$, such λ-constacyclic codes are called negacyclic codes.

Each codeword $c = (c_0, c_1, \dots, c_{n-1}) \in C$ is customarily identified with its polynomial representation $c(x) = c_0+c_1x+\cdots+c_{n-1}x^{n-1}$, and the code C is in turn identified with the set of all polynomial representations of its codewords. Then in the ring $R[x]/\langle x^n-\lambda\rangle$, $xc(x)$ corresponds to a λ-constacyclic shift of $c(x)$. From that, the following fact is well known (cf. [20], [23]) and straightforward:

Proposition 1: A linear code C of length n is λ-constacyclic over R if and only if C is an ideal of $R[x]/\langle x^n-\lambda\rangle$.

For a codeword $\boldsymbol{x} = (x_0, x_1, \dots, x_{n-1}) \in R^n$, the Hamming weight of $\boldsymbol{x}$, denoted by $wt_H(\boldsymbol{x})$, is the number of nonzero components of $\boldsymbol{x}$. The Hamming distance $d_H(\boldsymbol{x}, \boldsymbol{y})$ of two words $\boldsymbol{x}$ and $\boldsymbol{y}$ equals the number of components in which they differ, which is the Hamming weight $wt_H(\boldsymbol{x}-\boldsymbol{y})$ of $\boldsymbol{x}-\boldsymbol{y}$. For a nonzero linear code C, the Hamming weight $wt_H(C)$ and the Hamming distance $d_H(C)$ are the same and defined as the smallest Hamming weight of nonzero codewords of C:

$$d_H(C) = \min\{wt_H(\boldsymbol{x}) \mid 0 \neq \boldsymbol{x} \in C\}.$$

The zero code is conventionally said to have Hamming distance 0.

In this paper, let $\mathbb{F}_{p^m}$ be a finite field of p^m elements, where p is a prime number, and denote

$$\mathcal{R} = \mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m} (u^3 = 0).$$

The ring $\mathcal{R}$ can be expressed as $\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^3\rangle = \{a+bu+cu^2 \mid a, b, c \in \mathbb{F}_{p^m}\}$. It is easy to check that $\mathcal{R}$ is a local ring with maximal ideal $\langle u\rangle = u\mathbb{F}_{p^m}$. Therefore, it is a chain ring. Every invertible element in $\mathcal{R}$ is of the form: $a+bu+cu^2$ where $a, b, c \in \mathbb{F}_{p^m}$ and $a \neq 0$.

From now onwards, we shall focus our attention on γ-constacyclic codes of length p^s over $\mathcal{R}$, i.e., ideals of the ring

$$\mathcal{R}_\gamma = \mathcal{R}[x]/\langle x^{p^s}-\gamma\rangle,$$

where γ is a nonzero element of $\mathbb{F}_{p^m}$. By applying the Division Algorithm, there exist nonnegative integers γ_q, γ_r such that $s = \gamma_q m+\gamma_r$ with $0 \leq \gamma_r \leq m-1$. Let $\gamma_0 = \gamma^{p^{(\gamma_q+1)m-s}} = \gamma^{p^{m-\gamma_r}}$. Then $\gamma_0^{p^s} = \gamma^{p^{(\gamma_q+1)m}} = \gamma$.

In [22], Laaouine *et al.* classified all γ-constacyclic codes of length p^s over $\mathcal{R}$ and their detailed structures are also established.

Theorem 1 (cf. [22]): The ring $\mathcal{R}_\gamma$ is a local ring with maximal ideal $\langle u, x-\gamma_0\rangle$, but it is not a chain ring. The γ-constacyclic codes of length p^s over $\mathcal{R}$, i.e, ideals of the ring $\mathcal{R}_\gamma$, are

Type 1 $(\mathcal{C}_1)$:

$$\langle 0\rangle,\quad \langle 1\rangle.$$

Type 2 $(\mathcal{C}_2)$:

$$\mathcal{C}_2 = \langle u^2(x-\gamma_0)^\tau\rangle,\ \text{where } 0 \leq \tau \leq p^s-1.$$

Type 3 $(\mathcal{C}_3)$:

$$\mathcal{C}_3 = \langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x)\rangle,$$

where $0 \leq \mathsf{L} \leq \delta \leq p^s-1$, $0 \leq t < \mathsf{L}$, either $h(x)$ is 0 or $h(x)$ is a unit in $\mathcal{R}_\gamma$ of the form $\sum_{i=0}^{\mathsf{L}-t-1} h_i(x-\gamma_0)^i$ with $h_i \in \mathbb{F}_{p^m}$ and $h_0 \neq 0$. Here L is the smallest integer satisfying $u^2(x-\gamma_0)^{\mathsf{L}} \in \mathcal{C}_3$.

Type 4 $(\mathcal{C}_4)$:

$$\mathcal{C}_4 = \langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x), u^2(x-\gamma_0)^\omega\rangle,$$

where $0 \leq \omega < \mathsf{L} \leq \delta \leq p^s-1$, $0 \leq t < \omega$, either $h(x)$ is 0 or $h(x)$ is a unit in $\mathcal{R}_\gamma$ of the form $\sum_{i=0}^{\omega-t-1} h_i(x-\gamma_0)^i$ with $h_i \in \mathbb{F}_{p^m}$, $h_0 \neq 0$ and L is the smallest integer satisfying $u^2(x-\gamma_0)^{\mathsf{L}} \in \mathcal{C}_3$.

Type 5 $(\mathcal{C}_5)$:

$$\mathcal{C}_5 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle,$$

where $0 < \mathsf{V} \leq \mathsf{U} \leq a \leq p^s-1$, $0 \leq t_1 < \mathsf{U}$, $0 \leq t_2 < \mathsf{V}$, $h_1(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{\mathsf{U}-t_1-1} a_j(x-\gamma_0)^j$ with $a_j \in \mathbb{F}_{p^m}$, $a_0 \neq 0$ and $h_2(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{\mathsf{V}-t_2-1} b_j(x-\gamma_0)^j$ with $b_j \in \mathbb{F}_{p^m}$, $b_0 \neq 0$. Here U is the smallest integer satisfying $u(x-\gamma_0)^{\mathsf{U}}+u^2g(x) \in \mathcal{C}_5$, for some $g(x) \in \mathcal{R}_\gamma$ and V is the smallest integer such that $u^2(x-\gamma_0)^{\mathsf{V}} \in \mathcal{C}_5$.

Type 6 $(\mathcal{C}_6)$:

$$\mathcal{C}_6 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x) +u^2(x-\gamma_0)^{t_2}h_2(x), u^2(x-\gamma_0)^c\rangle,$$

where $0 \leq c < \mathsf{V} \leq \mathsf{U} \leq a \leq p^s-1, 0 \leq t_1 < \mathsf{U}, 0 \leq t_2 < c$, $h_1(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{\mathsf{U}-t_1-1} a_j(x-\gamma_0)^j$ with $a_j \in \mathbb{F}_{p^m}$, $a_0 \neq 0$, $h_2(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{c-t_2-1} b_j(x-\gamma_0)^j$ with $b_j \in \mathbb{F}_{p^m}$, $b_0 \neq 0$ and U is the smallest integer satisfying $u(x-\gamma_0)^{\mathsf{U}}+u^2g(x) \in \mathcal{C}_5$, for some $g(x) \in \mathcal{R}_\gamma$, V is the smallest integer such that $u^2(x-\gamma_0)^{\mathsf{V}} \in \mathcal{C}_5$.

Type 7 $(\mathcal{C}_7)$:

$$\mathcal{C}_7 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x) +u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x)\rangle,$$

where $0 \leq \mathsf{W} \leq b < \mathsf{U} \leq a \leq p^s-1$, $0 \leq t_1 < b$, $0 \leq t_2 < \mathsf{W}$, $0 \leq t_3 < \mathsf{W}$, $h_1(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{b-t_1-1} a_j(x-\gamma_0)^j$ with $a_j \in \mathbb{F}_{p^m}$, $a_0 \neq 0$, $h_2(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{\mathsf{W}-t_2-1} b_j(x-\gamma_0)^j$ with $b_j \in \mathbb{F}_{p^m}$, $b_0 \neq 0$ and $h_3(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{\mathsf{W}-t_3-1} c_j(x-\gamma_0)^j$ with $c_j \in \mathbb{F}_{p^m}$, $c_0 \neq 0$. Here W is the smallest integer satisfying $u^2(x-\gamma_0)^{\mathsf{W}} \in \mathcal{C}_7$ and U is the smallest integer satisfying $u(x-\gamma_0)^{\mathsf{U}}+u^2g(x) \in \mathcal{C}_5$, for some $g(x) \in \mathcal{R}_\gamma$.

Type 8 ($\mathcal{C}_8$) :

$$\mathcal{C}_8 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), \; u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x), \; u^2(x-\gamma_0)^c\rangle,$$

where $0 \leq c < \mathsf{W} \leq \mathsf{L}_1 \leq b < \mathsf{U} \leq a \leq p^s-1$, $0 \leq t_1 < b$, $0 \leq t_2 < c$, $0 \leq t_3 < c$, $h_1(x)$ is either 0 or a unit in $\mathcal{R}'$ of the form $\sum_{j=0}^{b-t_1-1} a_j(x-\gamma_0)^j$ with $a_j \in \mathbb{F}_{p^m}$, $a_0 \neq 0$, $h_2(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{c-t_2-1} b_j(x-\gamma_0)^i$ with $b_j \in \mathbb{F}_{p^m}$, $b_0 \neq 0$ and $h_3(x)$ is either 0 or a unit in $\mathcal{R}_\gamma$ of the form $\sum_{j=0}^{c-t_3-1} c_j(x-\gamma_0)^j$ with $c_j \in \mathbb{F}_{p^m}$, $c_0 \neq 0$. Here L_1 is the smallest integer such that $u^2(x-\gamma_0)^{\mathsf{L}_1} \in \langle u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x)\rangle$, U is the smallest integer satisfying $u(x-\gamma_0)^{\mathsf{U}}+u^2g(x) \in \mathcal{C}_5$, for some $g(x) \in \mathcal{R}_\gamma$ and W is the smallest integer such that $u^2(x-\gamma_0)^{\mathsf{W}} \in \mathcal{C}_7$.

Proposition 2 (cf. [22]): We have

$$\mathsf{L} = \begin{cases} \delta, & \text{if } h(x)=0,\\ \min\{\delta, p^s-\delta+t\}, & \text{if } h(x)\neq 0.\end{cases}$$

$$\mathsf{L}_1 = \begin{cases} b, & \text{if } h_3(x)=0,\\ \min\{b, p^s-b+t_3\}, & \text{if } h_3(x)\neq 0.\end{cases}$$

$$\mathsf{U} = \begin{cases} a, & \text{if } h_1(x)=0,\\ \min\{a, p^s-a+t_1\}, & \text{if } h_1(x)\neq 0.\end{cases}$$

$$\mathsf{V} = \begin{cases} a, & \text{if } h_1(x)=h_2(x)=0,\\ \min\{a, p^s-a+t_2\}, & \text{if } h_1(x)=0 \text{ and } h_2(x)\neq 0,\\ \min\{a, p^s-a+t_1\}, & \text{if } h_1(x)\neq 0.\end{cases}$$

$$\mathsf{W} = \begin{cases} b, & \textit{if } h_1(x)=h_2(x)=h_3(x)=0\\ & \quad \textit{or } h_1(x)\neq 0 \textit{ and } h_3(x)=0,\\ \min\{b, p^s-a+t_2\}, & \textit{if } h_1(x)=h_3(x)=0, h_2(x)\neq 0,\\ \min\{b, p^s-b+t_3\}, & \textit{if } h_1(x)=h_2(x)=0, h_3(x)\neq 0\\ & \quad \textit{or } h_1(x)\neq 0 \textit{ and } h_3(x)\neq 0,\\ \min\{b, p^s-a+t_2, p^s-b+t_3\}, & \textit{if } h_1(x)=0, h_2(x)\neq 0, h_3(x)\neq 0.\end{cases}$$

Theorem 2 (cf. [22]): Let $\mathcal{C}$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$. Then following the same notations as in Theorem 1, we have the following results:

- If $\mathcal{C} = \langle 0\rangle$, then $|\mathcal{C}|= 1$.
- If $\mathcal{C} = \langle 1\rangle$, then $|\mathcal{C}| = p^{3mp^s}$.
- If $\mathcal{C} = \langle u^2(x-\gamma_0)^\tau\rangle$ with $0 \leq \tau \leq p^s-1$, then

$$|\mathcal{C}| = p^{m(p^s-\tau)}.$$

- If $\mathcal{C} = \langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x)\rangle$ is of the Type 3, then

$$|\mathcal{C}| = p^{m(2p^s-\delta-\mathsf{L})} = \begin{cases} p^{2m(p^s-\delta)}, & \text{if } h(x)=0 \textit{ or } h(x)\neq 0, \text{ and } 0\leq \delta \leq \frac{p^s+t}{2},\\ p^{m(p^s-t)}, & \text{if } h(x)\neq 0 \textit{ and } \frac{p^s+t}{2} < \delta \leq p^s-1.\end{cases}$$

- If $\mathcal{C} = \langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x), u^2(x-\gamma_0)^\omega\rangle$ is of the Type 4, then

$$|\mathcal{C}| = p^{m(2p^s-\delta-\omega)}.$$

- If $\mathcal{C} = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle$ is of the Type 5, then

$$|\mathcal{C}| = p^{m(3p^s-a-\mathsf{U}-\mathsf{V})} = \begin{cases} p^{3m(p^s-a)}, & \textit{if } h_1(x)=h_2(x)=0\\ & \quad \textit{or } h_1(x)=0,\ h_2(x)\neq 0 \textit{ and } 0<a\leq \frac{p^s+t_2}{2}\\ & \quad \textit{or } h_1(x)\neq 0 \textit{ and } 0<a\leq \frac{p^s+t_1}{2},\\ p^{m(p^s+a-2t_1)}, & \textit{if } h_1(x)\neq 0, \textit{ and } \frac{p^s+t_1}{2}<a\leq p^s-1,\\ p^{m(2p^s-a-t_2)}, & \textit{if } h_1(x)=0, h_2(x)\neq 0, \textit{ and } \frac{p^s+t_2}{2}<a\leq p^s-1.\end{cases}$$

- If $\mathcal{C} = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u^2(x-\gamma_0)^c\rangle$ is of the Type 6, then

$$|\mathcal{C}| = p^{m(3p^s-a-\mathsf{U}-c)} = \begin{cases} p^{m(3p^s-2a-c)}, & \textit{if } h_1(x)=0 \textit{ or } h_1(x)\neq 0 \textit{ and } 0<a\leq \frac{p^s+t_1}{2},\\ p^{m(2p^s-t_1-c)}, & \textit{if } h_1(x)\neq 0 \textit{ and } \frac{p^s+t_1}{2}<a\leq p^s-1.\end{cases}$$

- If $\mathcal{C} = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x)\rangle$ is of the Type 7, then

$$|\mathcal{C}| = p^{m(3p^s-a-b-\mathsf{W})}$$

$$
= \begin{cases} p^{m(3p^s-a-2b)}, \ \text{if } h_1(x)=h_2(x)=h_3(x)=0, \\ \quad \text{or } h_1(x)\neq 0 \text{ and } h_3(x)=0, \\ \quad \text{or } h_1(x)=h_3(x)=0,\ h_2(x)\neq 0, \\ \qquad \text{and } 0\le b\le p^s-a+t_2, \\ \quad \text{or } h_1(x)=h_2(x)=0,\ h_3(x)\neq 0, \\ \qquad \text{and } 0\le b\le \frac{p^s+t_3}{2}, \\ \quad \text{or } h_1(x)\neq 0,\ h_3(x)\neq 0 \text{ and } 0\le b\le \frac{p^s+t_3}{2}, \\ \quad \text{or } h_1(x)=0, h_2(x)\neq 0,\ h_3(x)\neq 0, \\ \qquad \text{and } 0\le b\le \min\{p^s-a+t_2, \frac{p^s+t_3}{2}\}, \\ p^{m(2p^s-b-t_2)}, \ \text{if } h_1(x)=h_3(x)=0,\ h_2(x)\neq 0, \\ \qquad \text{and } p^s-a+t_2<b<p^s-1, \\ \quad \text{or } h_1(x)=0, h_2(x)\neq 0,\ h_3(x)\neq 0, \\ \qquad \text{and } p^s-a+t_2<b\le a+t_3-t_2, \\ p^{m(2p^s-a-t_3)}, \ \text{if } h_1(x)=h_2(x)=0,\ h_3(x)\neq 0, \\ \qquad \text{and } \frac{p^s+t_3}{2}<b<p^s-1, \\ \quad \text{or } h_1(x)\neq 0,\ h_3(x)\neq 0,\ \frac{p^s+t_3}{2}<b<p^s-1, \\ \quad \text{or } h_1(x)=0, h_2(x)\neq 0, h_3(x)\neq 0, \\ \qquad \text{and } \max\{a+t_3-t_2, \frac{p^s+t_3}{2}\}<b<p^s-1. \end{cases}
$$

- If $\mathcal{C} = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x), u^2(x-\gamma_0)^c\rangle$ is of the Type 8, then

$$|\mathcal{C}| = p^{m(3p^s-a-b-c)}.$$

III. HAMMING DISTANCE

In [7], [8] the algebraic structure and Hamming distances of γ-constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ were established and given by the following theorem.

Theorem 3 (cf. [7], [8]): Let $\mathcal{C}$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$. Then $\mathcal{C} = \langle (x-\gamma_0)^i\rangle \subseteq \mathbb{F}_{p^m}[x]/\langle x^{p^s}-\gamma\rangle$, for $i\in\{0,1,\ldots,p^s\}$, and its Hamming distance $d_H(\mathcal{C})$ is completely determined by:

$$
d_H(\mathcal{C}) = \begin{cases} \bullet\ 1, \ \text{if } i=0, \\ \bullet\ (n+1)p^k, \ \text{if} \\ p^s-pr+(n-1)r+1\le i\le p^s-pr+nr, \\ \quad \text{where } r=p^{s-k-1},\ 1\le n\le p-1 \\ \qquad \text{and } 0\le k\le s-1, \\ \bullet\ 0, \ \text{if } i=p^s. \end{cases}
$$

Note that $\mathbb{F}_{p^m}$ is a subring of $\mathcal{R}$, for a code $\mathcal{C}$ over $\mathcal{R}$, we denote $d_H(\mathcal{C}_{\mathbb{F}})$ as the Hamming distance of $\mathcal{C}|_{\mathbb{F}_{p^m}}$.

As we mentioned in Section II the γ-constacyclic codes of length p^s over $\mathcal{R}$ are precisely the ideals of the ring $\mathcal{R}_\gamma$. In order to compute the Hamming distances of all γ-constacyclic codes of length p^s over $\mathcal{R}$, we count the Hamming distance of the ideals of the ring $\mathcal{R}_\gamma$ as classified into 8 types in Theorem 1.

It is easy to see that $d_H(\mathcal{C}_1)=0$ when $\mathcal{C}_1=\{0\}$, and that $d_H(\mathcal{C}_1)=1$ when $\mathcal{C}_1=\{1\}$. For a code $\mathcal{C}_2=\langle u^2(x-\gamma_0)^\tau\rangle$ of Type 2, $0\le\tau\le p^s-1$, the codewords of $\mathcal{C}_2$ are precisely the codewords of the γ-constacyclic codes $\langle (x-\gamma_0)^\tau\rangle$ in $\mathbb{F}_{p^m}[x]/\langle x^{p^s}-\gamma\rangle$ multiplied by u^2. Therefore $d_H(\mathcal{C}_2) = d_H(\langle (x-\gamma_0)^\tau\rangle_{\mathbb{F}})$, which are given in Theorem 3.

Theorem 4: Let $\mathcal{C}_2=\langle u^2(x-\gamma_0)^\tau\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 2 (as classified in Theorem 1), where $0\le\tau\le p^s-1$. Then the Hamming distance of $\mathcal{C}_2$ is given by

$$
d_H(\mathcal{C}_2) = d_H(\langle (x-\gamma_0)^\tau\rangle_{\mathbb{F}}) = \begin{cases} \bullet\ 1, \ \text{if } \tau=0, \\ \bullet\ (n+1)p^k, \ \text{if} \\ p^s-pr+(n-1)r+1\le\tau\le p^s-pr+nr, \\ \quad \text{where } r=p^{s-k-1},\ 1\le n\le p-1 \\ \qquad \text{and } 0\le k\le s-1. \end{cases}
$$

In order to compute the Hamming distances of those codes for the rest cases (Type 3, 4, 5, 6, 7 and 8), we first observe that

$$wt_H(a(x)) \ge wt_H(ua(x)), \tag{1}$$

where $a(x)\in\mathcal{R}_\gamma$.

Theorem 5: Let $\mathcal{C}_3=\langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 3 (as classified in Theorem 1). Then the Hamming distance of $\mathcal{C}_3$ is given by

$$
d_H(\mathcal{C}_3) = d_H(\langle (x-\gamma_0)^{\mathsf{L}}\rangle_{\mathbb{F}}) = \begin{cases} \bullet\ 1, \ \text{if } \mathsf{L}=0, \\ \bullet\ (n+1)p^k, \ \text{if} \\ p^s-pr+(n-1)r+1\le \mathsf{L}\le p^s-pr+nr, \\ \quad \text{where } r=p^{s-k-1},\ 1\le n\le p-1 \\ \qquad \text{and } 0\le k\le s-1. \end{cases}
$$

Proof: First of all, since $u^2(x-\gamma_0)^{\mathsf{L}}\in\mathcal{C}_3$, it follows that

$$d_H(\mathcal{C}_3)\le d_H(\langle u^2(x-\gamma_0)^{\mathsf{L}}\rangle) = d_H(\langle (x-\gamma_0)^{\mathsf{L}}\rangle_{\mathbb{F}}).$$

Now, consider an arbitrary polynomial $c(x)\in\mathcal{C}_3$. Thus, by (1), we obtain that

$$
\begin{aligned} wt_H(c(x)) &\ge wt_H(uc(x)) \\ &\ge d_H(\langle u^2(x-\gamma_0)^\delta\rangle) \\ &= d_H(\langle (x-\gamma_0)^\delta\rangle_{\mathbb{F}}). \end{aligned}
$$

Since, $\langle (x-\gamma_0)^\delta\rangle\subseteq\langle (x-\gamma_0)^{\mathsf{L}}\rangle$, we have

$$d_H(\langle (x-\gamma_0)^\delta\rangle_{\mathbb{F}}) \ge d_H(\langle (x-\gamma_0)^{\mathsf{L}}\rangle_{\mathbb{F}}).$$

Hence, $d_H(\langle (x-\gamma_0)^{\mathsf{L}}\rangle_{\mathbb{F}})\le d_H(\mathcal{C}_3)$, forcing

$$d_H(\mathcal{C}_3) = d_H(\langle (x-\gamma_0)^{\mathsf{L}}\rangle_{\mathbb{F}}).$$

The rest of the proof follows from Theorem 3 and the discussion above. ∎

Theorem 6: Let $\mathcal{C}_4=\langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x), u^2(x-\gamma_0)^\omega\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 4 (as classified in Theorem 1). Then the Hamming distance of $\mathcal{C}_4$ is given by

$$d_H(\mathcal{C}_4) = d_H(\langle(x-\gamma_0)^{\omega}\rangle_{\mathbb{F}}) = \begin{cases} \bullet\ 1, \ if \ \ \omega = 0, \\ \bullet\ (n+1)p^k, \ \ if \\ p^s-pr+(n-1)r+1 \leq \omega \leq p^s-pr+nr, \\ \quad where\ r = p^{s-k-1},\ 1 \leq n \leq p-1 \\ \quad\quad and\ 0 \leq k \leq s-1. \end{cases}$$

Proof: First of all, since $u^2(x-\gamma_0)^{\omega} \in \mathcal{C}_4$, it follows that

$$d_H(\mathcal{C}_4) \leq d_H(\langle u^2(x-\gamma_0)^{\omega}\rangle) = d_H(\langle(x-\gamma_0)^{\omega}\rangle_{\mathbb{F}}).$$

Now, consider an arbitrary polynomial $c(x) \in \mathcal{C}_4\backslash\langle u^2(x-\gamma_0)^{\omega}\rangle$. Thus, by (1), we obtain that

$$\begin{aligned} wt_H(c(x)) &\geq wt_H(uc(x)) \\ &\geq d_H(\langle u^2(x-\gamma_0)^{\delta}\rangle) \\ &= d_H(\langle(x-\gamma_0)^{\delta}\rangle_{\mathbb{F}}) \\ &\geq d_H(\langle(x-\gamma_0)^{\omega}\rangle_{\mathbb{F}}). \end{aligned}$$

Hence, $d_H(\langle(x-\gamma_0)^{\omega}\rangle_{\mathbb{F}}) \leq d_H(\mathcal{C}_4)$, forcing

$$d_H(\mathcal{C}_4) = d_H(\langle(x-\gamma_0)^{\omega}\rangle_{\mathbb{F}}).$$

The rest of the proof follows from Theorem 3 and the discussion above. ■

Theorem 7: Let $\mathcal{C}_5 = \langle(x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 5 (as classified in Theorem 1). Then the Hamming distance of $\mathcal{C}_5$ is given by

$$\begin{aligned} d_H(\mathcal{C}_5) &= d_H(\langle(x-\gamma_0)^{\mathsf{V}}\rangle_{\mathbb{F}}) \\ &= (n+1)p^k, \end{aligned}$$

where $p^s-pr+(n-1)r+1 \leq \mathsf{V} \leq p^s-pr+nr$, $r = p^{s-k-1}$, $1 \leq n \leq p-1$ *and* $0 \leq k \leq s-1$.

Proof: First of all, since $u^2(x-\gamma_0)^{\mathsf{V}} \in \mathcal{C}_5$, it follows that

$$d_H(\mathcal{C}_5) \leq d_H(\langle u^2(x-\gamma_0)^{\mathsf{V}}\rangle) = d_H(\langle(x-\gamma_0)^{\mathsf{V}}\rangle_{\mathbb{F}}).$$

Now, consider an arbitrary polynomial $c(x) \in \mathcal{C}_5$. Thus, by (1), we obtain that

$$\begin{aligned} wt_H(c(x)) &\geq wt_H(u^2c(x)) \\ &\geq d_H(\langle u^2(x-\gamma_0)^{a}\rangle) \\ &= d_H(\langle(x-\gamma_0)^{a}\rangle_{\mathbb{F}}). \end{aligned}$$

Since, $\langle(x-\gamma_0)^a\rangle \subseteq \langle(x-\gamma_0)^{\mathsf{V}}\rangle$, we have

$$d_H(\langle(x-\gamma_0)^{a}\rangle_{\mathbb{F}}) \geq d_H(\langle(x-\gamma_0)^{\mathsf{V}}\rangle_{\mathbb{F}}).$$

Hence, $d_H(\langle(x-\gamma_0)^{\mathsf{V}}\rangle_{\mathbb{F}}) \leq d_H(\mathcal{C}_5)$, forcing

$$d_H(\mathcal{C}_5) = d_H(\langle(x-\gamma_0)^{\mathsf{V}}\rangle_{\mathbb{F}}).$$

The rest of the proof follows from Theorem 3 and the discussion above. ■

Theorem 8: Let $\mathcal{C}_6 = \langle(x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u^2(x-\gamma_0)^c\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 6 (as classified in Theorem 1). Then the Hamming distance of $\mathcal{C}_6$ is given by

$$d_H(\mathcal{C}_6) = d_H(\langle(x-\gamma_0)^{c}\rangle_{\mathbb{F}}) = \begin{cases} \bullet\ 1, \ if \ \ c = 0, \\ \bullet\ (n+1)p^k, \ \ if \\ p^s-pr+(n-1)r+1 \leq c \leq p^s-pr+nr, \\ \quad where\ r = p^{s-k-1},\ 1 \leq n \leq p-1 \\ \quad\quad and\ 0 \leq k \leq s-1. \end{cases}$$

Proof: First of all, since $u^2(x-\gamma_0)^{c} \in \mathcal{C}_6$, it follows that

$$d_H(\mathcal{C}_6) \leq d_H(\langle u^2(x-\gamma_0)^{c}\rangle) = d_H(\langle(x-\gamma_0)^{c}\rangle_{\mathbb{F}}).$$

Now, consider an arbitrary polynomial $c(x) \in \mathcal{C}_6\backslash\langle u^2(x-\gamma_0)^{c}\rangle$. Thus, by (1), we obtain that

$$\begin{aligned} wt_H(c(x)) &\geq wt_H(u^2c(x)) \\ &\geq d_H(\langle u^2(x-\gamma_0)^{a}\rangle) \\ &= d_H(\langle(x-\gamma_0)^{a}\rangle_{\mathbb{F}}) \\ &\geq d_H(\langle(x-\gamma_0)^{c}\rangle_{\mathbb{F}}). \end{aligned}$$

Hence, $d_H(\langle(x-\gamma_0)^{c}\rangle_{\mathbb{F}}) \leq d_H(\mathcal{C}_6)$, forcing

$$d_H(\mathcal{C}_6) = d_H(\langle(x-\gamma_0)^{c}\rangle_{\mathbb{F}}).$$

The rest of the proof follows from Theorem 3 and the discussion above. ■

Theorem 9: Let $\mathcal{C}_7 = \langle(x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 7 (as classified in Theorem 1). Then the Hamming distance of $\mathcal{C}_7$ is given by

$$d_H(\mathcal{C}_7) = d_H(\langle(x-\gamma_0)^{\mathsf{W}}\rangle_{\mathbb{F}}) = \begin{cases} \bullet\ 1, \ if \ \ \mathsf{W} = 0, \\ \bullet\ (n+1)p^k, \ \ if \\ p^s-pr+(n-1)r+1 \leq \mathsf{W} \leq p^s-pr+nr, \\ \quad where\ r = p^{s-k-1},\ 1 \leq n \leq p-1 \\ \quad\quad and\ 0 \leq k \leq s-1. \end{cases}$$

Proof: First of all, since $u^2(x-\gamma_0)^{\mathsf{W}} \in \mathcal{C}_7$, it follows that

$$d_H(\mathcal{C}_7) \leq d_H(\langle u^2(x-\gamma_0)^{\mathsf{W}}\rangle) = d_H(\langle(x-\gamma_0)^{\mathsf{W}}\rangle_{\mathbb{F}}).$$

Now, consider an arbitrary polynomial $c(x) \in \mathcal{C}_7$. We consider two cases.

$*$ Case 1: $c(x) \in \langle u\rangle$. In this case, by (1). We have

$$\begin{aligned} wt_H(c(x)) &\geq wt_H(uc(x)) \\ &\geq d_H(\langle u^2(x-\gamma_0)^{b}\rangle) \\ &= d_H(\langle(x-\gamma_0)^{b}\rangle_{\mathbb{F}}). \end{aligned}$$

$*$ Case 2: $c(x) \notin \langle u\rangle$. In this case, by (1). We have

$$\begin{aligned} wt_H(c(x)) &\geq wt_H(u^2c(x)) \\ &\geq d_H(\langle u^2(x-\gamma_0)^{a}\rangle) \\ &= d_H(\langle(x-\gamma_0)^{a}\rangle_{\mathbb{F}}). \end{aligned}$$

Since, $\langle (x-\gamma_0)^a\rangle \subseteq \langle (x-\gamma_0)^b\rangle \subseteq \langle (x-\gamma_0)^{\mathsf{W}}\rangle$, we have

$$d_H(\langle (x-\gamma_0)^a\rangle_{\mathbb{F}}) \geq d_H(\langle (x-\gamma_0)^b\rangle_{\mathbb{F}}) \geq d_H(\langle (x-\gamma_0)^{\mathsf{W}}\rangle_{\mathbb{F}}).$$

Hence, $d_H(\langle (x-\gamma_0)^{\mathsf{W}}\rangle_{\mathbb{F}}) \leq d_H(\mathcal{C}_7)$, forcing

$$d_H(\mathcal{C}_7) = d_H(\langle (x-\gamma_0)^{\mathsf{W}}\rangle_{\mathbb{F}}).$$

The rest of the proof follows from Theorem 3 and the discussion above. ∎

Theorem 10: Let $\mathcal{C}_8 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x), u^2(x-\gamma_0)^c\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 8 (as classified in Theorem 1). Then the Hamming distance of $\mathcal{C}_8$ is given by

$$d_H(\mathcal{C}_8) = d_H(\langle (x-\gamma_0)^c\rangle_{\mathbb{F}}) = \begin{cases} \bullet\ 1, \ \textit{if} \ \ c=0, \\ \bullet\ (n+1)p^k, \ \ \textit{if} \\ p^s-pr+(n-1)r+1 \leq c \leq p^s-pr+nr, \\ \quad \textit{where}\ r=p^{s-k-1},\ 1\leq n\leq p-1 \\ \quad\quad \textit{and}\ 0\leq k\leq s-1. \end{cases}$$

Proof: First of all, since $u^2(x-\gamma_0)^c \in \mathcal{C}_8$, it follows that

$$d_H(\mathcal{C}_8) \leq d_H(\langle u^2(x-\gamma_0)^c\rangle) = d_H(\langle (x-\gamma_0)^c\rangle_{\mathbb{F}}).$$

Now, consider an arbitrary polynomial $c(x) \in \mathcal{C}_8\backslash\langle u^2(x-\gamma_0)^c\rangle$. We consider two cases.

$*$ Case 1: $c(x) \in \langle u\rangle$. In this case, by (1). We have

$$\begin{aligned} wt_H(c(x)) &\geq wt_H(uc(x)) \\ &\geq d_H(\langle u^2(x-\gamma_0)^b\rangle) \\ &= d_H(\langle (x-\gamma_0)^b\rangle_{\mathbb{F}}) \\ &\geq d_H(\langle (x-\gamma_0)^c\rangle_{\mathbb{F}}). \end{aligned}$$

$*$ Case 2: $c(x) \notin \langle u\rangle$. In this case, by (1). We have

$$\begin{aligned} wt_H(c(x)) &\geq wt_H(u^2c(x)) \\ &\geq d_H(\langle u^2(x-\gamma_0)^a\rangle) \\ &= d_H(\langle (x-\gamma_0)^a\rangle_{\mathbb{F}}) \\ &\geq d_H(\langle (x-\gamma_0)^c\rangle_{\mathbb{F}}). \end{aligned}$$

Hence, $d_H(\langle (x-\gamma_0)^c\rangle_{\mathbb{F}}) \leq d_H(\mathcal{C}_8)$, forcing

$$d_H(\mathcal{C}_8) = d_H(\langle (x-\gamma_0)^c\rangle_{\mathbb{F}}).$$

The rest of the proof follows from Theorem 3 and the discussion above. ∎

IV. MAXIMUM DISTANCE SEPARABLE CODES WITH RESPECT TO HAMMING DISTANCE

In [24], Norton *et al.* discussed the Singleton bound for finite chain ring $\mathcal{R}$ with respect to the Hamming distance $d_H(C)$ and is given as $|C| \leq |\mathcal{R}|^{(n-d_H(C)+1)}$. Maximum Distance Separable (MDS) codes are classified as an important class of linear codes that meet the Singleton bound. They have high error correction capability as compared to non MDS codes.

Theorem 11 (Singleton Bound With Respect to Hamming Distance [24]): Let C be a linear code of length n over $\mathcal{R}$ with Hamming distance $d_H(C)$. Then, the Singleton bound with respect to the Hamming distance $d_H(C)$ is given by $|C| \leq p^{3m(n-d_H(C)+1)}$.

Definition 1: Let C be a linear code of length n over $\mathcal{R}$. Then, C is said to be a maximum distance separable (MDS) code with respect to the Hamming distance if it attains the Singleton bound.

In this section, we identify the MDS codes for each type of γ-constacyclic codes one by one. First, we consider the γ-constacyclic codes of length p^s of Type 1.

Theorem 12: Let $\mathcal{C}_1$ be a γ-constacyclic code of length p^s over $\mathcal{R}$ of Type 1 (as classified in Theorem 1), then the only MDS code is $\langle 1\rangle$.

Proof: *Case 1*: If $\mathcal{C}_1 = \langle 0\rangle$, then the Hamming distance is $d_H(\mathcal{C}_1) = 0$. For $\mathcal{C}_1$ to be MDS we must have, $|\mathcal{C}_1| = p^{3m(p^s-d_H(\mathcal{C}_1)+1)}$, i.e., $1 = p^{3m(p^s+1)}$, i.e., $p^s+1 = 0$, which is not true for any p and s.

Case 2: If $\mathcal{C}_1 = \langle 1\rangle$, then $d_H(\mathcal{C}_1) = 1$. For $\mathcal{C}_1$ to be MDS we must have, $|\mathcal{C}_1| = p^{3m(p^s-d_H(\mathcal{C}_1)+1)}$, i.e., $p^{3mp^s} = p^{3m(p^s-1+1)}$, which is true for all p and s. Thus, the code $\mathcal{C}_1$ is MDS in this case. ∎

Now we examine the MDS condition for Type 2 γ-constacyclic codes.

Theorem 13: Let $\mathcal{C}_2 = \langle u^2(x-\gamma_0)^\tau\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 2 (as classified in Theorem 1), where $0 \leq \tau \leq p^s-1$. Then no MDS codes exist.

Proof: Here, we have $|\mathcal{C}_2| = p^{m(p^s-\tau)}$. So, $\mathcal{C}_2$ is a MDS code if and only if $|\mathcal{C}_2| = p^{3m(p^s-d_H(\mathcal{C}_2)+1)}$, i.e., $p^{m(p^s-\tau)} = p^{3m(p^s-d_H(\mathcal{C}_2)+1)}$, i.e., $\tau = 3\,d_H(\mathcal{C}_2)-2p^s-3$. We consider two cases as follows:

Case 1: If $\tau = 0$, then $d_H(\mathcal{C}_2) = 1$. For $\mathcal{C}_2$ to be MDS we must have, $p^s = 0$, which is not true for any p and s. Thus, $\mathcal{C}_2$ is not MDS for $\tau = 0$.

Case 2: If $p^s-pr+(n-1)r+1 \leq \tau \leq p^s-pr+nr$, where $r = p^{s-k-1}$, $1 \leq n \leq p-1$ and $0 \leq k \leq s-1$. Then we have Hamming distance $d_H(\mathcal{C}_2) = (n+1)p^k$.

Now,

$$\begin{aligned} \tau &\geq p^s-p^{s-k}+(n-1)p^{s-k-1}+1 \\ &= p^{s-k}(3p^k-1)-2p^s+(n-1)p^{s-k-1}+1 \\ &\geq p(3p^k-1)-2p^s+(n-1)+1 \\ &\geq (n+1)(3p^k-1)-2p^s+n \\ &= 3(n+1)p^k-2p^s-1 \\ &> 3(n+1)p^k-2p^s-3 \\ &= 3\,d_H(\mathcal{C}_2)-2p^s-3. \end{aligned}$$

Since, $\tau > 3\,d_H(\mathcal{C}_2)-2p^s-3$, no MDS code exists in this case. ∎

Here, we consider the γ-constacyclic codes of Type 3 to verify the MDS condition for these codes. Here, we have $|\mathcal{C}_3| = p^{m(2p^s-\delta-\mathsf{L})}$. So, $\mathcal{C}_3$ is a MDS code if and only if $|\mathcal{C}_3| = p^{3m(p^s-d_H(\mathcal{C}_3)+1)}$, i.e., $p^{m(2p^s-\delta-\mathsf{L})} = p^{3m(p^s-d_H(\mathcal{C}_3)+1)}$, i.e., $\mathsf{L} = 3\,d_H(\mathcal{C}_3)-p^s-\delta-3$. Hence, follows the theorem.

Theorem 14: Let $\mathcal{C}_3 = \langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 3 (as classified in Theorem 1). Then, there is no MDS code.

Proof: We consider two cases as follows:

Case 1: If $\mathsf{L}=0$, then $d_H(\mathcal{C}_3)=1$. For $\mathcal{C}_3$ to be MDS we must have, $\delta=-p^s$, which is not true for any p and s. Thus, $\mathcal{C}_3$ is not MDS for $\mathsf{L}=0$.

Case 2: If $p^s-pr+(n-1)r+1 \le \mathsf{L} \le p^s-pr+nr$, where $r=p^{s-k-1}$, $1\le n\le p-1$ and $0\le k\le s-1$. Then we have Hamming distance $d_H(\mathcal{C}_3)=(n+1)p^k$.

Now,

$$\begin{aligned}\mathsf{L} &\ge p^s-p^{s-k}+(n-1)p^{s-k-1}+1\\ &= p^{s-k}(3p^k-1)-2p^s+(n-1)p^{s-k-1}+1\\ &\ge p(3p^k-1)-2p^s+(n-1)+1\\ &\quad \times(\textit{equality when } k=s-1, \textit{or } \ s=1)\\ &\ge (n+1)(3p^k-1)-2p^s+n\\ &\quad \times(\textit{equality when } n=p-1)\\ &= 3(n+1)p^k-2p^s-1\\ &= 3\ d_H(\mathcal{C}_3)-2p^s-1.\end{aligned}$$

Now, $\mathsf{L}\ge 3\ d_H(\mathcal{C}_3)-p^s-\delta-3$ if and only if $\delta\ge p^s-2$, i.e., equality when $\delta=p^s-2$. Thus, equality occurs when $n=p-1, k=s-1, \delta=p^s-2$, i.e., $\mathsf{L}=p^s-1$, which is a contradiction, since $\mathsf{L}\le\delta$. Thus, there is no MDS code in this case. ∎

Now we examine the MDS condition for Type 4 γ-constacyclic codes.

Theorem 15: Let $\mathcal{C}_4=\langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x), u^2(x-\gamma_0)^\omega\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 4 (as classified in Theorem 1). Then, there is no MDS code.

Proof: Here, we have $|\mathcal{C}_4|=p^{m(2p^s-\delta-\omega)}$. So, $\mathcal{C}_4$ is a MDS code if and only if $|\mathcal{C}_4|=p^{3m(p^s-d_H(\mathcal{C}_4)+1)}$, i.e., $p^{m(2p^s-\delta-\omega)}=p^{3m(p^s-d_H(\mathcal{C}_4)+1)}$, i.e., $\omega=3\ d_H(\mathcal{C}_4)-p^s-3-\delta$. We consider two cases as follows:

Case 1: If $\omega=0$, then $d_H(\mathcal{C}_4)=1$. For $\mathcal{C}_4$ to be MDS we must have, $\delta=-p^s$, which is a contradiction, since $1\le\delta\le p^s-1$. Thus, $\mathcal{C}_4$ is not MDS for $\omega=0$.

Case 2: If $p^s-pr+(n-1)r+1\le\omega\le p^s-pr+nr$, where $r=p^{s-k-1}$, $1\le n\le p-1$ and $0\le k\le s-1$. Then we have Hamming distance $d_H(\mathcal{C}_4)=(n+1)p^k$. For $\mathcal{C}_4$ to be MDS we must have $\omega=3\ d_H(\mathcal{C}_4)-p^s-3-\delta$. Let $\delta=p^s-m$, where $1\le m\le p^s-1$. Thus, the condition for $\mathcal{C}_4$ to be a MDS constacyclic code becomes $\omega=3d_H(\mathcal{C}_4)-2p^s-3+m$.

Now,

$$\begin{aligned}\omega &\ge p^s-p^{s-k}+(n-1)p^{s-k-1}+1\\ &= p^{s-k}(3p^k-1)-2p^s+(n-1)p^{s-k-1}+1\\ &\ge p(3p^k-1)-2p^s+(n-1)+1\\ &\quad \times(\textit{equality when } k=s-1, \textit{or } \ s=1)\\ &\ge (n+1)(3p^k-1)-2p^s+n\\ &\quad \times(\textit{equality when } n=p-1)\\ &= 3(n+1)p^k-2p^s-1\\ &= 3\ d_H(\mathcal{C}_4)-2p^s-1.\end{aligned}$$

Now, $\omega\ge 3\ d_H(\mathcal{C}_4)-2p^s-3+m$ if and only if $2\ge m$, i.e., equality when $m=2$. Thus, equality occurs when $n=p-1, k=s-1, m=2$, i.e., $\delta=p^s-2$ and $\omega=p^s-1$, which is a contradiction, since $\omega<\delta$. Thus, there is no MDS code in this case. ∎

Now we examine the MDS condition for Type 5 γ-constacyclic codes. Here, we have $|\mathcal{C}_5|=p^{m(3p^s-a-\mathsf{U}-\mathsf{V})}$. So, $\mathcal{C}_5$ is a MDS code if and only if $|\mathcal{C}_5|=p^{3m(p^s-d_H(\mathcal{C}_5)+1)}$, i.e., $p^{m(3p^s-a-\mathsf{U}-\mathsf{V})}=p^{3m(p^s-d_H(\mathcal{C}_5)+1)}$, i.e., $\mathsf{V}=3d_H(\mathcal{C}_5)-a-\mathsf{U}-3$. Thus, we get the following cases:

Case 1: When $h_1(x)=h_2(x)=0$ then, $\mathsf{V}=\mathsf{U}=a$. For $\mathcal{C}_5$ to be MDS we must have $a=d_H(\mathcal{C}_5)-1$. Hence, the MDS codes for Type 5 ideals are similar to the MDS γ-constacyclic codes over $\mathbb{F}_{p^m}$ [15, Corollary 13]. Hence, we have the following theorem:

Theorem 16: Let $\mathcal{C}_5=\langle(x-\gamma_0)^a\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 5 (as classified in Theorem 1). Then $\mathcal{C}_5$ is a MDS code if and only if one of the following conditions holds:

- If $s=1$ then $a=n$ for $1\le n\le p-1$, in such case, $d_H(\mathcal{C}_5)=n+1$.
- If $s\ge 2$, then
  - ∗ $a=1$, in such case, $d_H(\mathcal{C}_5)=2$,
  - ∗ $a=p^s-1$, in such case, $d_H(\mathcal{C}_5)=p^s$.

Case 2: When $h_1(x)=0, h_2(x)\ne 0$ and $1\le a\le\frac{p^s+t_2}{2}$ then, $\mathsf{V}=\mathsf{U}=a$. For $\mathcal{C}_5$ to be MDS we must have $a=d_H(\mathcal{C}_5)-1$, which is similar to the result in case 1. But we have $1\le a\le\frac{p^s+t_2}{2}$ and $0\le t_2<a$, which implies that $\max\{2a-p^s,0\}\le t_2<a$. Hence, we conclude the following theorem.

Theorem 17: Let $\mathcal{C}_5=\langle(x-\gamma_0)^a+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 5 (as classified in Theorem 1), where $h_2(x)\ne 0$ and $1\le a\le\frac{p^s+t_2}{2}$. Then $\mathcal{C}_5$ is a MDS code if and only if one of the following conditions holds:

- If $s=1$, $a=n$, $1\le n\le p-1$ and $\max\{2n-p,0\}\le t_2<n$, then $d_H(\mathcal{C}_5)=n+1$.
- If $s\ge 2$,
  - ∗ $a=1$ and $t_2=0$, then $d_H(\mathcal{C}_5)=2$,
  - ∗ $a=p^s-1$ and $t_2=p^s-2$, then $d_H(\mathcal{C}_5)=p^s$.

Case 3: When $h_1(x)\ne 0$ and $1\le a\le\frac{p^s+t_1}{2}$ then, $\mathsf{V}=\mathsf{U}=a$. For $\mathcal{C}_5$ to be MDS we must have $a=d_H(\mathcal{C}_5)-1$, which is similar to the result in case 1. But we have $1\le a\le\frac{p^s+t_1}{2}$ and $0\le t_1<a$, which implies that $\max\{2a-p^s,0\}\le t_1<a$. Hence, we conclude the following theorem.

Theorem 18: Let $\mathcal{C}_5=\langle(x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 5 (as classified in Theorem 1), where $h_1(x)\ne 0$ and $1\le a\le\frac{p^s+t_1}{2}$. Then $\mathcal{C}_5$ is a MDS code if and only if one of the following conditions holds:

- If $s=1$, $a=n$, $1\le n\le p-1$ and $\max\{2n-p,0\}\le t_1<n$, then $d_H(\mathcal{C}_5)=n+1$.
- If $s\ge 2$,
  - ∗ $a=1$ and $t_1=0$, then $d_H(\mathcal{C}_5)=2$,
  - ∗ $a=p^s-1$ and $t_1=p^s-2$, then $d_H(\mathcal{C}_5)=p^s$.

Case 4: When $h_1(x) \neq 0$ and $\frac{p^s+t_1}{2} < a \leq p^s-1$ then, $\mathsf{V} = \mathsf{U} = p^s-a+t_1$. For $\mathcal{C}_5$ to be MDS we must have $a = 2p^s-3d_H(\mathcal{C}_5)+2t_1+3$. Hence, follows the theorem.

Theorem 19: Let $\mathcal{C}_5 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 5 (as classified in Theorem 1), where $h_1(x) \neq 0$ and $\frac{p^s+t_1}{2} < a \leq p^s-1$. Then, there is no MDS code.

Proof: When $h_1(x) \neq 0$ and $\frac{p^s+t_1}{2} < a \leq p^s-1$, then $\mathsf{V} = p^s-a+t_1$. If $p^s-pr+(n-1)r+1 \leq p^s-a+t_1 \leq p^s-pr+nr$, i.e., $t_1+pr-nr \leq a \leq t_1+pr-(n-1)r-1$, where $r = p^{s-k-1}$, $1 \leq n \leq p-1$ and $0 \leq k \leq s-1$. Then we have Hamming distance $d_H(\mathcal{C}_5) = (n+1)p^k$. We get MDS code for $a = 2p^s-3d_H(\mathcal{C}_5)+2t_1+3$.

Now,

$$\begin{aligned} a &\geq t_1+p^{s-k}-np^{s-k-1} \\ &= t_1+p^{s-k-1}(p-n) \\ &\geq t_1+(p-n) \\ &\quad \times(\textit{equality when } k = s-1, \textit{or} \;\; s = 1) \\ &\geq t_1+1 \\ &\quad \times(\textit{equality when } n = p-1) \\ &= -3(n+1)p^k+t_1+1+3(n+1)p^k \\ &\geq -3(n+1)p^k+t_1+1+3(n+1) \\ &\quad \times(\textit{equality when } k = 0) \\ &\geq -3(n+1)p^k+t_1+7 \\ &\quad \times(\textit{equality when } n = 1) \\ &= -3d_H(\mathcal{C}_5)+t_1+7. \end{aligned}$$

Now, $a \geq 2p^s-3d_H(\mathcal{C}_5)+2t_1+3$ if and only if $-2p^s+4 \geq t_1$, i.e., equality when $t_1 = -2p^s+4$. Thus, equality occurs when $n = 1, k = 0, s = 1, p = 2$ and $t_1 = 0$, i.e., $a = 1$, which is a contradiction, since $1 = \frac{2^1+0}{2} < a$. Thus, there is no MDS code in this case. ∎

Case 5: When $h_1(x) = 0$, $h_2(x) \neq 0$ and $\frac{p^s+t_2}{2} < a \leq p^s-1$ then, $\mathsf{V} = p^s-a+t_2$ and $\mathsf{U} = a$. For $\mathcal{C}_5$ to be MDS we must have $a = 3d_H(\mathcal{C}_5)-p^s-t_2-3$. Hence, follows the theorem.

Theorem 20: Let $\mathcal{C}_5 = \langle (x-\gamma_0)^a+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 5 (as classified in Theorem 1), where $h_2(x) \neq 0$ and $\frac{p^s+t_2}{2} < a \leq p^s-1$. Then, there is no MDS code.

Proof: When $h_1(x) = 0$, $h_2(x) \neq 0$ and $\frac{p^s+t_2}{2} < a \leq p^s-1$, then $\mathsf{V} = p^s-a+t_2$. If $p^s-pr+(n-1)r+1 \leq p^s-a+t_2 \leq p^s-pr+nr$, i.e., $t_2+pr-nr \leq a \leq t_2+pr-(n-1)r-1$, where $r = p^{s-k-1}$, $1 \leq n \leq p-1$ and $0 \leq k \leq s-1$. Then we have Hamming distance $d_H(\mathcal{C}_5) = (n+1)p^k$. We get MDS code for $a = 3d_H(\mathcal{C}_5)-p^s-t_2-3$.

Now,

$$\begin{aligned} a &\geq t_2+p^{s-k}-np^{s-k-1} \\ &= t_2+p^{s-k-1}(p-n) \\ &\geq t_2+1 \\ &= 3(n+1)p^k+t_2+1-3(n+1)p^k \\ &\geq 3(n+1)p^k+t_2+1-3p^s \\ &= 3d_H(\mathcal{C}_5)+t_2+1-3p^s. \end{aligned}$$

Now, $a \geq 3d_H(\mathcal{C}_5)-p^s-t_2-3$ if and only if $t_2 \geq p^s-2$, i.e., equality when $t_2 = p^s-2$, i.e., $\frac{p^s+p^s-2}{2} < a \leq p^s-1$, i.e., $p^s-1 < a \leq p^s-1$, which is a contradiction. Thus, there is no MDS code in this case. ∎

Here, we consider the γ-constacyclic codes of Type 6 to verify the MDS condition for these codes.

Theorem 21: Let $\mathcal{C}_6 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u^2(x-\gamma_0)^c\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 6 (as classified in Theorem 1). Then, there is no MDS code.

Proof: Here, we have $|\mathcal{C}_6| = p^{m(3p^s-a-\mathsf{U}-c)}$. So, $\mathcal{C}_6$ is a MDS code if and only if $|\mathcal{C}_6| = p^{3m(p^s-d_H(\mathcal{C}_6)+1)}$, i.e., $p^{m(3p^s-a-\mathsf{U}-c)} = p^{3m(p^s-d_H(\mathcal{C}_6)+1)}$, i.e., $c = 3\,d_H(\mathcal{C}_6)-\mathsf{U}-a-3$. We consider two cases as follows:

Case 1: If $c = 0$, then $d_H(\mathcal{C}_6) = 1$. For $\mathcal{C}_6$ to be MDS we must have, $a = -\mathsf{U}$, which is contradiction, since $0 < \mathsf{U} \leq a$.

Case 2: If $p^s-pr+(n-1)r+1 \leq c \leq p^s-pr+nr$, where $r = p^{s-k-1}$, $1 \leq n \leq p-1$ and $0 \leq k \leq s-1$. Then we have Hamming distance $d_H(\mathcal{C}_6) = (n+1)p^k$. We have the following subcases:

Subcase 2.1: When $h_1(x) = 0$ or $h_1(x) \neq 0$ and $0 < a \leq \frac{p^s+t_1}{2}$, then $\mathsf{U} = a$. So, $\mathcal{C}_6$ is a MDS code if and only if $c = 3\,d_H(\mathcal{C}_6)-2a-3$. Now,

$$\begin{aligned} c &\geq p^s-p^{s-k}+(n-1)p^{s-k-1}+1 \\ &= p^{s-k}(3p^k-1)-2p^s+(n-1)p^{s-k-1}+1 \\ &\geq p(3p^k-1)-2p^s+(n-1)+1 \\ &\quad \times(\textit{equality when } k = s-1, \textit{or} \;\; s = 1) \\ &\geq (n+1)(3p^k-1)-2p^s+n \\ &\quad \times(\textit{equality when } n = p-1) \\ &= 3(n+1)p^k-2p^s-1 \\ &= 3\,d_H(\mathcal{C}_6)-2p^s-1. \end{aligned}$$

Now, $c \geq 3\,d_H(\mathcal{C}_6)-2a-3$ if and only if $a \geq p^s-1$, i.e., equality when $a = p^s-1$. Thus, equality occurs when $n = p-1, k = s-1, a = p^s-1$, i.e., $c = p^s-1$, which is a contradiction, since $c < a$. Thus, there is no MDS code in this case.

Subcase 2.2: When $h_1(x) \neq 0$ and $\frac{p^s+t_1}{2} < a \leq p^s-1$, then $\mathsf{U} = p^s-a+t_1$. So, $\mathcal{C}_6$ is a MDS code if and only if $c = 3\,d_H(\mathcal{C}_6)-p^s-t_1-3$.

Now,

$$\begin{aligned} c &\geq p^s-p^{s-k}+(n-1)p^{s-k-1}+1 \\ &= p^{s-k}(3p^k-1)-2p^s+(n-1)p^{s-k-1}+1 \\ &\geq p(3p^k-1)-2p^s+(n-1)+1 \\ &\quad \times(\textit{equality when } k = s-1, \textit{or} \;\; s = 1) \\ &\geq (n+1)(3p^k-1)-2p^s+n \\ &\quad \times(\textit{equality when } n = p-1) \\ &= 3(n+1)p^k-2p^s-1 \\ &= 3\,d_H(\mathcal{C}_6)-2p^s-1. \end{aligned}$$

Now, $c \geq 3\,d_H(\mathcal{C}_6)-p^s-t_1-3$ if and only if $t_1 \geq p^s-2$, i.e., equality when $t_1 = p^s-2$, i.e., $\frac{p^s+p^s-2}{2} < a \leq p^s-1$, i.e., $p^s-1 < a \leq p^s-1$, which is a contradiction. Thus, there is no MDS code in this case. ■

Now we examine the MDS condition for Type 7 γ-constacyclic codes. Here, we have $|\mathcal{C}_7| = p^{m(3p^s-a-b-\mathsf{W})}$. So, $\mathcal{C}_7$ is a MDS code if and only if $|\mathcal{C}_7| = p^{3m(p^s-d_H(\mathcal{C}_7)+1)}$, i.e., $p^{m(3p^s-a-b-\mathsf{W})} = p^{3m(p^s-d_H(\mathcal{C}_7)+1)}$, i.e., $\mathsf{W} = 3\, d_H(\mathcal{C}_7)-b-a-3$.

Theorem 22: Let $\mathcal{C}_7 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 7 (as classified in Theorem 1). Then, there is no MDS code.

Proof: *Case 1*: If $\mathsf{W} = 0$, then $d_H(\mathcal{C}_7) = 1$. For $\mathcal{C}_7$ to be MDS we must have, $a = -b$, which is contradiction, since $0 \leq b < a$. Thus, $\mathcal{C}_7$ is not MDS for $\mathsf{W} = 0$.

Case 2: If $p^s-pr+(n-1)r+1 \leq \mathsf{W} \leq p^s-pr+nr$, where $r = p^{s-k-1}$, $1 \leq n \leq p-1$ and $0 \leq k \leq s-1$. Then we have Hamming distance $d_H(\mathcal{C}_7) = (n+1)p^k$.

Now,

$$\begin{aligned}
\mathsf{W} &\geq p^s-p^{s-k}+(n-1)p^{s-k-1}+1\\
&= p^{s-k}(3p^k-1)-2p^s+(n-1)p^{s-k-1}+1\\
&\geq p(3p^k-1)-2p^s+(n-1)+1\\
&\quad \times(\textit{equality when } k = s-1, \textit{ or } \ s = 1)\\
&\geq (n+1)(3p^k-1)-2p^s+n\\
&\quad \times(\textit{equality when } n = p-1)\\
&= 3(n+1)p^k-2p^s-1\\
&= 3d_H(\mathcal{C}_7)-2p^s-1.
\end{aligned}$$

Now, $\mathsf{W} \geq 3\, d_H(\mathcal{C}_7)-b-a-3$ if and only if $a+b \geq 2p^s-2$, i.e., equality when $a+b = 2p^s-2$. Thus, equality occurs when $n = p-1$, $k = s-1$, $a+b = 2p^s-2$, i.e., $\mathsf{W} = p^s-1$, which is a contradiction, since $\mathsf{W} < p^s-1$. Thus, there is no MDS code in this case. ■

Finally, we explore the MDS γ-constacyclic codes of Type 8.

Theorem 23: Let $\mathcal{C}_8 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x), u^2(x-\gamma_0)^c\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 8 (as classified in Theorem 1). Then, there is no MDS code.

Proof: Here, we have $|\mathcal{C}_8| = p^{m(3p^s-a-b-c)}$. So, $\mathcal{C}_8$ is a MDS code if and only if $|\mathcal{C}_8| = p^{3m(p^s-d_H(\mathcal{C}_8)+1)}$, i.e., $p^{m(3p^s-a-b-c)} = p^{3m(p^s-d_H(\mathcal{C}_8)+1)}$, i.e., $c = 3\, d_H(\mathcal{C}_8)-b-a-3$. We consider two cases as follows:

Case 1: If $c = 0$, then $d_H(\mathcal{C}_8) = 1$. For $\mathcal{C}_8$ to be MDS we must have $a = -b$, which is contradiction, since $0 \leq b < a$. Thus, $\mathcal{C}_8$ is not MDS for $c = 0$.

Case 2: If $p^s-pr+(n-1)r+1 \leq c \leq p^s-pr+nr$, where $r = p^{s-k-1}$, $1 \leq n \leq p-1$ and $0 \leq k \leq s-1$. Then we have Hamming distance $d_H(\mathcal{C}_8) = (n+1)p^k$. Now,

$$\begin{aligned}
c &\geq p^s-p^{s-k}+(n-1)p^{s-k-1}+1\\
&= p^{s-k}(3p^k-1)-2p^s+(n-1)p^{s-k-1}+1\\
&\geq p(3p^k-1)-2p^s+(n-1)+1\\
&\quad \times(\textit{equality when } k = s-1, \textit{ or } \ s = 1)\\
&\geq (n+1)(3p^k-1)-2p^s+n\\
&\quad \times(\textit{equality when } n = p-1)\\
&= 3(n+1)p^k-2p^s-1\\
&= 3\, d_H(\mathcal{C}_8)-2p^s-1.
\end{aligned}$$

Now, $c \geq 3\, d_H(\mathcal{C}_8)-b-a-3$ if and only if $a+b \geq 2p^s-2$, i.e., equality when $a+b = 2p^s-2$. Thus, equality occurs when $n = p-1$, $k = s-1$, $a+b = 2p^s-2$, i.e., $c = p^s-1$, which is a contradiction, since $c < p^s-1$. Thus, there is no MDS code in this case. ■

Consequently, we have the list of all MDS γ-constacyclic codes of length p^s over $\mathcal{R} = \mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$.

Theorem 24: All MDS γ-constacyclic codes of length p^s over $\mathcal{R}$ are determined as follows:

- Type 1 (trivial ideals): $\mathcal{C}_1 = \langle 1\rangle$ is the only MDS code with $d_H(\mathcal{C}_1) = 1$.
- Type 2: $\mathcal{C}_2 = \langle u^2(x-\gamma_0)^\tau\rangle$, where $0 \leq \tau \leq p^s-1$. No MDS constacyclic codes can be obtained in this case.
- Type 3: $\mathcal{C}_3 = \langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x)\rangle$, where $0 \leq \delta \leq p^s-1$, $0 \leq t < \delta$, either $h(x)$ is 0 or a unit in $\mathcal{R}_\gamma$. No MDS constacyclic code can be obtained in this case.
- Type 4: $\mathcal{C}_4 = \langle u(x-\gamma_0)^\delta+u^2(x-\gamma_0)^t h(x), u^2(x-\gamma_0)^\omega\rangle$, where $0 \leq \omega < \delta \leq p^s-1$, $0 \leq t < \omega$, either $h(x)$ is 0 or a unit in $\mathcal{R}_\gamma$. No MDS constacyclic code can be obtained in this case.
- Type 5: $\mathcal{C}_5 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x)\rangle$, where $1 \leq a \leq p^s-1$, $0 \leq t_1 < a$, $0 \leq t_2 < a$, either $h_1(x)$, $h_2(x)$ are 0 or are units in $\mathcal{R}_\gamma$.
  - When $h_1(x) = h_2(x) = 0$, then $\mathcal{C}_5$ is a MDS code if and only if one of the following conditions holds:
    - If $s = 1$, $a = n$ for $1 \leq n \leq p-1$, in such case, $d_H(\mathcal{C}_5) = n+1$.
    - If $s \geq 2$,
      - $a = 1$ in such case, $d_H(\mathcal{C}_5) = 2$,
      - $a = p^s-1$ in such case, $d_H(\mathcal{C}_5) = p^s$.
  - When $h_1(x) = 0$, $h_2(x) \neq 0$ and $1 \leq a \leq \frac{p^s+t_2}{2}$. Then $\mathcal{C}_5$ is a MDS code if and only if one of the following conditions holds:
    - If $s = 1$, $a = n$, $1 \leq n \leq p-1$ and $\max\{2n-p, 0\} \leq t_2 < n$, then $d_H(\mathcal{C}_5) = n+1$.
    - If $s \geq 2$,
      - $a = 1$ and $t_2 = 0$, then $d_H(\mathcal{C}_5) = 2$,
      - $a = p^s-1$ and $t_2 = p^s-2$, then $d_H(\mathcal{C}_5) = p^s$.
  - When $h_1(x) \neq 0$ and $1 \leq a \leq \frac{p^s+t_1}{2}$. Then $\mathcal{C}_5$ is a MDS code if and only if one of the following conditions holds:
    - If $s = 1$, $a = n$, $1 \leq n \leq p-1$ and $\max\{2n-p, 0\} \leq t_1 < n$, then $d_H(\mathcal{C}_5) = n+1$.
    - If $s \geq 2$,
      - $a = 1$ and $t_1 = 0$, then $d_H(\mathcal{C}_5) = 2$,
      - $a = p^s-1$ and $t_1 = p^s-2$, then $d_H(\mathcal{C}_5) = p^s$.
  - When $h_1(x) \neq 0$ and $\frac{p^s+t_1}{2} < a \leq p^s-1$ (or when $h_1(x) = 0$, $h_2(x) \neq 0$ and $\frac{p^s+t_2}{2} < a \leq p^s-1$). Then, there is no MDS code.

- Type 6: $\mathcal{C}_6 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u^2(x-\gamma_0)^c\rangle$, where $0 \le c < a \le p^s-1$, $0 \le t_1 < a$, $0 \le t_2 < c$, either $h_1(x), h_2(x)$ are 0 or are units in $\mathcal{R}_\gamma$. No MDS constacyclic code can be obtained in this case.
- Type 7: $\mathcal{C}_7 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x)\rangle$, where $0 \le b < a \le p^s-1$, $0 \le t_1 < b$, $0 \le t_2 < b$, $0 \le t_3 < b$, either $h_1(x), h_2(x), h_3(x)$ are 0 or are units in $\mathcal{R}_\gamma$. No MDS constacyclic code can be obtained in this case.
- Type 8: $\mathcal{C}_8 = \langle (x-\gamma_0)^a+u(x-\gamma_0)^{t_1}h_1(x)+u^2(x-\gamma_0)^{t_2}h_2(x), u(x-\gamma_0)^b+u^2(x-\gamma_0)^{t_3}h_3(x), u^2(x-\gamma_0)^c\rangle$, where $0 \le c < b < a \le p^s-1$, $0 \le t_1 < b$, $0 \le t_2 < c$, $0 \le t_3 < c$, either $h_1(x), h_2(x), h_3(x)$ are 0 or are units in $\mathcal{R}_\gamma$. No MDS constacyclic code can be obtained in this case.

V. EXAMPLES

In this section, we present some examples of constacyclic codes of length p^s over $\mathcal{R} = \mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}(u^3=0)$.

Example 1: γ-constacyclic codes of length 3 over the chain ring $\mathcal{R} = \mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$ are precisely the ideals of $\mathcal{R}[x]/\langle x^3-\gamma\rangle$, where $\gamma \in \{1,2\}$.

In the following, we list all distinct γ-constacyclic codes of length 3 over the chain ring $\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$. There are 82 distinct γ-constacyclic codes listed below. In all codes we have $h_0, a_0, b_0, c_0 \in \{1,2\}$ and $b_1 \in \{0,1,3\}$.

Using the results in Sections III and IV, we list all Hamming distances d_H of such codes and the number of codewords $|\mathcal{C}|$ in each of those constacyclic codes. We also give all MDS and non-MDS codes (Table 1).

Among these 82 codes, 31 of them are MDS codes.

∗ Type 1 ($\mathcal{C}_1$): $\langle 0\rangle$, $\langle 1\rangle$.

∗ Type 2 ($\mathcal{C}_2$):
- → $\tau = 0$: $\langle u^2\rangle$,
- → $\tau = 1$: $\langle u^2(x-\gamma)\rangle$,
- → $\tau = 2$: $\langle u^2(x-\gamma)^2\rangle$.

∗ Type 3 ($\mathcal{C}_3$):
- → $h(x) = 0$ and $\delta = 0$: $\langle u\rangle$,
- → $h(x) = 0$ and $\delta = 1$: $\langle u(x-\gamma)\rangle$,
- → $h(x) = 0$ and $\delta = 2$: $\langle\langle u(x-\gamma)^2\rangle$,
- → $h(x) \neq 0$, $\delta = 1$ and $t = 0$: $\langle u(x-\gamma)+h_0u^2\rangle$,
- → $h(x) \neq 0$, $\delta = 2$ and $t = 0$: $\langle u(x-\gamma)^2+h_0u^2\rangle$,
- → $h(x) \neq 0$, $\delta = 2$ and $t = 1$: $\langle u(x-\gamma)^2+h_0u^2(x-\gamma)\rangle$.

∗ Type 4 ($\mathcal{C}_4$):
- → $h(x) = 0$, $\delta = 1$ and $\omega = 0$: $\langle u(x-\gamma), u^2\rangle$,
- → $h(x) = 0$, $\delta = 2$ and $\omega = 0$: $\langle u(x-\gamma)^2, u^2\rangle$,
- → $h(x) = 0$, $\delta = 2$ and $\omega = 1$: $\langle u(x-\gamma)^2, u^2(x-\gamma)\rangle$.

∗ Type 5 ($\mathcal{C}_5$):
- → $h_1(x) = h_2(x) = 0$ and $a = 1$: $\langle (x-\gamma)\rangle$,
- → $h_1(x) = h_2(x) = 0$ and $a = 2$: $\langle (x-\gamma)^2\rangle$,
- → $h_1(x) = 0$, $h_2(x) \neq 0$, $a = 1$ and $t_2 = 0$: $\langle (x-\gamma)+b_0u^2\rangle$,
- → $h_1(x) = 0$, $h_2(x) \neq 0$, $a = 2$ and $t_2 = 0$: $\langle (x-\gamma)^2+b_0u^2\rangle$,
- → $h_1(x) = 0$, $h_2(x) \neq 0$, $a = 2$ and $t_2 = 1$: $\langle (x-\gamma)^2+b_0u^2(x-\gamma)\rangle$,
- → $h_1(x) \neq 0$, $h_2(x) = 0$, $a = 1$ and $t_1 = 0$: $\langle (x-\gamma)+a_0u\rangle$,
- → $h_1(x) \neq 0$, $h_2(x) = 0$, $a = 2$ and $t_1 = 0$: $\langle (x-\gamma)^2+a_0u\rangle$,
- → $h_1(x) \neq 0$, $h_2(x) = 0$, $a = 2$ and $t_1 = 1$: $\langle (x-\gamma)^2+a_0u(x-\gamma)\rangle$,
- → $h_1(x) \neq 0$, $h_2(x) \neq 0$, $a = 1$ and $t_1 = t_2 = 0$: $\langle (x-\gamma)+a_0u+b_0u^2\rangle$,

TABLE 1. γ-constacyclic codes of length 3 over the chain ring $\mathbb{F}_3 \mathcal{C} u\mathbb{F}_3 \mathcal{C} u^2\mathbb{F}_3$.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
→ Type 1 :			
$*\langle 0\rangle$	0	1	No
$*\langle 1\rangle$	1	3^9	Yes
→ Type 2 :			
$*\langle u^2\rangle$	1	3^3	No
$*\langle u^2(x-\gamma)\rangle$	2	3^2	No
$*\langle u^2(x-\gamma)^2\rangle$	3	3	No
→ Type 3 :			
$*\langle u\rangle$	1	3^6	No
$*\langle u(x-\gamma)\rangle$	2	3^4	No
$*\langle u(x-\gamma)^2\rangle$	3	3^2	No
$*\langle u(x-\gamma)+h_0u^2\rangle$	2	3^4	No
$*\langle u(x-\gamma)^2+h_0u^2\rangle$	2	3^3	No
$*\langle u(x-\gamma)^2+h_0u^2(x-\gamma)\rangle$	3	3^2	No
→ Type 4 :			
$*\langle u(x-\gamma), u^2\rangle$	1	3^5	No
$*\langle u(x-\gamma)^2, u^2\rangle$	1	3^4	No
$*\langle u(x-\gamma)^2, u^2(x-\gamma)\rangle$	2	3^3	No
→ Type 5 :			
$*\langle (x-\gamma)\rangle$	2	3^6	Yes
$*\langle (x-\gamma)^2\rangle$	3	3^3	Yes
$*\langle (x-\gamma)+b_0u^2\rangle$	2	3^6	Yes
$*\langle (x-\gamma)^2+b_0u^2\rangle$	2	3^4	No
$*\langle (x-\gamma)^2+b_0u^2(x-\gamma)\rangle$	3	3^3	Yes
$*\langle (x-\gamma)+a_0u\rangle$	2	3^6	Yes
$*\langle (x-\gamma)^2+a_0u\rangle$	2	3^5	No
$*\langle (x-\gamma)^2+a_0u(x-\gamma)\rangle$	3	3^3	Yes
$*\langle (x-\gamma)+a_0u+b_0u^2\rangle$	2	3^6	Yes
$*\langle (x-\gamma)^2+a_0u+b_0u^2\rangle$	2	3^5	No
$*\langle (x-\gamma)^2+a_0u(x-\gamma)+b_0u^2+b_1u^2(x-\gamma)\rangle$	3	3^3	Yes
$*\langle (x-\gamma)^2+a_0u(x-\gamma)+b_0u^2(x-\gamma)\rangle$	3	3^3	Yes
→ Type 6 :			
$*\langle (x-\gamma), u^2\rangle$	1	3^7	No
$*\langle (x-\gamma)^2, u^2\rangle$	1	3^5	No
$*\langle (x-\gamma)^2, u^2(x-\gamma)\rangle$	2	3^4	No
$*\langle (x-\gamma)+a_0u, u^2\rangle$	1	3^7	No
$*\langle (x-\gamma)^2+a_0u, u^2\rangle$	1	3^6	No
$*\langle (x-\gamma)^2+a_0u(x-\gamma), u^2\rangle$	1	3^5	No
$*\langle (x-\gamma)^2+a_0u(x-\gamma), u^2(x-\gamma)\rangle$	2	3^4	No
$*\langle (x-\gamma)^2+a_0u(x-\gamma)+b_0u^2, u^2(x-\gamma)\rangle$	2	3^4	No
→ Type 7 :			
$*\langle (x-\gamma), u\rangle$	1	3^8	No
$*\langle (x-\gamma)^2, u\rangle$	1	3^7	No
$*\langle (x-\gamma)^2, u(x-\gamma)\rangle$	2	3^5	No
$*\langle (x-\gamma)^2, u(x-\gamma)+c_0u^2\rangle$	2	3^5	No
$*\langle (x-\gamma)^2+b_0u^2, u(x-\gamma)\rangle$	2	3^5	No
$*\langle (x-\gamma)^2+b_0u^2, u(x-\gamma)+c_0u^2\rangle$	2	3^5	No
→ Type 8 :			
$*\langle (x-\gamma)^2, u(x-\gamma), u^2\rangle$	1	3^6	No

→ $h_1(x) \neq 0$, $h_2(x) \neq 0$, $a = 2$ and $t_1 = t_2 = 0$: $\langle (x-\gamma)^2+a_0u+b_0u^2\rangle$,
→ $h_1(x) \neq 0$, $h_2(x) \neq 0$, $a = 2$, $t_1 = 1$ and $t_2 = 0$: $\langle (x-\gamma)^2+a_0u(x-\gamma)+b_0u^2+b_1u^2(x-\gamma)\rangle$,
→ $h_1(x) \neq 0$, $h_2(x) \neq 0$, $a = 2$ and $t_1 = t_2 = 1$: $\langle (x-\gamma)^2+a_0u(x-\gamma)+b_0u^2(x-\gamma)\rangle$.

* Type 6 ($\mathcal{C}_6$):
  → $h_1(x) = h_2(x) = 0$, $a = 1$ and $c = 0$: $\langle (x-\gamma), u^2\rangle$,
  → $h_1(x) = h_2(x) = 0$, $a = 2$ and $c = 0$: $\langle (x-\gamma)^2, u^2\rangle$,
  → $h_1(x) = h_2(x) = 0$, $a = 2$ and $c = 1$: $\langle (x-\gamma)^2, u^2(x-\gamma)\rangle$,
  → $h_1(x) \neq 0$, $h_2(x) = 0$, $a = 1$, $t_1 = 0$ and $c = 0$: $\langle (x-\gamma)+a_0u, u^2\rangle$,
  → $h_1(x) \neq 0$, $h_2(x) = 0$, $a = 2$, $t_1 = 0$ and $c = 0$: $\langle (x-\gamma)^2+a_0u, u^2\rangle$,
  → $h_1(x) \neq 0$, $h_2(x) = 0$, $a = 2$, $t_1 = 1$ and $c = 0$: $\langle (x-\gamma)^2+a_0u(x-\gamma), u^2\rangle$,
  → $h_1(x) \neq 0$, $h_2(x) = 0$, $a = 2$, $t_1 = 1$ and $c = 1$: $\langle (x-\gamma)^2+a_0u(x-\gamma), u^2(x-\gamma)\rangle$,
  → $h_1(x) \neq 0$, $h_2(x) \neq 0$, $a = 2$, $t_1 = 1$, $c = 1$ and $t_2 = 0$: $\langle (x-\gamma)^2+a_0u(x-\gamma)+b_0u^2, u^2(x-\gamma)\rangle$.
* Type 7 ($\mathcal{C}_7$):
  → $h_1(x) = h_2(x) = h_3(x) = 0$, $a = 1$ and $b = 0$: $\langle (x-\gamma), u\rangle$,
  → $h_1(x) = h_2(x) = h_3(x) = 0$, $a = 2$ and $b = 0$: $\langle (x-\gamma)^2, u\rangle$,
  → $h_1(x) = h_2(x) = h_3(x) = 0$, $a = 2$ and $b = 1$: $\langle (x-\gamma)^2, u(x-\gamma)\rangle$,
  → $h_1(x) = h_2(x) = 0$, $h_3(x) \neq 0$, $a = 2$, $b = 1$ and $t_3 = 0$: $\langle (x-\gamma)^2, u(x-\gamma)+c_0u^2\rangle$,
  → $h_1(x) = 0$, $h_2(x) \neq 0$, $h_3(x) = 0$, $a = 2$, $b = 1$ and $t_2 = 0$: $\langle (x-\gamma)^2+b_0u^2, u(x-\gamma)\rangle$,
  → $h_1(x) = 0$, $h_2(x) \neq 0$, $h_3(x) \neq 0$, $a = 2$, $b = 1$, $t_2 = 0$ and $t_3 = 0$: $\langle (x-\gamma)^2+b_0u^2, u(x-\gamma)+c_0u^2\rangle$.
* Type 8 ($\mathcal{C}_8$):
  → $h_1(x) = h_2(x) = h_3(x) = 0$, $a = 2$, $b = 1$ and $c = 0$: $\langle (x-\gamma)^2, u(x-\gamma), u^2\rangle$.

Example 2: We obtain cyclic codes corresponding to the unit $\gamma = 1$. cyclic codes of length 8 over the chain ring $\mathcal{R} = \mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ are precisely the ideals of $\mathcal{R}[x]/\langle x^8-1\rangle$.

The following Tables 2, 3, 4, 5, 6 and 7 shows the representation of all cyclic codes $\mathcal{C}$ of length 8 over the chain ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ of Type 1, 2 and 3 (of Type 4, of Type 5 $\{h_1(x) = h_2(x) = 0, h_1(x) = 0$ and $h_2(x) \neq 0\}$, of Type 5 $\{h_1(x) \neq 0$ and $h_2(x) = 0, h_1(x) \neq 0$ and $h_2(x) \neq 0\}$, of Type 6 $\{h_1(x) = h_2(x) = 0\}$, of Type 7 $\{h_1(x) = h_2(x) = h_3(x) = 0\}$, and of Type 8 $\{h_1(x) = h_2(x) = h_3(x) = 0\}$ respectively), together with the Hamming distances d_H of such codes and the number of codewords $|\mathcal{C}|$ in each of those cyclic codes. We also give all MDS and non-MDS codes. In all codes we have $h_i, a_i, b_i \in \{0, 1\}$ and $b_0 = 1$.

Example 3: γ-constacyclic codes of length 49 over the chain ring $\mathcal{R} = \mathbb{F}_7+u\mathbb{F}_7+u^2\mathbb{F}_7$ are precisely the ideals of $\mathcal{R}[x]/\langle x^{49}-\gamma\rangle$, where $\gamma \in \{1, 2, 3, 4, 5, 6\}$. Different generators of the constacyclic codes and their corresponding conditions to be MDS codes are given as follows:

- Type 1 ($\mathcal{C}_1$): $\langle 0\rangle$, $\langle 1\rangle$. For these codes the condition for MDS code are given by $3 = d_H(\mathcal{C}_1)$ and $1 = d_H(\mathcal{C}_1)$. As mentioned in Section IV, the only MDS constacyclic codes in this case is $\langle 1\rangle$.
- Type 2: $\mathcal{C}_2 = \langle u^2(x-\gamma)^\tau\rangle$, where $0 \leq \tau \leq 48$. The condition for MDS code is given by $\tau = 3d_H(\mathcal{C}_2)-101$. MDS constacyclic codes are non-existent in this case.

TABLE 2. Cyclic codes of length 8 over the chain ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ of Type 1, 2 and 3.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
→ Type 1:			
$*\langle 0\rangle$	0	1	No
$*\langle 1\rangle$	1	2^{24}	Yes
→ Type 2:			
$*\langle u^2\rangle$	1	2^8	No
$*\langle u^2(x-1)\rangle$	2	2^7	No
$*\langle u^2(x-1)^2\rangle$	2	2^6	No
$*\langle u^2(x-1)^3\rangle$	2	2^5	No
$*\langle u^2(x-1)^4\rangle$	2	2^4	No
$*\langle u^2(x-1)^5\rangle$	4	2^3	No
$*\langle u^2(x-1)^6\rangle$	4	2^2	No
$*\langle u^2(x-1)^7\rangle$	8	2	No
→ Type 3:			
$*\langle u\rangle$	1	2^{16}	No
$*\langle u(x-1)\rangle$	2	2^{14}	No
$*\langle u(x-1)^2\rangle$	2	2^{12}	No
$*\langle u(x-1)^3\rangle$	2	2^{10}	No
$*\langle u(x-1)^4\rangle$	2	2^8	No
$*\langle u(x-1)^5\rangle$	4	2^6	No
$*\langle u(x-1)^6\rangle$	4	2^4	No
$*\langle u(x-1)^7\rangle$	8	2^2	No
$*\langle u(x-1)+u^2\rangle$	2	2^{14}	No
$*\langle u(x-1)^2+u^2+h_1u^2(x-1)\rangle$	2	2^{12}	No
$*\langle u(x-1)^2+u^2(x-1)\rangle$	2	2^{12}	No
$*\langle u(x-1)^3+u^2+h_1u^2(x-1)+h_2u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle u(x-1)^3+u^2(x-1)+h_1u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle u(x-1)^3+u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle u(x-1)^4+u^2+h_1u^2(x-1)+h_3u^2(x-1)^3\rangle$	2	2^8	No
$*\langle u(x-1)^4+u^2(x-1)+h_1u^2(x-1)^2+h_2u^2(x-1)^3\rangle$	2	2^8	No
$*\langle u(x-1)^4+u^2(x-1)^2+h_1u^2(x-1)^3\rangle$	2	2^8	No
$*\langle u(x-1)^4+u^2(x-1)^3\rangle$	2	2^8	No
$*\langle u(x-1)^5+u^2+h_1u^2(x-1)+h_2u^2(x-1)^2\rangle$	2	2^8	No
$*\langle u(x-1)^5+u^2(x-1)+h_1u^2(x-1)^2+h_2u^2(x-1)^3\rangle$	2	2^7	No
$*\langle u(x-1)^5+u^2(x-1)^2+h_1u^2(x-1)^3+h_2u^2(x-1)^4\rangle$	4	2^6	No
$*\langle u(x-1)^5+u^2(x-1)^3+h_1u^2(x-1)^4\rangle$	4	2^6	No
$*\langle u(x-1)^5+u^2(x-1)^4\rangle$	4	2^6	No
$*\langle u(x-1)^6+u^2+h_1u^2(x-1)\rangle$	2	2^8	No
$*\langle u(x-1)^6+u^2(x-1)+h_1u^2(x-1)^2\rangle$	2	2^7	No
$*\langle u(x-1)^6+u^2(x-1)^2+h_1u^2(x-1)^3\rangle$	2	2^6	No
$*\langle u(x-1)^6+u^2(x-1)^3+h_1u^2(x-1)^4\rangle$	4	2^5	No
$*\langle u(x-1)^6+u^2(x-1)^4+h_1u^2(x-1)^5\rangle$	4	2^4	No
$*\langle u(x-1)^6+u^2(x-1)^5\rangle$	4	2^4	No
$*\langle u(x-1)^7+u^2\rangle$	2	2^8	No
$*\langle u(x-1)^7+u^2(x-1)\rangle$	2	2^7	No
$*\langle u(x-1)^7+u^2(x-1)^2\rangle$	2	2^6	No
$*\langle u(x-1)^7+u^2(x-1)^3\rangle$	2	2^5	No
$*\langle u(x-1)^7+u^2(x-1)^4\rangle$	4	2^4	No
$*\langle u(x-1)^7+u^2(x-1)^5\rangle$	4	2^3	No
$*\langle u(x-1)^7+u^2(x-1)^6\rangle$	8	2^2	No

TABLE 3. Cyclic codes of length 8 over the chain ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ of Type 4.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
$\to$ Type 4 :			
$*\langle u(x-1), u^2\rangle$	1	2^{15}	No
$*\langle u(x-1)^2, u^2\rangle$	1	2^{14}	No
$*\langle u(x-1)^2, u^2(x-1)\rangle$	2	2^{13}	No
$*\langle u(x-1)^3, u^2\rangle$	1	2^{13}	No
$*\langle u(x-1)^3, u^2(x-1)\rangle$	2	2^{12}	No
$*\langle u(x-1)^3, u^2(x-1)^2\rangle$	2	2^{11}	No
$*\langle u(x-1)^4, u^2\rangle$	1	2^{12}	No
$*\langle u(x-1)^4, u^2(x-1)\rangle$	2	2^{11}	No
$*\langle u(x-1)^4, u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle u(x-1)^4, u^2(x-1)^3\rangle$	2	2^{9}	No
$*\langle u(x-1)^5, u^2\rangle$	1	2^{11}	No
$*\langle u(x-1)^5, u^2(x-1)\rangle$	2	2^{10}	No
$*\langle u(x-1)^5, u^2(x-1)^2\rangle$	2	2^{9}	No
$*\langle u(x-1)^5, u^2(x-1)^3\rangle$	2	2^{8}	No
$*\langle u(x-1)^5, u^2(x-1)^4\rangle$	2	2^{7}	No
$*\langle u(x-1)^6, u^2\rangle$	1	2^{10}	No
$*\langle u(x-1)^6, u^2(x-1)\rangle$	2	2^{9}	No
$*\langle u(x-1)^6, u^2(x-1)^2\rangle$	2	2^{8}	No
$*\langle u(x-1)^6, u^2(x-1)^3\rangle$	2	2^{7}	No
$*\langle u(x-1)^6, u^2(x-1)^4\rangle$	2	2^{6}	No
$*\langle u(x-1)^6, u^2(x-1)^5\rangle$	4	2^{5}	No
$*\langle u(x-1)^7, u^2\rangle$	1	2^{9}	No
$*\langle u(x-1)^7, u^2(x-1)\rangle$	2	2^{8}	No
$*\langle u(x-1)^7, u^2(x-1)^2\rangle$	2	2^{7}	No
$*\langle u(x-1)^7, u^2(x-1)^3\rangle$	2	2^{6}	No
$*\langle u(x-1)^7, u^2(x-1)^4\rangle$	2	2^{5}	No
$*\langle u(x-1)^7, u^2(x-1)^5\rangle$	4	2^{4}	No
$*\langle u(x-1)^7, u^2(x-1)^6\rangle$	4	2^{3}	No
$*\langle u(x-1)^2+u^2, u^2(x-1)\rangle$	2	2^{13}	No
$*\langle u(x-1)^3+u^2, u^2(x-1)\rangle$	2	2^{12}	No
$*\langle u(x-1)^3+u^2+h_1u^2(x-1), u^2(x-1)^2\rangle$	2	2^{11}	No
$*\langle u(x-1)^3+u^2(x-1), u^2(x-1)^2\rangle$	2	2^{11}	No
$*\langle u(x-1)^4+u^2, u^2(x-1)\rangle$	2	2^{11}	No
$*\langle u(x-1)^4+u^2+h_1u^2(x-1), u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle u(x-1)^4+u^2(x-1), u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle u(x-1)^4+u^2+h_1u^2(x-1)+h_2u^2(x-1)^2, u^2(x-1)^3\rangle$	2	2^{9}	No
$*\langle u(x-1)^4+u^2(x-1)+h_1u^2(x-1)^2, u^2(x-1)^3\rangle$	2	2^{9}	No
$*\langle u(x-1)^4+u^2(x-1)^2, u^2(x-1)^3\rangle$	2	2^{9}	No
$*\langle u(x-1)^5+u^2, u^2(x-1)\rangle$	2	2^{10}	No
$*\langle u(x-1)^5+u^2, u^2(x-1)^2\rangle$	2	2^{9}	No
$*\langle u(x-1)^5+u^2(x-1), u^2(x-1)^2\rangle$	2	2^{9}	No
$*\langle u(x-1)^5+u^2(x-1)+u^2h_1(x-1)^2, u^2(x-1)^3\rangle$	2	2^{8}	No
$*\langle u(x-1)^5+u^2(x-1)^2, u^2(x-1)^3\rangle$	2	2^{8}	No
$*\langle u(x-1)^5+u^2(x-1)^2+u^2h_1(x-1)^3, u^2(x-1)^4\rangle$	2	2^{7}	No
$*\langle u(x-1)^5+u^2(x-1)^3, u^2(x-1)^4\rangle$	2	2^{7}	No
$*\langle u(x-1)^6+u^2, u^2(x-1)\rangle$	2	2^{9}	No
$*\langle u(x-1)^6+u^2(x-1), u^2(x-1)^2\rangle$	2	2^{8}	No
$*\langle u(x-1)^6+u^2(x-1)^2, u^2(x-1)^3\rangle$	2	2^{7}	No
$*\langle u(x-1)^6+u^2(x-1)^3, u^2(x-1)^4\rangle$	2	2^{6}	No
$*\langle u(x-1)^6+u^2(x-1)^4, u^2(x-1)^5\rangle$	4	2^{5}	No

- Type 3: $\mathcal{C}_3 = \langle u(x-\gamma)^\delta+u^2(x-\gamma)^t h(x)\rangle$, where $0 \le \delta \le 48$, $0 \le t < \delta$, either $h(x)$ is 0 or a unit in $\mathcal{R}_\gamma$. The condition for MDS code is given by $\mathsf{L} = 3d_H(\mathcal{C}_3)-\delta-52$. No MDS constacyclic code can be obtained in this case.
- Type 4: $\mathcal{C}_4 = \langle u(x-\gamma)^\delta+u^2(x-\gamma)^t h(x), u^2(x-\gamma)^\omega\rangle$, where $0 \le \omega < \delta \le 48$, $0 \le t < \omega$, either $h(x)$ is 0 or a unit in $\mathcal{R}_\gamma$. The condition for MDS code is given by $\omega = 3d_H(\mathcal{C}_4)-\delta-52$. No MDS constacyclic code can be obtained in this case.

TABLE 4. Cyclic codes of length 8 over the chain ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ of Type 5 $\{h_1(x) = h_2(x) = 0, h_1(x) = 0$ and $h_2(x) \neq 0\}$.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
$\to$ Type 5 $\{h_1(x) = h_2(x) = 0,\ h_1(x) = 0$ and $h_2(x) \neq 0\}$:			
$*\langle (x-1)\rangle$	2	2^{21}	Yes
$*\langle (x-1)^2\rangle$	2	2^{18}	No
$*\langle (x-1)^3\rangle$	2	2^{15}	No
$*\langle (x-1)^4\rangle$	2	2^{12}	No
$*\langle (x-1)^5\rangle$	4	2^{9}	No
$*\langle (x-1)^6\rangle$	4	2^{6}	No
$*\langle (x-1)^7\rangle$	8	2^{3}	Yes
$*\langle (x-1)+u^2\rangle$	2	2^{21}	Yes
$*\langle (x-1)^2+u^2+b_1u^2(x-1)\rangle$	2	2^{18}	No
$*\langle (x-1)^2+u^2(x-1)\rangle$	2	2^{18}	No
$*\langle (x-1)^3+u^2+b_1u^2(x-1)+b_2u^2(x-1)^2\rangle$	2	2^{15}	No
$*\langle (x-1)^3+u^2(x-1)+b_1u^2(x-1)^2\rangle$	2	2^{15}	No
$*\langle (x-1)^3+u^2(x-1)^2\rangle$	2	2^{15}	No
$*\langle (x-1)^4+u^2+b_1u^2(x-1)+b_2u^2(x-1)^2+b_3u^2(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u^2(x-1)+b_1u^2(x-1)^2+b_2u^2(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u^2(x-1)^2+b_1u^2(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u^2(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^5+u^2+b_1u^2(x-1)+b_2u^2(x-1)^2\rangle$	2	2^{11}	No
$*\langle (x-1)^5+u^2(x-1)+b_1u^2(x-1)^2+b_2u^2(x-1)^3\rangle$	2	2^{10}	No
$*\langle (x-1)^5+u^2(x-1)^2+b_1u^2(x-1)^3+b_2u^2(x-1)^4\rangle$	4	2^{9}	No
$*\langle (x-1)^5+u^2(x-1)^3+b_1u^2(x-1)^4\rangle$	4	2^{9}	No
$*\langle (x-1)^5+u^2(x-1)^4\rangle$	4	2^{9}	No
$*\langle (x-1)^6+u^2+b_1u^2(x-1)\rangle$	2	2^{10}	No
$*\langle (x-1)^6+u^2(x-1)+b_1u^2(x-1)^2\rangle$	2	2^{9}	No
$*\langle (x-1)^6+u^2(x-1)^2+b_1u^2(x-1)^3\rangle$	2	2^{8}	No
$*\langle (x-1)^6+u^2(x-1)^3+b_1u^2(x-1)^4\rangle$	4	2^{7}	No
$*\langle (x-1)^6+u^2(x-1)^4+b_1u^2(x-1)^5\rangle$	4	2^{6}	No
$*\langle (x-1)^6+u^2(x-1)^5\rangle$	4	2^{6}	No
$*\langle (x-1)^7+u^2\rangle$	2	2^{9}	No
$*\langle (x-1)^7+u^2(x-1)\rangle$	2	2^{8}	No
$*\langle (x-1)^7+u^2(x-1)^2\rangle$	2	2^{7}	No
$*\langle (x-1)^7+u^2(x-1)^3\rangle$	2	2^{6}	No
$*\langle (x-1)^7+u^2(x-1)^4\rangle$	4	2^{5}	No
$*\langle (x-1)^7+u^2(x-1)^5\rangle$	4	2^{4}	No
$*\langle (x-1)^7+u^2(x-1)^6\rangle$	8	2^{3}	Yes

- Type 5: $\mathcal{C}_5 = \langle (x-\gamma)^a+u(x-\gamma)^{t_1}h_1(x)+u^2(x-\gamma)^{t_2}h_2(x)\rangle$, where $1 \le a \le 48$, $0 \le t_1 < a$, $0 \le t_2 < a$, either $h_1(x)$, $h_2(x)$ are 0 or are units in $\mathcal{R}_\gamma$. The condition for MDS code is given by $\mathsf{V} = 3d_H(\mathcal{C}_5)-a-\mathsf{U}-3$ and all the distinct MDS codes are given by:
  - $\langle (x-\gamma)\rangle$
  - $\langle (x-\gamma)^{48}\rangle$,
  - $\langle (x-\gamma)+b_0u^2\rangle$,
  - $\langle (x-\gamma)^{48}+b_0u^2(x-\gamma)^{47}\rangle$,
  - $\langle (x-\gamma)+a_0u\rangle$,
  - $\langle (x-\gamma)+a_0u+b_0u^2\rangle$,
  - $\langle (x-\gamma)^{48}+a_0u(x-\gamma)^{47}\rangle$,
  - $\langle (x-\gamma)^{48}+a_0u(x-\gamma)^{47}+u^2(x-\gamma)^{t_2}\sum_{j=0}^{47-t_2} b_j(x-\gamma)^j\rangle$,

  where $0 \le t_2 \le 47$, $a_0, b_0 \in \{1, \ldots, 6\}$ and $b_j \in \{0, 1, \ldots, 6\}$.

  We present γ-constacyclic codes of Type 5 $\{h_1(x) = h_2(x) = 0\}$ in Table 8, together with the Hamming

TABLE 5. Cyclic codes of length 8 over the chain ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ of Type 5 $\{h_1(x)\neq 0$ and $h_2(x)=0$, $h_1(x)\neq 0$ and $h_2(x)\neq 0\}$.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
$\to$ Type 5 $\{h_1(x)\neq 0$ and $h_2(x)=0$, $h_1(x)\neq 0$ and $h_2(x)\neq 0\}$:			
$*\langle (x-1)+u\rangle$	2	2^{21}	Yes
$*\langle (x-1)^2+u+a_1u(x-1)\rangle$	2	2^{18}	No
$*\langle (x-1)^2+u(x-1)\rangle$	2	2^{18}	No
$*\langle (x-1)^3+u+a_1u(x-1)+a_2u(x-1)^2\rangle$	2	2^{15}	No
$*\langle (x-1)^3+u(x-1)+a_1u(x-1)^2\rangle$	2	2^{15}	No
$*\langle (x-1)^3+u(x-1)^2\rangle$	2	2^{15}	No
$*\langle (x-1)^4+u+a_1u(x-1)+a_2u(x-1)^2+a_3u(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u(x-1)+a_1u(x-1)^2+a_2u(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u(x-1)^2+a_1u(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^5+u+a_1u(x-1)+a_2u(x-1)^2\rangle$	2	2^{13}	No
$*\langle (x-1)^5+u(x-1)+a_1u(x-1)^2+a_2u(x-1)^3\rangle$	2	2^{11}	No
$*\langle (x-1)^5+u(x-1)^2+a_1u(x-1)^3+a_2u(x-1)^4\rangle$	2	2^{9}	No
$*\langle (x-1)^5+u(x-1)^3+a_1u(x-1)^4\rangle$	4	2^{9}	No
$*\langle (x-1)^5+u(x-1)^4\rangle$	4	2^{9}	No
$*\langle (x-1)^6+u+a_1u(x-1)\rangle$	2	2^{14}	No
$*\langle (x-1)^6+u(x-1)+a_1u(x-1)^2\rangle$	2	2^{12}	No
$*\langle (x-1)^6+u(x-1)^2+a_1u(x-1)^3\rangle$	2	2^{10}	No
$*\langle (x-1)^6+u(x-1)^3+a_1u(x-1)^4\rangle$	4	2^{8}	No
$*\langle (x-1)^6+u(x-1)^4+a_1u(x-1)^5\rangle$	4	2^{6}	No
$*\langle (x-1)^6+u(x-1)^5\rangle$	4	2^{6}	No
$*\langle (x-1)^7+u\rangle$	2	2^{15}	No
$*\langle (x-1)^7+u(x-1)\rangle$	2	2^{13}	No
$*\langle (x-1)^7+u(x-1)^2\rangle$	2	2^{11}	No
$*\langle (x-1)^7+u(x-1)^3\rangle$	2	2^{9}	No
$*\langle (x-1)^7+u(x-1)^4\rangle$	4	2^{7}	No
$*\langle (x-1)^7+u(x-1)^5\rangle$	4	2^{5}	No
$*\langle (x-1)^7+u(x-1)^6\rangle$	8	2^{3}	Yes
$*\langle (x-1)+u+u^2\rangle$	2	2^{21}	Yes
$*\langle (x-1)^2+u+a_1u(x-1)+\mathsf{B}_1\rangle$	2	2^{18}	No
$*\langle (x-1)^2+u(x-1)+\mathsf{B}_1\rangle$	2	2^{18}	No
$*\langle (x-1)^3+u+a_1u(x-1)+a_2u(x-1)^2+\mathsf{B}_2\rangle$	2	2^{15}	No
$*\langle (x-1)^3+u(x-1)+a_1u(x-1)^2+\mathsf{B}_2\rangle$	2	2^{15}	No
$*\langle (x-1)^3+u(x-1)^2+\mathsf{B}_2\rangle$	2	2^{15}	No
$*\langle (x-1)^4+u+a_1u(x-1)+a_2u(x-1)^2+a_3u(x-1)^3+\mathsf{B}_3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u(x-1)+a_1u(x-1)^2+a_2u(x-1)^3+\mathsf{B}_3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u(x-1)^2+a_1u(x-1)^3+\mathsf{B}_3\rangle$	2	2^{12}	No
$*\langle (x-1)^4+u(x-1)^3+\mathsf{B}_3\rangle$	2	2^{12}	No
$*\langle (x-1)^5+u+a_1u(x-1)+a_2u(x-1)^2+\mathsf{B}_2\rangle$	2	2^{13}	No
$*\langle (x-1)^5+u(x-1)+a_1u(x-1)^2+a_2u(x-1)^3+\mathsf{B}_3\rangle$	2	2^{11}	No
$*\langle (x-1)^5+u(x-1)^2+a_1u(x-1)^3+a_2u(x-1)^4+\mathsf{B}_4\rangle$	4	2^{9}	No
$*\langle (x-1)^5+u(x-1)^3+a_1u(x-1)^4+\mathsf{B}_4\rangle$	4	2^{9}	No
$*\langle (x-1)^5+u(x-1)^4+\mathsf{B}_4\rangle$	4	2^{9}	
$*\langle (x-1)^6+u+a_1u(x-1)+\mathsf{B}_1\rangle$	2	2^{14}	No
$*\langle (x-1)^6+u(x-1)+a_1u(x-1)^2+\mathsf{B}_2\rangle$	2	2^{12}	No
$*\langle (x-1)^6+u(x-1)^2+a_1u(x-1)^3+\mathsf{B}_3\rangle$	2	2^{10}	No
$*\langle (x-1)^6+u(x-1)^3+a_1u(x-1)^4+\mathsf{B}_4\rangle$	4	2^{8}	No
$*\langle (x-1)^6+u(x-1)^4+a_1u(x-1)^5+\mathsf{B}_5\rangle$	4	2^{6}	No
$*\langle (x-1)^6+u(x-1)^5+\mathsf{B}_5\rangle$	4	2^{6}	No
$*\langle (x-1)^7+u+u^2\rangle$	2	2^{15}	No
$*\langle (x-1)^7+u(x-1)+\mathsf{B}_1\rangle$	2	2^{13}	No
$*\langle (x-1)^7+u(x-1)^2+\mathsf{B}_2\rangle$	2	2^{11}	No
$*\langle (x-1)^7+u(x-1)^3+\mathsf{B}_3\rangle$	2	2^{9}	No
$*\langle (x-1)^7+u(x-1)^4+\mathsf{B}_4\rangle$	4	2^{7}	No
$*\langle (x-1)^7+u(x-1)^5+\mathsf{B}_5\rangle$	4	2^{5}	No
$*\langle (x-1)^7+u(x-1)^6+\mathsf{B}_6\rangle$	8	2^{3}	Yes
where $\mathsf{B}_i=u^2(x-1)^{t_2}\sum_{j=0}^{i-t_2} b_j(x-1)^j$ and $0\leq t_2\leq i$.			

TABLE 6. Cyclic codes of length 8 over the chain ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ of Type 6 and 7.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
$\to$ Type 6 $\{h_1(x)=h_2(x)=0\}$:			
$*\langle (x-1),u^2\rangle$	1	2^{22}	No
$*\langle (x-1)^2,u^2\rangle$	1	2^{20}	No
$*\langle (x-1)^2,u^2(x-1)\rangle$	2	2^{19}	No
$*\langle (x-1)^3,u^2\rangle$	1	2^{18}	No
$*\langle (x-1)^3,u^2(x-1)\rangle$	2	2^{17}	No
$*\langle (x-1)^3,u^2(x-1)^2\rangle$	2	2^{16}	No
$*\langle (x-1)^4,u^2\rangle$	1	2^{16}	No
$*\langle (x-1)^4,u^2(x-1)\rangle$	2	2^{15}	No
$*\langle (x-1)^4,u^2(x-1)^2\rangle$	2	2^{14}	No
$*\langle (x-1)^4,u^2(x-1)^3\rangle$	2	2^{13}	No
$*\langle (x-1)^5,u^2\rangle$	1	2^{14}	No
$*\langle (x-1)^5,u^2(x-1)\rangle$	2	2^{13}	No
$*\langle (x-1)^5,u^2(x-1)^2\rangle$	2	2^{12}	No
$*\langle (x-1)^5,u^2(x-1)^3\rangle$	2	2^{11}	No
$*\langle (x-1)^5,u^2(x-1)^4\rangle$	2	2^{10}	No
$*\langle (x-1)^6,u^2\rangle$	1	2^{12}	No
$*\langle (x-1)^6,u^2(x-1)\rangle$	2	2^{11}	No
$*\langle (x-1)^6,u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle (x-1)^6,u^2(x-1)^3\rangle$	2	2^{9}	No
$*\langle (x-1)^6,u^2(x-1)^4\rangle$	2	2^{8}	No
$*\langle (x-1)^6,u^2(x-1)^5\rangle$	4	2^{7}	No
$*\langle (x-1)^7,u^2\rangle$	1	2^{10}	No
$*\langle (x-1)^7,u^2(x-1)\rangle$	2	2^{9}	No
$*\langle (x-1)^7,u^2(x-1)^2\rangle$	2	2^{8}	No
$*\langle (x-1)^7,u^2(x-1)^3\rangle$	2	2^{7}	No
$*\langle (x-1)^7,u^2(x-1)^4\rangle$	2	2^{6}	No
$*\langle (x-1)^7,u^2(x-1)^5\rangle$	4	2^{5}	No
$*\langle (x-1)^7,u^2(x-1)^6\rangle$	4	2^{4}	No
$\to$ Type 7 $\{h_1(x)=h_2(x)=h_3(x)=0\}$:			
$*\langle (x-1),u\rangle$	1	2^{23}	No
$*\langle (x-1)^2,u\rangle$	1	2^{22}	No
$*\langle (x-1)^2,u(x-1)\rangle$	2	2^{20}	No
$*\langle (x-1)^3,u\rangle$	1	2^{21}	No
$*\langle (x-1)^3,u(x-1)\rangle$	2	2^{19}	No
$*\langle (x-1)^3,u(x-1)^2\rangle$	2	2^{17}	No
$*\langle (x-1)^4,u\rangle$	1	2^{20}	No
$*\langle (x-1)^4,u(x-1)\rangle$	2	2^{18}	No
$*\langle (x-1)^4,u(x-1)^2\rangle$	2	2^{16}	No
$*\langle (x-1)^4,u(x-1)^3\rangle$	2	2^{14}	No
$*\langle (x-1)^5,u\rangle$	1	2^{19}	No
$*\langle (x-1)^5,u(x-1)\rangle$	2	2^{17}	No
$*\langle (x-1)^5,u(x-1)^2\rangle$	2	2^{15}	No
$*\langle (x-1)^5,u(x-1)^3\rangle$	2	2^{13}	No
$*\langle (x-1)^5,u(x-1)^4\rangle$	2	2^{11}	No
$*\langle (x-1)^6,u\rangle$	1	2^{18}	No
$*\langle (x-1)^6,u(x-1)\rangle$	2	2^{16}	No
$*\langle (x-1)^6,u(x-1)^2\rangle$	2	2^{14}	No
$*\langle (x-1)^6,u(x-1)^3\rangle$	2	2^{12}	No
$*\langle (x-1)^6,u(x-1)^4\rangle$	2	2^{10}	No
$*\langle (x-1)^6,u(x-1)^5\rangle$	4	2^{8}	No
$*\langle (x-1)^7,u\rangle$	1	2^{17}	No
$*\langle (x-1)^7,u(x-1)\rangle$	2	2^{15}	No
$*\langle (x-1)^7,u(x-1)^2\rangle$	2	2^{13}	No
$*\langle (x-1)^7,u(x-1)^3\rangle$	2	2^{11}	No
$*\langle (x-1)^7,u(x-1)^4\rangle$	2	2^{9}	No
$*\langle (x-1)^7,u(x-1)^5\rangle$	4	2^{7}	No
$*\langle (x-1)^7,u(x-1)^6\rangle$	4	2^{5}	No

distances d_H of such codes and the number of codewords $|\mathcal{C}|$ in each of those constacyclic codes. We also give all MDS and non-MDS codes.

- Type 6: $\mathcal{C}_6 = \langle (x-\gamma)^a+u(x-\gamma)^{t_1}h_1(x)+u^2(x-\gamma)^{t_2}h_2(x),\ u^2(x-\gamma)^c\rangle$, where $0\leq c<a\leq 48$, $0\leq t_1<a$, $0\leq t_2<c$, either $h_1(x), h_2(x)$ are 0 or are

TABLE 7. Cyclic codes of length 8 over the chain ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ of Type 8 $\{h_1(x)=h_2(x)=h_3(x)=0\}$.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
$\to$ Type 8 $\{h_1(x)=h_2(x)=h_3(x)=0\}$:			
$*\langle(x-1)^2, u(x-1), u^2\rangle$	1	2^{21}	No
$*\langle(x-1)^3, u(x-1), u^2\rangle$	1	2^{20}	No
$*\langle(x-1)^3, u(x-1)^2, u^2\rangle$	1	2^{19}	No
$*\langle(x-1)^3, u(x-1)^2, u^2(x-1)\rangle$	2	2^{18}	No
$*\langle(x-1)^4, u(x-1), u^2\rangle$	1	2^{19}	No
$*\langle(x-1)^4, u(x-1)^2, u^2\rangle$	1	2^{18}	No
$*\langle(x-1)^4, u(x-1)^2, u^2(x-1)\rangle$	2	2^{17}	No
$*\langle(x-1)^4, u(x-1)^3, u^2\rangle$	1	2^{17}	No
$*\langle(x-1)^4, u(x-1)^3, u^2(x-1)\rangle$	2	2^{16}	No
$*\langle(x-1)^4, u(x-1)^3, u^2(x-1)^2\rangle$	2	2^{15}	No
$*\langle(x-1)^5, u(x-1), u^2\rangle$	1	2^{18}	No
$*\langle(x-1)^5, u(x-1)^2, u^2\rangle$	1	2^{17}	No
$*\langle(x-1)^5, u(x-1)^2, u^2(x-1)\rangle$	2	2^{16}	No
$*\langle(x-1)^5, u(x-1)^3, u^2\rangle$	1	2^{16}	No
$*\langle(x-1)^5, u(x-1)^3, u^2(x-1)\rangle$	2	2^{15}	No
$*\langle(x-1)^5, u(x-1)^3, u^2(x-1)^2\rangle$	2	2^{14}	No
$*\langle(x-1)^5, u(x-1)^4, u^2\rangle$	1	2^{15}	No
$*\langle(x-1)^5, u(x-1)^4, u^2(x-1)\rangle$	2	2^{14}	No
$*\langle(x-1)^5, u(x-1)^4, u^2(x-1)^2\rangle$	2	2^{13}	No
$*\langle(x-1)^5, u(x-1)^4, u^2(x-1)^3\rangle$	2	2^{12}	No
$*\langle(x-1)^6, u(x-1), u^2\rangle$	1	2^{17}	No
$*\langle(x-1)^6, u(x-1)^2, u^2\rangle$	1	2^{16}	No
$*\langle(x-1)^6, u(x-1)^2, u^2(x-1)\rangle$	2	2^{15}	No
$*\langle(x-1)^6, u(x-1)^3, u^2\rangle$	1	2^{15}	No
$*\langle(x-1)^6, u(x-1)^3, u^2(x-1)\rangle$	2	2^{14}	No
$*\langle(x-1)^6, u(x-1)^3, u^2(x-1)^2\rangle$	2	2^{13}	No
$*\langle(x-1)^6, u(x-1)^4, u^2\rangle$	1	2^{14}	No
$*\langle(x-1)^6, u(x-1)^4, u^2(x-1)\rangle$	2	2^{13}	No
$*\langle(x-1)^6, u(x-1)^4, u^2(x-1)^2\rangle$	2	2^{12}	No
$*\langle(x-1)^6, u(x-1)^4, u^2(x-1)^3\rangle$	2	2^{11}	No
$*\langle(x-1)^6, u(x-1)^5, u^2\rangle$	1	2^{13}	No
$*\langle(x-1)^6, u(x-1)^5, u^2(x-1)\rangle$	2	2^{12}	No
$*\langle(x-1)^6, u(x-1)^5, u^2(x-1)^2\rangle$	2	2^{11}	No
$*\langle(x-1)^6, u(x-1)^5, u^2(x-1)^3\rangle$	2	2^{10}	No
$*\langle(x-1)^6, u(x-1)^5, u^2(x-1)^4\rangle$	2	2^{9}	No
$*\langle(x-1)^7, u(x-1), u^2\rangle$	1	2^{16}	No
$*\langle(x-1)^7, u(x-1)^2, u^2\rangle$	1	2^{15}	No
$*\langle(x-1)^7, u(x-1)^2, u^2(x-1)\rangle$	2	2^{14}	No
$*\langle(x-1)^7, u(x-1)^3, u^2\rangle$	1	2^{14}	No
$*\langle(x-1)^7, u(x-1)^3, u^2(x-1)\rangle$	2	2^{13}	No
$*\langle(x-1)^7, u(x-1)^3, u^2(x-1)^2\rangle$	2	2^{12}	No
$*\langle(x-1)^7, u(x-1)^4, u^2\rangle$	1	2^{13}	No
$*\langle(x-1)^7, u(x-1)^4, u^2(x-1)\rangle$	2	2^{12}	No
$*\langle(x-1)^7, u(x-1)^4, u^2(x-1)^2\rangle$	2	2^{11}	No
$*\langle(x-1)^7, u(x-1)^4, u^2(x-1)^3\rangle$	2	2^{10}	No
$*\langle(x-1)^7, u(x-1)^5, u^2\rangle$	1	2^{12}	No
$*\langle(x-1)^7, u(x-1)^5, u^2(x-1)\rangle$	2	2^{11}	No
$*\langle(x-1)^7, u(x-1)^5, u^2(x-1)^2\rangle$	2	2^{10}	No
$*\langle(x-1)^7, u(x-1)^5, u^2(x-1)^3\rangle$	2	2^{9}	No
$*\langle(x-1)^7, u(x-1)^5, u^2(x-1)^4\rangle$	2	2^{8}	No
$*\langle(x-1)^7, u(x-1)^6, u^2\rangle$	1	2^{11}	No
$*\langle(x-1)^7, u(x-1)^6, u^2(x-1)\rangle$	2	2^{10}	No
$*\langle(x-1)^7, u(x-1)^6, u^2(x-1)^2\rangle$	2	2^{9}	No
$*\langle(x-1)^7, u(x-1)^6, u^2(x-1)^3\rangle$	2	2^{8}	No
$*\langle(x-1)^7, u(x-1)^6, u^2(x-1)^4\rangle$	2	2^{7}	No
$*\langle(x-1)^7, u(x-1)^6, u^2(x-1)^5\rangle$	4	2^{6}	No

units in $\mathcal{R}_\gamma$. The condition for MDS code is given by $c=3d_H(\mathcal{C}_6)-a-\mathsf{U}-3$. No MDS constacyclic code can be obtained in this case.

- Type 7: $\mathcal{C}_7 = \langle(x-\gamma)^a+u(x-\gamma)^{t_1}h_1(x)+u^2(x-\gamma)^{t_2}h_2(x),\ u(x-\gamma)^b+u^2(x-\gamma)^{t_3}h_3(x)\rangle$, where $0\le b<a\le 48$, $0\le t_1<b$, $0\le t_2<b$, $0\le t_3<b$, either $h_1(x), h_2(x), h_3(x)$ are 0 or are units in $\mathcal{R}_\gamma$. The condition for MDS code is given by $\mathsf{W}=3d_H(\mathcal{C}_7)-b-a-3$. No MDS constacyclic code can be obtained in this case.

TABLE 8. γ-constacyclic codes of length 49 over the chain ring $\mathbb{F}_7+u\mathbb{F}_7+u^2\mathbb{F}_7$ of Type 5 $\{h_1(x)=h_2(x)=0\}$.

Ideal ($\mathcal{C}$)	d_H	$\lvert\mathcal{C}\rvert$	MDS
$\to$ Type 5 $\{h_1(x)=h_2(x)=0\}$:			
$*\langle(x-\gamma)\rangle$	2	7^{144}	Yes
$*\langle(x-\gamma)^2\rangle$	2	7^{141}	No
$*\langle(x-\gamma)^3\rangle$	2	7^{138}	No
$*\langle(x-\gamma)^4\rangle$	2	7^{135}	No
$*\langle(x-\gamma)^5\rangle$	2	7^{132}	No
$*\langle(x-\gamma)^6\rangle$	2	7^{129}	No
$*\langle(x-\gamma)^7\rangle$	2	7^{126}	No
$*\langle(x-\gamma)^8\rangle$	3	7^{123}	No
$*\langle(x-\gamma)^9\rangle$	3	7^{120}	No
$*\langle(x-\gamma)^{10}\rangle$	3	7^{117}	No
$*\langle(x-\gamma)^{11}\rangle$	3	7^{114}	No
$*\langle(x-\gamma)^{12}\rangle$	3	7^{111}	No
$*\langle(x-\gamma)^{13}\rangle$	3	7^{108}	No
$*\langle(x-\gamma)^{14}\rangle$	3	7^{105}	No
$*\langle(x-\gamma)^{15}\rangle$	4	7^{102}	No
$*\langle(x-\gamma)^{16}\rangle$	4	7^{99}	No
$*\langle(x-\gamma)^{17}\rangle$	4	7^{96}	No
$*\langle(x-\gamma)^{18}\rangle$	4	7^{93}	No
$*\langle(x-\gamma)^{19}\rangle$	4	7^{90}	No
$*\langle(x-\gamma)^{20}\rangle$	4	7^{87}	No
$*\langle(x-\gamma)^{21}\rangle$	4	7^{84}	No
$*\langle(x-\gamma)^{22}\rangle$	5	7^{81}	No
$*\langle(x-\gamma)^{23}\rangle$	5	7^{78}	No
$*\langle(x-\gamma)^{24}\rangle$	5	7^{75}	No
$*\langle(x-\gamma)^{25}\rangle$	5	7^{72}	No
$*\langle(x-\gamma)^{26}\rangle$	5	7^{69}	No
$*\langle(x-\gamma)^{27}\rangle$	5	7^{66}	No
$*\langle(x-\gamma)^{28}\rangle$	5	7^{63}	No
$*\langle(x-\gamma)^{29}\rangle$	6	7^{60}	No
$*\langle(x-\gamma)^{30}\rangle$	6	7^{57}	No
$*\langle(x-\gamma)^{31}\rangle$	6	7^{54}	No
$*\langle(x-\gamma)^{32}\rangle$	6	7^{51}	No
$*\langle(x-\gamma)^{33}\rangle$	6	7^{48}	No
$*\langle(x-\gamma)^{34}\rangle$	6	7^{45}	No
$*\langle(x-\gamma)^{35}\rangle$	6	7^{42}	No
$*\langle(x-\gamma)^{36}\rangle$	7	7^{39}	No
$*\langle(x-\gamma)^{37}\rangle$	7	7^{36}	No
$*\langle(x-\gamma)^{38}\rangle$	7	7^{33}	No
$*\langle(x-\gamma)^{39}\rangle$	7	7^{30}	No
$*\langle(x-\gamma)^{40}\rangle$	7	7^{27}	No
$*\langle(x-\gamma)^{41}\rangle$	7	7^{24}	No
$*\langle(x-\gamma)^{42}\rangle$	7	7^{21}	No
$*\langle(x-\gamma)^{43}\rangle$	14	7^{18}	No
$*\langle(x-\gamma)^{44}\rangle$	21	7^{15}	No
$*\langle(x-\gamma)^{45}\rangle$	28	7^{12}	No
$*\langle(x-\gamma)^{46}\rangle$	35	7^{9}	No
$*\langle(x-\gamma)^{47}\rangle$	42	7^{6}	No
$*\langle(x-\gamma)^{48}\rangle$	49	7^{3}	Yes

- Type 8: $\mathcal{C}_8 = \langle(x-\gamma)^a+u(x-\gamma)^{t_1}h_1(x)+u^2(x-\gamma)^{t_2}h_2(x),\ u(x-\gamma)^b+u^2(x-\gamma)^{t_3}h_3(x),\ u^2(x-\gamma)^c\rangle$, where $0\le c<b<a\le 48$, $0\le t_1<b$, $0\le t_2<c$, $0\le t_3<c$, either $h_1(x), h_2(x), h_3(x)$ are 0 or are units in $\mathcal{R}_\gamma$. The condition for MDS code is given by $c=3d_H(\mathcal{C}_8)-b-a-3$. No MDS constacyclic code can be obtained in this case.

VI. CONCLUSION AND FUTURE WORK

Let p be a prime, s, m be positive integers, and let $\mathcal{R}=\mathbb{F}_{p^m}[u]/\langle u^3\rangle$ be the finite commutative chain ring with unity.

Let γ be an any nonzero element of the finite field $\mathbb{F}_{p^m}$. It is well known that the γ-constacyclic codes of length p^s over $\mathcal{R}$ are ideals of the ring $\mathcal{R}[x]/\langle x^{p^s}-\gamma\rangle$ which is a local ring with the maximal ideal $\langle u, x-\gamma_0\rangle$, but it is not a chain ring. Determining the Hamming distances of constacyclic codes and obtaining MDS constacyclic codes are very important in coding theory. Motivated by this, in this research article, we completed the problem of determining the Hamming distances of all γ-constacyclic codes by study their classifications of 8 types. Using these distances, we then obtain all MDS codes among such codes. We also give some examples in which we discuss the parameters of some MDS constacyclic codes for different values of p and s in Tables 1, 2, 3, 4, 5, 6, 7 and 8.

For future work, it would be interesting to determine the symbol-pair distances of γ-constacyclic codes of length of length p^s over $\mathcal{R}$, and to determine MDS symbol-pair γ-constacyclic codes of length p^s over $\mathcal{R}$.

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