

A Novel Approach Toward Roughness of Bipolar Soft Sets and Their Applications in MCGDM

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ABSTRACT The uncertainty in the data is an obstacle in decision-making (DM) problems. In order to solve problems with a variety of uncertainties a number of useful mathematical approaches together with fuzzy sets, rough sets, soft sets, bipolar soft sets have been developed. The rough set theory is an effective technique to study the uncertainty in data, while bipolar soft sets have the ability to handle the vagueness, as well as bipolarity of the data in a variety of situations. This study develops a new methodology, which we call the theory of dominance-based bipolar soft rough sets (DB-BSRSs), which will be used to propose a new technique to solve decision-making problems. The idea introduced in this study has never been discussed earlier. Furthermore, this concept has been explored by means of a detailed study of the structural properties. Moreover, some important measures like *the accuracy measure*, *the measure of precision*, and *the measure of quality* for DB-BSRSs are also provided. Finally, an application of the DB-BSRSs in multi-criteria group decision-making (MCGDM) problem is presented and an algorithm for this application is proposed, supported by an example, which yields the best decision, as well as, the worst decision between some objects. In comparison with some existing results, we also present some advantages of our proposed method.

INDEX TERMS Rough set, bipolar soft rough set, DB-BSRSs, decision-making.

I. INTRODUCTION

Cantor's set theory is essential for the entire of mathematics. But one issue related to the notion of a set is the concept of uncertainty. This uncertainty has been an issue for a long time for researchers and mathematicians to solve complicated problems in various fields, like social sciences, economics, medical sciences, management sciences, engineering, decision making, and artificial intelligence, etc.

To get rid of this uncertainty, researchers in mathematics, computer science, and many other related fields have proposed various theories like Probability theory, Fuzzy set theory [68], Rough set (RS) theory [52], Vague set theory, Graph theory, Decision-Making theory, etc. But, all of these theories have their own internal problems, which might be due to the inadequacy of the parameterization tools of the theories as discussed in [48].

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In 1999, Molodtsov [48] initiated the idea of the soft set (SS) as a new mathematical approach for handling uncertainty and vagueness. SS theory has rich potential for applications in many areas. Many interesting applications of SS theory can be seen in [12]–[14], [27], [29], [31], [42], [69].

There has been a growing interest in SS theory. Maji *et al.* [42] introduced some operations on soft sets and made a theoretical study of soft sets. Based on [42], Ali *et al.* [4] proposed a number of new operations on soft sets and improve the concept of compliment of the soft set. Fatimah *et al.* [16] introduced the concept of N-soft set as an extension of SS. Feng *et al.* [17], [18], proposed the relationships among soft sets, rough sets, and fuzzy sets, obtaining three types of hybrid models: rough soft sets, soft rough sets, and soft-rough fuzzy sets. Shabir *et al.* [57] reclassified a model of a soft rough set known as a *modified soft rough set (MRS-set)*. Malik and Shabir [46] introduced the idea of soft mapping and studied some of its properties. They applied soft mapping in medical diagnosis. Maji *et al.* [43] proposed the notion of fuzzy soft sets by combining fuzzy sets and

soft sets. Babitha and Sunil [11] introduced the concept of soft relations and soft functions. Then Qin *et al.* [54] pointed out that Babitha and Sunil's definition contradicts Cantor's set theory. Therefore they redefined the notion of soft relations and soft functions.

SS theory [48] and RS theory [52] are regarded as effective mathematical approaches to address uncertainty. In 2011, Feng *et al.* [17] established a relationship among these two theories and introduced the concept of a new hybrid version of *the soft rough sets (SRSs)*, that can give better approximations over Pawlak's RS theory in some cases ([17], Example 4.7). This approach can be viewed as a generalization of RS theory.

The idea of dominance-based rough sets was proposed by Greco *et al.* [21]–[24], [65]. It is an extension of RS theory. Du and Hu [15] presented the notion of a dominance-based rough fuzzy set. In 2019, Shaheen *et al.* [64] put forth the idea of dominance-based SRSs and discuss their application in decision analysis. Jun and Ahn [30] discussed the double-framed soft sets with applications in BCK/BCI-algebras.

A. LITERATURE REVIEW

In numerous sorts of data analysis, the bipolarity of the data is a key component to be taken into consideration while developing mathematical models for some issues. Bipolarity discusses the positive and negative aspects of the data. The positive data addresses what is assumed to be possible, while the negative data addresses what is not possible or certainly false. The concept that lies behind the presence of bipolar information is that a wide assortment of human decision-making depends on bipolar judgemental thinking. For example, sweetness and sourness of food, participation and rivalry, friendship and hostility, effects and side effects of drugs are the two different aspects of information in decision-making and coordination. The soft sets, the fuzzy sets, and the rough sets are not appropriate tools to handle this bipolarity.

Because of the significance of giving positive and negative aspects of data, Shabir and Naz [59] in 2013, initiated the concept of bipolar soft sets (BSSs) and its set-theoretic operations such as union, intersection, and complement and discuss its application to DM problems. After this research, BSSs have become increasingly popular with researchers. In 2015, Karaaslan and Karataş [33] reclassified a version of BSSs with another approximation giving opportunity to examine topological structures of BSSs. They also provide a DM methodology using BSSs with the help of an example. Later on, Karaaslan *et al.* [35] presented the notion of bipolar soft groups. In addition, Naz and Shabir [49] initiated the notion of fuzzy BSSs and studied their algebraic structures. In 2017, Shabir and Bakhtawar [60] first proposed the notion of bipolar soft connected, bipolar soft disconnected, and bipolar soft compact spaces. Then, Öztürk [51] further discussed the notions of interior and closure operators, basis, and subspaces in the bipolar soft topological spaces. Abdullah *et al.* [1] brought the idea of bipolar fuzzy soft

sets by hybridizing the soft sets and the bipolar fuzzy sets and applied this idea in DM problem. Alkouri *et al.* [8] initiated the idea of bipolar complex fuzzy sets and discussed their application in DM problems. In 2018, Karaaslan and Çağman [36] proposed the concept of the bipolar soft rough sets (BSRSs) which is a combination of RS theory and BSSs. They also provide applications of BSRSs in DM. Shabir and Gul [61] introduced the notion of modified rough bipolar soft sets (MRBS-sets) and discussed their application in MCGDM. Mahmood *et al.* [41] introduced a novel complex fuzzy N-soft sets and their DM algorithm. Malik and Shabir [45] initiated the idea of rough fuzzy BSSs and applied this idea to solve DM problems. Also, Malik and Shabir [46] developed a consensus model based on rough bipolar fuzzy approximations. Mahmood [40] redefined a version of BSSs which is called *T-bipolar soft sets* and applied this idea in DM problems. Shabir *et al.* [62] initiated the idea of rough approximations of BSSs by soft relations and applied this idea in DM. Al-Shami [9] came up with the concept of belong and nonbelong relations between a BSS and an ordinary point. Riaz and Tehrim [55] initiated the concept of bipolar fuzzy soft mappings and discuss their applications to bipolar disorders. Kamacı and Petchimuthu [32] proposed the idea of bipolar N-soft set, which is an extension of N-soft set, and discuss its applications in DM problems. Akram and Ali [2] developed a hybrid model for DM under rough Pythagorean fuzzy bipolar soft information.

B. MOTIVATION

If we summarize all the above arguments, then we have noticed that the BSSs have the capability to deal with the bipolarity of the information about certain objects with the help of two mappings. One mapping handles the positivity of the information, while the other mapping measures the negativity. Keeping in view the association between RSs and BSSs, two attempts have been made to study the roughness of BSSs: one by Karaaslan and Çağman [36], and the other by Shabir and Gul [61]. This is the main motivation for us to introduce and study the novel approach of the roughness of BSSs by using dominance-based bipolar soft rough sets (DB-BSRSs) and discuss their application in decision-making.

C. AIM OF THE PROPOSED STUDY

The main goal of this study is to present another interesting and novel version of BSRSs by utilizing DB-BSRSs.

We highlight the article by the following pioneering work:

- A novel concept known as DB-BSRSs is proposed.
- Some important structural properties of DB-BSRSs are investigated in detail.
- Some important measures associated with DB-BSRSs are proposed.
- A comprehensive MAGDM method in the framework of DB-BSRSs is introduced and the validity of this approach is also verified by a practical example.

- The detailed comparative analysis with other existing methods is done in order to show the advantages of the proposed methodology.

D. ORGANIZATION OF THE PAPER

The article has been organized in the following manner. Section 2 gives an overview of some basic ideas, which is required for the understanding of our research work. Section 3 starts by characterizing some dominance-based bipolar soft operators. Further, we discuss the relationship between these operators and their properties. Moreover, based upon these operators, we proposed the notion of DB-BSRSs. The notion is further investigated by considering its important structural properties in detail. In section 4, we discuss some important measures associated with DB-BSRSs and examine their properties. In section 5, we present a general framework of the MCGDM technique based on DB-BSRSs to choose the best element among the alternatives. Section 6 states our proposed DM algorithm for selecting the best alternative. After that, we give an illustrative example of the proposed DM technique to show that the technique can be effectively applied to some real-life problems in section 7. In section 8, a comparison analysis is made between the proposed model and some other well-known DM techniques. At the last, section 9 concludes with a summary of the present work and a suggestion for further research.

II. PRELIMINARIES

This section is dedicated to recall some essential ideas and notions that will be utilized in the coming sections. All through this article, we will use \mathfrak{U} to denote the *initial universe*, \mathcal{A} to denote the *parameters set*, and $2^{\mathfrak{U}}$ to denote the *power set* of \mathfrak{U} , except if expressed something else.

Definition 1 [52]: Assume that \mathfrak{U} be a non-empty finite universe, and Ω be an equivalence relation on \mathfrak{U} . Then $\mathbb{P} = (\mathfrak{U}, \Omega)$ is named an *approximation space*.

For a non-empty set $X \subseteq \mathfrak{U}$, the lower and upper approximations of X with respect to $\mathbb{P} = (\mathfrak{U}, \Omega)$ are respectively characterized as follows:

$$\Omega_*(X) = \{x \in \mathfrak{U} : [x]_{\Omega} \subseteq X\}, \tag{1}$$

$$\Omega^*(X) = \{x \in \mathfrak{U} : [x]_{\Omega} \cap X \neq \emptyset\}, \tag{2}$$

where

$$[x]_{\Omega} = \{u' \in \mathfrak{U} : (x, u') \in \Omega\}. \tag{3}$$

Moreover, the set

$$Bnd_{\Omega}(X) = \Omega^*(X) - \Omega_*(X) \tag{4}$$

is regarded as the Ω -*boundary region* of X .

Subsequently set X is called *definable* with respect to Ω if $\Omega_*(X) = \Omega^*(X)$; equivalently $Bnd_{\Omega}(X) = \emptyset$. Furthermore, set X is called *undefinable (rough)* with respect to Ω if $\Omega_*(X) \neq \Omega^*(X)$; equivalently $Bnd_{\Omega}(X) \neq \emptyset$.

Definition 2 [48]: Let \mathfrak{U} be a non-empty finite universe of objects and \mathcal{A} be a non-empty set of parameters associated

with the objects of \mathfrak{U} . Then a pair (f, \mathcal{A}) is named a *SS* over \mathfrak{U} , where f is a set-valued mapping given by $f : \mathcal{A} \rightarrow 2^{\mathfrak{U}}$.

Thus, a *SS* over the universe \mathfrak{U} offers a parameterized family of subsets of the universe \mathfrak{U} . For $e \in \mathcal{A}$, $f(e)$ could also be considered as the set of e -approximate elements of \mathfrak{U} by the *SS* (f, \mathcal{A}) . A *SS* over \mathfrak{U} may also be expressed by the set of ordered pairs:

$$(f, \mathcal{A}) = \{(e, f(e)) : e \in \mathcal{A}, f(e) \in 2^{\mathfrak{U}}\}. \tag{5}$$

Definition 3 [42]: By a *NOT* set of parameters of \mathcal{A} , we mean a set of the form $\tilde{\mathcal{A}} = \{\neg e : e \in \mathcal{A}\}$ in which $\neg e = \text{not } e$ for $e \in \mathcal{A}$.

Definition 4 [59]: Let f and g are two mappings, given by $f : \mathcal{A} \rightarrow 2^{\mathfrak{U}}$ and $g : \tilde{\mathcal{A}} \rightarrow 2^{\mathfrak{U}}$ such that $f(e) \cap g(\neg e) = \emptyset$ for all $e \in \mathcal{A}$. Then, a triplet $(f, g : \mathcal{A})$ is called a *BSS* over the universe \mathfrak{U} .

Subsequently, a *BSS* over the universe \mathfrak{U} offers a couple of parameterized families of subsets of \mathfrak{U} and $f(e) \cap g(\neg e) = \emptyset$ for all $e \in \mathcal{A}$, $\neg e \in \tilde{\mathcal{A}}$, is used as a consistency constraint. A *BSS* can also be represented through the set of triples as follows:

$$(f, g : \mathcal{A}) = \{(e, f(e), g(\neg e)) : e \in \mathcal{A}, \neg e \in \tilde{\mathcal{A}} \text{ and } f(e) \cap g(\neg e) = \emptyset\}. \tag{6}$$

From now onward, the set of all *BSSs* over the universe \mathfrak{U} are going to be represented by $\mathcal{BPSS}(\mathfrak{U})$.

Definition 5 [36]: A *BSS* $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathfrak{U})$ is said to be a *full BSS* if the following two conditions are satisfied:

- (1) $\bigcup_{e \in \mathcal{A}} f(e) = \mathfrak{U}$
- (2) $\bigcup_{\neg e \in \tilde{\mathcal{A}}} g(\neg e) = \mathfrak{U}$.

Definition 6 [36]: For $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathfrak{U})$, the object of the form $\Upsilon = (\mathfrak{U}, (f, g : \mathcal{A}))$ is known as a *BSA-space (bipolar soft approximation space)*. Based on Υ , the subsequent four operators are defined for any $X \subseteq \mathfrak{U}$ as follows:

$$\left. \begin{aligned} \underline{\mathcal{S}}_{\Upsilon^P}(X) &= \{x \in \mathfrak{U} : \exists e \in \mathcal{A}, [x \in f(e) \subseteq X]\}, \\ \underline{\mathcal{S}}_{\Upsilon^N}(X) &= \{x \in \mathfrak{U} : \exists \neg e \in \tilde{\mathcal{A}}, [x \in g(\neg e), \\ &\quad g(\neg e) \cap X^c \neq \emptyset]\}, \\ \overline{\mathcal{S}}_{\Upsilon^P}(X) &= \{x \in \mathfrak{U} : \exists e \in \mathcal{A}, [x \in f(e), \\ &\quad f(e) \cap X \neq \emptyset]\}, \\ \overline{\mathcal{S}}_{\Upsilon^N}(X) &= \{x \in \mathfrak{U} : \exists \neg e \in \tilde{\mathcal{A}}, [x \in g(\neg e) \subseteq X^c]\} \end{aligned} \right\} \tag{7}$$

are regarded as *soft Υ -lower positive, soft Υ -lower negative, soft Υ -upper positive, and soft Υ -upper negative approximations* of X , respectively. Generally, the ordered pairs are given as:

$$\left. \begin{aligned} \underline{\mathcal{BS}}_{\Upsilon}(X) &= (\underline{\mathcal{S}}_{\Upsilon^P}(X), \underline{\mathcal{S}}_{\Upsilon^N}(X)), \\ \overline{\mathcal{BS}}_{\Upsilon}(X) &= (\overline{\mathcal{S}}_{\Upsilon^P}(X), \overline{\mathcal{S}}_{\Upsilon^N}(X)) \end{aligned} \right\} \tag{8}$$

are called *bipolar soft rough approximations (BSR-approximations)* of X with respect to \mathcal{BSA} -space. Moreover, when $\underline{\mathcal{BS}}_{\Upsilon}(X) \neq \overline{\mathcal{BS}}_{\Upsilon}(X)$, then X is referred to as *bipolar soft rough set (BSRS)*; in any case, X is referred to as bipolar soft Υ -definable. Further, the boundary region of the *BSRS* is characterized as follows:

$$\text{Bnd}_{\Upsilon}(X) = \left(\overline{\mathcal{S}}_{\Upsilon^P}(X) - \underline{\mathcal{S}}_{\Upsilon^P}(X), \underline{\mathcal{S}}_{\Upsilon^N}(X) - \overline{\mathcal{S}}_{\Upsilon^N}(X) \right). \tag{9}$$

The properties satisfied by *BSR-approximations* of $X \subseteq \mathcal{U}$ can be found in [36].

To preserve the underlying idea of the *BSSs* as a parameterization tool and the capacity of RS theory to accommodate with vague ideas because of indiscernibility in data, Shabir and Gul [61] offered the notions of modified rough bipolar soft sets (*MRBS-sets*) as follows:

Definition 7 [61]: Assume that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$, where the mappings f and g are given by $f : \mathcal{A} \rightarrow 2^{\mathcal{U}}$ and $g : \tilde{\mathcal{A}} \rightarrow 2^{\mathcal{U}}$. Let us characterize two different mappings Φ and Ψ as follows:

$$\begin{aligned} \Phi : \mathcal{U} &\rightarrow 2^{\mathcal{A}} \\ \Phi(x) &= \{e : x \in f(e)\} \end{aligned}$$

and

$$\begin{aligned} \Psi : \mathcal{U} &\rightarrow 2^{\tilde{\mathcal{A}}} \\ \Psi(x) &= \{-e : x \in g(-e)\} \text{ for all } x \in \mathcal{U}. \end{aligned}$$

Then $\Theta = (\mathcal{U}, (\Phi, \Psi))$ is named as *MRBSA-space (modified rough bipolar soft approximation space)*.

For any $M \subseteq \mathcal{U}$, the *lower* and the *upper modified bipolar pairs* with respect to Θ are respectively characterized as follows:

$$\frac{\underline{\mathcal{MBS}}_{\Theta}(M)}{\overline{\mathcal{MBS}}_{\Theta}(M)} = \left(\frac{\underline{M}_{\Phi^+}, \underline{M}_{\Psi^-}}{\overline{M}_{\Phi^+}, \overline{M}_{\Psi^-}} \right), \tag{10}$$

where

$$\left. \begin{aligned} \underline{M}_{\Phi^+} &= \{x \in M : \Phi(x) \neq \Phi(y) \text{ for all } y \in M^c\}, \\ \overline{M}_{\Phi^+} &= \{x \in \mathcal{U} : \Phi(x) = \Phi(y) \text{ for some } y \in M\}, \\ \underline{M}_{\Psi^-} &= \{x \in \mathcal{U} : \Psi(x) = \Psi(y) \text{ for some } y \in M\}, \\ \overline{M}_{\Psi^-} &= \{x \in M : \Psi(x) \neq \Psi(y) \text{ for all } y \in M^c\}. \end{aligned} \right\} \tag{11}$$

Here $M^c = \mathcal{U} - M$. Generally \underline{X}_{Φ^+} , \overline{X}_{Φ^+} , \underline{X}_{Ψ^-} and \overline{X}_{Ψ^-} are called Φ -lower positive, Φ -upper positive, Ψ -lower negative and Ψ -upper negative *MRBS-approximations* of $M \subseteq \mathcal{U}$, respectively. Moreover, if $\underline{\mathcal{MBS}}_{\Theta}(M) \neq \overline{\mathcal{MBS}}_{\Theta}(M)$, then M is stated to be *MRBS-set*; otherwise, M is called *MRBS-definable*. Furthermore, the boundary region of the *MRBS-set* is described as follows:

$$\mathcal{MBnd}_{\Theta}(M) = \left(\overline{M}_{\Phi^+} \setminus \underline{M}_{\Phi^+}, \underline{M}_{\Psi^-} \setminus \overline{M}_{\Psi^-} \right). \tag{12}$$

The properties satisfied by *MRBS-approximations* can be found in [61].

III. DOMINANCE-BASED BIPOLAR SOFT ROUGH SETS (DB-BSRSs)

BSRSs were initially presented by Karaaslan and Çağman [36] to deal with the roughness through BSSs which were subsequently altered and upgraded by Shabir and Gul [61]. In the current section, we introduce another novel version of BSRSs by using DB-BSRSs.

Definition 8: Assume that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. For any object, $u \in \mathcal{U}$, define the \mathcal{A} -dominating set $f^+(u)$ and $\tilde{\mathcal{A}}$ -dominating set $g^+(u)$ as follows:

$$\begin{aligned} f^+(u) &= \left\{ \begin{array}{ll} \left\{ \begin{array}{l} v \in \mathcal{U} : v \notin f(e) \\ \text{for all } e \in \mathcal{A} \end{array} \right\} & \text{if } u \notin f(e) \\ \bigcap_{u \in f(e)} f(e) & \text{if } u \in f(e) \end{array} \right. \\ &\quad \text{for some } e \in \mathcal{A}, \\ g^+(u) &= \left\{ \begin{array}{ll} \left\{ \begin{array}{l} v \in \mathcal{U} : v \notin g(-e) \\ \text{for all } -e \in \tilde{\mathcal{A}} \end{array} \right\} & \text{if } u \notin g(-e) \\ \bigcap_{u \in g(-e)} g(-e) & \text{if } u \in g(-e) \end{array} \right. \\ &\quad \text{for some } -e \in \tilde{\mathcal{A}}. \end{array} \tag{13}$$

Similarly, \mathcal{A} -dominated set $f^-(u)$ and $\tilde{\mathcal{A}}$ -dominated set $g^-(u)$ are defined as follows:

$$\begin{aligned} f^-(u) &= \{v \in \mathcal{U} : u \in f^+(v)\}, \\ g^-(u) &= \{v \in \mathcal{U} : u \in g^+(v)\}. \end{aligned} \tag{14}$$

And, \mathcal{A} -equivalent sets $f^{\pm}(u)$ and $\tilde{\mathcal{A}}$ -equivalent set $g^{\pm}(u)$ are defined as follows:

$$\begin{aligned} f^{\pm}(u) &= f^+(u) \cap f^-(u), \\ g^{\pm}(u) &= g^+(u) \cap g^-(u). \end{aligned} \tag{15}$$

Generally, the operators f^+ , g^+ , f^- , g^- , f^{\pm} and g^{\pm} are called *\mathcal{A} -dominating*, *$\tilde{\mathcal{A}}$ -dominating*, *\mathcal{A} -dominated*, *$\tilde{\mathcal{A}}$ -dominated*, *\mathcal{A} -equivalent* and *$\tilde{\mathcal{A}}$ -equivalent bipolar soft operators*, respectively.

Remark 1: From the above definition, we have f^+ , f^- , $f^{\pm} : \mathcal{U} \rightarrow 2^{\mathcal{U}}$ and g^+ , g^- , $g^{\pm} : \mathcal{U} \rightarrow 2^{\mathcal{U}}$.

Remark 2: From the above definition, we can see that:

- (1) The sets $f^+(u)$ and $g^+(u)$ consists of all those elements that are not inferior to u subject to given parameters in \mathcal{A} and $\tilde{\mathcal{A}}$, respectively.
- (2) The sets $f^-(u)$ and $g^-(u)$ consists of all those elements that are not superior to u subject to given parameters in \mathcal{A} and $\tilde{\mathcal{A}}$.
- (3) The sets $f^{\pm}(u)$ and $g^{\pm}(u)$, is the intersection of the preceding two, consists of all those elements that have the similar parameters as u .
- (4) The objects that do not have any parameter from the given set of parameters are going to be mapped under the six operators to all those objects that have a similar behavior subject to the given parameter set.

Definition 9: Let $x, y \in \mathcal{U}$, then preference among the objects of \mathcal{U} is taken in the following sense:

- We say that x is preferred over y with respect to the parameter e , represented as $x \succeq_e y$, if x is at least as good as y with respect to the parameter e .
- Similarly, x is preferred over y with respect to the not parameter $\neg e$, represented as $x \preceq_{\neg e} y$, if x is at least as bad as y with respect to the not parameter $\neg e$.

Equivalently,

$$x \succeq_e y \iff x, y \notin f(e) \text{ or } x, y \in f(e) \text{ or } x \in f(e) \text{ but } y \notin f(e),$$

and

$$x \preceq_{\neg e} y \iff x, y \notin g(\neg e) \text{ or } x, y \in g(\neg e) \text{ or } x \in g(\neg e) \text{ but } y \notin g(\neg e).$$

In the framework of BSSs, the relation $x \succeq_e y$ implies that either x and y both have the parameter e or x possesses this parameter but y does not, or x and y both do not have the parameter e . Similarly, the relation $x \preceq_{\neg e} y$ means that either x and y both have the not parameter $\neg e$ or x possesses this not parameter but y does not or x and y both does not have the not parameter $\neg e$.

Proposition 1: Suppose that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. Then for any object $u \in \mathcal{U}$, we have

- (1) $f^\pm(u)$ is the greatest set which is contained in both $f^+(u)$ and $f^-(u)$.
- (2) $g^\pm(u)$ is the greatest set which is contained in both $g^+(u)$ and $g^-(u)$.

Proof 1: Straightforward.

To understand the ideas of bipolar soft operators given in Definition 8, here, we consider an example.

Example 1: Suppose that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$, where $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{A} = \{e_1, e_2, e_3\}$ and $\tilde{\mathcal{A}} = \{\neg e_1, \neg e_2, \neg e_3\}$. The maps f and g are characterized as follows:

$$f : \mathcal{A} \longrightarrow 2^{\mathcal{U}},$$

$$e \mapsto \begin{cases} \{u_1, u_3, u_5\} & \text{if } e = e_1, \\ \{u_2, u_3, u_5\} & \text{if } e = e_2, \\ \{u_2, u_5\} & \text{if } e = e_3, \end{cases}$$

$$g : \tilde{\mathcal{A}} \longrightarrow 2^{\mathcal{U}}$$

$$\neg e \mapsto \begin{cases} \{u_2, u_4\} & \text{if } \neg e = \neg e_1, \\ \{u_4, u_6\} & \text{if } \neg e = \neg e_2, \\ \{u_3, u_4, u_6\} & \text{if } \neg e = \neg e_3. \end{cases}$$

Now, according to Definition 8, the \mathcal{A} -dominating, the $\tilde{\mathcal{A}}$ -dominating, the \mathcal{A} -dominated, the $\tilde{\mathcal{A}}$ -dominated, the \mathcal{A} -equivalent and the $\tilde{\mathcal{A}}$ -equivalent bipolar soft operators for the elements of the universe \mathcal{U} can be calculated as follows:

$$f^+(u_1) = \{u_1, u_3, u_5\},$$

$$f^+(u_2) = f(e_2) \cap f(e_3) = \{u_2, u_5\},$$

$$f^+(u_3) = f(e_1) \cap f(e_2) = \{u_3, u_5\},$$

$$f^+(u_4) = \{u_4, u_6\},$$

$$f^+(u_5) = f(e_1) \cap f(e_2) \cap f(e_3) = \{u_5\},$$

$$f^+(u_6) = \{u_4, u_6\},$$

$$f^-(u_1) = \{u_1\},$$

$$f^-(u_2) = \{u_2\},$$

$$f^-(u_3) = \{u_1, u_3\},$$

$$f^-(u_4) = \{u_4, u_6\},$$

$$f^-(u_5) = \{u_1, u_2, u_3, u_5\}$$

$$f^-(u_6) = \{u_4, u_6\},$$

$$f^\pm(u_1) = \{u_1\}$$

$$f^\pm(u_2) = \{u_2\},$$

$$f^\pm(u_3) = \{u_3\},$$

$$f^\pm(u_4) = \{u_4, u_6\},$$

$$f^\pm(u_5) = \{u_5\},$$

$$f^\pm(u_6) = \{u_4, u_6\},$$

and

$$g^+(u_1) = \{u_1, u_5\},$$

$$g^+(u_2) = \{u_2, u_4\},$$

$$g^+(u_3) = \{u_3, u_4, u_6\},$$

$$g^+(u_4) = g(\neg e_1) \cap g(\neg e_2) \cap g(\neg e_3) = \{u_4\},$$

$$g^+(u_5) = \{u_1, u_5\},$$

$$g^+(u_6) = g(\neg e_2) \cap g(\neg e_3) = \{u_4, u_6\},$$

$$g^-(u_1) = \{u_1, u_5\},$$

$$g^-(u_2) = \{u_2\},$$

$$g^-(u_3) = \{u_3\},$$

$$g^-(u_4) = \{u_2, u_3, u_4, u_6\},$$

$$g^-(u_5) = \{u_1, u_5\},$$

$$g^-(u_6) = \{u_3, u_6\},$$

$$g^\pm(u_1) = \{u_1, u_5\},$$

$$g^\pm(u_2) = \{u_2\},$$

$$g^\pm(u_3) = \{u_3\},$$

$$g^\pm(u_4) = \{u_4\},$$

$$g^\pm(u_5) = \{u_1, u_5\},$$

$$g^\pm(u_6) = \{u_6\}.$$

Similarly, we can calculate

$$x \succeq_{e_1} y = \left\{ \begin{array}{l} (x, y) : x, y \notin f(e_1) \text{ or } x, y \in f(e_1) \\ \text{or } x \in f(e_1) \text{ but } y \notin f(e_1) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (u_2, u_4), (u_2, u_6), (u_4, u_2), (u_4, u_6), (u_6, u_2), \\ (u_6, u_4), (u_1, u_3), (u_1, u_5), (u_3, u_1), (u_3, u_5), \\ (u_5, u_1), (u_5, u_3), (u_1, u_2), (u_1, u_4), (u_1, u_6), \\ (u_3, u_2), (u_3, u_4), (u_3, u_6), (u_5, u_2), (u_5, u_4), \\ (u_5, u_6) \end{array} \right\}.$$

$$x \succeq_{e_2} y = \left\{ \begin{array}{l} (x, y) : x, y \notin f(e_2) \text{ or } x, y \in f(e_2) \\ \text{or } x \in f(e_2) \text{ but } y \notin f(e_2) \end{array} \right\}$$

$$\begin{aligned}
 &= \left\{ \begin{array}{l} (u_1, u_4), (u_1, u_6), (u_4, u_1), (u_4, u_6), (u_6, u_1), \\ (u_6, u_4), (u_2, u_3), (u_2, u_5), (u_3, u_2), (u_3, u_5), \\ (u_5, u_2), (u_5, u_3), (u_2, u_1), (u_2, u_4), (u_2, u_6), \\ (u_3, u_1), (u_3, u_4), (u_3, u_6), (u_5, u_1), (u_5, u_4), \\ (u_5, u_6) \end{array} \right\} \\
 x \succeq_{e_3} y &= \left\{ \begin{array}{l} (x, y) : x, y \notin f(e_3) \text{ or } x, y \in f(e_3) \\ \text{or } x \in f(e_3) \text{ but } y \notin f(e_3) \end{array} \right\} \\
 &= \left\{ \begin{array}{l} (u_1, u_3), (u_1, u_4), (u_1, u_6), (u_3, u_1), (u_3, u_4), \\ (u_3, u_6), (u_4, u_1), (u_4, u_3), (u_4, u_6), (u_6, u_1), \\ (u_6, u_3), (u_6, u_4), (u_2, u_5), (u_5, u_2), (u_2, u_1), \\ (u_2, u_3), (u_2, u_4), (u_2, u_6), (u_5, u_1), (u_5, u_3), \\ (u_5, u_4), (u_5, u_6) \end{array} \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 x \preceq_{-e_1} y &= \left\{ \begin{array}{l} (x, y) : x, y \notin g(\neg e_1) \text{ or } x, y \in g(\neg e_1) \\ \text{or } x \in g(\neg e_1) \text{ but } y \notin g(\neg e_1) \end{array} \right\} \\
 &= \left\{ \begin{array}{l} (u_1, u_3), (u_1, u_5), (u_1, u_6), (u_3, u_1), (u_3, u_5), \\ (u_3, u_6), (u_5, u_1), (u_5, u_3), (u_5, u_6), (u_6, u_1), \\ (u_6, u_3), (u_6, u_5), (u_2, u_4), (u_4, u_2), (u_2, u_1), \\ (u_2, u_3), (u_2, u_5), (u_2, u_6), (u_4, u_1), (u_4, u_3), \\ (u_5, u_4), (u_4, u_6) \end{array} \right\} \\
 x \preceq_{-e_2} y &= \left\{ \begin{array}{l} (x, y) : x, y \notin g(\neg e_2) \text{ or } x, y \in g(\neg e_2) \\ \text{or } x \in g(\neg e_2) \text{ but } y \notin g(\neg e_2) \end{array} \right\} \\
 &= \left\{ \begin{array}{l} (u_1, u_2), (u_1, u_3), (u_1, u_5), (u_2, u_1), (u_2, u_3), \\ (u_2, u_5), (u_3, u_1), (u_3, u_2), (u_3, u_5), (u_5, u_1), \\ (u_5, u_2), (u_5, u_3), (u_4, u_6), (u_6, u_4), (u_4, u_1), \\ (u_4, u_2), (u_4, u_3), (u_4, u_5), (u_6, u_1), (u_6, u_2), \\ (u_6, u_3), (u_6, u_5) \end{array} \right\} \\
 x \preceq_{-e_3} y &= \left\{ \begin{array}{l} (x, y) : x, y \notin g(\neg e_3) \text{ or } x, y \in g(\neg e_3) \\ \text{or } x \in g(\neg e_3) \text{ but } y \notin g(\neg e_3) \end{array} \right\} \\
 &= \left\{ \begin{array}{l} (u_1, u_2), (u_1, u_5), (u_2, u_1), (u_2, u_5), (u_5, u_1), \\ (u_5, u_2), (u_3, u_4), (u_3, u_6), (u_4, u_3), (u_4, u_6), \\ (u_6, u_3), (u_6, u_4), (u_3, u_1), (u_3, u_2), (u_3, u_5), \\ (u_4, u_1), (u_4, u_2), (u_4, u_5), (u_6, u_1), (u_6, u_2), \\ (u_6, u_5) \end{array} \right\}
 \end{aligned}$$

Proposition 2: Assume that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. Then for any $u_1, u_2 \in \mathcal{U}$, the following axioms are true.

- (1a) $u_2 \in f^+(u_1)$ if and only if $f^+(u_2) \subseteq f^+(u_1)$,
- (1b) $u_2 \in g^+(u_1)$ if and only if $g^+(u_2) \subseteq g^+(u_1)$.
- (2a) $u_2 \in f^-(u_1)$ if and only if $f^-(u_2) \subseteq f^-(u_1)$,
- (2b) $u_2 \in g^-(u_1)$ if and only if $g^-(u_2) \subseteq g^-(u_1)$,
- (3a) $u_2 \in f^\pm(u_1)$ if and only if $f^\pm(u_2) = f^\pm(u_1)$,
- (3b) $u_2 \in g^\pm(u_1)$ if and only if $g^\pm(u_2) = g^\pm(u_1)$,
- (4a) $f^\pm(u_1) = \{u_2 \in \mathcal{U} : f^+(u_1) = f^+(u_2)\} = \{u_2 \in \mathcal{U} : f^-(u_1) = f^-(u_2)\}$,
- (4b) $g^\pm(u_1) = \{u_2 \in \mathcal{U} : g^+(u_1) = g^+(u_2)\} = \{u_2 \in \mathcal{U} : g^-(u_1) = g^-(u_2)\}$.

Proof 2: Obvious.

Proposition 2 is better explained through Example 1. Suppose, for instance

$u_5 \in f^+(u_2)$ implies that $f^+(u_5) = \{u_5\} \subseteq \{u_2, u_5\} = f^+(u_2)$ and $u_4 \in g^+(u_2)$ implies that $g^+(u_4) = \{u_4\} \subseteq \{u_2, u_4\} = g^+(u_2)$.

Similarly,

$u_1 \in f^-(u_3)$ implies that $f^-(u_1) = \{u_1\} \subseteq \{u_1, u_3\} = f^-(u_3)$ and $u_3 \in g^-(u_6)$ implies that $g^-(u_3) = \{u_3\} \subseteq \{u_3, u_6\} = g^-(u_6)$.

Also,

$u_4 \in f^\pm(u_6)$ implies that $f^\pm(u_4) = \{u_4, u_6\} = f^\pm(u_6)$ and $u_1 \in g^\pm(u_5)$ implies that $g^\pm(u_1) = \{u_1, u_5\} = g^\pm(u_5)$.

Finally, consider

$f^\pm(u_4) = \{u_4, u_6\}$. It contains two objects which satisfies $f^+(u_4) = f^+(u_6) = \{u_4, u_6\} = f^-(u_4) = f^-(u_6)$. Also, $g^\pm(u_1) = \{u_1, u_5\}$. It contains two objects which satisfies $g^+(u_1) = g^+(u_5) = \{u_1, u_5\} = g^-(u_1) = g^-(u_5)$.

Now, we give the definition of bipolar soft dominating rough set, bipolar soft dominated rough set, and bipolar soft equivalent rough set with the help of soft operators given in Definition 8.

Definition 10: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. Then for any $X \subseteq \mathcal{U}$, the bipolar lower and upper approximations of X under the \mathcal{A} -dominating bipolar soft operator f^+ and the $\tilde{\mathcal{A}}$ -dominating bipolar soft operator g^+ can be defined, respectively, by the following two pairs:

$$\begin{aligned}
 \underline{DBS}_{(f^+, g^+)}(X) &= (\underline{X}_{f^+}, \underline{X}_{g^+}), \\
 \overline{DBS}_{(f^+, g^+)}(X) &= (\overline{X}_{f^+}, \overline{X}_{g^+}),
 \end{aligned} \tag{16}$$

where,

$$\begin{aligned}
 \underline{X}_{f^+} &= \{u \in \mathcal{U} : f^+(u) \subseteq X\}, \\
 \underline{X}_{g^+} &= \{u \in \mathcal{U} : g^+(u) \cap X^c \neq \emptyset\}, \\
 \overline{X}_{f^+} &= \{u \in \mathcal{U} : f^+(u) \cap X \neq \emptyset\}, \\
 \overline{X}_{g^+} &= \{u \in \mathcal{U} : g^+(u) \subseteq X^c\}.
 \end{aligned}$$

If $\underline{DBS}_{(f^+, g^+)}(X) \neq \overline{DBS}_{(f^+, g^+)}(X)$ then X is called bipolar soft dominating rough set, otherwise, it is called bipolar soft dominating definable set.

Similarly, the bipolar lower and upper approximations of X under the \mathcal{A} -dominated bipolar soft operator f^- and the $\tilde{\mathcal{A}}$ -dominated bipolar soft operator g^- can be defined by the following two pairs:

$$\begin{aligned}
 \underline{DBS}_{(f^-, g^-)}(X) &= (\underline{X}_{f^-}, \underline{X}_{g^-}), \\
 \overline{DBS}_{(f^-, g^-)}(X) &= (\overline{X}_{f^-}, \overline{X}_{g^-}),
 \end{aligned} \tag{17}$$

where,

$$\begin{aligned}
 \underline{X}_{f^-} &= \{u \in \mathcal{U} : f^-(u) \subseteq X\}, \\
 \underline{X}_{g^-} &= \{u \in \mathcal{U} : g^-(u) \cap X^c \neq \emptyset\}, \\
 \overline{X}_{f^-} &= \{u \in \mathcal{U} : f^-(u) \cap X \neq \emptyset\}, \\
 \overline{X}_{g^-} &= \{u \in \mathcal{U} : g^-(u) \subseteq X^c\}.
 \end{aligned}$$

If $\underline{DBS}_{(f^-,g^-)}(X) \neq \overline{DBS}_{(f^-,g^-)}(X)$ then X is called bipolar soft dominated rough set, otherwise, it is called bipolar soft dominated definable set.

Moreover, the bipolar lower and upper approximations of X under the \mathcal{A} -equivalent bipolar soft operator f^\pm and the $\tilde{\mathcal{A}}$ -equivalent bipolar soft operator g^\pm can be defined by the following two pairs:

$$\left. \begin{aligned} \underline{DBS}_{(f^\pm,g^\pm)}(X) &= (\underline{X}_{f^\pm}, \underline{X}_{g^\pm}), \\ \overline{DBS}_{(f^\pm,g^\pm)}(X) &= (\overline{X}_{f^\pm}, \overline{X}_{g^\pm}), \end{aligned} \right\} \quad (18)$$

where,

$$\begin{aligned} \underline{X}_{f^\pm} &= \{u \in \mathcal{U} : f^\pm(u) \subseteq X\}, \\ \underline{X}_{g^\pm} &= \{u \in \mathcal{U} : g^\pm(u) \cap X^c \neq \emptyset\}, \\ \overline{X}_{f^\pm} &= \{u \in \mathcal{U} : f^\pm(u) \cap X \neq \emptyset\}, \\ \overline{X}_{g^\pm} &= \{u \in \mathcal{U} : g^\pm(u) \subseteq X^c\}. \end{aligned}$$

If $\underline{DBS}_{(f^\pm,g^\pm)}(X) \neq \overline{DBS}_{(f^\pm,g^\pm)}(X)$ then X is called bipolar soft equivalent rough set, otherwise, it is named bipolar soft equivalent definable set.

The approximations in Equations (16) to (18) are collectively called *dominance-based bipolar soft rough approximations (DB-BSR-approximations)* of $X \subseteq \mathcal{U}$.

Remark 3: In the framework of the decision-making process:

- (1) \underline{X}_{f^+} may be considered as the collection of most favored alternatives in X under f^+ , while \overline{X}_{g^+} may be considered as the collection of most favored alternatives in X^c under g^+ .
- (2) \underline{X}_{f^-} may be considered as the collection of least favored alternatives in X under f^- , while \overline{X}_{g^-} may be considered as the collection of least favored alternatives in X^c under g^- .
- (3) \overline{X}_{f^+} may be considered as the collection of all possible alternatives that could be favored over the elements of X under f^+ , while \underline{X}_{g^+} can be considered as the collection of all possible alternatives that could be favored over the elements of X^c under g^+ .
- (4) \overline{X}_{f^-} may be interpreted as having elements over which the elements of X could possibly be favored under f^- , while \underline{X}_{g^-} may be interpreted as having elements over which the elements of X^c could possibly be favored under g^- .

Definition 11: The regions listed below:

- (1) $\text{POS}_{(f^+,g^+)}(X) = (\underline{X}_{f^+}, \overline{X}_{g^+})$,
- (2) $\text{BND}_{(f^+,g^+)}(X) = (\overline{X}_{f^+} - \underline{X}_{f^+}, \underline{X}_{g^+} - \overline{X}_{g^+})$,
- (3) $\text{NEG}_{(f^+,g^+)}(X) = (\mathcal{U}, \mathcal{U}) - (\overline{X}_{f^+}, \underline{X}_{g^+})$,

are called *the bipolar positive dominating region, the bipolar boundary dominating region and the bipolar negative dominating region*, respectively. The same terminologies can be defined for bipolar dominated region and bipolar equivalent region.

Corollary 1: From the above definition, we immediately have that $X \subseteq \mathcal{U}$ is a dominance-based bipolar

soft definable set if and only if $\text{BND}_{(f^+,g^+)}(X) = \text{BND}_{(f^-,g^-)}(X) = \text{BND}_{(f^\pm,g^\pm)}(X) = (\emptyset, \emptyset)$.

Example 2: To illustrate the notion of *DB-BSR-approximations*, let us revisit Example 1, where $X = \{u_1, u_3, u_5\} \subseteq \mathcal{U}$ and $X^c = \{u_2, u_4, u_6\}$. Now using Definition 10, we get

$$\begin{aligned} \underline{X}_{f^+} &= \{u_1, u_3, u_5\}, & \underline{X}_{g^+} &= \{u_2, u_3, u_4, u_6\}, \\ \overline{X}_{f^+} &= \{u_1, u_2, u_3, u_5\}, & \overline{X}_{g^+} &= \{u_2, u_4, u_6\}. \end{aligned}$$

Thus, bipolar soft dominating rough approximations of $X \subseteq \mathcal{U}$ are:

$$\begin{aligned} \underline{DBS}_{(f^+,g^+)}(X) &= (\{u_1, u_3, u_5\}, \{u_2, u_3, u_4, u_6\}), \\ \overline{DBS}_{(f^+,g^+)}(X) &= (\{u_1, u_2, u_3, u_5\}, \{u_2, u_4, u_6\}). \end{aligned}$$

Similarly,

$$\begin{aligned} \underline{X}_{f^-} &= \{u_1, u_3\}, & \underline{X}_{g^-} &= \{u_2, u_4, u_6\}, \\ \overline{X}_{f^-} &= \{u_1, u_3, u_5\}, & \overline{X}_{g^-} &= \{u_2\}. \end{aligned}$$

Therefore, bipolar soft dominated rough approximations of $X \subseteq \mathcal{U}$ are:

$$\begin{aligned} \underline{DBS}_{(f^-,g^-)}(X) &= (\{u_1, u_3\}, \{u_2, u_4, u_6\}), \\ \overline{DBS}_{(f^-,g^-)}(X) &= (\{u_1, u_3, u_5\}, \{u_2\}). \end{aligned}$$

Also,

$$\begin{aligned} \underline{X}_{f^\pm} &= \{u_1, u_3, u_5\}, & \underline{X}_{g^\pm} &= \{u_2, u_4, u_6\}, \\ \overline{X}_{f^\pm} &= \{u_1, u_3, u_5\}, & \overline{X}_{g^\pm} &= \{u_2, u_4, u_6\}. \end{aligned}$$

So, bipolar soft equivalent rough approximations of $X \subseteq \mathcal{U}$ are:

$$\begin{aligned} \underline{DBS}_{(f^\pm,g^\pm)}(X) &= (\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}), \\ \overline{DBS}_{(f^\pm,g^\pm)}(X) &= (\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}). \end{aligned}$$

Moreover, the bipolar positive dominating region, the bipolar boundary dominating region and the bipolar negative dominating region can be calculated as:

$$\begin{aligned} \text{POS}_{(f^+,g^+)}(X) &= (\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}), \\ \text{BND}_{(f^+,g^+)}(X) &= (\{u_2\}, \{u_3\}), \\ \text{NEG}_{(f^+,g^+)}(X) &= (\{u_4, u_6\}, \{u_1, u_5\}). \end{aligned}$$

Similarly, the bipolar positive dominated region, the bipolar boundary dominated region and the bipolar negative dominated region can be calculated as:

$$\begin{aligned} \text{POS}_{(f^-,g^-)}(X) &= (\{u_1, u_3\}, \{u_2\}), \\ \text{BND}_{(f^-,g^-)}(X) &= (\{u_5\}, \{u_4, u_6\}), \\ \text{NEG}_{(f^-,g^-)}(X) &= (\{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}). \end{aligned}$$

Also, the bipolar positive equivalent region, the bipolar boundary equivalent region and the bipolar negative equivalent region can be calculated as:

$$\begin{aligned} \text{POS}_{(f^\pm,g^\pm)}(X) &= (\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}), \\ \text{BND}_{(f^\pm,g^\pm)}(X) &= (\emptyset, \emptyset), \\ \text{NEG}_{(f^\pm,g^\pm)}(X) &= (\{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}). \end{aligned}$$

Since $\mathbb{BND}_{(f^+,g^+)}(X) \neq (\emptyset, \emptyset)$, $\mathbb{BND}_{(f^-,g^-)}(X) \neq (\emptyset, \emptyset)$ and $\mathbb{BND}_{(f^\pm,g^\pm)}(X) = (\emptyset, \emptyset)$, so X is a bipolar soft dominating (as well as bipolar soft dominated) rough set and a bipolar soft equivalent definable set.

Remark 4: Assume that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. Then:

(1) For any $u \in \mathcal{U}$ such that $u \notin f(e)$ for all $e \in \mathcal{A}$, we have

$$f^+(u) = f^-(u) = f^\pm(u) = \left\{ \begin{array}{l} v \in \mathcal{U} : v \notin f(e) \\ \text{for all } e \in \mathcal{A} \end{array} \right\}.$$

In this situation, we express the set by $f^*(u)$. That is, $f^+(u) = f^-(u) = f^\pm(u) = f^*(u)$.

(2) For any $u \in \mathcal{U}$ such that $u \notin g(\neg e)$ for all $\neg e \in \tilde{\mathcal{A}}$, we have

$$g^+(u) = g^-(u) = g^\pm(u) = \left\{ \begin{array}{l} v \in \mathcal{U} : v \notin g(\neg e) \\ \text{for all } \neg e \in \tilde{\mathcal{A}} \end{array} \right\}.$$

In this situation, we express the set by $g^*(u)$. That is, $g^+(u) = g^-(u) = g^\pm(u) = g^*(u)$.

For instance, in Example 1, $u_4, u_6 \notin f(e)$ for all $e \in \mathcal{A}$, so $f^*(u_4) = f^*(u_6) = \{u_4, u_6\}$. Similarly, $u_1, u_5 \notin g(\neg e)$ for all $\neg e \in \tilde{\mathcal{A}}$, so $g^*(u_1) = g^*(u_5) = \{u_1, u_5\}$.

Definition 12: For $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$, an object $u \in \mathcal{U}$ is said to be a *redundant object* with respect to f if $u \notin f(e)$ for all $e \in \mathcal{A}$. Similarly, an object $u \in \mathcal{U}$ is said to be a *redundant object* with respect to g if $u \notin g(\neg e)$ for all $\tilde{\mathcal{A}}$.

Proposition 3: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. Then $(f, g : \mathcal{A})$ is a full BSS if and only if $(f, g : \mathcal{A})$ has no redundant object with respect to f and g .

Proof 3: Straightforward.

Proposition 4: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$.

(1) If $u \in \mathcal{U}$ is a redundant object with respect to f , then $u \in f^*(u)$.

(2) If $u \in \mathcal{U}$ is a redundant object with respect to g , then $u \in g^*(u)$.

Proof 4: Straightforward.

Example 3: We can observe that in Example 1, $u_4, u_6 \in \mathcal{U}$ are redundant objects with respect to f and $u_1, u_5 \in \mathcal{U}$ are redundant objects with respect to g . Thus, $u_4, u_6 \in f^*(u_4) = f^*(u_6) = \{u_4, u_6\}$. Similarly, $u_1, u_5 \in g^*(u_1) = g^*(u_5) = \{u_1, u_5\}$.

In order to discover the relationship between the *DB-BSR-approximations* of a single set and the *DB-BSR-approximations* of two sets under the operators f^+, g^+, f^- and g^- , the following properties are given.

Theorem 1: Assume that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. Then for any $X, Y \subseteq \mathcal{U}$, the *DB-BSR-approximations* fulfill the following axioms:

- (1a) $\underline{X}_{f^+} \subseteq X \subseteq \overline{X}_{f^+}$,
- (1b) $\underline{X}_{f^-} \subseteq X \subseteq \overline{X}_{f^-}$,
- (2a) $\underline{\emptyset}_{f^+} = \emptyset = \overline{\emptyset}_{f^+}$,
- (2b) $\underline{\emptyset}_{f^-} = \emptyset = \overline{\emptyset}_{f^-}$,
- (3a) $\underline{\mathcal{U}}_{f^+} = \mathcal{U} = \overline{\mathcal{U}}_{f^+}$,
- (3b) $\underline{\mathcal{U}}_{f^-} = \mathcal{U} = \overline{\mathcal{U}}_{f^-}$,
- (4a) $X \subseteq Y \implies \underline{X}_{f^+} \subseteq \underline{Y}_{f^+}$,

- (4b) $X \subseteq Y \implies \underline{X}_{f^-} \subseteq \underline{Y}_{f^-}$,
- (5a) $X \subseteq Y \implies \overline{X}_{f^+} \subseteq \overline{Y}_{f^+}$,
- (5b) $X \subseteq Y \implies \overline{X}_{f^-} \subseteq \overline{Y}_{f^-}$,
- (6a) $\overline{(X \cap Y)}_{f^+} \subseteq \overline{X}_{f^+} \cap \overline{Y}_{f^+}$,
- (6b) $\overline{(X \cap Y)}_{f^-} \subseteq \overline{X}_{f^-} \cap \overline{Y}_{f^-}$,
- (7a) $\underline{(X \cap Y)}_{f^+} = \underline{X}_{f^+} \cap \underline{Y}_{f^+}$,
- (7b) $\underline{(X \cap Y)}_{f^-} = \underline{X}_{f^-} \cap \underline{Y}_{f^-}$,
- (8a) $\overline{(X \cup Y)}_{f^+} = \overline{X}_{f^+} \cup \overline{Y}_{f^+}$,
- (8b) $\overline{(X \cup Y)}_{f^-} = \overline{X}_{f^-} \cup \overline{Y}_{f^-}$,
- (9a) $\underline{(X \cup Y)}_{f^+} \supseteq \underline{X}_{f^+} \cup \underline{Y}_{f^+}$,
- (9b) $\underline{(X \cup Y)}_{f^-} \supseteq \underline{X}_{f^-} \cup \underline{Y}_{f^-}$,
- (10a) $\overline{X}_{f^+} = (\underline{X^c}_{f^+})^c$,
- (10b) $\overline{X}_{f^-} = (\underline{X^c}_{f^-})^c$,
- (11a) $\underline{X}_{f^+} = (\overline{(X^c)_{f^+}})^c$,
- (11b) $\underline{X}_{f^-} = (\overline{(X^c)_{f^-}})^c$,
- (12a) $\overline{(\underline{X}_{f^+})_{f^+}} = \overline{X}_{f^+} \subseteq \overline{(\overline{X}_{f^+})_{f^+}}$,
- (12b) $\overline{(\underline{X}_{f^-})_{f^-}} = \overline{X}_{f^-} \subseteq \overline{(\overline{X}_{f^-})_{f^-}}$,
- (13a) $\overline{(\overline{X}_{f^+})_{f^+}} \subseteq \overline{X}_{f^+} = \overline{(\overline{\overline{X}_{f^+}})_{f^+}}$,
- (13b) $\overline{(\overline{X}_{f^-})_{f^-}} \subseteq \overline{X}_{f^-} = \overline{(\overline{\overline{X}_{f^-}})_{f^-}}$,
- (14a) $\overline{X}_{g^+} \subseteq X^c \subseteq \underline{X}_{g^+}$,
- (14b) $\overline{X}_{g^-} \subseteq X^c \subseteq \underline{X}_{g^-}$,
- (15a) $\underline{\emptyset}_{g^+} = \emptyset = \overline{\emptyset}_{g^+}$,
- (15b) $\underline{\emptyset}_{g^-} = \emptyset = \overline{\emptyset}_{g^-}$,
- (16a) $\underline{\mathcal{U}}_{g^+} = \mathcal{U} = \overline{\mathcal{U}}_{g^+}$,
- (16b) $\underline{\mathcal{U}}_{g^-} = \mathcal{U} = \overline{\mathcal{U}}_{g^-}$,
- (17a) $X \subseteq Y \implies \underline{Y}_{g^+} \subseteq \underline{X}_{g^+}$,
- (17b) $X \subseteq Y \implies \underline{Y}_{g^-} \subseteq \underline{X}_{g^-}$,
- (18a) $X \subseteq Y \implies \overline{Y}_{g^+} \subseteq \overline{X}_{g^+}$,
- (18b) $X \subseteq Y \implies \overline{Y}_{g^-} \subseteq \overline{X}_{g^-}$,
- (19a) $\overline{(X \cap Y)}_{g^+} \supseteq \overline{X}_{g^+} \cap \overline{Y}_{g^+}$,
- (19b) $\overline{(X \cap Y)}_{g^-} \supseteq \overline{X}_{g^-} \cap \overline{Y}_{g^-}$,
- (20a) $\underline{(X \cap Y)}_{g^+} \supseteq \underline{X}_{g^+} \cap \underline{Y}_{g^+}$,
- (20b) $\underline{(X \cap Y)}_{g^-} \supseteq \underline{X}_{g^-} \cap \underline{Y}_{g^-}$,
- (21a) $\overline{(X \cup Y)}_{g^+} \subseteq \overline{X}_{g^+} \cup \overline{Y}_{g^+}$,
- (21b) $\overline{(X \cup Y)}_{g^-} \subseteq \overline{X}_{g^-} \cup \overline{Y}_{g^-}$,
- (22a) $\underline{(X \cup Y)}_{g^+} \subseteq \underline{X}_{g^+} \cup \underline{Y}_{g^+}$,
- (22b) $\underline{(X \cup Y)}_{g^-} \subseteq \underline{X}_{g^-} \cup \underline{Y}_{g^-}$,
- (23a) $\underline{X}_{g^+} = (\overline{(X^c)_{g^+}})^c$,
- (23b) $\underline{X}_{g^-} = (\overline{(X^c)_{g^-}})^c$,
- (24a) $\overline{X}_{g^+} = (\underline{(X^c)_{g^+}})^c$,
- (24b) $\overline{X}_{g^-} = (\underline{(X^c)_{g^-}})^c$,
- (25a) $\overline{(\overline{X}_{g^+})_{g^+}} = \overline{(X^c)_{g^+}} \subseteq \overline{(\overline{\overline{X}_{g^+}})_{g^+}}$,
- (25b) $\overline{(\overline{X}_{g^-})_{g^-}} = \overline{(X^c)_{g^-}} \subseteq \overline{(\overline{\overline{X}_{g^-}})_{g^-}}$,
- (26a) $\underline{(\underline{X}_{g^+})_{g^+}} \subseteq \underline{(X^c)_{g^+}} = \underline{(\overline{\underline{X}_{g^+}})_{g^+}}$,
- (26b) $\underline{(\underline{X}_{g^-})_{g^-}} \subseteq \underline{(X^c)_{g^-}} = \underline{(\overline{\underline{X}_{g^-}})_{g^-}}$.

Proof 5: Follows from Definition 10.

In order to discover the connection between the *DB-BSR-approximations* of $X \subseteq \mathcal{U}$ under the operators $f^+, g^+, f^-, g^-, f^\pm$ and g^\pm , the subsequent results are given.

Proposition 5: Suppose that $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$. Then the following properties hold for any $X \subseteq \mathcal{U}$;

- (1) $\underline{X}_{f^-} \subseteq \underline{X}_{f^\pm}$ and $\overline{X}_{f^\pm} \subseteq \overline{X}_{f^-}$.
- (2) $\underline{X}_{f^+} \subseteq \underline{X}_{f^\pm}$ and $\overline{X}_{f^\pm} \subseteq \overline{X}_{f^+}$.
- (3) $\underline{X}_{g^\pm} \subseteq \underline{X}_{g^-}$ and $\overline{X}_{g^-} \subseteq \overline{X}_{g^\pm}$.
- (4) $\underline{X}_{g^\pm} \subseteq \underline{X}_{g^+}$ and $\overline{X}_{g^+} \subseteq \overline{X}_{g^\pm}$.

Proof 6:

- (1) Let $u \in \underline{X}_{f^-}$. Then by Definition 10, we have $f^-(u) \subseteq X$. But from Definition 8, it follows that $f^\pm(u) \subseteq f^-(u)$. This implies that $f^\pm(u) \subseteq X$. Thus, $u \in \underline{X}_{f^\pm}$ and hence $\underline{X}_{f^-} \subseteq \underline{X}_{f^\pm}$.
Now, assume that $u \in \overline{X}_{f^\pm}$. Then by Definition 10, we have $f^\pm(u) \cap X \neq \emptyset$. But according to Definition 8, $f^\pm(u) \subseteq f^-(u)$. Thus, $f^-(u) \cap X \neq \emptyset$ which implies that $u \in \overline{X}_{f^-}$ and hence $\overline{X}_{f^\pm} \subseteq \overline{X}_{f^-}$.
- (2) Analogous to the proof of part (1).
- (3) Let $u \in \underline{X}_{g^\pm}$. Then by Definition 10, we have $g^\pm(u) \cap X \neq \emptyset$. But from Definition 8, it follows that $g^\pm(u) \subseteq g^-(u)$. This implies that $g^-(u) \cap X \neq \emptyset$ which implies that $u \in \underline{X}_{g^-}$. Hence, $\underline{X}_{g^\pm} \subseteq \underline{X}_{g^-}$.
Similarly, assume that $u \in \overline{X}_{g^-}$. Then by Definition 10, we have $g^-(u) \subseteq X^c$. But from Definition 8, we have $g^\pm(u) \subseteq g^-(u)$. This gives $g^\pm(u) \subseteq X^c$. Thus $u \in \overline{X}_{g^\pm}$ and hence, $\overline{X}_{g^-} \subseteq \overline{X}_{g^\pm}$.
- (4) Analogous to the proof of part (3).

Remark 5: If we compare the DB-BSR-approximations with BSR-approximations given in [36], we conclude the following points.

- (1) To show $\underline{\mathcal{S}}_{\gamma^P}(X \cap Y) = \underline{\mathcal{S}}_{\gamma^P}(X) \cap \underline{\mathcal{S}}_{\gamma^P}(Y)$, there is a strong condition on the positive SS to be the intersection complete. However, in the case of DB-BSR-approximations, no such condition is needed to prove $(X \cap Y)_{f^+} = \underline{X}_{f^+} \cap \underline{Y}_{f^+}$ or $(X \cap Y)_{f^-} = \underline{X}_{f^-} \cap \underline{Y}_{f^-}$.
- (2) Similarly, to show $\underline{\mathcal{S}}_{\gamma^P}(\mathcal{U}) = \mathcal{U} = \overline{\mathcal{S}}_{\gamma^P}(\mathcal{U})$, the BSS $(f, g : \mathcal{A})$ must be full. However in DB-BSR-approximations no such condition is required for $\underline{u}_{f^+} = \mathcal{U} = \overline{u}_{f^+}$, $\underline{u}_{f^-} = \mathcal{U} = \overline{u}_{f^-}$, $\underline{u}_{g^+} = \mathcal{U} = \overline{u}_{g^+}$ and $\underline{u}_{g^-} = \mathcal{U} = \overline{u}_{g^-}$.
- (3) Also to show $\underline{\mathcal{S}}_{\gamma^P}(X) \cap \underline{\mathcal{S}}_{\gamma^N}(X) = \emptyset$ and $\overline{\mathcal{S}}_{\gamma^P}(X) \cap \overline{\mathcal{S}}_{\gamma^N}(X) = \emptyset$ in [36] Karaaslan and Çağman imposed a constraint on the BSS $(f, g : \mathcal{A})$ to be semi-intersection. But in DB-BSR-approximations, we have $\underline{X}_{f^+} \cap \underline{Y}_{f^-} \neq \emptyset$, $\overline{X}_{f^+} \cap \overline{Y}_{f^-} \neq \emptyset$, $\underline{X}_{g^+} \cap \underline{Y}_{g^-} \neq \emptyset$ and $\overline{X}_{g^+} \cap \overline{Y}_{g^-} \neq \emptyset$ in any case.

IV. MEASURES ASSOCIATED WITH DB-BSRS

Generally, the uncertainty of a set is because of the presence of the boundary region. The broader the boundary region of a set is, the lower the exactness of the set is. To present the concept accurately, in this section, we propose a few significant measures related with DB-BSRSs and examine their properties.

Pawlak [52] proposed the notions of accuracy measure (AM) and roughness measures (RM) related to

RS approximations. The AM is the ratio of the cardinality of lower approximation to the cardinality of upper approximation while the RM is the complement of the AM. The aim of introducing these measures is to express the degree of completeness of information about set X or to communicate the quality of an approximation. The RM is considered as the degree of incompleteness of information about set X . As an extension of these measures, we introduce AM and RM using DB-BSR-approximations as follows:

Definition 13: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$, $\Upsilon = \langle \mathcal{U}, (f, g : \mathcal{A}) \rangle$ be \mathcal{BSA} -space and $X \subseteq \mathcal{U}$. Then the *accuracy measure* for DB-BSRS with respect to X under dominating bipolar soft operators f^+ and g^+ is denoted by $\mathbb{AM}(X^+)$ and is defined by an ordered pair:

$$\mathbb{AM}(X^+) = (\mathfrak{A}_X^{f^+}, \mathfrak{A}_X^{g^+}), \tag{19}$$

where $\mathfrak{A}_X^{f^+} = \frac{|\underline{X}_{f^+}|}{|\overline{X}_{f^+}|}$ and $\mathfrak{A}_X^{g^+} = \frac{|\overline{X}_{g^+}|}{|\underline{X}_{g^+}|}$ provided $\overline{X}_{f^+} \neq 0$, $\underline{X}_{g^+} \neq 0$. Here $|\bullet|$ denote the cardinality of the set.

Similarly, the *roughness measure* for DB-BSRS with respect to X under dominating bipolar soft operators f^+ and g^+ denoted by $\mathbb{RM}(X^+)$ and is defined as:

$$\mathbb{RM}(X^+) = (1, 1) - \mathbb{AM}(X^+) = (1 - \mathfrak{A}_X^{f^+}, 1 - \mathfrak{A}_X^{g^+}). \tag{20}$$

Obviously, $0 \leq \mathfrak{A}_X^{f^+} \leq 1$ and $0 \leq \mathfrak{A}_X^{g^+} \leq 1$ for any $\emptyset \neq X \subseteq \mathcal{U}$.

Also the *accuracy measures* for DB-BSRS with respect to X under dominated bipolar soft operators f^- and g^- and equivalent bipolar soft operators f^\pm and g^\pm are respectively given as:

$$\mathbb{AM}(X^-) = (\mathfrak{A}_X^{f^-}, \mathfrak{A}_X^{g^-}), \tag{21}$$

$$\mathbb{AM}(X^\pm) = (\mathfrak{A}_X^{f^\pm}, \mathfrak{A}_X^{g^\pm}), \tag{22}$$

where $\mathfrak{A}_X^{f^-} = \frac{|\underline{X}_{f^-}|}{|\overline{X}_{f^-}|}$, $\mathfrak{A}_X^{g^-} = \frac{|\overline{X}_{g^-}|}{|\underline{X}_{g^-}|}$, $\mathfrak{A}_X^{f^\pm} = \frac{|\underline{X}_{f^\pm}|}{|\overline{X}_{f^\pm}|}$ and $\mathfrak{A}_X^{g^\pm} = \frac{|\overline{X}_{g^\pm}|}{|\underline{X}_{g^\pm}|}$ provided $\overline{X}_{f^-} \neq 0$, $\underline{X}_{g^-} \neq 0$, $\overline{X}_{f^\pm} \neq 0$, $\underline{X}_{g^\pm} \neq 0$.

The corresponding *roughness measures* are:

$$\mathbb{RM}(X^-) = (1, 1) - \mathbb{AM}(X^-) = (1 - \mathfrak{A}_X^{f^-}, 1 - \mathfrak{A}_X^{g^-}), \tag{23}$$

$$\mathbb{RM}(X^\pm) = (1, 1) - \mathbb{AM}(X^\pm) = (1 - \mathfrak{A}_X^{f^\pm}, 1 - \mathfrak{A}_X^{g^\pm}). \tag{24}$$

Clearly, $0 \leq \mathfrak{A}_X^{f^-} \leq 1$, $0 \leq \mathfrak{A}_X^{g^-} \leq 1$, $0 \leq \mathfrak{A}_X^{f^\pm} \leq 1$ and $0 \leq \mathfrak{A}_X^{g^\pm} \leq 1$ for any $\emptyset \neq X \subseteq \mathcal{U}$.

Proposition 6: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$ and $\Upsilon = \langle \mathcal{U}, (f, g : \mathcal{A}) \rangle$ be \mathcal{BSA} -space. For any $X \subseteq \mathcal{U}$, $\mathbb{AM}(X^+)$

under dominating bipolar soft operators f^+ and g^+ satisfies the following properties.

- (1) $\mathbb{AM}(X^+) = (1, 1)$ if and only if $\underline{X}_{f^+} = \overline{X}_{f^+}$ and $\underline{X}_{g^+} = \overline{X}_{g^+}$.
- (2) $\mathbb{AM}(X^+) = (0, 0)$ if and only if $\underline{X}_{f^+} = \emptyset$ and $\overline{X}_{g^+} = \emptyset$.
- (3) $\mathbb{AM}(X^+) = (1, 1)$ if and only if $X = \mathcal{U}$.

Proof 7: Straightforward.

Similar results can be proved for $\mathbb{AM}(X^-)$ and $\mathbb{AM}(X^\pm)$.

Proposition 7: Let $\emptyset \neq X, Y \subseteq \mathcal{U}$ and $\mathbb{AM}(X^+) = (\mathfrak{A}_X^{f^+}, \mathfrak{A}_X^{g^+})$, $\mathbb{AM}(Y^+) = (\mathfrak{A}_Y^{f^+}, \mathfrak{A}_Y^{g^+})$ be the accuracy measures of X and Y under dominating bipolar soft operators f^+ and g^+ respectively. If $X \subseteq Y$, then $\mathfrak{A}_X^{f^+} \leq \mathfrak{A}_Y^{f^+}$ and $\mathfrak{A}_X^{g^+} \geq \mathfrak{A}_Y^{g^+}$.

Proof 8: Obvious.

In 2001, Gediga and Düntsch [19] described the idea of *the measure of the precision* for $\emptyset \neq X \subseteq \mathcal{U}$, which is the ratio of the cardinality of lower approximation of X to the cardinality of X . In the framework of DB-BSR-approximations it can be characterized in the following manner:

Definition 14: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$ and $\Upsilon = (\mathcal{U}, (f, g : \mathcal{A}))$ be \mathcal{BSA} -space. Then for any $\emptyset \neq X \subseteq \mathcal{U}$ *the measure of the precision* for DB-BSRS with respect to X under dominating bipolar soft operators f^+ and g^+ is expressed as $\mathbb{MIP}(X^+)$ and is characterized by an ordered pair:

$$\mathbb{MIP}(X^+) = (\pi_X^{f^+}, \pi_X^{g^+}), \tag{25}$$

where $\pi_X^{f^+} = \frac{|\underline{X}_{f^+}|}{|X|}$ and $\pi_X^{g^+} = \frac{|\overline{X}_{g^+}|}{|X^c|}$.

Here $X^c = \mathcal{U} - X$. Clearly, $\mathbb{MIP}(X^+) \geq \mathbb{AM}(X^+)$. It is noteworthy that $\mathbb{MIP}(X^+)$ needs complete knowledge about the set X ; while $\mathbb{AM}(X^+)$ does not.

Also, $0 \leq \pi_X^{f^+} \leq 1$ and $0 \leq \pi_X^{g^+} \leq 1$ for any $\emptyset X \subseteq \mathcal{U}$. Moreover, it can be noticed that $\mathbb{MIP}(X^+) = (1, 1)$ if and only if $|\underline{X}_{f^+}| = |X|$ and $|\overline{X}_{g^+}| = |X^c|$.

Same terminologies of measure of the precision can be defined for DB-BSRS with respect to X under dominated and equivalent bipolar soft operators.

Proposition 8: Let $X, Y \subseteq \mathcal{U}$ and $\mathbb{MIP}(X^+) = (\pi_X^{f^+}, \pi_X^{g^+})$, $\mathbb{MIP}(Y^+) = (\pi_Y^{f^+}, \pi_Y^{g^+})$ be the measures of the precision of X and Y under dominating bipolar soft operators f^+ and g^+ , respectively. If $X \subseteq Y$, then $\pi_X^{f^+} \geq \pi_Y^{f^+}$ and $\pi_X^{g^+} \leq \pi_Y^{g^+}$.

Proof 9: Straightforward.

In 2010, Yao [67] proposed another measure related to RS approximations known as *the measure of quality*, which is described in the following way:

$$\alpha_\Omega(X) = \frac{|\Omega_*(X)| + |\Omega_*(X^c)|}{|\mathcal{U}|}. \tag{26}$$

In the framework of the DB-BSR-approximations it can be characterized in the following manner:

Definition 15: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$ and $\Upsilon = (\mathcal{U}, (f, g : \mathcal{A}))$ be \mathcal{BSA} -space. Then for any $\emptyset \neq X \subseteq \mathcal{U}$ *the measure of quality* for DB-BSRS with respect to X under

dominating bipolar soft operators f^+ and g^+ is expressed as $\mathbb{MQ}(X^+)$ and is characterized by means of an ordered pair:

$$\mathbb{MQ}(X^+) = (\alpha^{f^+}, \beta^{g^+}), \tag{27}$$

where

$$\alpha^{f^+} = \frac{|\underline{X}_{f^+}| + |\underline{X}_{f^+}^c|}{|\mathcal{U}|} \tag{28}$$

and

$$\beta^{g^+} = \frac{|\overline{X}_{g^+}| + |\overline{X}_{g^+}^c|}{|\mathcal{U}|}. \tag{29}$$

Obviously, $0 \leq \alpha^{f^+} \leq 1$ and $0 \leq \beta^{g^+} \leq 1$ for any $\emptyset \neq X \subseteq \mathcal{U}$.

The corresponding measure of roughness is defined as:

$$\mathbb{RQ}(X^+) = (1, 1) - \mathbb{MQ}(X^+) = (1 - \alpha^{f^+}, 1 - \beta^{g^+}). \tag{30}$$

Same terminologies of measure of quality can be defined for DB-BSRS with respect to X under dominated and equivalent bipolar soft operators.

Proposition 9: Let $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$, $\Upsilon = (\mathcal{U}, (f, g : \mathcal{A}))$ a \mathcal{BSA} -space and $\emptyset \neq X \subseteq \mathcal{U}$. Then $\mathbb{MQ}(X) = (1, 1)$ if and only if $X = \mathcal{U}$ or $X = \emptyset$.

Proof 10: Straightforward.

In order to understand the aforementioned terminologies, here we employ an example.

Example 4: Let us consider $(f, g : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$ as given in Example 1, where $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{A} = \{e_1, e_2, e_3\}$, $\tilde{\mathcal{A}} = \{-e_1, -e_2, -e_3\}$ and the maps f and g describe as:

$$\begin{aligned} f : \mathcal{A} &\longrightarrow 2^{\mathcal{U}}, \\ e &\mapsto \begin{cases} \{u_1, u_3, u_5\} & \text{if } e = e_1, \\ \{u_2, u_3, u_5\} & \text{if } e = e_2, \\ \{u_2, u_5\} & \text{if } e = e_3, \end{cases} \\ g : \tilde{\mathcal{A}} &\longrightarrow 2^{\mathcal{U}} \\ -e &\mapsto \begin{cases} \{u_2, u_4\} & \text{if } -e = -e_1, \\ \{u_4, u_6\} & \text{if } -e = -e_2, \\ \{u_3, u_4, u_6\} & \text{if } -e = -e_3. \end{cases} \end{aligned}$$

We already know that DB-BSR-approximations of $X = \{u_1, u_3, u_5\} \subseteq \mathcal{U}$ under f^+ and g^+ are:

$$\begin{aligned} \underline{X}_{f^+} &= \{u_1, u_3, u_5\}, \\ \underline{X}_{g^+} &= \{u_2, u_3, u_4, u_6\}, \\ \overline{X}_{f^+} &= \{u_1, u_2, u_3, u_5\}, \\ \overline{X}_{g^+} &= \{u_2, u_4, u_6\}, \\ \overline{X}_{f^+}^c &= \{u_6\}, \\ \overline{X}_{g^+}^c &= \{u_1, u_5\}. \end{aligned}$$

So the values of accuracy measure, measure of precision and measure of quality are respectively given as:

$$AM(X^+) = (\alpha_X^{f^+}, \alpha_X^{g^+}) = \left(\frac{3}{4}, \frac{3}{4}\right) = (0.75, 0.75),$$

$$MP(X^+) = (\pi_X^{f^+}, \pi_X^{g^+}) = \left(\frac{3}{3}, \frac{3}{3}\right) = (1, 1),$$

$$MQ(X^+) = (\alpha^{f^+}, \beta^{g^+}) = \left(\frac{3+1}{6}, \frac{3+2}{6}\right) = (0.666, 0.833).$$

Hence, $\alpha_X^{f^+}$ and $\alpha_X^{g^+}$ depicts the elements of \mathcal{U} precisely up to the degree 0.75. While $\pi_X^{f^+}$ and $\pi_X^{g^+}$ depict the elements of \mathcal{U} precisely up to the degree 1. Moreover, α^{f^+} and β^{g^+} depicts the elements of \mathcal{U} precisely up to the degree 0.666 and 0.833, respectively.

V. MCGDM USING DB-BSRSs

The growing complexity of the socio-economic environment, operational research, and industrial engineering force humans to address problems crossing many fields. Group decision-making (GDM), as one of the successful techniques to cope with complex DM problems, is characterized as a decision problem in which numerous experts give their judgment over a set of alternatives. The purpose is to reconcile (or compromise) variations of opinion expressed by individual experts to discover an alternative (or set of alternatives) that is most acceptable by the group of experts as a whole. In a complex society, GDM methods must inevitably take many criteria (or factors) under consideration. Thus, research on GDM that explicitly incorporates multiple criteria has been a major perspective and has made significant advancement with the rapid development of operations research, management science, systems engineering, and other fields. Hwang and Lin [28] first look at to investigate systematically how multiple criteria could be used in GDM.

In general, MCGDM is a procedure in which a group of experts (decision-makers) collaborates to select the best alternative from a set of feasible alternatives that are classified according to their attributes in a given situation. In this section, we develop a novel MCGDM approach using the DB-BSRSs. We give a brief description of a MCGDM problem under the environment of the DB-BSRSs, and afterward, give

a general DM methodology for the MCGDM problem by using the DB-BSRSs.

A. PROBLEM DESCRIPTION

We firstly provide the basic description of the considered MCGDM problem in this section.

Let $\mathcal{U} = \{u_1, u_2, \dots, u_n\}$ be the non-empty finite universe of n objects (alternatives) and $\mathcal{A} = \{e_1, e_2, \dots, e_m\}$ be the finite collection of all possible parameters of objects. Suppose that $\mathcal{H} = \{p_1, p_2, \dots, p_k\}$ is a set of k independent experts (decision-makers), X_1, X_2, \dots, X_k are non-empty subsets of \mathcal{U} , represent results of primary assessments of experts p_1, p_2, \dots, p_k , respectively and $T_1, T_2, \dots, T_r \in \mathcal{BPSS}(\mathcal{U})$ are the actual results that previously received for problems in various periods or various locations. Then the decision-making for this MCGDM problem is: ‘‘how to obtain the evaluation of these particular experts so that the selected object (alternative) is optimal for all criteria’’.

B. METHODOLOGY OF DECISION-MAKING

In this subsection, we propose the mathematical formulation and the strategy of the MCGDM approach based on the DB-BSRSs.

Definition 16: Assume that $\underline{DBS}^+_{T_q}(X_j) = (\underline{X}_{j_{f^+}}, \underline{X}_{j_{g^+}})$ and $\overline{DBS}^+_{T_q}(X_j) = (\overline{X}_{j_{f^+}}, \overline{X}_{j_{g^+}})$ be lower and upper dominating soft rough approximations of X_j ; ($j = 1, 2, \dots, k$) related to $T_q = (f_q, g_q : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$; ($q = 1, 2, \dots, r$). Then, (31) and (32), as shown at the bottom of the page, are called the dominating bipolar soft lower approximation matrix and the dominating bipolar soft upper approximation matrix, respectively. Here

$$\underline{X}_{j_{f^+}} = \left(\underline{u}_{1j_{f^+}}, \underline{u}_{2j_{f^+}}, \dots, \underline{u}_{nj_{f^+}} \right), \tag{33}$$

$$\underline{X}_{j_{g^+}} = \left(\underline{u}_{1j_{g^+}}, \underline{u}_{2j_{g^+}}, \dots, \underline{u}_{nj_{g^+}} \right), \tag{34}$$

$$\overline{X}_{j_{f^+}} = \left(\overline{u}_{1j_{f^+}}, \overline{u}_{2j_{f^+}}, \dots, \overline{u}_{nj_{f^+}} \right), \tag{35}$$

$$\overline{X}_{j_{g^+}} = \left(\overline{u}_{1j_{g^+}}, \overline{u}_{2j_{g^+}}, \dots, \overline{u}_{nj_{g^+}} \right). \tag{36}$$

$$\underline{[\mathcal{D}]}_{(f^+, g^+)} = \begin{pmatrix} \langle \underline{X}_{1_{f^+}}, \underline{X}_{1_{g^+}} \rangle & \langle \underline{X}_{2_{f^+}}, \underline{X}_{2_{g^+}} \rangle & \cdots & \langle \underline{X}_{k_{f^+}}, \underline{X}_{k_{g^+}} \rangle \\ \langle \underline{X}_{1_{f^+}}, \underline{X}_{1_{g^+}} \rangle & \langle \underline{X}_{2_{f^+}}, \underline{X}_{2_{g^+}} \rangle & \cdots & \langle \underline{X}_{k_{f^+}}, \underline{X}_{k_{g^+}} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \underline{X}_{1_{f^+}}, \underline{X}_{1_{g^+}} \rangle & \langle \underline{X}_{2_{f^+}}, \underline{X}_{2_{g^+}} \rangle & \cdots & \langle \underline{X}_{k_{f^+}}, \underline{X}_{k_{g^+}} \rangle \end{pmatrix}, \tag{31}$$

$$\overline{[\mathcal{D}]}_{(f^+, g^+)} = \begin{pmatrix} \langle \overline{X}_{1_{f^+}}, \overline{X}_{1_{g^+}} \rangle & \langle \overline{X}_{2_{f^+}}, \overline{X}_{2_{g^+}} \rangle & \cdots & \langle \overline{X}_{k_{f^+}}, \overline{X}_{k_{g^+}} \rangle \\ \langle \overline{X}_{1_{f^+}}, \overline{X}_{1_{g^+}} \rangle & \langle \overline{X}_{2_{f^+}}, \overline{X}_{2_{g^+}} \rangle & \cdots & \langle \overline{X}_{k_{f^+}}, \overline{X}_{k_{g^+}} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \overline{X}_{1_{f^+}}, \overline{X}_{1_{g^+}} \rangle & \langle \overline{X}_{2_{f^+}}, \overline{X}_{2_{g^+}} \rangle & \cdots & \langle \overline{X}_{k_{f^+}}, \overline{X}_{k_{g^+}} \rangle \end{pmatrix} \tag{32}$$

where,

$$\underline{u}_{ij_{f_q^+}} = \begin{cases} 1 & \text{if } u_i \in X_{j_{f_q^+}}, \\ 0 & \text{if } u_i \notin X_{j_{f_q^+}}. \end{cases} \quad (37)$$

$$\underline{u}_{ij_{g_q^+}} = \begin{cases} -1/2 & \text{if } u_i \in X_{j_{g_q^+}}, \\ 0 & \text{if } u_i \notin X_{j_{g_q^+}}. \end{cases} \quad (38)$$

$$\overline{u}_{ij_{f_q^+}} = \begin{cases} 1/2 & \text{if } u_i \in \overline{X}_{j_{f_q^+}}, \\ 0 & \text{if } u_i \notin \overline{X}_{j_{f_q^+}}. \end{cases} \quad (39)$$

$$\overline{u}_{ij_{g_q^+}} = \begin{cases} -1 & \text{if } u_i \in \overline{X}_{j_{g_q^+}}, \\ 0 & \text{if } u_i \notin \overline{X}_{j_{g_q^+}}. \end{cases} \quad (40)$$

In the same style, let $\underline{DBS}^-_{T_q}(X_j) = (\underline{X}_{j_{f^-}}, \underline{X}_{j_{g^-}})$ and $\overline{DBS}^-_{T_q}(X_j) = (\overline{X}_{j_{f^-}}, \overline{X}_{j_{g^-}})$ be lower and upper dominated soft rough approximations of X_j ; ($j = 1, 2, \dots, k$) related to $T_q = (f_q, g_q : \mathcal{A}) \in \mathcal{BPSS}(\mathcal{U})$; ($q = 1, 2, \dots, r$). Then, (41) and (42), as shown at the bottom of the page, are referred to as *the dominated bipolar soft lower and the dominated bipolar soft upper approximation matrix*, respectively. Here,

$$\underline{X}_{j_{f^-}} = (\underline{u}_{1j_{f^-}}, \underline{u}_{2j_{f^-}}, \dots, \underline{u}_{nj_{f^-}}), \quad (43)$$

$$\underline{X}_{j_{g^-}} = (\underline{u}_{1j_{g^-}}, \underline{u}_{2j_{g^-}}, \dots, \underline{u}_{nj_{g^-}}), \quad (44)$$

$$\overline{X}_{j_{f^-}} = (\overline{u}_{1j_{f^-}}, \overline{u}_{2j_{f^-}}, \dots, \overline{u}_{nj_{f^-}}), \quad (45)$$

$$\overline{X}_{j_{g^-}} = (\overline{u}_{1j_{g^-}}, \overline{u}_{2j_{g^-}}, \dots, \overline{u}_{nj_{g^-}}). \quad (46)$$

where,

$$\underline{u}_{ij_{f^-}} = \begin{cases} 1 & \text{if } u_i \in X_{j_{f^-}}, \\ 0 & \text{if } u_i \notin X_{j_{f^-}}. \end{cases} \quad (47)$$

$$\underline{u}_{ij_{g^-}} = \begin{cases} -1/2 & \text{if } u_i \in X_{j_{g^-}}, \\ 0 & \text{if } u_i \notin X_{j_{g^-}}. \end{cases} \quad (48)$$

$$\overline{u}_{ij_{f^-}} = \begin{cases} 1/2 & \text{if } u_i \in \overline{X}_{j_{f^-}}, \\ 0 & \text{if } u_i \notin \overline{X}_{j_{f^-}}. \end{cases} \quad (49)$$

$$\overline{u}_{ij_{g^-}} = \begin{cases} -1 & \text{if } u_i \in \overline{X}_{j_{g^-}}, \\ 0 & \text{if } u_i \notin \overline{X}_{j_{g^-}}. \end{cases} \quad (50)$$

Definition 17: Let $[\underline{\mathcal{D}}]_{(f^+, g^+)}$, $[\overline{\mathcal{D}}]_{(f^+, g^+)}$, $[\underline{\mathcal{D}}]_{(f^-, g^-)}$ and $[\overline{\mathcal{D}}]_{(f^-, g^-)}$ be the dominating bipolar soft lower, dominating bipolar soft upper, dominated bipolar soft lower and dominated bipolar soft upper approximation matrices, respectively. Then

$$(\underline{\mathcal{V}})_{(f^+, g^+)} = \sum_{j=1}^k \sum_{q=1}^r (X_{j_{f_q^+}} \oplus X_{j_{g_q^+}}) \quad (51)$$

$$(\overline{\mathcal{V}})_{(f^+, g^+)} = \sum_{j=1}^k \sum_{q=1}^r (\overline{X}_{j_{f_q^+}} \oplus \overline{X}_{j_{g_q^+}}). \quad (52)$$

are called *the dominating bipolar soft lower approximation vector* and *the dominating bipolar soft upper approximation vector*, respectively.

Similarly,

$$(\underline{\mathcal{V}})_{(f^-, g^-)} = \sum_{j=1}^k \sum_{q=1}^r (X_{j_{f_q^-}} \oplus X_{j_{g_q^-}}) \quad (53)$$

$$(\overline{\mathcal{V}})_{(f^-, g^-)} = \sum_{j=1}^k \sum_{q=1}^r (\overline{X}_{j_{f_q^-}} \oplus \overline{X}_{j_{g_q^-}}). \quad (54)$$

are called *the dominated bipolar soft lower approximation vector* and *the dominated bipolar soft upper approximation vector*, respectively.

Here the operations \sum and \oplus represent the vector addition.

Definition 18: Suppose that $(\underline{\mathcal{V}})_{(f^+, g^+)}$, $(\overline{\mathcal{V}})_{(f^+, g^+)}$, $(\underline{\mathcal{V}})_{(f^-, g^-)}$ and $(\overline{\mathcal{V}})_{(f^-, g^-)}$ be the dominating bipolar soft lower, and dominating bipolar soft upper, dominated bipolar soft lower and dominated bipolar soft upper approximation vectors, respectively. Then

$$\begin{aligned} \mathcal{V}_d &= (\underline{\mathcal{V}})_{(f^+, g^+)} \oplus (\overline{\mathcal{V}})_{(f^+, g^+)} \oplus (\underline{\mathcal{V}})_{(f^-, g^-)} \oplus (\overline{\mathcal{V}})_{(f^-, g^-)} \\ &= (v_1, v_2, \dots, v_n) \end{aligned} \quad (55)$$

is known as the *decision vector*.

Definition 19: For a decision vector $\mathcal{V}_d = (v_1, v_2, \dots, v_n)$, each v_i is known as *the weighted number (WN) of $u_i \in \mathcal{U}$* .

$$[\underline{\mathcal{D}}]_{(f^-, g^-)} = \begin{pmatrix} \langle \underline{X}_{1_{f^-}}, \underline{X}_{1_{g^-}} \rangle & \langle \underline{X}_{2_{f^-}}, \underline{X}_{2_{g^-}} \rangle & \dots & \langle \underline{X}_{k_{f^-}}, \underline{X}_{k_{g^-}} \rangle \\ \langle \underline{X}_{1_{f^-}}, \underline{X}_{2_{g^-}} \rangle & \langle \underline{X}_{2_{f^-}}, \underline{X}_{2_{g^-}} \rangle & \dots & \langle \underline{X}_{k_{f^-}}, \underline{X}_{k_{g^-}} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \underline{X}_{1_{f^-}}, \underline{X}_{1_{g^-}} \rangle & \langle \underline{X}_{2_{f^-}}, \underline{X}_{2_{g^-}} \rangle & \dots & \langle \underline{X}_{k_{f^-}}, \underline{X}_{k_{g^-}} \rangle \end{pmatrix}, \quad (41)$$

$$[\overline{\mathcal{D}}]_{(f^-, g^-)} = \begin{pmatrix} \langle \overline{X}_{1_{f^-}}, \overline{X}_{1_{g^-}} \rangle & \langle \overline{X}_{2_{f^-}}, \overline{X}_{2_{g^-}} \rangle & \dots & \langle \overline{X}_{k_{f^-}}, \overline{X}_{k_{g^-}} \rangle \\ \langle \overline{X}_{1_{f^-}}, \overline{X}_{2_{g^-}} \rangle & \langle \overline{X}_{2_{f^-}}, \overline{X}_{2_{g^-}} \rangle & \dots & \langle \overline{X}_{k_{f^-}}, \overline{X}_{k_{g^-}} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \overline{X}_{1_{f^-}}, \overline{X}_{1_{g^-}} \rangle & \langle \overline{X}_{2_{f^-}}, \overline{X}_{2_{g^-}} \rangle & \dots & \langle \overline{X}_{k_{f^-}}, \overline{X}_{k_{g^-}} \rangle \end{pmatrix} \quad (42)$$

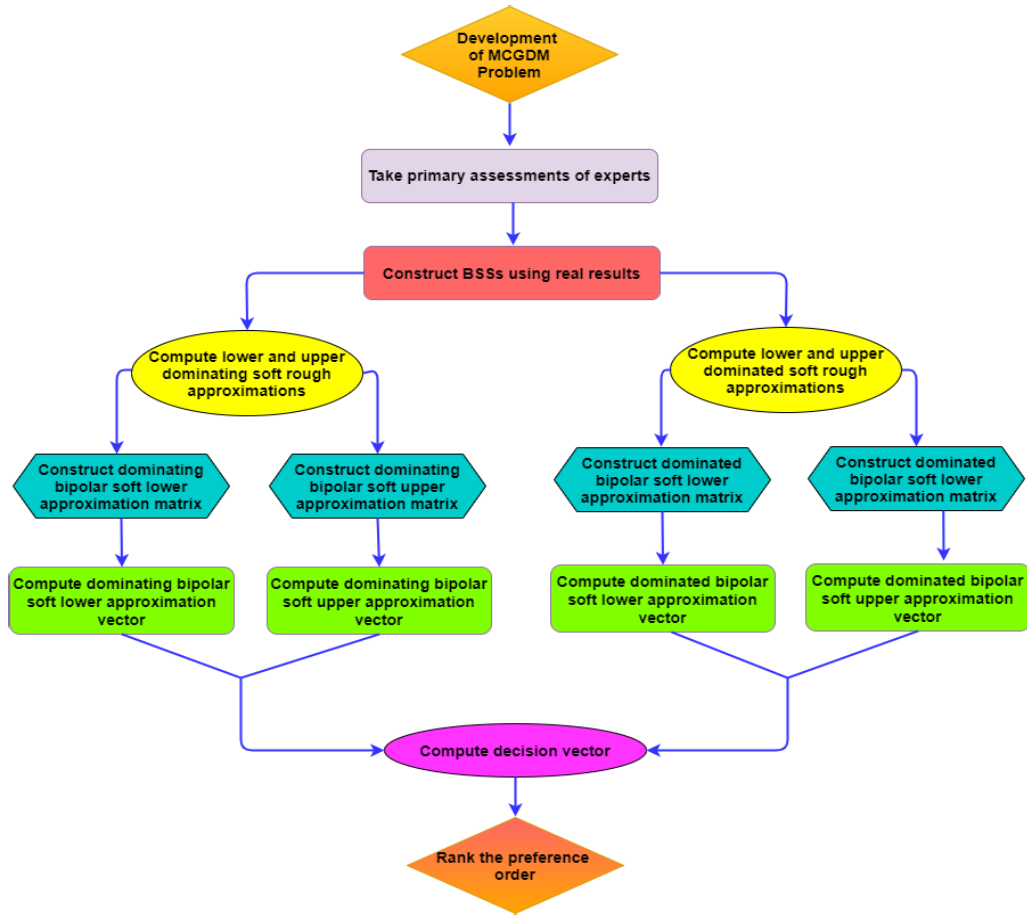


FIGURE 1. (Procedure of the proposed technique for MCGDM).

- (1) An object $u_i \in \mathcal{U}$ is referred to as *an optimal object* of \mathcal{U} if its WN is a maximum of v_i for all $i = 1, 2, \dots, n$.
- (2) An object $u_i \in \mathcal{U}$ is referred to as *the worst element* of \mathcal{U} if its WN is a minimum of v_i for all $i = 1, 2, \dots, n$.

If there is more than one optimal/worst element of \mathcal{U} , then pick out any individual of them.

VI. PROPOSED ALGORITHM FOR MCGDM PROBLEM

In the current section, we give an algorithm for the established strategy of the MCGDM problem being considered in section V. The corresponding steps of the algorithm are given as follows:

- Step 1:** Take primary assessments X_1, X_2, \dots, X_k of experts p_1, p_2, \dots, p_k .
- Step 2:** Construct $T_1, T_2, \dots, T_r \in \mathcal{BPSS}(\mathcal{U})$ using the actual results.
- Step 3:** By using Definition 10, calculate $\overline{DBS}^+_{T_q}(X_j)$, $\overline{DBS}^-_{T_q}(X_j)$, $\underline{DBS}^+_{T_q}(X_j)$ and $\underline{DBS}^-_{T_q}(X_j)$ for all $j = 1, 2, \dots, k$ and $q = 1, 2, \dots, r$.
- Step 4:** Calculate $[\mathcal{D}]_{(f^+,g^+)}$, $[\overline{\mathcal{D}}]_{(f^+,g^+)}$, $[\mathcal{D}]_{(f^-,g^-)}$ and $[\overline{\mathcal{D}}]_{(f^-,g^-)}$ by using Equations (31), (32), (41) and (42).

- Step 5:** Calculate $(\mathcal{V})_{(f^+,g^+)}$, $(\overline{\mathcal{V}})_{(f^+,g^+)}$, $(\mathcal{V})_{(f^-,g^-)}$ and $(\overline{\mathcal{V}})_{(f^-,g^-)}$ by using Definition 17.
- Step 6:** Compute decision vector \mathcal{V}_d by using Definition 18.
- Step 7:** Calculate $\max_i v_i; i = 1, 2, \dots, n$. An alternative with maximum WN should be chosen for the final decision.

Flow chart portrayal of the above algorithm is displayed in Figure 1.

VII. APPLICATION: A DESCRIPTIVE EXAMPLE

To illustrate the potential of the above-proposed technique of DM, here we consider a real-life example

Example 5 (Selection of a Best Company for Investment): Suppose there are three investors (decision-makers) p_1, p_2 , and p_3 who want to make an investment in a specific company. Assume that $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$ be the universe consisting of five different companies, where $u_1 =$ Automobile company, $u_2 =$ Food company, $u_3 =$ Computer company, $u_4 =$ Arms company and $u_5 =$ Medicine company. Consider $\mathcal{A} = \{e_1, e_2, e_3\}$ is the set of parameters (qualities of companies), where $e_1 =$ appreciation, $e_2 =$ economical growth and $e_3 =$ yearly benefit. Then $\tilde{\mathcal{A}} = \{-e_1 =$ depreciation, $-e_2 =$ economical decay, $-e_3 =$ yearly loss}.

Step 1: Primary assessments of investors p_1, p_2 and p_3 are:

$$X_1 = \{u_1, u_2, u_5\}, X_2 = \{u_1, u_3, u_5\} \text{ and } X_3 = \{u_2, u_4, u_5\}.$$

Step 2: Actual results in three distinct locations and periods are represented as BSSs $T_1 = (f_1, g_1 : \mathcal{A}), T_2 = (f_2, g_2 : \mathcal{A})$ and $T_3 = (f_3, g_3 : \mathcal{A})$ as follows:

$$\begin{aligned} f_1 : \mathcal{A} &\longrightarrow 2^{\mathcal{U}}, \\ e &\mapsto \begin{cases} \{u_1\} & \text{if } e = e_1, \\ \{u_1, u_5\} & \text{if } e = e_2, \\ \{u_4, u_5\} & \text{if } e = e_3, \end{cases} \\ g_1 : \tilde{\mathcal{A}} &\longrightarrow 2^{\mathcal{U}} \\ \neg e &\mapsto \begin{cases} \{u_3, u_5\} & \text{if } \neg e = \neg e_1, \\ \{u_3\} & \text{if } \neg e = \neg e_2, \\ \{u_1, u_3\} & \text{if } \neg e = \neg e_3. \end{cases} \\ f_2 : \mathcal{A} &\longrightarrow 2^{\mathcal{U}}, \\ e &\mapsto \begin{cases} \{u_2\} & \text{if } e = e_1, \\ \{u_2, u_4\} & \text{if } e = e_2, \\ \{u_3, u_4\} & \text{if } e = e_3, \end{cases} \\ g_2 : \tilde{\mathcal{A}} &\longrightarrow 2^{\mathcal{U}} \\ \neg e &\mapsto \begin{cases} \{u_1, u_4\} & \text{if } \neg e = \neg e_1, \\ \{u_5\} & \text{if } \neg e = \neg e_2, \\ \{u_1, u_5\} & \text{if } \neg e = \neg e_3. \end{cases} \end{aligned}$$

and

$$\begin{aligned} f_3 : \mathcal{A} &\longrightarrow 2^{\mathcal{U}}, \\ e &\mapsto \begin{cases} \{u_3, u_5\} & \text{if } e = e_1, \\ \{u_2\} & \text{if } e = e_2, \\ \{u_2, u_5\} & \text{if } e = e_3, \end{cases} \\ g_3 : \tilde{\mathcal{A}} &\longrightarrow 2^{\mathcal{U}} \\ \neg e &\mapsto \begin{cases} \{u_1, u_2\} & \text{if } \neg e = \neg e_1, \\ \{u_4\} & \text{if } \neg e = \neg e_2, \\ \{u_1, u_3\} & \text{if } \neg e = \neg e_3. \end{cases} \end{aligned}$$

Step 3: Using Definition 8 to calculate the operators f^+, g^+, f^- and g^- for $T_1 = (f_1, g_1 : \mathcal{A})$:

$$\begin{aligned} f_1^+(u_1) &= \{u_1\}, & g_1^+(u_1) &= \{u_1, u_3\}, \\ f_1^+(u_2) &= \{u_2, u_3\}, & g_1^+(u_2) &= \{u_2, u_4\}, \\ f_1^+(u_3) &= \{u_2, u_3\}, & g_1^+(u_3) &= \{u_3\}, \\ f_1^+(u_4) &= \{u_4, u_5\}, & g_1^+(u_4) &= \{u_2, u_4\}, \\ f_1^+(u_5) &= \{u_5\}, & g_1^+(u_5) &= \{u_3, u_5\}, \\ f_1^-(u_1) &= \{u_1\}, & g_1^-(u_1) &= \{u_1\}, \\ f_1^-(u_2) &= \{u_2, u_3\}, & g_1^-(u_2) &= \{u_2, u_4\}, \\ f_1^-(u_3) &= \{u_2, u_3\}, & g_1^-(u_3) &= \{u_1, u_3, u_5\}, \\ f_1^-(u_4) &= \{u_4\}, & g_1^-(u_4) &= \{u_2, u_4\}, \\ f_1^-(u_5) &= \{u_4, u_5\}, & g_1^-(u_5) &= \{u_5\}. \end{aligned}$$

Similarly, the operators f^+, g^+, f^- and g^- for $T_2 = (f_2, g_2 : \mathcal{A})$:

$$\begin{aligned} f_2^+(u_1) &= \{u_1, u_5\}, & g_2^+(u_1) &= \{u_1\}, \\ f_2^+(u_2) &= \{u_2\}, & g_2^+(u_2) &= \{u_2, u_3\}, \\ f_2^+(u_3) &= \{u_3, u_4\}, & g_2^+(u_3) &= \{u_2, u_3\}, \\ f_2^+(u_4) &= \{u_4\}, & g_2^+(u_4) &= \{u_1, u_4\}, \\ f_2^+(u_5) &= \{u_1, u_5\}, & g_2^+(u_5) &= \{u_5\}, \\ f_2^-(u_1) &= \{u_1, u_5\}, & g_2^-(u_1) &= \{u_1, u_4\}, \\ f_2^-(u_2) &= \{u_2\}, & g_2^-(u_2) &= \{u_2, u_3\}, \\ f_2^-(u_3) &= \{u_3\}, & g_2^-(u_3) &= \{u_2, u_3\}, \\ f_2^-(u_4) &= \{u_3, u_4\}, & g_2^-(u_4) &= \{u_4\}, \\ f_2^-(u_5) &= \{u_1, u_5\}, & g_2^-(u_5) &= \{u_5\}. \end{aligned}$$

And the operators f^+, g^+, f^- and g^- for $T_3 = (f_3, g_3 : \mathcal{A})$: we have

$$\begin{aligned} f_3^+(u_1) &= \{u_1, u_4\}, & g_3^+(u_1) &= \{u_1\}, \\ f_3^+(u_2) &= \{u_2\}, & g_3^+(u_2) &= \{u_1, u_2\}, \\ f_3^+(u_3) &= \{u_3, u_5\}, & g_3^+(u_3) &= \{u_1, u_3\}, \\ f_3^+(u_4) &= \{u_1, u_4\}, & g_3^+(u_4) &= \{u_4\}, \\ f_3^+(u_5) &= \{u_5\}, & g_3^+(u_5) &= \{u_5\}, \\ f_3^-(u_1) &= \{u_1, u_4\}, & g_3^-(u_1) &= \{u_1, u_2, u_3\}, \\ f_3^-(u_2) &= \{u_2\}, & g_3^-(u_2) &= \{u_2\}, \\ f_3^-(u_3) &= \{u_3\}, & g_3^-(u_3) &= \{u_3\}, \\ f_3^-(u_4) &= \{u_1, u_4\}, & g_3^-(u_4) &= \{u_4\}, \\ f_3^-(u_5) &= \{u_3, u_5\}, & g_3^-(u_5) &= \{u_5\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \underline{DBS}^+_{T_1}(X_1) &= (\{u_1, u_5\}, \{u_1, u_2, u_3, u_4, u_5\}), \\ \underline{DBS}^+_{T_1}(X_2) &= (\{u_1, u_5\}, \{u_2, u_5\}), \\ \underline{DBS}^+_{T_1}(X_3) &= (\{u_4, u_5\}, \{u_1, u_3, u_5\}). \\ \overline{DBS}^+_{T_1}(X_1) &= (\{u_1, u_2, u_3, u_4, u_5\}, \{u_3\}), \\ \overline{DBS}^+_{T_1}(X_2) &= (\{u_1, u_2, u_3, u_4, u_5\}, \{u_2, u_4\}), \\ \overline{DBS}^+_{T_1}(X_3) &= (\{u_2, u_3, u_4, u_5\}, \{u_1, u_3\}). \end{aligned}$$

Similarly,

$$\begin{aligned} \underline{DBS}^+_{T_2}(X_1) &= (\{u_1, u_2, u_5\}, \{u_2, u_3, u_4\}), \\ \underline{DBS}^+_{T_2}(X_2) &= (\{u_5\}, \{u_2, u_3, u_4\}), \\ \underline{DBS}^+_{T_2}(X_3) &= (\{u_2, u_4\}, \{u_1, u_2, u_3\}). \\ \overline{DBS}^+_{T_2}(X_1) &= (\{u_1, u_2, u_5\}, \{\}), \\ \overline{DBS}^+_{T_2}(X_2) &= (\{u_1, u_3, u_5\}, \{\}), \\ \overline{DBS}^+_{T_2}(X_3) &= (\{u_1, u_2, u_3, u_4, u_5\}, \{u_1\}). \end{aligned}$$

Also,

$$\begin{aligned} \underline{DBS}^+_{T_3}(X_1) &= (\{u_2, u_5\}, \{u_3, u_4\}), \\ \underline{DBS}^+_{T_3}(X_2) &= (\{u_3, u_5\}, \{u_2, u_4\}), \\ \underline{DBS}^+_{T_3}(X_3) &= (\{u_2, u_5\}, \{u_1, u_2, u_3\}). \end{aligned}$$

$$\begin{aligned} \overline{DBS^+}_{T_3}(X_1) &= (\{u_1, u_2, u_3, u_4, u_5\}, \{u_4\}), \\ \overline{DBS^+}_{T_3}(X_2) &= (\{u_1, u_3, u_4, u_5\}, \{u_4\}), \\ \overline{DBS^+}_{T_3}(X_3) &= (\{u_1, u_2, u_3, u_4, u_5\}, \{u_1, u_3\}). \end{aligned}$$

In the similar way, we have

$$\begin{aligned} \overline{DBS^-}_{T_1}(X_1) &= (\{u_1\}, \{u_2, u_3, u_4\}), \\ \overline{DBS^-}_{T_1}(X_2) &= (\{u_1\}, \{u_2, u_4\}), \\ \overline{DBS^-}_{T_1}(X_3) &= (\{u_4, u_5\}, \{u_1, u_3\}). \\ \overline{DBS^-}_{T_1}(X_1) &= (\{u_1, u_2, u_3, u_5\}, \{\}), \\ \overline{DBS^-}_{T_1}(X_2) &= (\{u_1, u_2, u_3, u_5\}, \{u_2, u_4\}), \\ \overline{DBS^-}_{T_1}(X_3) &= (\{u_2, u_3, u_4, u_5\}, \{u_1\}). \end{aligned}$$

And

$$\begin{aligned} \overline{DBS^-}_{T_2}(X_1) &= (\{u_1, u_2, u_5\}, \{u_1, u_2, u_3, u_4\}), \\ \overline{DBS^-}_{T_2}(X_2) &= (\{u_1, u_3, u_5\}, \{u_1, u_2, u_3, u_4\}), \\ \overline{DBS^-}_{T_2}(X_3) &= (\{u_2\}, \{u_1, u_2, u_3\}). \\ \overline{DBS^-}_{T_2}(X_1) &= (\{u_1, u_3, u_5\}, \{u_4\}), \\ \overline{DBS^-}_{T_2}(X_2) &= (\{u_1, u_3, u_4, u_5\}, \{u_4\}), \\ \overline{DBS^-}_{T_2}(X_3) &= (\{u_1, u_2, u_4, u_5\}, \{\}). \end{aligned}$$

Also,

$$\begin{aligned} \overline{DBS^-}_{T_3}(X_1) &= (\{u_2\}, \{u_1, u_3, u_4\}), \\ \overline{DBS^-}_{T_3}(X_2) &= (\{u_3, u_5\}, \{u_1, u_2, u_4\}), \\ \overline{DBS^-}_{T_3}(X_3) &= (\{u_2\}, \{u_1, u_3\}). \\ \overline{DBS^-}_{T_3}(X_1) &= (\{u_1, u_2, u_4, u_5\}, \{u_3, u_4\}), \\ \overline{DBS^-}_{T_3}(X_2) &= (\{u_1, u_3, u_5\}, \{u_2, u_4\}), \\ \overline{DBS^-}_{T_3}(X_3) &= (\{u_1, u_2, u_4, u_5\}, \{u_3\}). \end{aligned}$$

Step 4: $[\underline{\mathcal{D}}]_{(f^+, g^+)}$, $[\overline{\mathcal{D}}]_{(f^+, g^+)}$, $[\underline{\mathcal{D}}]_{(f^-, g^-)}$ and $[\overline{\mathcal{D}}]_{(f^-, g^-)}$, as shown at the bottom of the next page are obtained by using Equations (31), (32), (41) and (42) as follows:

Step 5: Using Definitions 17, $(\underline{\mathcal{V}})_{(f^+, g^+)}$, $(\overline{\mathcal{V}})_{(f^+, g^+)}$, $(\underline{\mathcal{V}})_{(f^-, g^-)}$ and $(\overline{\mathcal{V}})_{(f^-, g^-)}$ can be obtained as follows:

$$\begin{aligned} (\underline{\mathcal{V}})_{(f^+, g^+)} &= (1, -0.5, -2.5, -0.5, 6.5), \\ (\overline{\mathcal{V}})_{(f^+, g^+)} &= (1, 2, 1.5, 0.5, 4.5), \\ (\underline{\mathcal{V}})_{(f^-, g^-)} &= (0.5, 1, -1.5, -2, 4), \\ (\overline{\mathcal{V}})_{(f^-, g^-)} &= (3, 1, 1, -2.5, 4.5). \end{aligned}$$

Step 6: By using Definition 18, the decision vector is obtained as:

$$\mathcal{V}_d = (5.5, 3.5, -1.5, -4.5, 19.5).$$

Step 7: As $\max_{i \in I_n} u_i = u_5 = 19.5$. So u_5 (Medicine company) is the optimal element and $\min_{i \in I_n} u_i = u_4 = -4.5$, so u_4 (Arms company) is the worst element.

Moreover the ranking among the elements of the universe \mathcal{U} is given as: $u_5 \geq u_1 \geq u_2 \geq u_3 \geq u_4$. The pictorial portrayal for the ranking of the companies is shown in Figure 2.

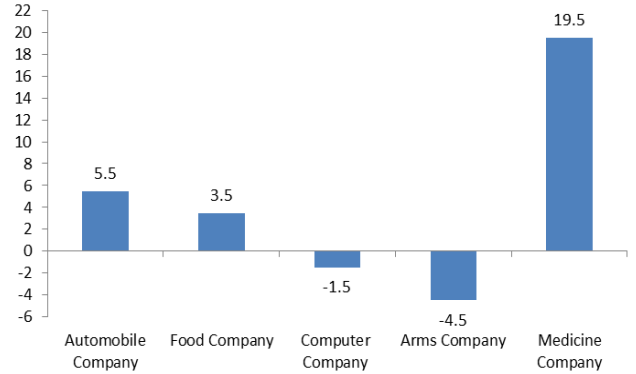


FIGURE 2. (Ranking of companies).

VIII. DISCUSSION AND COMPARATIVE ANALYSIS

In this section, we discuss the validity of the proposed method, its advantages, and disadvantages, and finally a comparison of the proposed technique with some existing methodologies.

A. VALIDITY OF THE PROPOSED MODEL

As we know that, the preferences or opinions of all decision-makers, aggregation is the crucial step for the classical GDM methods. In our proposed DM approach, every decision-maker expressed their opinion as a BSS, and afterward, all opinions given by decision-makers are aggregated through the usage the DB-BSR-approximations, and then a compromise optimal proposal is acquired. So, the DB-BSRSs approach to MCGDM provides a different strategy to aggregate the preferences of decision-makers. Therefore, the proposed DM approach is valid and offers a novel technique and perspective to investigate GDM problems in real life.

B. ADVANTAGES OF THE PROPOSED TECHNIQUE

In general, real-world MCDM and MCGDM problems arise in a complicated environment under uncertain and imprecise data, which is hard to address. The proposed technique is exceptionally appropriate for the scenario when the data is complex, vague, and uncertain. Especially, when the existing data is depending on the bipolar information by decision-makers. A few benefits of the proposed technique are listed below:

- (i) The proposed approach considers positive and negative aspects of each individual alternative in the form of a BSS. This hybrid model is more generalized and appropriate to deal with aggressive DM problems.
- (ii) This technique is additionally ideal on the grounds that in this strategy the decision-makers are liberated from any external conditions and prerequisites.

- (iii) Our proposed technique is effective in solving MCGDM problems when the weights information of criteria is completely unknown.
- (iv) The proposed MCGDM technique is more effective for discrete data problems.
- (v) The proposed method not only considers the opinions of key decision-makers but also incorporates the past experiences by DB-BSR-approximations in actual scenarios. Hence it is a more comprehensive method for a better interpretation of available information and thus making decisions using artificial intelligence.
- (vi) The proposed MCGDM approach is easy to understand and can be applied to DM problems in real life.

C. DISADVANTAGES OF THE PROPOSED TECHNIQUE

Some minor flaws are there in the proposed technique, inclusive of its complicated structure, the large data in the form of bipolar information. Such large data is hard to deal with, due to massive calculations, which are not so natural to perform. However, one could create a MATLAB programming code to make these complicated calculations simpler.

D. COMPARISON WITH SOME EXISTING TECHNIQUES

There are numerous techniques in the literature helpful for tackling MCGDM problems. All these techniques of

MCGDM have their own merits and demerits. The capability of every technique relies on the problem under consideration. In this subsection, we make a set-based comparison of the proposed MCGDM technique with some current MCGDM techniques in the fuzzy and bipolar fuzzy environments and see the significance of the proposed MCGDM strategy.

We talk about comparative analysis of proposed strategy with dominance-based soft rough sets [64], fuzzy soft set [3], dominance-based rough fuzzy set [15], intuitionistic fuzzy rough sets [66], picture fuzzy set [10], generalized hesitant fuzzy rough sets [63] and generalized intuitionistic fuzzy soft sets [37]. All these techniques have their own value in the literature. If we compare all these techniques with our proposed strategy, we investigate the following points.

- (i) The previously-mentioned techniques cannot catch bipolarity in DM which is a fundamental aspect of human thinking and behavior.
- (ii) Besides, these techniques do not ensure harmony in the opinions of decision-makers.
- (iii) If we compare our proposed model with the methods presented in [10], [37], we have seen that these methods require a weight vector that represents the weight of each alternative, but in our proposed model there is no need for a weight vector.

$$\begin{aligned}
 & \overline{[\mathcal{Q}]}_{(f^+, g^+)} \\
 = & \begin{pmatrix} \langle (1, 0, 0, 0, 1), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \rangle & \langle (1, 0, 0, 0, 1), (0, -\frac{1}{2}, 0, 0, -\frac{1}{2}) \rangle & \langle (0, 0, 0, 1, 1), (-\frac{1}{2}, 0, -\frac{1}{2}, 0, -\frac{1}{2}) \rangle \\ \langle (1, 1, 0, 0, 1), (0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0) \rangle & \langle (0, 0, 0, 0, 1), (0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0) \rangle & \langle (0, 1, 0, 1, 0), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0) \rangle \\ \langle (0, 1, 0, 0, 1), (0, 0, -\frac{1}{2}, -\frac{1}{2}, 0) \rangle & \langle (0, 0, 1, 0, 1), (0, -\frac{1}{2}, 0, -\frac{1}{2}, 0) \rangle & \langle (0, 1, 0, 0, 1), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0) \rangle \end{pmatrix} \\
 & \overline{[\mathcal{Q}]}_{(f^+, g^+)} \\
 = & \begin{pmatrix} \langle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, 0, -1, 0, 0) \rangle & \langle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, -1, 0, -1, 0) \rangle & \langle (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-1, 0, -1, 0, 0) \rangle \\ \langle (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}), (0, 0, 0, 0, 0) \rangle & \langle (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}), (0, 0, 0, 0, 0) \rangle & \langle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-1, 0, 0, 0, 0) \rangle \\ \langle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, 0, 0, -1, 0) \rangle & \langle (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, 0, 0, -1, 0) \rangle & \langle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-1, 0, -1, 0, 0) \rangle \end{pmatrix} \\
 & \overline{[\mathcal{Q}]}_{(f^-, g^-)} \\
 = & \begin{pmatrix} \langle (1, 0, 0, 0, 0), (0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0) \rangle & \langle (1, 0, 0, 0, 0), (0, -\frac{1}{2}, 0, -\frac{1}{2}, 0) \rangle & \langle (0, 0, 0, 1, 1), (-\frac{1}{2}, 0, -\frac{1}{2}, 0, 0) \rangle \\ \langle (1, 1, 0, 0, 1), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0) \rangle & \langle (1, 0, 1, 0, 1), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0) \rangle & \langle (0, 1, 0, 0, 0), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0) \rangle \\ \langle (0, 1, 0, 0, 0), (-\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0) \rangle & \langle (0, 0, 1, 0, 1), (-\frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, 0) \rangle & \langle (0, 1, 0, 0, 0), (-\frac{1}{2}, 0, -\frac{1}{2}, 0, 0) \rangle \end{pmatrix} \\
 & \overline{[\mathcal{Q}]}_{(f^-, g^-)} \\
 = & \begin{pmatrix} \langle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}), (0, 0, 0, 0, 0) \rangle & \langle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}), (0, -1, 0, -1, 0) \rangle & \langle (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-1, 0, 0, 0, 0) \rangle \\ \langle (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}), (0, 0, 0, -1, 0) \rangle & \langle (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, 0, 0, -1, 0) \rangle & \langle (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}), (0, 0, 0, 0, 0) \rangle \\ \langle (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}), (0, 0, -1, -1, 0) \rangle & \langle (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}), (0, -1, 0, -1, 0) \rangle & \langle (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}), (0, 0, -1, 0, 0) \rangle \end{pmatrix}
 \end{aligned}$$

TABLE 1. The results obtained using different methods for Example 5.

Methods	The final ranking	The best alternative	The worst alternative
Karaaslan and Çağman [36]	$u_5 \succ u_2 \succ u_4 \succ u_3 \succ u_1$	u_5	u_1
Shabir and Gul [61]	$u_5 \succ u_4 \succ u_1 = u_2 \succ u_3$	u_5	u_3
Our proposed method	$u_5 \succ u_1 \succ u_2 \succ u_3 \succ u_4$	u_5	u_4

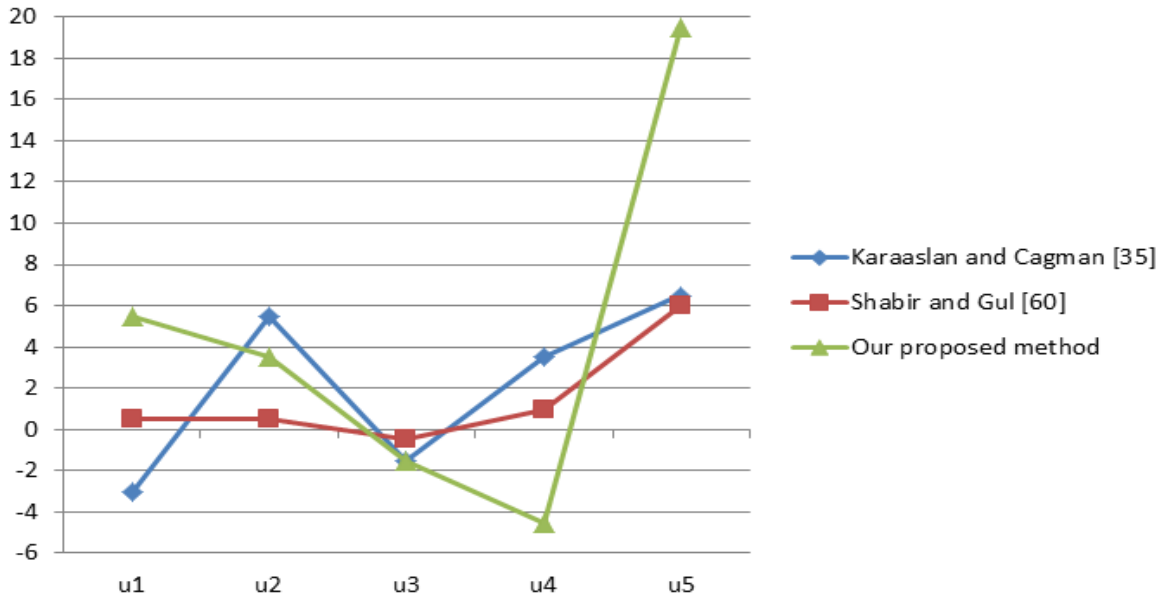


FIGURE 3. (Ranking of companies using different methods for Example 5).

- (iv) It is well known that the models presented in [10], [37], [66] can manage some DM problems to describe the idea of decision-makers through a crisp number. However, because of the uncertainty of the objective world and the complexity of the decision-making problems, they fail to handle some GDM problems. For example, several experts argue the membership degree of an element to a set and cannot compromise each other. One wants to assign 0.3, but the other tends to assign 0.7. In this case, DB-BSRSs can be a very good solution to this problem.
- (v) If we compare our proposed result with the method presented in [9], we have seen that in this method the optimal alternative is obtained just by using the tabular form of BSSs, while in our proposed model the optimal alternative is obtained by using the DB-BSR-approximations.
- (vi) In [40], Mahmood uses the tabular form of the T-BSSs and presented two types of algorithms with the help of score function to obtain the optimal alternative. But in our proposed model there is no need for the score function.
- (vii) If we apply the recent approaches proposed in [36] and [61] to our Example 5, we get the following ranking among the alternatives (shown in Table 1)

and the corresponding pictorial depiction is given in Figure 3.

IX. CONCLUSION AND FUTURE DIRECTIONS

The RS theory is arising as an incredible theory and has different applications in numerous fields. On the other hand, the BSSs are the appropriate mathematical model to deal with the uncertainty as well as the bipolarity of the data. In this study, we have initiated a novel technique of roughness of BSSs known as “dominance-based bipolar soft rough sets (DB-BSRSs)”. For a given BSS, we first define some dominance-based bipolar soft operators. Using these operators, DB-BSR-approximations have been defined. Some important structural properties of DB-BSR-approximations have also been studied in detail with examples. Additionally, some significant measures related to DB-BSRSs like the accuracy measure, the measure of precision, and the measure of quality are also provided. In the meantime, we provide a general methodology of the MCGDM approach based on the DB-BSRSs. An algorithm of decision-making is also proposed which has the following two main advantages. Firstly, it manipulates the bipolarity of the data, endowed with uncertainty. Secondly, this algorithm accommodates the opinions of any (finite) number of decision-makers about any (finite)

number of alternatives. Furthermore, the legitimacy of this methodology is illustrated by a practical application. Lastly, a comparison analysis of the proposed model is performed.

In the future, on the basis of the characterized ideas and operations in this paper, researchers may additionally look at the algebraic structures of DB-BSRSs. Modeling the supported natural phenomenon is our subsequent objective. Another perspective route is to have a look at the topological axioms and similarity measures of DB-BSRSs in order to seek a strong basis of the research studies and enhancement of working techniques. Also, the idea of DB-BSRSs could also be extended to dominance-based bipolar fuzzy soft rough set and successful decision-making strategies can be developed. The notions of the DB-BSRSs can also be extended to multi-granulation DB-BSRSs. For more complicated decision-making problems, the technique requires further investigation. The different types of correlation coefficients can also be studied in the framework of DB-BSRSs. Some important aggregation operators such as Hammy mean operators, weighted aggregation operators, arithmetic and harmonic aggregation operators, power aggregation operators, etc. can also be developed in the follow-up work. Also, the technique is more practical for discrete data problems. For continuous scaled data sets, this approach requires further research. Moreover, we will concentrate on the implementation of the proposed strategy in tackling a more extensive scope of selection problems, like TOPSIS, VIKOR, ELECTRE, AHP, and PROMETHEE. We may also be searching on the possible fuzzification of the proposed strategy to obtain more accuracy in outcomes and applying these procedures to certifiable problems with large data sets. In this way, we can acquire and demonstrate the usefulness of our proposed model.

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