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Spreading Dynamics of SHIPR Pyramid Scheme Model on Scale-Free Networks

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ABSTRACT Nowadays, pyramid schemes have caused extremely negative effects on people's lives and seriously damaged the social economy. With the rapid development of network and communication technology, people's direct or indirect social interaction is more frequent, which makes the phenomenon of pyramid schemes more serious. Therefore, it is necessary to study transmission mechanisms and transmission rules of pyramid schemes. In order to study the influence of government management and social interaction topology on the spreading of pyramid schemes, a novel *SHIPR* (susceptible-hesitator-involved-punished-resister) pyramid scheme spreading model is proposed on scale-free networks. The spreading dynamics of pyramid schemes are analyzed in detail by mean-field theory. Then, the basic reproduction number R_0 and equilibria are got. Theoretical analysis shows that the basic reproduction number R_0 has a great correlation with government crackdown intensity for involved individuals, the coverage rate of government anti-pyramid scheme publicity for susceptible individuals and hesitator, and social interaction topology. Furthermore, the local asymptotic stability of fraud-elimination equilibrium is analyzed based on the Routh-Hurwitz criterion, the global asymptotic stability of the fraud-elimination equilibrium is discussed by the Lyapunov function, the global attractivity of fraud-prevailing equilibrium is proved in detail by comparison principle. Finally, numerical simulations verify the theoretical analysis results.

INDEX TERMS Pyramid schemes, SHIPR model, government management, global attractivity, scale-free networks.

I. INTRODUCTION

With the development of network and communication technology, people's life and work are more convenient, but various pyramid scheme organizations emerge in endlessly due to the imperfect internet supervision mechanism. Nowadays, pyramid schemes present new characteristics, and the speed of diffusion is gradually accelerated [1]–[3]. The number of people participating in pyramid schemes is growing exponentially, the economic losses caused by pyramid schemes are increasingly serious [4], [5]. It also has caused serious damages to the development of network finance and social stability [6], [7]. Therefore, it is necessary to take effective control measures and establish a pyramid scheme model to study the spreading dynamics of pyramid schemes. It is helpful to deeply analyze transmission mechanisms and transmission rules of the pyramid scheme, and provide an important basis for effective control of the spread of the pyramid scheme.

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In recent years, the research of pyramid schemes has attracted the attention of many scholars [8]–[12]. For instance, Keep and Nat [13] studied the evolution from direct selling to pyramid schemes and analyzed the overlap between the legal MLM and illegal pyramid schemes. Schiffauer [14] studied how ordinary people become involved in a financial hoax. Constantin [15] studied the main features of legal marketing and analyzed how independent distributors avoid cooperation with illegal pyramid schemes. However, they have not studied the dynamics behavior of the spread of pyramid schemes. It is very significant to study the spread dynamics of pyramid schemes. The study of the spread dynamics of pyramid schemes can comprehensively learn about transmission mechanisms and transmission rules of pyramid schemes, which is conducive to controlling the spread of pyramid schemes and thus reducing the damage of pyramid schemes to society.

At present, the research on the spreading dynamics of pyramid schemes is relatively less. In the spreading dynamics of complex networks, scholars have done

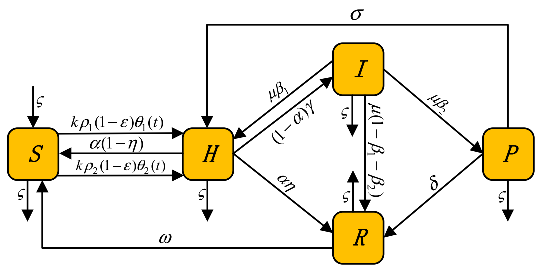

FIGURE 1. Schematic diagram of the SHIPR model.

more in-depth research on the spreading dynamics of disease and information [16]–[19]. For instance, Li *et al.* [20] proposed a SIQRS (susceptible-infected-quarantined-recovered-susceptible) model to investigate the influence heterogeneity of the underlying networks and quarantine strategy on epidemic spreading. Liu *et al.* [21] proposed a SIALS (susceptible-infected-acknowledged-loath-susceptible) model on scale-free networks and analyzed the dynamics behavior of Word-of-mouth (WOM) communication. Meanwhile, some researchers found that the scale-free property is an important characteristic of social networks [22]–[24]. Obviously, the spread of pyramid schemes is based on social networks. In addition, government management also play a key role in controlling the spread of pyramid schemes. The government anti-pyramid scheme publicity can improve people's awareness of preventing pyramid schemes, help people learn about the harm of pyramid schemes and prevent them from joining pyramid scheme organizations; The government crackdown on pyramid scheme organizations can effectively crack down on criminals, make them withdraw from pyramid scheme organizations and reduce the damage of pyramid schemes to society. Based on the above analysis, considering government management and social interaction topology, we present a novel *SHIPR* pyramid scheme spreading model on scale-free networks.

The rest of the paper is structured as follows: The *SHIPR* pyramid scheme spreading model is presented in Section 2. The basic reproduction number R_0, fraud-elimination equilibrium E_0 and fraud-prevailing equilibrium E^+ are proved in Section 3. the local and global asymptotic stability of fraud-elimination equilibrium is analyzed in Section 4. Then, the global attractivity of fraud-prevailing equilibrium is studied in Section 5. Numerical simulations verify the theoretical analysis results in Section 6. We conclude this paper in Section 7.

II. MODEL FORMULATION

In this section, we propose a novel *SHIPR* pyramid scheme spreading model on scale-free networks. The schematic diagram of the model is shown in Figure 1. In this model, we divide the whole population into five groups: Susceptible individuals (S), Hesitator (H), Involved individuals (I), Punished individuals (P), Resister (R). S represents the people who are not involved in pyramid scheme organizations and can be influenced by the pyramid scheme; H represents the people who know the phenomenon of pyramid schemes, hesitate whether join in pyramid scheme organizations and can use fraudulent information to influence others; I represents the people who join in pyramid scheme organizations and can use fraudulent information to recruit new participants; P represents the people who have been punished by law for participating in pyramid scheme; R represents the people who resist pyramid scheme.

Considering the heterogeneous structure of the networks, we denote the density of nodes of susceptible individuals, hesitator, involved individuals, punished individuals and resister with degree k at time t respectively by $S_k(t)$, $H_k(t)$, $I_k(t)$, $P_k(t)$, $R_k(t)$. In the process of pyramid scheme spreading, if a susceptible individual who does not learn about the harm of pyramid schemes come into contact with a hesitator or an involved individual, then he or she will become a hesitator. The hesitator may learn about the harm of pyramid schemes through the government anti-pyramid scheme publicity, so that they give up joining pyramid scheme organizations and become a susceptible individual or a resister. The hesitater will become an involved individual due to they did not learn about the harm of pyramid schemes. The involved individual will become a hesitator, a punished individual or a resister under government crackdown. The punished individual will become an involved individual or a resister. Because of psychological factors of the resister, such as forgetting and so on, resister will become a susceptible individual. To simplify the subjects we study, we assume that the population immigration rate is equal to the population emigration rate.

Definition of parameters of the model in Table 1:

TABLE 1. Definition of parameters.

Parameter	Definition
ρ_1	The probability of susceptible individuals become hesitator
ρ_2	The probability of susceptible individuals become hesitator
μ	The intensity of government crackdown for involved individuals
ε	The coverage rate of government anti-pyramid publicity for susceptible individuals
α	The coverage rate of government anti-pyramid publicity for hesitator
σ	The probability of punished individuals become hesitator
η	The probability of hesitator become resister
β_1	The probability of involved individuals become hesitator
β_2	The probability of involved individuals become punished individuals
δ	The probability of punished individuals become resister
ω	The probability of resister become susceptible individuals
ς	The rate of immigration and emigration

As described above, the dynamic mean-field equations of the *SHIPR* model as follows:

$$\begin{cases} \frac{dS_k(t)}{dt} = \varsigma - k(1-\varepsilon)(\rho_1\theta_1(t)+\rho_2\theta_2(t))S_k(t) \\ \qquad +\alpha(1-\eta)H_k(t)+\omega R_k(t)-\varsigma S_k(t), \\ \frac{dH_k(t)}{dt} = k(1-\varepsilon)(\rho_1\theta_1(t)+\rho_2\theta_2(t))S_k(t)+\mu\beta_1 I_k(t) \\ \qquad +\sigma P_k(t)-(\alpha+(1-\alpha)\gamma+\varsigma)H_k(t), \\ \frac{dI_k(t)}{dt} = (1-\alpha)\gamma H_k(t)-\mu I_k(t)-\varsigma I_k(t), \\ \frac{dP_k(t)}{dt} = \mu\beta_2 I_k(t)-\sigma P_k(t)-\delta P_k(t)-\varsigma P_k(t), \\ \frac{dR_k(t)}{dt} = \alpha\eta H_k(t)+\mu(1-\beta_1-\beta_2)I_k(t)+\delta P_k(t) \\ \qquad -\varsigma R_k(t)-\omega R_k(t). \end{cases} \tag{1}$$

where $\theta_1(t)$ is the probability that a link is connected to a hesitator at time t and satisfies

$$\theta_1(t) = \frac{\sum_k kG(k)H_k(t)}{\sum_m mG(m)} = \frac{1}{\langle k\rangle}\sum_k kG(k)H_k(t).$$

And $\theta_2(t)$ is the probability that a link is connected to an involved individual at time t and satisfies

$$\theta_2(t) = \frac{\sum_k kG(k)I_k(t)}{\sum_m mG(m)} = \frac{1}{\langle k\rangle}\sum_k kG(k)I_k(t).$$

Here, $\langle k\rangle = \sum_m mG(m)$ is the average degree, and $G(k)$ denotes the degree distribution. Let $H(t) = \sum_k G(k)H_k(t)$, which refers to the density of all the hesitator, and $I(t) = \sum_k G(k)I_k(t)$, which refers to the density of all involved individuals.

According to the above description, these variables obey the normalization condition, we can obtain

$$S_k(t)+H_k(t)+I_k(t)+P_k(t)+R_k(t) = 1, \quad k = 1,\ 2, 3, \cdots.$$

The initial values for the system (1) can be given as follows:

$$\begin{cases} S_k(0) = 1-H_k(0)-I_k(0)-P_k(0)-R_k(0) \geq 0, \\ H_k(0), I_k(0) \geq 0. \end{cases}$$

III. THE BASIC REPRODUCTION NUMBER AND EQUILIBRIA

In this section, we calculate the basic reproduction number and two equilibria of the *SHIPR* model.

Definition 1: For system (1),

1) Clearly, $S_k(t) = 1$ and $H_k(t) = I_k(t) = P_k(t) = R_k(t) = 0$ is an equilibrium status of the system (1), we define it as the fraud-elimination $E_0(1, 0, 0, 0, 0)$.

2) If an equilibrium satisfies $\lim_{t\to\infty} H_k(t) = H_k^* > 0$, $\lim_{t\to\infty} I_k(t) = I_k^* > 0$, we define the equilibrium status as the fraud-prevailing equilibrium $E^+(S_k^*,\ H_k^*, I_k^*, P_k^*, R_k^*)$.

Theorem 1: For system (1), the basic reproduction number is defined as, R_0, as shown at the bottom of the page.

There is a fraud-elimination equilibrium $E_0(1, 0, 0, 0, 0)$ when $R_0 < 1$, and a fraud-prevailing equilibrium $E^+(S_k^*, H_k^*, I_k^*, P_k^*, R_k^*)$ when $R_0 > 1$.

Proof: For simplicity, it is denoted that $G_i = iG(i)/\langle k\rangle$ and $n = k_{\max}$ in this paper. Considering the normalization conditions $S_k(t) = 1-H_k(t)-I_k(t)-P_k(t)-R_k(t)$. According to the system (1), we can obtain

$$\begin{cases} \frac{dH_k(t)}{dt} = k(1-\varepsilon)(\rho_1\theta_1(t)+\rho_2\theta_2(t))(1-H_k(t)-I_k(t) \\ \qquad -P_k(t)-R_k(t))+\mu\beta_1 I_k(t)+\sigma P_k(t) \\ \qquad -(\alpha+(1-\alpha)\gamma+\varsigma)H_k(t), \\ \frac{dI_k(t)}{dt} = (1-\alpha)\gamma H_k(t)-(\mu+\varsigma)I_k(t), \\ \frac{dP_k(t)}{dt} = \mu\beta_2 I_k(t)-(\sigma+\delta+\varsigma)P_k(t), \\ \frac{dR_k(t)}{dt} = \alpha\eta H_k(t)+\mu(1-\beta_1-\beta_2)I_k(t)+\delta P_k(t) \\ \qquad -(\varsigma+\omega)R_k(t). \end{cases} \tag{2}$$

Using the next-generation matrix method [25], we have

$$x = [H_k, I_k, P_k, R_k]^T \text{ and } \frac{dx}{dt} = q(x)-c(x),$$

where

$$q(x) = \begin{pmatrix} k(1-\varepsilon)(\rho_1\theta_1+\rho_2\theta_2)(1-H_k-I_k-P_k-R_k) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$c(x) = \begin{pmatrix} (\alpha+(1-\alpha)\gamma+\varsigma)H_k-\mu\beta_1 I_k-\sigma P_k \\ (\mu+\varsigma)I_k-(1-\alpha)\gamma H_k \\ (\sigma+\delta+\varsigma)P_k-\mu\beta_2 I_k \\ -\alpha\eta H_k-\mu(1-\beta_1-\beta_2)I_k-\delta P_k+(\varsigma+\omega)R_k \end{pmatrix},$$

For the fraud-elimination $E_0(1, 0, 0, 0, 0)$, the Jacobian matrices of $q(x)$ and $c(x)$ are as follows:

$$Q = Jq(E0) = \begin{pmatrix} Q_{11} & Q_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = Jc(E0) = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & 0 & 0 \\ 0 & C_{32} & C_{33} & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix}.$$

$$R_0 = \frac{\langle k^2\rangle}{\langle k\rangle}\cdot\frac{(1-\varepsilon)(\sigma+\delta+\varsigma)(\rho_1(\mu+\varsigma)+\rho_2(1-\alpha)\gamma)}{((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma}.$$

where

$$Q_{11} = \frac{\rho_1(1-\varepsilon)}{\langle k\rangle}\begin{bmatrix}1\\2\\\vdots\\n\end{bmatrix}[G(1), 2G(2), \cdots, nG(n)],$$

$$Q_{12} = \frac{\rho_2(1-\varepsilon)}{\langle k\rangle}\begin{bmatrix}1\\2\\\vdots\\n\end{bmatrix}[G(1), 2G(2), \cdots, nG(n)],$$

$$C_{11} = (\alpha+(1-\alpha)\gamma+\varsigma)I,$$
$$C_{12} = -\mu\beta_1 I, \quad C_{13} = -\sigma I,$$
$$C_{21} = -(1-\alpha)\gamma I, \ C_{22} = (\mu+\varsigma)I,$$
$$C_{32} = -\mu\beta_2 I, \ C_{33} = (\sigma+\delta+\varsigma)I,$$
$$C_{41} = -\alpha\eta I, \ C_{42} = -\mu(1-\beta_1-\beta_2)I,$$
$$C_{43} = -\delta(1-\beta_1-\beta_2)I, \ C_{44} = (\varsigma+\omega)I.$$

where I represents the identity matrix. Then, based on Theorem 2 [25], we can obtain the basic reproduction number, R_0, as shown at the bottom of the page, where $\langle k^2\rangle = \sum_k k^2G(k)$ and $((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma \geq 0$.

To obtain the fraud-prevailing equilibrium, let the system (1) equal to zero

$$\frac{dS_k(t)}{dt} = 0, \quad \frac{dH_k(t)}{dt} = 0,$$
$$\frac{dI_k(t)}{dt} = 0, \quad \frac{dP_k(t)}{dt} = 0, \quad \frac{dR_k(t)}{dt} = 0.$$

we can get

$$\begin{cases} \varsigma+\alpha(1-\eta)-k(1-\varepsilon)(\rho_1\theta_1^*+\rho_2\theta_2^*)S_k^* \\ \qquad -\varsigma S_k^*+\omega R_k^* = 0, \\ k(1-\varepsilon)(\rho_1\theta_1^*+\rho_2\theta_2^*)S_k^*+\mu\beta_1 I_k^*+\sigma P_k^* \\ \qquad -(\alpha+(1-\alpha)\gamma+\varsigma)H_k^* = 0, \\ (1-\alpha)\gamma H_k^*-(\mu+\varsigma)I_k^* = 0, \\ \mu\beta_2 I_k^*-(\sigma+\delta+\varsigma)P_k^* = 0, \\ \alpha\eta H_k^*+\mu(1-\beta_1-\beta_2)I_k^*+\delta P_k^*-(\varsigma+\omega)R_k^* = 0. \end{cases}$$

For simplicity, let $\Theta(t) = \rho_1\theta_1(t)+\rho_2\theta_2(t)$, we get, (3), as shown at the bottom of the page.

Consider the normalization condition, we can get, I_k^*, as shown at the bottom of the page, here

$$\begin{aligned} F_x =\ & (\varsigma+\omega)(\mu+\varsigma)k(1-\varepsilon)\Theta(\sigma+\delta+\varsigma) \\ & +k(1-\varepsilon)\Theta(1-\alpha)\gamma(\sigma+\delta+\varsigma)(\varsigma+\omega) \\ & +\mu\beta_2(\varsigma+\omega)k(1-\varepsilon)\Theta(1-\alpha)\gamma \\ & +k(1-\varepsilon)\Theta(\alpha\eta(\mu+\varsigma)(\sigma+\delta+\varsigma) \\ & +\mu(1-\beta_1-\beta_2)(1-\alpha)\gamma(\sigma+\delta+\varsigma)+\delta\mu\beta_2(1-\alpha)\gamma). \end{aligned} \tag{4}$$

Because of $\Theta(t) = \frac{\sum_k kG(k)}{\langle k\rangle}(\rho_1 H_k(t)+\rho_2 I_k(t))$, we have

$$\begin{aligned} \Theta(\infty) &= \rho_1\theta_1(\infty)+\rho_2\theta_2(\infty) \\ &= \frac{\sum_k kG(k)}{\langle k\rangle}(\rho_1 H_k^*+\rho_2 I_k^*) \\ &= \frac{\sum_k kG(k)}{\langle k\rangle}\left(\frac{\mu+\varsigma}{(1-\alpha)\gamma}\rho_1+\rho_2\right)I_k^* \\ &= f\left(\Theta\left(\infty\right)\right). \end{aligned} \tag{5}$$

Obviously, $\Theta(\infty) = 0$ satisfies equation (5), if equation (5) has a non-trivial solution satisfied that $\frac{d}{d\Theta(\infty)}\left(f\left(\Theta\left(\infty\right)\right)\right)\Big|_{\Theta(\infty)=0} > 1$ and $f(1) \leq 1$.

We get the basic reproduction number, R_0, as shown at the bottom of the next page, where $\langle k^2\rangle = \sum_k k^2G(k)$.

By inserting the nontrivial solution of equation (5) into equation (4), we can easily get I_k^*. By equation (5), we can obtain $0 < S_k^* < 1$, $0 < H_k^* < 1$, $0 < I_k^* < 1$, $0 < P_k^* < 1$, $0 < R_k^* < 1$. Hence, only one positive equilibrium $E^+(S_k^*, H_k^*, I_k^*, P_k^*, R_k^*)$. This completes the proof.

IV. THE STABILITY OF EQUILIBRIA

Theorem 2: The fraud-elimination equilibrium E_0 (1, 0, 0, 0, 0) is locally asymptotically stable If $R_0 < 1$, and unstable if $R_0 > 1$.

Proof: According to the system (2), we obtain Jacobian matrix at fraud-elimination equilibrium E_0 as, $J(E_0)$, as shown at the bottom of the next page.

$$\begin{aligned} R_0 &= \rho(QC^{-1}) \\ &= \frac{\langle k^2\rangle}{\langle k\rangle}\cdot\frac{(1-\varepsilon)(\sigma+\delta+\varsigma)(\rho_1(\mu+\varsigma)+\rho_2(1-\alpha)\gamma)}{((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma}, \end{aligned}$$

$$\begin{cases} S_k^* = \dfrac{(\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)(\sigma+\delta+\varsigma)-\mu\beta_1(1-\alpha)\gamma(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma}{k(1-\varepsilon)\Theta(1-\alpha)\gamma(\sigma+\delta+\varsigma)}I_k^*, \\ H_k^* = \dfrac{\mu+\varsigma}{(1-\alpha)\gamma}I_k^*, \\ P_k^* = \dfrac{\mu\beta_2}{\sigma+\delta+\varsigma}I_k^*, \\ R_k^* = \dfrac{\alpha\eta(\mu+\varsigma)(\sigma+\delta+\varsigma)+\mu(1-\beta_1-\beta_2)(1-\alpha)\gamma(\sigma+\delta+\varsigma)+\delta\mu\beta_2(1-\alpha)\gamma}{(\varsigma+\omega)(1-\alpha)\gamma(\sigma+\delta+\varsigma)}I_k^*. \end{cases} \tag{3}$$

$$I_k^* = \frac{k(1-\varepsilon)\Theta(1-\alpha)\gamma(\sigma+\delta+\varsigma)(\varsigma+\omega)}{(\varsigma+\omega)((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)(\sigma+\delta+\varsigma)-\mu\beta_1(1-\alpha)\gamma(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma)+F_x},$$

where

$$Q_{11}=\frac{\rho_1(1-\varepsilon)}{\langle k\rangle}\begin{bmatrix} G(1) & G(2) & \cdots & G(n)\\ 2G(1) & 2^2G(2) & \cdots & 2nG(n)\\ \vdots & \vdots & \ddots & \vdots\\ nG(1) & 2nG(2) & \cdots & n^2G(n)\end{bmatrix},$$

$$Q_{12}=\frac{\rho_2(1-\varepsilon)}{\langle k\rangle}\begin{bmatrix} G(1) & G(2) & \cdots & G(n)\\ 2G(1) & 2^2G(2) & \cdots & 2nG(n)\\ \vdots & \vdots & \ddots & \vdots\\ nG(1) & 2nG(2) & \cdots & n^2G(n)\end{bmatrix}.$$

The characteristic polynomial of $J(E_0)$ is, $|\lambda-J(E_0)|$, as shown at the bottom of the page.

So, the characteristic polynomial of $J(E_0)$ is $f(\lambda)=b_4\lambda^4+b_3\lambda^3+b_2\lambda^2+b_1\lambda+b_0$. We can obtain

$$\begin{aligned} b_4 &= 1,\\ b_3 &= (\alpha+(1-\alpha)\gamma+\varsigma)-Q_{11}+\mu+\sigma+\delta+\omega+3\varsigma,\\ b_2 &= (\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma\\ &\quad -(\mu+\varsigma)Q_{11}-(1-\alpha)\gamma Q_{12}\\ &\quad +(\mu+\varsigma)(\sigma+\delta+\omega+2\varsigma)+(\sigma+\delta+\varsigma)(\varsigma+\omega)\\ &\quad +(\alpha+(1-\alpha)\gamma+\varsigma)(\sigma+\delta+\varsigma)-(\sigma+\delta+\varsigma)Q_{11}\\ &\quad +(\alpha+(1-\alpha)\gamma+\varsigma)(\varsigma+\omega)-(\varsigma+\omega)Q_{11},\\ b_1 &= (\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)(\sigma+\delta+\varsigma)\\ &\quad -\mu\beta_1(1-\alpha)\gamma(\sigma+\delta+\varsigma)-\sigma(1-\alpha)\gamma\mu\beta_2\\ &\quad -(\mu+\varsigma)(\sigma+\delta+\varsigma)Q_{11}-(1-\alpha)\gamma(\sigma+\delta+\varsigma)Q_{12}\\ &\quad +(\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)(\varsigma+\omega)-(1-\alpha)\gamma(\varsigma+\omega)Q_{12}\\ &\quad -\mu\beta_1(1-\alpha)\gamma(\varsigma+\omega)-(\mu+\varsigma)(\varsigma+\omega)Q_{11}\\ &\quad +(\alpha+(1-\alpha)\gamma+\varsigma)(\sigma+\delta+\varsigma)(\varsigma+\omega)\\ &\quad -(\sigma+\delta+\varsigma)(\varsigma+\omega)Q_{11}+(\mu+\varsigma)(\sigma+\delta+\varsigma)(\varsigma+\omega),\\ b_0 &= (\varsigma+\omega)((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)(\sigma+\delta+\varsigma)\\ &\quad -\mu\beta_1(1-\alpha)\gamma(\sigma+\delta+\varsigma)-\sigma(1-\alpha)\gamma\mu\beta_2\\ &\quad -(1-\alpha)\gamma(\sigma+\delta+\varsigma)Q_{12}-(\mu+\varsigma)(\sigma+\delta+\varsigma)Q_{11}),\end{aligned}$$

and, R_0, as shown at the bottom of the next page.

Based on the Routh-Hurwitz criterion, the Routh-Hurwitz table of the system (2) are shown in Table 2:

TABLE 2. **The Routh-Hurwitz table.**

λ^4	b_4	b_2	b_0
λ^3	b_3	b_1	
λ^2	a_2	b_0	
λ^1	a_1		
λ^0	b_0		

where $a_2=\frac{b_2b_3-b_4b_1}{b_3}$, $a_1=\frac{b_1a_2-b_3b_0}{a_2}$.

We can get $b_4>0$, $b_3>0$, $b_2>0$, $b_1>0$, $b_0>0$ when $R_0<1$. By adjusting the parameters, we can make $b_2b_3>b_4b_1$, $b_1a_2>b_3b_0$, this is, $a_2>0$, $a_1>0$. Thus, the conditions that all roots have negative real parts. According to the theorem [26], we can know the fraud-elimination equilibrium $E_0(1,0,0,0,0)$ is locally asymptotically stable when $R_0<1$. And if $R_0>1$, we can find $b_0<0$. It means that the fraud-elimination equilibrium $E_0(1,0,0,0,0)$ is unstable. The proof is completed.

Theorem 3: If $R_0<1$, the fraud-elimination equilibrium E_0 is globally asymptotically stable.

Proof: Let's set up a Lyapunov function $V(t)$ as follows:

$$V(t)=\sum_k kG(k)\begin{bmatrix}(\rho_1(\mu+\varsigma)(\sigma+\delta+\varsigma)+\rho_2(1-\alpha)\\ \gamma(\sigma+\delta+\varsigma))H_k(t)\\ +(\rho_1(\mu\beta 2+(\sigma+\delta+\varsigma)+\rho_2(\sigma+\delta+\varsigma)\\ (\alpha+(1-\alpha)\gamma+\varsigma))I_k(t)\\ +(\rho_1(\mu+\varsigma)\sigma+\rho_2(1-\alpha)\gamma\sigma)P_k(t)\end{bmatrix}$$

Then, by calculating the derivative of $V(t)$ along the solution of the system (1), we have, $\frac{dV(t)}{dt}$, as shown at the bottom of the next page.

When $R_0<1$, we can get $V'(t)\le 0$, $V(t)\ge 0$. If and only if $\rho_1\theta_1(t)+\rho_2\theta_2(t)=0$, we can get $V'(t)=0$, that is, $H_k=0$ and $I_k=0$. Finally, according to LaSalle Invariant

$$R_0=\frac{\langle k^2\rangle}{\langle k\rangle}\cdot\frac{(1-\varepsilon)(\sigma+\delta+\varsigma)(\rho_1(\mu+\varsigma)+\rho_2(1-\alpha)\gamma)}{((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma}>1,$$

$$J(E_0)=\begin{pmatrix} -(\alpha+(1-\alpha)\gamma+\varsigma)+Q_{11} & \mu\beta_1+Q_{12} & \sigma & 0\\ (1-\alpha)\gamma & -(\mu+\varsigma) & 0 & 0\\ 0 & \mu\beta_2 & -(\sigma+\delta+\varsigma) & 0\\ \alpha\eta & \mu(1-\beta_1-\beta_2) & \delta & -(\varsigma+\omega)\end{pmatrix},$$

$$|\lambda-J(E_0)|=\begin{vmatrix} \lambda+(\alpha+(1-\alpha)\gamma+\varsigma)-Q_{11} & -(\mu\beta_1+Q_{12}) & -\sigma & 0\\ -(1-\alpha)\gamma & \lambda+(\mu+\varsigma) & 0 & 0\\ 0 & -\mu\beta_2 & \lambda+(\sigma+\delta+\varsigma) & 0\\ -\alpha\eta & -\mu(1-\beta_1-\beta_2) & -\delta & \lambda+(\varsigma+\omega)\end{vmatrix}.$$

Set Theorem [27], we can know the fraud-elimination equilibrium E_0 is globally asymptotically stable.

V. THE GLOBAL ATTRACTIVITY OF FRAUD-PREVAILING EQUILIBRIUM

In this section, we prove the global attractivity of fraud-prevailing equilibrium by a novel monotonic iterative algorithm [28].

Theorem 4: Suppose that $(H_k(t), I_k(t), P_k(t), R_k(t))$ is a solution of system (2) satisfying initial conditions $H_k(t) > 0$ or $I_k(t) > 0$. If $R0 > 1$, then

$$\lim_{t\to\infty}(H_k(t), I_k(t), P_k(t), R_k(t)) = (H_k^*(t), I_k^*(t), P_k^*(t), R_k^*(t)),$$

where $E^+(S_k^*, H_k^*, I_k^*, P_k^*, R_k^*)$ is the fraud-prevailing equilibrium of system (2) for $k = 1, 2, 3\cdots$.

Proof: Assuming k is any positive integer between 1 and n. By theorem 4, there exists a positive number $\psi > 0$ and a sufficiently larger constant $T > 0$, such that $H_k(t) \geq \psi$, $I_k(t) \geq \psi$ for $t > T$. So, there is $\Theta(t) > \psi(\rho_1 + \rho_2)$ for $t > T$. From the first equation of the system (2), we get

$$dH_k(t)/dt \leq (k(1-\varepsilon)(\rho_1+\rho_2)+\mu\beta_1+\sigma) *(1-H_k(t))-(\alpha+(1-\alpha)\gamma+\varsigma)H_k(t), \quad t > T.$$

By the comparison theorem of differential equation theory [29], for any given positive number

$$0 < \psi_1 < \frac{\alpha+(1-\alpha)\gamma+\varsigma}{2(k(1-\varepsilon)(\rho_1+\rho_2)+\mu\beta_1+\sigma+(\alpha+(1-\alpha)\gamma+\varsigma))},$$

there exists a $t_1 > T$ such that $H_k(t) \leq X_k^{(1)} - \psi_1$ for $t > t_1$, where

$$X_k^{(1)} = \frac{k(1-\varepsilon)(\rho_1+\rho_2)+\mu\beta_1+\sigma}{k(1-\varepsilon)(\rho_1+\rho_2)+\mu\beta_1+\sigma+(\alpha+(1-\alpha)\gamma+\varsigma)}+2\psi_1 < 1.$$

From the second equation of the system (2), we get

$$dI_k(t)/dt \leq (1-\alpha)\gamma(1-I_k(t))-(\mu+\varsigma)I_k(t), \quad t > t_1.$$

Hence, for any given positive number $0 < \psi_2 < \min\left\{\frac{1}{2}, \psi_1, \frac{\mu+\varsigma}{2((1-\alpha)\gamma+(\mu+\varsigma))}\right\}$, there exists a $t_2 > t_1$ such that $I_k(t) \leq Y_k^{(1)} - \psi_2$ for $t > t_2$, where

$$Y_k^{(1)} = \frac{(1-\alpha)\gamma}{(1-\alpha)\gamma+(\mu+\varsigma)}+2\psi_2 < 1.$$

From the third equation of the system (2), we get

$$dP_k(t)/dt \leq \mu\beta_2(1-P_k(t))-(\sigma+\delta+\varsigma)P_k(t), \quad t > t_2.$$

Consequently, for any given positive number $0 < \psi_3 < \min\left\{\frac{1}{3}, \psi_2, \frac{\sigma+\delta+\varsigma}{2(\mu\beta_2+(\sigma+\delta+\varsigma))}\right\}$, there exists a $t_3 > t_2$ such that

$$\begin{aligned} R_0 &= \frac{\langle k^2\rangle}{\langle k\rangle}\cdot\frac{(1-\varepsilon)(\sigma+\delta+\varsigma)(\rho_1(\mu+\varsigma)+\rho_2(1-\alpha)\gamma)}{((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma} \\ &= \frac{(\sigma+\delta+\varsigma)((\mu+\varsigma)Q_{11}+(1-\alpha)\gamma Q_{12})}{((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma}. \end{aligned}$$

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_k kG(k)\begin{bmatrix} (\rho_1(\mu+\varsigma)(\sigma+\delta+\varsigma) \\ +\rho_2(1-\alpha)\gamma(\sigma+\delta+\varsigma))(k(1-\varepsilon)(\rho_1\theta_1(t)+\rho_2\theta_2(t))S_k(t) \\ +\mu\beta_1 I_k(t)+\sigma P_k(t)-\alpha H_k(t)-(1-\alpha)\gamma H_k(t)-\varsigma H_k(t)) \end{bmatrix} \\ &\quad + \sum_k kG(k)\begin{bmatrix} (\rho_1(\mu\beta_2+(\sigma+\delta+\varsigma) \\ +\rho_2(\sigma+\delta+\varsigma)(\alpha+(1-\alpha)\gamma+\varsigma))((1-\alpha)\gamma H_k(t)-\mu I_k(t)-\varsigma I_k(t)) \end{bmatrix} \\ &\quad + \sum_k kG(k)\left[(\rho_1(\mu+\varsigma)\sigma+\rho_2(1-\alpha)\gamma\sigma)(\mu\beta_2 I_k(t)-\sigma P_k(t)-\delta P_k(t)-\varsigma P_k(t))\right] \\ &= \sum_k kG(k)\begin{bmatrix} (\rho_1(\mu+\varsigma)(\sigma+\delta+\varsigma) \\ +\rho_2(1-\alpha)\gamma(\sigma+\delta+\varsigma))(k(1-\varepsilon)(\rho_1\theta_1(t)+\rho_2\theta_2(t))S_k(t) \\ +((1-\alpha)\gamma(\sigma\mu\beta_2+(\sigma+\delta+\varsigma)\mu\beta_1) \\ -(\mu+\varsigma)(\sigma+\delta+\varsigma)(\alpha+(1-\alpha)\gamma+\varsigma))(\rho_1 H_k(t)+\rho_2 I_k(t)) \end{bmatrix} \\ &\leq \sum_k kG(k)\begin{bmatrix} (\rho_1(\mu+\varsigma)(\sigma+\delta+\varsigma) \\ +\rho_2(1-\alpha)\gamma(\sigma+\delta+\varsigma))(k(1-\varepsilon)(\rho_1\theta_1(t)+\rho_2\theta_2(t)) \\ +((1-\alpha)\gamma(\sigma\mu\beta_2+(\sigma+\delta+\varsigma)\mu\beta_1) \\ -(\mu+\varsigma)(\sigma+\delta+\varsigma)(\alpha+(1-\alpha)\gamma+\varsigma))(\rho_1 H_k(t)+\rho_2 I_k(t)) \end{bmatrix} \\ &= \langle k\rangle\begin{bmatrix} (((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma) \\ *(\frac{\langle k^2\rangle}{\langle k\rangle}\cdot\frac{(1-\varepsilon)(\sigma+\delta+\varsigma)(\rho_1(\mu+\varsigma)+\rho_2(1-\alpha)\gamma)}{((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma}-1) \\ *(\rho_1\theta_1(t)+\rho_2\theta_2(t)) \end{bmatrix} \\ &= \langle k\rangle\,(((\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)-\mu\beta_1(1-\alpha)\gamma)(\sigma+\delta+\varsigma)-\sigma\mu\beta_2(1-\alpha)\gamma)(R_0-1)(\rho_1\theta_1(t)+\rho_2\theta_2(t)) \end{aligned}$$

$P_k(t) \le Z_k^{(1)} - \psi_3$ for $t > t_3$, where

$$Z_k^{(1)} = \frac{\mu\beta_2}{\mu\beta_2 + (\sigma+\delta+\varsigma)} + 2\psi_3 < 1.$$

From the fourth equation of the system (2), we get

$$dR_k(t)/dt \le \alpha\eta + \mu(1-\beta_1-\beta_2)+\delta)(1-R_k(t)) - (\varsigma+\omega)R_k(t), \quad t > t_3.$$

Therefore, for any given positive number $0 < \psi_4 < \min\left\{\frac{1}{4}, \psi_3, \frac{\varsigma+\omega}{2((\alpha\eta+\mu(1-\beta_1-\beta_2)+\delta)+(\varsigma+\omega))}\right\}$, there exists a $t_4 > t_3$ such that $R_k(t) \le Q_k^{(1)} - \psi_4$ for $t > t_4$, where

$$Q_k^{(1)} = \frac{\alpha\eta + \mu(1-\beta_1-\beta_2)+\delta}{(\alpha\eta+\mu(1-\beta_1-\beta_2)+\delta)+(\varsigma+\omega)} + 2\psi_4 < 1.$$

Next, we replace $I_k(t) \le Y_k^{(1)}$, $P_k(t) \le Z_k^{(1)}$ and $R_k(t) \le Q_k^{(1)}$ into the first equation of the system (2), the results are as follows:

$$\begin{aligned} dH_k(t)/dt &\ge k\psi(1-\varepsilon)(\rho_1+\rho_2)(1-H_k(t)-I_k(t)-P_k(t)-R_k(t)) \\ &\quad -(\alpha+(1-\alpha)\gamma+\varsigma)H_k(t) \\ &\ge k\psi(1-\varepsilon)(\rho_1+\rho_2)(1-Y_k^{(1)}-Z_k^{(1)}-Q_k^{(1)}) \\ &\quad -(k\psi(1-\varepsilon)(\rho_1+\rho_2)+\alpha+(1-\alpha)\gamma+\varsigma)H_k(t), \ t > T. \end{aligned}$$

Hence, for any given positive number

$$0 < \psi_5 < \min\left\{\frac{1}{5}, \psi_4, \frac{k\psi(1-\varepsilon)(\rho_1+\rho_2)(1-Y_k^{(1)}-Z_k^{(1)}-Q_k^{(1)})}{2(k\psi(1-\varepsilon)(\rho_1+\rho_2)+\alpha+(1-\alpha)\gamma+\varsigma)}\right\},$$

there exists a $t_5 > t_4$ such that $H_k(t) \ge x_k^{(1)} + \psi_5$ for $t > t_5$, where

$$x_k^{(1)} = \frac{k\psi(1-\varepsilon)(\rho_1+\rho_2)(1-Y_k^{(1)}-Z_k^{(1)}-Q_k^{(1)})}{k\psi(1-\varepsilon)(\rho_1+\rho_2)+\alpha+(1-\alpha)\gamma+\varsigma} - 2\psi_5 > 0.$$

From the second equation of the system (2), we get

$$dI_k(t)/dt \ge (1-\alpha)\gamma x_k^{(1)} - (\mu+\varsigma)I_k(t), \quad t > t_5.$$

Consequently, for any given positive number

$$0 < \psi_6 < \min\left\{\frac{1}{6}, \psi_5, \frac{(1-\alpha)\gamma x_k^{(1)}}{2(\mu+\varsigma)}\right\},$$

there exists a $t_6 > t_5$ such that $I_k(t) \ge y_k^{(1)} + \psi_6$ for $t > t_6$, where

$$y_k^{(1)} = \frac{(1-\alpha)\gamma x_k^{(1)}}{\mu+\varsigma} - 2\psi_6 > 0.$$

From the third equation of the system (2), we get

$$dP_k(t)/dt \ge \mu\beta_2 y_k^{(1)} - (\sigma+\delta+\varsigma)P_k(t), \quad t > t_6.$$

Therefore, for any given positive number $0 < \psi_7 < \min\left\{\frac{1}{7}, \psi_6, \frac{\mu\beta_2 y_k^{(1)}}{2(\sigma+\delta+\varsigma)}\right\}$, there exists a $t_7 > t_6$ such that $P_k(t) \ge z_k^{(1)} + \psi_7$ for $t > t_7$, where

$$z_k^{(1)} = \frac{\mu\beta_2 y_k^{(1)}}{\sigma+\delta+\varsigma} - 2\psi_7 > 0.$$

From the fourth equation of the system (2), we get

$$dR_k(t)/dt \ge \alpha\eta x_k^{(1)} + \mu(1-\beta_1-\beta_2)y_k^{(1)} + \delta z_k^{(1)} - (\varsigma+\omega)R_k(t), \quad t > t_7.$$

So, for any given positive number

$$0 < \psi_8 < \min\left\{\frac{1}{8}, \psi_7, \frac{\alpha\eta x_k^{(1)} + \mu(1-\beta_1-\beta_2)y_k^{(1)} + \delta z_k^{(1)}}{2(\varsigma+\omega)}\right\},$$

there exists a $t_8 > t_7$ such that $R_k(t) \ge q_k^{(1)} + \psi_8$ for $t > t_8$, where

$$q_k^{(1)} = \frac{\alpha\eta x_k^{(1)} + \mu(1-\beta_1-\beta_2)y_k^{(1)} + \delta z_k^{(1)}}{\varsigma+\omega} - 2\psi_8 > 0.$$

Because ψ is a small positive number, it has that $0 < x_k^{(1)} < H_k < X_k^{(1)} < 1$, $0 < y_k^{(1)} < I_k < Y_k^{(1)} < 1$, $0 < z_k^{(1)} < P_k < Z_k^{(1)} < 1$ and $0 < q_k^{(1)} < R_k < Q_k^{(1)} < 1$. Let

$$\begin{cases} m^{(j)} = \sum_{k=1}^{n} G_k\left(\rho_1 x_k^{(j)} + \rho_2 y_k^{(j)}\right) \\ M^{(j)} = \sum_{k=1}^{n} G_k\left(\rho_1 X_k^{(j)} + \rho_2 Y_k^{(j)}\right), \end{cases} \quad j = 1, 2, \ldots,$$

From the above discussion, we can get

$$0 < m^{(1)} \le \Theta(t) \le M^{(1)} < \rho_1 + \rho_2, \quad t > t_8.$$

Again, we substitute $I_k(t) \ge y_k^{(1)} + \psi_6$, $P_k(t) \ge z_k^{(1)} + \psi_7$ and $R_k(t) \ge q_k^{(1)} + \psi_8$ into the first equation of the system (2), the results are as follows:

$$\begin{aligned} &dH_k(t)/dt \\ &\le k(1-\varepsilon)M^{(1)}(1-H_k(t)-y_k^{(1)}-z_k^{(1)}-q_k^{(1)}) \\ &\quad +\mu\beta_1 Y_k^{(1)} + \sigma Z_k^{(1)} - (\alpha+(1-\alpha)\gamma+\varsigma)H_k(t), \quad t > t8. \end{aligned}$$

Hence, for any given positive number $0 < \psi_9 < \min\{1/9, \psi_8\}$, there exists a $t_9 > t_8$, such that, $H_k(t) \le X_k^{(2)}$, as shown at the bottom of the next page.

From the second equation of the system (2), we get

$$dI_k(t)/dt \le (1-\alpha)\gamma X_k^{(1)} - (\mu+\varsigma)I_k(t), \quad t > t_9.$$

Consequently, for any given positive number $0 < \psi_{10} < \min\{1/10, \psi_9\}$, there exists a $t_{10} > t_9$ such that

$$I_k(t) \le Y_k^{(2)} \min\left\{Y_k^{(1)} - \psi_2, \frac{(1-\alpha)\gamma X_k^{(1)}}{\mu+\varsigma} + \psi_{10}\right\}, \ t > t_{10}.$$

From the third equation of the system (2), we get

$$dP_k(t)/dt \le \mu\beta_2 Y_k^{(2)} - (\sigma+\delta+\varsigma)P_k(t), \quad t > t_{10}.$$

Therefore, for any given positive number $0 < \psi_{11} < \min\{1/11, \psi_{10}\}$, there exists a $t_{11} > t_{10}$ such that

$$P_k(t) \le Z_k^{(2)} \min\left\{Z_k^{(1)} - \psi_3, \frac{\mu\beta_2 Y_k^{(2)}}{\sigma+\delta+\varsigma} + \psi_{11}\right\}, \quad t > t_{11}.$$

From the fourth equation of the system (2), we get

$$dR_k(t)/dt \le \alpha\eta X_k^{(2)}+\mu(1-\beta_1-\beta_2)Y_k^{(2)}+\delta Z_k^{(2)} \\ -(\varsigma+\omega)R_k(t), \quad t > t_{11}.$$

Thus, for any given positive number $0 < \psi_{12} < \min\{1/12, \psi_{11}\}$, there exists a $t_{12} > t_{11}$ such that

$$R_k(t) \le Q_k^{(2)} \\ \min\left\{Q_k^{(1)}-\psi_4, \frac{\alpha\eta X_k^{(2)}+\mu(1-\beta_1-\beta_2)Y_k^{(2)}+\delta Z_k^{(2)}}{\varsigma+\omega}+\psi_{12}\right\}, \\ t > t_{12}.$$

Back to the system (2), we substitute $I_k(t) \le Y_k^{(2)}$, $P_k(t) \le Z_k^{(2)}$ and $R_k(t) \le Q_k^{(2)}$ into the first equation of the system (2), we get

$$dH_k(t)/dt \\ \ge k(1-\varepsilon)m^{(1)}(1-H_k(t)-Y_k^{(2)}-Z_k^{(2)}-Q_k^{(2)}) \\ +\mu\beta_1 y_k^{(2)}+\sigma z_k^{(2)}-(\alpha+(1-\alpha)\gamma+\varsigma)H_k(t), \quad t > t_{12}.$$

Hence, for any given positive number, equation as shown at the bottom of the page, there exists a $t_{13} > t_{12}$ such that $H_k(t) \ge x_k^{(2)}+\psi_{13}$ for $t > t_{13}$, where, $x_k^{(2)}$, as shown at the bottom of the page,

From the second equation of the system (2), we get

$$dI_k(t)/dt \ge (1-\alpha)\gamma x_k^{(2)}-(\mu+\varsigma)I_k(t), \quad t > t_{13}.$$

Hence, for any given positive number $0 < \psi_{14} < \min\left\{\frac{1}{14}, \psi_{13}, \frac{(1-\alpha)\gamma x_k^{(2)}}{2(\mu+\varsigma)}\right\}$, there exists a $t_{14} > t_{13}$ such that $I_k(t) \ge y_k^{(2)}+\psi_{14}$ for $t > t_{14}$, where

$$y_k^{(2)} = \max\left\{y_k^{(1)}+\psi_6, \frac{(1-\alpha)\gamma x_k^{(2)}}{\mu+\varsigma}-2\psi_{14}\right\}.$$

From the third equation of the system (2), we get

$$dP_k(t)/dt \ge \mu\beta_2 y_k^{(2)}-(\sigma+\delta+\varsigma)P_k(t), \quad t > t_{14}.$$

Hence, for any given positive number $0 < \psi_{15} < \min\left\{\frac{1}{15}, \psi_{14}, \frac{\mu\beta_2 y_k^{(2)}}{2(\sigma+\delta+\varsigma)}\right\}$, there exists a $t_{15} > t_{14}$ such that $P_k(t) \ge z_k^{(2)}+\psi_{15}$ for $t > t_{15}$, where

$$z_k^{(2)} = \max\left\{z_k^{(1)}+\psi_7, \frac{\mu\beta_2 y_k^{(2)}}{\sigma+\delta+\varsigma}-2\psi_{15}\right\}.$$

From the third equation of the system (2), we get

$$dR_k(t)/dt \ge \alpha\eta x_k^{(2)}+\mu(1-\beta_1-\beta_2)y_k^{(2)}+\delta z_k^{(2)} \\ -(\varsigma+\omega)R_k(t), \quad t > t_{15}.$$

So, for any given positive number

$$0<\psi_{16}<\min\left\{\frac{1}{16}, \psi_{15}, \frac{\alpha\eta x_k^{(2)}+\mu(1-\beta_1-\beta_2)y_k^{(2)}+\delta z_k^{(2)}}{2(\varsigma+\omega)}\right\},$$

there exists a $t_{16} > t_{15}$ such that $R_k(t) \le q_k^{(2)}+\psi_{16}$ for $t > t_{16}$, where

$$q_k^{(2)} \\ =\max\left\{q_k^{(1)}+\psi_8, \frac{\alpha\eta x_k^{(2)}+\mu(1-\beta_1-\beta_2)y_k^{(2)}+\delta z_k^{(2)}}{\varsigma+\omega}-2\psi_{16}\right\}.$$

Similarly, we repeat the above calculation steps to get eight sequences $\{X_k^{(l)}\}$, $\{Y_k^{(l)}\}$, $\{Z_k^{(l)}\}$, $\{Q_k^{(l)}\}$, $\{x_k^{(l)}\}$, $\{y_k^{(l)}\}$, $\{z_k^{(l)}\}$ and $\{q_k^{(l)}\}$ where $l = 1, 2, 3\cdots$. Since the first four sequences are strictly monotonically decreasing and the last four sequences are monotonically increasing, there exists a very large positive integer ζ, when there is $l \ge \zeta$, we can obtain, equation as shown at the bottom of the next page.

Obviously,

$$x_k^{(l)} \le H_k(t) \le X_k^{(l)},\ y_k^{(l)} \le I_k(t) \le Y_k^{(l)}, \\ z_k^{(l)} \le P_k(t) \le Z_k^{(l)},\ q_k^{(l)} \le R_k(t) \le Q_k^{(l)}, \quad t > t_{8l}. \tag{6}$$

Since the above equation limit exists, let $\lim_{l\to\infty} \Omega_k^{(l)} = \Omega_k$, where

$$\Omega_k^{(l)} \in \left\{X_k^{(l)}, Y_k^{(l)}, Z_k^{(l)}, Q_k^{(l)}, x_k^{(l)}, y_k^{(l)}, z_k^{(l)}, q_k^{(l)}, M_k^{(l)}, m_k^{(l)}\right\}, \\ \Omega_k \in \{X_k, Y_k, Z_k, Q_k, x_k, y_k, z_k, q_k, M_k, m_k\}.$$

$$H_k(t) \le X_k^{(2)} \triangleq \min\left\{X_k^{(1)}-\psi_1, \frac{k(1-\varepsilon)M^{(1)}(1-y_k^{(1)}-z_k^{(1)}-q_k^{(1)})+\mu\beta_1 Y_k^{(1)}+\sigma Z_k^{(1)}}{k(1-\varepsilon)M^{(1)}+(\alpha+(1-\alpha)\gamma+\varsigma)}+\psi_9\right\}, \quad t > t_9.$$

$$0 < \psi_{13} < \min\left\{\frac{1}{13}, \psi_{12}, \frac{k(1-\varepsilon)m^{(1)}(1-Y_k^{(2)}-Z_k^{(2)}-Q_k^{(2)})+\mu\beta_1 y_k^{(2)}+\sigma z_k^{(2)}}{2(k(1-\varepsilon)m^{(1)}+(\alpha+(1-\alpha)\gamma+\varsigma))}\right\},$$

$$x_k^{(2)} = \max\left\{x_k^{(1)}+\psi_5, \frac{k(1-\varepsilon)m^{(1)}(1-Y_k^{(2)}-Z_k^{(2)}-Q_k^{(2)})+\mu\beta_1 y_k^{(2)}+\sigma z_k^{(2)}}{k(1-\varepsilon)m^{(1)}+(\alpha+(1-\alpha)\gamma+\varsigma)}-2\psi_{13}\right\}.$$

Since $0 < \psi_l < 1/l$, it has $\psi_l \to 0$ as $l \to 0$. Supposing $l \to 0$, it follows that

$$\begin{cases} X_k = \dfrac{k(1-\varepsilon)M(1-y_k-z_k-q_k)+\mu\beta_1 Y_k+\sigma Z_k}{k(1-\varepsilon)M+(\alpha+(1-\alpha)\gamma+\varsigma)}, \\ Y_k = \dfrac{(1-\alpha)\gamma X_k}{\mu+\varsigma}, \\ Z_k = \dfrac{\mu\beta_2 Y_k}{\sigma+\delta+\varsigma}, \\ Q_k = \dfrac{\alpha\eta X_k+\mu(1-\beta_1-\beta_2)Y_k+\delta Z_k}{\varsigma+\omega}, \\ x_k = \dfrac{k(1-\varepsilon)m(1-Y_k-Z_k-Q_k)+\mu\beta_1 y_k+\sigma z_k}{k(1-\varepsilon)m+(\alpha+(1-\alpha)\gamma+\varsigma)}, \\ y_k = \dfrac{(1-\alpha)\gamma x_k}{\mu+\varsigma}, \\ z_k = \dfrac{\mu\beta_2 y_k}{\sigma+\delta+\varsigma}, \\ q_k = \dfrac{\alpha\eta x_k+\mu(1-\beta_1-\beta_2)y_k+\delta z_k}{\varsigma+\omega}, \end{cases} \tag{7}$$

where

$$\begin{cases} m = \sum_{k=1}^{n} G_k\left(\rho_1 x_k+\rho_2 y_k\right), \\ M = \sum_{k=1}^{n} G_k\left(\rho_1 X_k+\rho_2 Y_k\right). \end{cases}$$

Further,

$$\begin{cases} x_k = \dfrac{(\mu+\varsigma)(\sigma+\delta+\varsigma)(\varsigma+\omega)}{k(1-\varepsilon)MJ_k^2}\xi_M, \\ X_k = \dfrac{(\mu+\varsigma)(\sigma+\delta+\varsigma)(\varsigma+\omega)}{k(1-\varepsilon)mJ_k^2}\xi_m. \end{cases}$$

$$\begin{cases} y_k = \dfrac{(1-\alpha)\gamma(\sigma+\delta+\varsigma)(\varsigma+\omega)}{k(1-\varepsilon)MJ_k^2}\xi_M, \\ Y_k = \dfrac{(1-\alpha)\gamma(\sigma+\delta+\varsigma)(\varsigma+\omega)}{k(1-\varepsilon)mJ_k^2}\xi_m. \end{cases}$$

where

$$\xi_M = \begin{bmatrix} k(1-\varepsilon)M(\mu\beta_2+\sigma+\delta+\mu) \\ +(1-\alpha)\gamma(\varsigma+\omega)(\mu\beta_1(\sigma+\delta+\varsigma)+\sigma\mu\beta_2) \\ -(\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)(\sigma+\delta+\varsigma)(\varsigma+\omega) \\ +k(1-\varepsilon)M(\sigma+\delta+\varsigma)(\mu+\varsigma)\alpha\eta \\ +k(1-\varepsilon)M(1-\alpha)\gamma(\mu\beta_2\delta+(\sigma+\delta+\varsigma)\mu(1-\beta_1-\beta_2)) \end{bmatrix},$$

$$\xi_m = \begin{bmatrix} k(1-\varepsilon)m(\mu\beta_2+\sigma+\delta+\mu) \\ +(1-\alpha)\gamma(\varsigma+\omega)(\mu\beta_1(\sigma+\delta+\varsigma)+\sigma\mu\beta_2) \\ -(\alpha+(1-\alpha)\gamma+\varsigma)(\mu+\varsigma)(\sigma+\delta+\varsigma)(\varsigma+\omega) \\ +k(1-\varepsilon)m(\sigma+\delta+\varsigma)(\mu+\varsigma)\alpha\eta \\ +k(1-\varepsilon)m(1-\alpha)\gamma(\mu\beta_2\delta+(\sigma+\delta+\varsigma)\mu(1-\beta_1-\beta_2)) \end{bmatrix},$$

$$\begin{aligned} J_k &= (1-\alpha)\gamma(\varsigma+\omega)(\mu\beta_2+\sigma+\delta+\varsigma) \\ &\quad +(1-\alpha)\gamma(\mu\beta_2\delta+(\sigma+\delta+\varsigma)\mu(1-\beta_1-\beta_2)) \\ &\quad +(\sigma+\delta+\varsigma)(\mu+\varsigma)\alpha\eta. \end{aligned}$$

Substituting x_k, X_k, y_k and Y_k into M and m, respectively, one has

$$1 = \frac{\sum_{i=1}^{n} G_i\left(\rho_1 x_i+\rho_2 y_i\right)}{m}, \tag{8}$$

$$1 = \frac{\sum_{i=1}^{n} G_i\left(\rho_1 X_i+\rho_2 Y_i\right)}{M}. \tag{9}$$

$$\begin{cases} X_k^{(l)} = \dfrac{k(1-\varepsilon)M^{(l-1)}(1-y_k^{(l-1)}-z_k^{(l-1)}-q_k^{(l-1)})+\mu\beta_1 Y_k^{(l-1)}+\sigma Z_k^{(l-1)}}{k(1-\varepsilon)M^{(l-1)}+(\alpha+(1-\alpha)\gamma+\varsigma)}+2\psi_{8l-7}, \\ Y_k^{(l)} = \dfrac{(1-\alpha)\gamma X_k^{(l)}}{\mu+\varsigma}+2\psi_{8l-6}, \\ Z_k^{(l)} = \dfrac{\mu\beta_2 Y_k^{(l)}}{\sigma+\delta+\varsigma}+2\psi_{8l-5}, \\ Q_k^{(l)} = \dfrac{\alpha\eta X_k^{(l)}+\mu(1-\beta_1-\beta_2)Y_k^{(l)}+\delta Z_k^{(l)}}{\varsigma+\omega}+2\psi_{8l-4}, \\ x_k^{(l)} = \dfrac{k(1-\varepsilon)m^{(l-1)}(1-Y_k^{(l)}-Z_k^{(l)}-Q_k^{(l)})+\mu\beta_1 y_k^{(l)}+\sigma z_k^{(l)}}{k(1-\varepsilon)m^{(l-1)}+(\alpha+(1-\alpha)\gamma+\varsigma)}-2\psi_{8l-3}, \\ y_k^{(l)} = \dfrac{(1-\alpha)\gamma x_k^{(l)}}{\mu+\varsigma}-2\psi_{8l-2}, \\ z_k^{(l)} = \dfrac{\mu\beta_2 y_k^{(l)}}{\sigma+\delta+\varsigma}-2\psi_{8l-1}, \\ q_k^{(l)} = \dfrac{\alpha\eta x_k^{(l)}+\mu(1-\beta_1-\beta_2)y_k^{(l)}+\delta z_k^{(l)}}{\varsigma+\omega}-2\psi_{8l}. \end{cases}$$

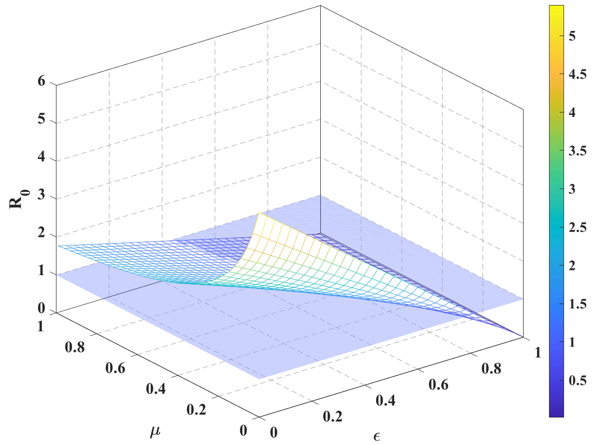

FIGURE 2. **The relationship between the basic reproduction number R_0 and the parameters μ and ε.**

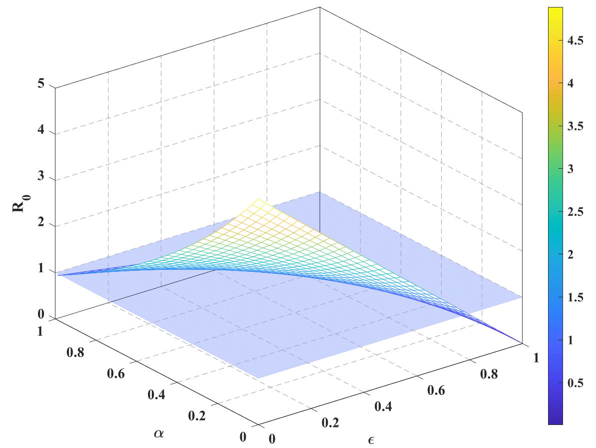

FIGURE 3. **The relationship between the basic reproduction number R_0 and the parameters ε and α.**

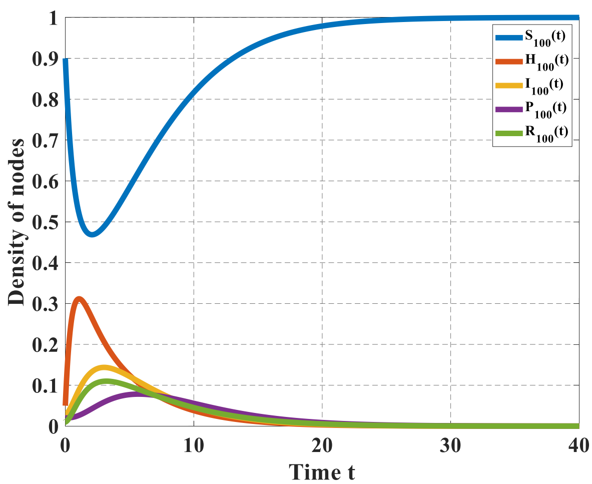

FIGURE 4. **The time series and orbits of five states when $R_0 < 1$.**

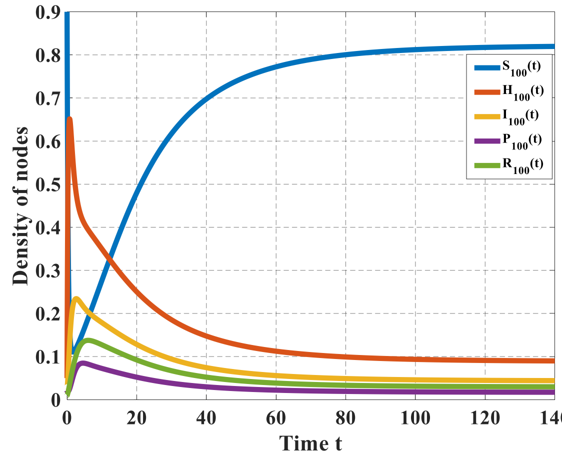

FIGURE 5. **The time series and orbits of five states when $R_0 > 1$.**

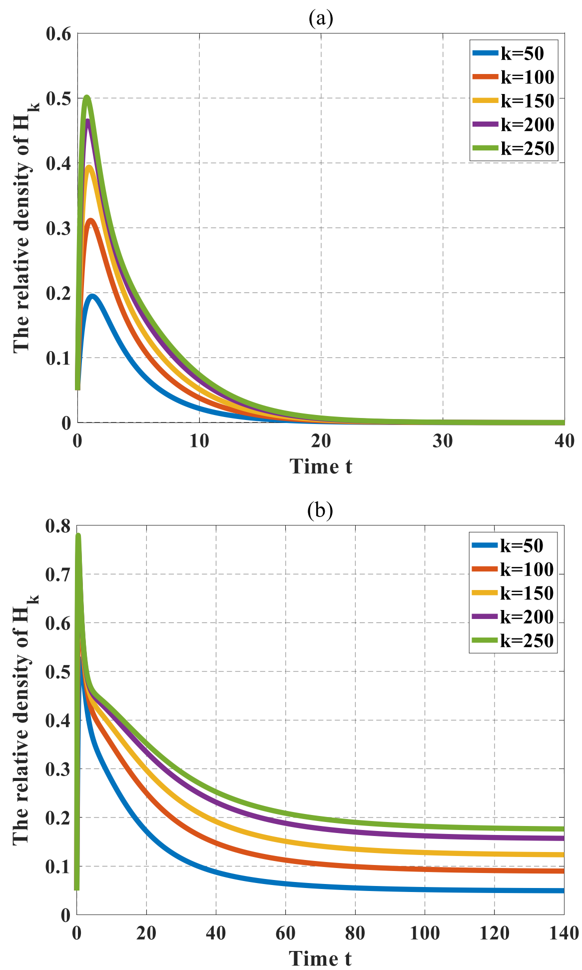

FIGURE 6. **Dynamics behavior of hesitator with different degree when $R_0 < 1$ and $R_0 > 1$.**

Subtract equation (8) from equation (9), we can obtain

$$(M-m)k(1-\varepsilon)(\sigma+\delta+\varsigma)(\varsigma+\omega) \times \sum_{i=1}^{n} G_i \frac{(\rho_1(\mu+\varsigma)+\rho_2(1-\alpha)\gamma)}{k(1-\varepsilon)mMJ_k^2}\varpi \equiv 0 \quad (10)$$

where

$$\varpi = \begin{bmatrix} (\mu\beta_2+\sigma+\delta+\mu) \\ +(1-\alpha)\gamma(\varsigma+\omega)(\mu\beta_1(\sigma+\delta+\varsigma)+\sigma\mu\beta_2) \\ +(\sigma+\delta+\varsigma)(\mu+\varsigma)\alpha\eta \\ +(1-\alpha)\gamma(\mu\beta_2\delta+(\sigma+\delta+\varsigma)\mu(1-\beta_1-\beta_2)) \end{bmatrix}.$$

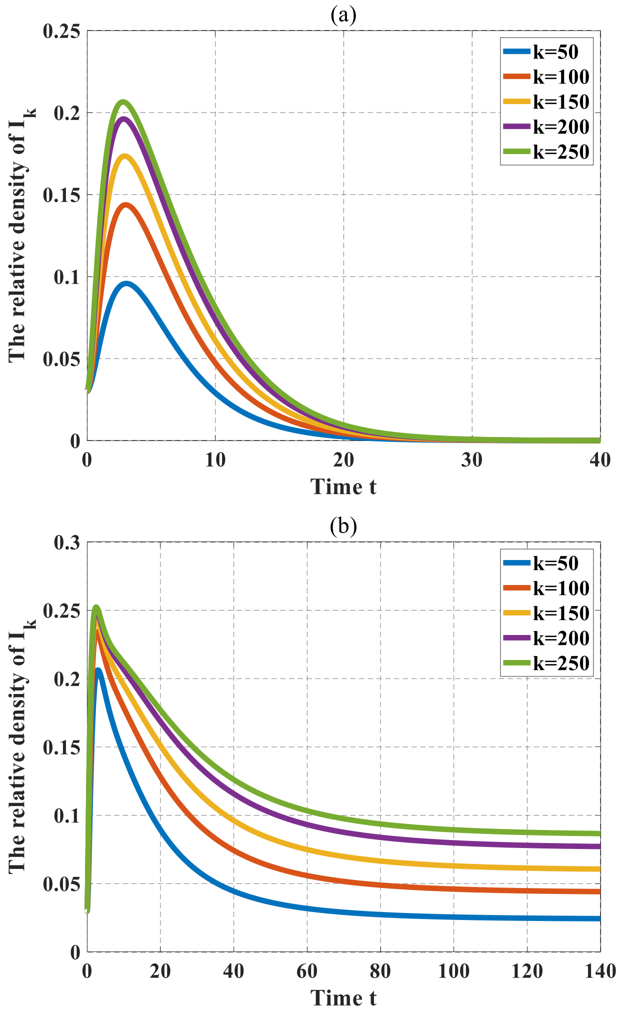

FIGURE 7. Dynamics behavior of involved individuals with different degree when $R_0 < 1$ and $R_0 > 1$.

From equation (10), we can get $M = m$. Thus, $\sum_{i=1}^{n} P_i[\rho_1(x_i - X_i) + \rho_2(y_i - Y_i)] = 0$, which is equivalent to $x_i = X_i$ and $y_i = Y_i$ for $1 \le i \le n$.

Then, from the system (6) and (7), it shows that

$$\lim_{t\to\infty} H_k(t) = X_k = x_k,$$
$$\lim_{t\to\infty} I_k(t) = Y_k = y_k,$$
$$\lim_{t\to\infty} P_k(t) = Z_k = z_k,$$
$$\lim_{t\to\infty} R_k(t) = Q_k = q_k.$$

Therefore, $X_k = H_k^*$, $Y_k = I_k^*$, $Z_k = P_k^*$ and $Q_k = R_k^*$, The proof is completed.

VI. NUMERICAL SIMULATIONS

In this section, we verify the theoretical analysis results by using numerical simulations. We will consider the system (1) has a degree distribution $P(k) = \omega k^{-3}$ on the scale-free networks, where the parameter ω satisfies $\sum_{k=0}^{n} \omega k^{-3} = 1$, $n = 1000$.

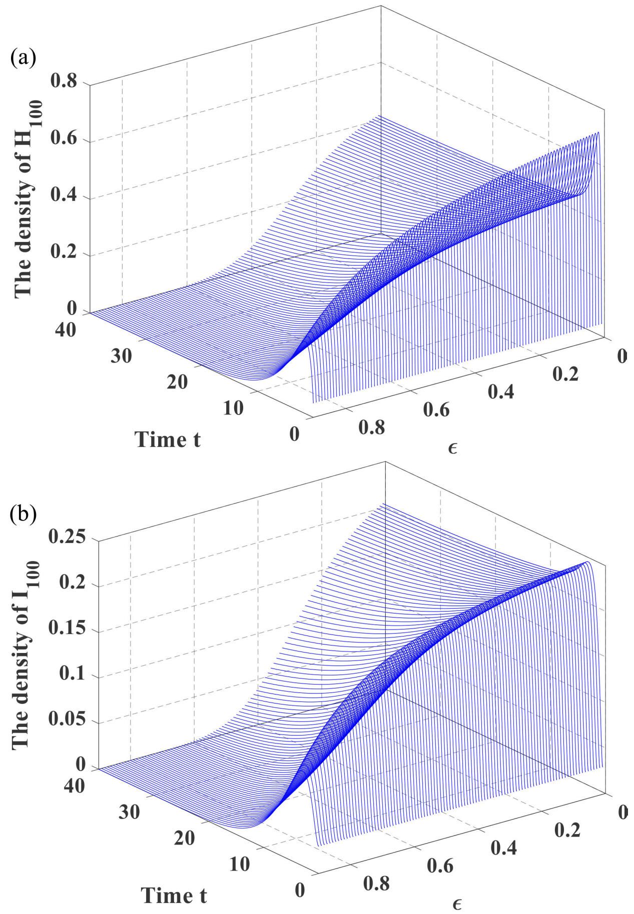

FIGURE 8. Dynamics behavior of H_{100} (a) and I_{100} (b) with different government management intensity ε.

In Figure 2 and Figure 3, we can find that the basic reproduction number R_0 decreases with the increase of parameters μ, α and ε. Compared with the parameters μ and α, the parameter ε has a greater impact on R_0. Therefore, increasing the government crackdown for involved individuals and the coverage rate of government anti-pyramid scheme publicity for hesitator and susceptible individuals can effectively curb the phenomenon of pyramid scheme spreading, especially the management on susceptible people is more effective.

In Figure 4, we choose $\varepsilon = 0.63$, $\varsigma = 0.16$, $\rho_1 = 0.24$, $\rho_2 = 0.48$, $\alpha = 0.52$, $\gamma = 0.71$, $\beta_1 = 0.09$, $\beta_2 = 0.74$, $\mu = 0.33$, $\delta = 0.13$, $\sigma = 0.05$, $\eta = 0.51$, $\omega = 0.46$, thus $R_0 = 0.9907 < 1$. The density of hesitator and involved individuals will eventually equal zero in the whole network, the phenomenon of pyramid scheme spreading will eventually disappear.

In Figure 5, we choose $\varepsilon = 0.32$, $\varsigma = 0.16$, $\rho_1 = 0.63$, $\rho_2 = 0.66$, $\alpha = 0.24$, $\gamma = 0.51$, $\beta_1 = 0.28$, $\beta_2 = 0.52$, $\mu = 0.63$, $\delta = 0.45$, $\sigma = 0.23$, $\eta = 0.24$, $\omega = 0.46$, thus $R_0 = 4.4915 > 1$. The density of hesitator and involved individuals will maintain a positive state in the whole network, the phenomenon of pyramid scheme spreading will exist forever, the spread of pyramid scheme is persistent.

Figure 6 and Figure 7 show the dynamics behavior of hesitator and involved individuals with different degree when

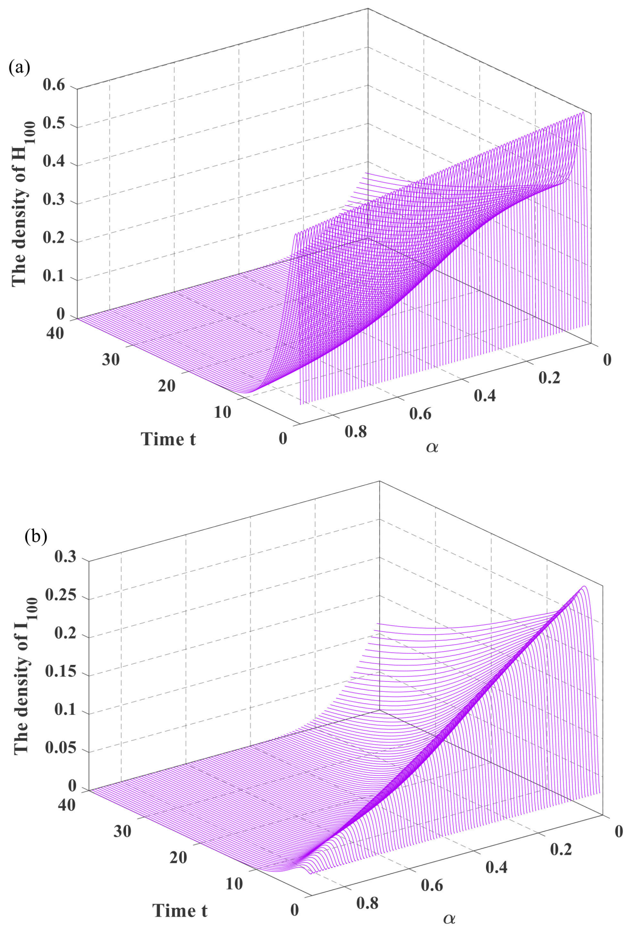

FIGURE 9. Dynamics behavior of H_{100} (a) and I_{100} (b) with different government management intensity α.

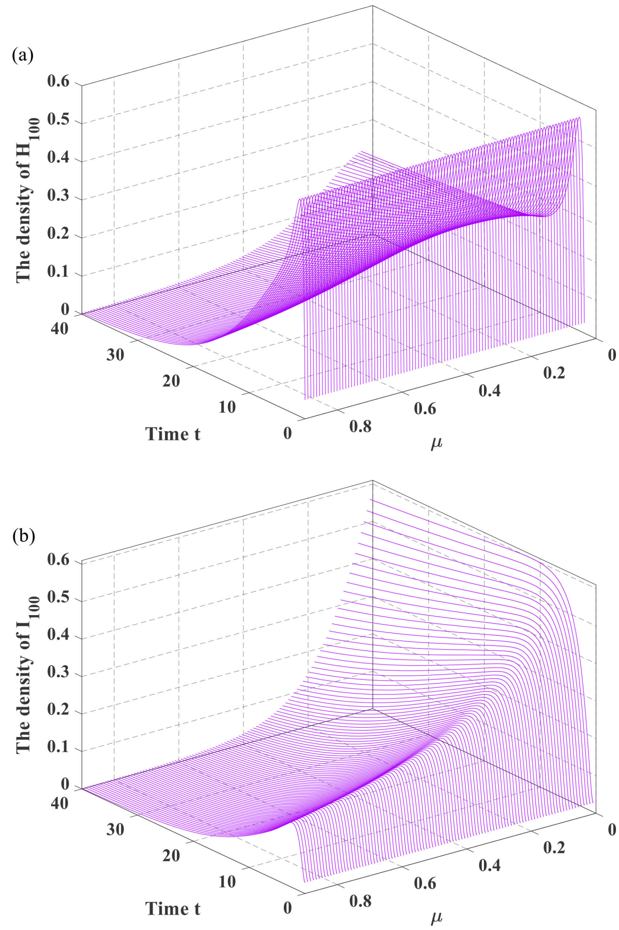

FIGURE 10. Dynamics behavior of H_{100} (a) and I_{100} (b) with different government management intensity μ.

the reproduction number $R0 < 1$ and $R0 > 1$. In Figure 6, the parameters $\varepsilon = 0.63$, $\varsigma = 0.16$, $\rho_1 = 0.24$, $\rho_2 = 0.48$, $\alpha = 0.52$, $\gamma = 0.71$, $\beta_1 = 0.09$, $\beta_2 = 0.74$, $\mu = 0.33$, $\delta = 0.13$, $\sigma = 0.05$, $\eta = 0.51$, $\omega = 0.46$ are chosen when $R_0 = 0.9907 < 1$; In Figure 7, the parameters $\varepsilon = 0.32$, $\varsigma = 0.16$, $\rho_1 = 0.63$, $\rho_2 = 0.66$, $\alpha = 0.24$, $\gamma = 0.51$, $\beta_1 = 0.28$, $\beta_2 = 0.52$, $\mu = 0.63$, $\delta = 0.45$, $\sigma = 0.23$, $\eta = 0.24$, $\omega = 0.46$ are chosen when $R_0 = 4.4915 > 1$. We find that the larger degree leads to the faster spread of pyramid schemes.

In Figure 8, we choose $\varsigma = 0.16$, $\rho_1 = 0.63$, $\rho_2 = 0.66$, $\alpha = 0.24$, $\gamma = 0.51$, $\beta_1 = 0.28$, $\beta_2 = 0.52$, $\mu = 0.63$, $\delta = 0.45$, $\sigma = 0.23$, $\eta = 0.51$, $\omega = 0.46$ and ε from 0 to 0.9. The density of hesitator and involved individuals is regarded as a function of time and parameter ε. The results show that the density of hesitator and involved individuals will gradually converge to a positive constant that is greater than zero. Apparently, increasing ε can decrease the constant to zero, which means that increasing the coverage rate of government anti-pyramid scheme publicity for susceptible individuals can effectively control the phenomenon of pyramid scheme spreading.

In Figure 9, we choose $\varsigma = 0.16$, $\rho_1 = 0.63$, $\rho_2 = 0.66$, $\varepsilon = 0.32$, $\gamma = 0.51$, $\beta_1 = 0.28$, $\beta_2 = 0.52$, $\mu = 0.63$, $\delta = 0.45$, $\sigma = 0.23$, $\eta = 0.51$, $\omega = 0.46$ and α from 0 to 0.9. The density of hesitator and involved individuals is regarded as a function of time and parameter α. We can find that the density of hesitator and involved individuals will gradually converge to a positive constant that is greater than zero. Apparently, increasing α can decrease the constant to zero, which means that increasing the coverage rate of government anti-pyramid scheme publicity for hesitator can effectively curb the phenomenon of pyramid scheme spreading.

In Figure 10, we choose $\varsigma = 0.16$, $\rho_1 = 0.43$, $\rho_2 = 0.46$, $\alpha = 0.24$, $\gamma = 0.51$, $\beta_1 = 0.28$, $\beta_2 = 0.52$, $\varepsilon = 0.32$, $\delta = 0.45$, $\sigma = 0.23$, $\eta = 0.51$, $\omega = 0.46$ and μ from 0 to 0.9. The result shows that the density of hesitator and involved individuals gradually become a constant. With the increase of government crackdown intensity on pyramid schemes, the constant eventually becomes zero, which means that the government crackdown for the involved individuals can control the phenomenon of pyramid scheme spreading.

VII. CONCLUSION

In this paper, we have proposed a novel *SHIPR* pyramid scheme spreading model on scale-free networks to study transmission mechanisms and transmission rules of pyramid schemes. Using the mean-field theory, we have obtained the basic reproduction number R_0 and two equilibria. By the Routh-Hurwitz criterion, comparison principle and Lyapunov function, we have proved that the fraud-elimination equilibrium is locally and globally asymptotically stable when $R0 < 1$; the fraud-prevailing equilibrium is globally stable when $R0 > 1$. We draw conclusions: the phenomenon of pyramid scheme spreading will not disappear and the spread of pyramid scheme is persistent if $R0 > 1$, thus the total amounts of hesitator and involved individuals will converge to a positive constant that is greater than zero; the spread of pyramid scheme will eventually stop and the phenomenon of pyramid scheme spreading will disappear ultimately if $R0 < 1$, thus the total amounts of hesitator and involved individuals will become zero. Besides, the influence of government management and social interaction topology on pyramid schemes have been analyzed in detail. That is, increasing government crackdown intensity for involved individuals, the coverage rate of government anti-pyramid scheme publicity for susceptible individuals and hesitator can hasten the disappearance of pyramid schemes. This research has important guiding significance for controlling the spread of pyramid schemes.

REFERENCES

[1] G. Albaum and R. A. Peterson, "Multilevel (network) marketing: An objective view," *Marketing Rev.*, vol. 11, no. 4, pp. 347–361, Dec. 2011.

[2] J. A. Muncy, "Ethical issues in multilevel marketing: Is it a legitimate business or just another pyramid scheme?" *Marketing Educ. Rev.*, vol. 14, no. 3, pp. 47–53, Oct. 2004.

[3] P. Feng, X. Lu, Z. Gong, B. Li, and D. Sun, "Social network analysis model for research on organizational structure of the pyramid scheme communication network," *MethodsX*, vol. 8, Jan. 2021, Art. no. 101259.

[4] R. R. Babu and P. Anand, "Legal aspects of multilevel marketing in India: Negotiating through murky waters," *Decision*, vol. 42, no. 4, pp. 359–378, Oct. 2015.

[5] P. Herbig and R. Yelkurm, "A review of the multilevel marketing phenomenon," *J. Marketing Channels*, vol. 6, no. 1, pp. 17–33, Sep. 1997.

[6] J. Xiong, "A method of mining key accounts from internet pyramid selling data," *Tehniki Vjesnik*, vol. 26, no. 3, pp. 728–735, 2019.

[7] D. Krige, "Fields of dreams, fields of schemes: Ponzi finance and multi-level marketing in South Africa," *Africa*, vol. 82, Feb. 2021, Art. no. 101259.

[8] C. J. Snook, "Is network marketing and pyramid selling the same thing? A layperson's guide to evaluating the pros and cons of this often misunderstood business model," Dept. Business Admin., San Diego State Univ., San Diego, CA, USA, Tech. Rep., 2005.

[9] K.-F. Lee, T.-C. Lau, and K.-Y. Loi, "Driving distributors' satisfaction in multilevel marketing (MLM) companies," *Int. J. Academic Res. Bus. Social Sci.*, vol. 6, no. 2, pp. 105–122, Mar. 2016.

[10] I. Constantin, "What is MLM system," *Manager*, vol. 2007, no. 5, pp. 107–117, 2007.

[11] I. Lahiri and M. K. Das, "Distributors' inclination towards MLM: An analysis," *Siddhant J. Decis. Making*, vol. 12, no. 1, pp. 26–37, Jan. 2012.

[12] S. Yaakob, B. Kartika, M. A. Jamaludin, M. A. Razali, and F. F. P. Perdana, "A critical analysis of halal marketing in Malaysia's multi-level marketing (MLM) industry," *J. Halal Ind. Services*, vol. 3, no. 1, Sep. 2020.

[13] W. W. Keep and P. J. V. Nat, "Multilevel marketing and pyramid schemes in the united states," *J. Historical Res. Marketing*, vol. 6, no. 2, pp. 188–210, May 2014.

[14] L. Schiffauer, "Dangerous speculation: The appeal of pyramid schemes in rural Siberia," *Focaal*, vol. 2018, no. 81, pp. 58–71, Jun. 2018.

[15] C. Constantin, "Multi-level marketing—A tool of relationship marketing," *Bull. Transilvania Univ. Brasov. Econ. Sci. Ser.*, vol. 2, no. 2, p. 31, 2009.

[16] Y. Lei, T. Li, Y. Wang, G. Ye, S. Sun, and Z. Xia, "Spreading dynamics of a CPFB group booking preferential information model on scale-free networks," *IEEE Access*, vol. 7, pp. 156287–156300, 2019.

[17] X. Liu, T. Li, X. Cheng, W. Liu, and H. Xu, "Spreading dynamics of a preferential information model with hesitation psychology on scale-free networks," *Adv. Difference Equ.*, vol. 2019, no. 1, pp. 1–19, Jul. 2019.

[18] L. Huo, T. Lin, C. Fan, C. Liu, and J. Zhao, "Optimal control of a rumor propagation model with latent period in emergency event," *Adv. Difference Equ.*, vol. 2015, no. 1, pp. 1–19, Feb. 2015.

[19] C. Li, "A study on time-delay rumor propagation model with saturated control function," *Adv. Difference Equ.*, vol. 2017, no. 1, pp. 1–22, Aug. 2017.

[20] T. Li, Y. Wang, and Z. H. Guan, "Spreading dynamics of a SIQRS epidemic model on scale-free networks," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 19, no. 3, pp. 686–692, 2014.

[21] W. Liu, T. Li, X. Liu, and H. Xu, "Spreading dynamics of a word-of-mouth model on scale-free networks," *IEEE Access*, vol. 6, pp. 65563–65572, 2018.

[22] R. Albert and A. L. Barabási, "Statistical mechanics of complex networks," *Rev. Mod. Phys.*, vol. 74, no. 1, p. 47, Jan. 2002.

[23] M. Liu, G. Sun, Z. Jin, and T. Zhou, "An analysis of transmission dynamics of drug-resistant disease on scale-free networks," *Appl. Math. Comput.*, vol. 222, pp. 177–189, Oct. 2013.

[24] R. Pastor-Satorras and A. Vespignani, "Epidemic spreading in scale-free networks," *Phys. Rev. Lett.*, vol. 86, no. 14, p. 3200, Apr. 2021.

[25] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," *Math. Biosci.*, vol. 180, nos. 1–2, pp. 29–48, Nov. 2002.

[26] J. Aweya, M. Ouellette, and D. Y. Montuno, "Design and stability analysis of a rate control algorithm using the Routh–Hurwitz stability criterion," *IEEE/ACM Trans. Netw.*, vol. 12, no. 4, pp. 719–732, Aug. 2004.

[27] L. Wang and G.-Z. Dai, "Global stability of virus spreading in complex heterogeneous networks," *SIAM J. Appl. Math.*, vol. 68, no. 5, pp. 1495–1502, 2008.

[28] G. Zhu, X. Fu, and G. Chen, "Spreading dynamics and global stability of a generalized epidemic model on complex heterogeneous networks," *Appl. Math. Model.*, vol. 36, no. 12, pp. 5808–5817, 2012.

[29] F. Chen, "On a nonlinear nonautonomous predator–prey model with diffusion and distributed delay," *J. Comput. Appl. Math.*, vol. 180, no. 1, pp. 33–49, Aug. 2005.

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