IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received September 8, 2021, accepted September 21, 2021, date of publication September 23, 2021, date of current version October 4, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3115444

Optimum Synthesis of Four-Bar Mechanism by Using Relative Angle Method: A Comparative Performance Study

LUIS ERNESTO VALENCIA-SEGURA [1,2],
MIGUEL GABRIEL VILLARREAL-CERVANTES [1], **(Member, IEEE)**,
LEONEL GERMÁN CORONA-RAMÍREZ [2],
FRANCISCO CUENCA-JIMÉNEZ [3],
AND ROBERTO CASTRO-MEDINA [1]

[1] Instituto Politécnico Nacional (IPN), Centro de Innovación y Desarrollo Tecnológico en Cómputo (CIDETEC), Mexico City 07700, Mexico
[2] Instituto Politécnico Nacional (IPN), Unidad Profesional Interdisciplinaria en Ingeniería y Tecnologías Avanzadas (UPIITA), Mexico City 07340, Mexico.
[3] Faculty of Engineering, Universidad Nacional Autónoma de México (UNAM), Mexico City 04510, Mexico

Corresponding author: Miguel Gabriel Villarreal-Cervantes (mvillarrealc@ipn.mx)

This work was supported in part by the Secretaría de Investigación y Posgrado (SIP) under Project 20210374 and Project 20210314, and in part by the Comisión de Operación y Fomento de Actividades Académicas (COFAA) of the Instituto Politécnico Nacional (IPN). The work of Luis Ernesto Valencia-Segura was supported by the Mexican Consejo Nacional de Ciencia y Tecnología (CONACyT) through a scholarship to pursue graduate studies at IPN.

ABSTRACT In this paper, the dimensional synthesis of the four-bar mechanism for path generation is formulated using the relative angle motion analysis and the link geometry parameterization with Cartesian coordinates. The Optimum Dimensional Synthesis using Relative Angles and the Cartesian space link Parameterization (ODSRA+CP) is stated as an optimization problem, and the solution is given by the differential evolution variant DE/best/1/bin. This study investigates the behavior and performance of such formulation and performs a comparative empirical study with the well-known synthesis method based on vector-loop equation motion analysis where different modifications in the metaheuristic algorithms are established in the literature to improve the obtained solution. Five study cases of dimensional synthesis for path generation with and without prescribed timing are solved and analyzed. The empirical results show that the way of stating the optimization problem in the ODSRA+CP significantly improves the search process for finding promising solutions in the optimizer without requiring algorithm modifications. Therefore, it is confirmed that the optimizer search process in the optimal synthesis of mechanisms is not the only way of improving the obtained solutions, but also the optimization problem formulation has a significant influence on the search for better solutions.

INDEX TERMS Mechanism synthesis, four-bar mechanism, optimization, differential evolution.

I. INTRODUCTION

One of the most recurrent mechanisms in the development of machines and systems is the four-bar mechanism since it can be used for endless tasks. Some examples where these mechanisms are applied are in [1], where the four-bar mechanism is used to design an exoskeleton that helps in gait rehabilitation. Likewise, in [2], this mechanism is used to generate tasks related to the natural movement of the upper limb to use in the rehabilitation process of patients. Other applications of the four-bar mechanism are in the design of the under-actuated mechanical fingers to perform natural movements, and self-adaptive grips [3], and in the transplantation of rice seedlings [4]. In all previously mentioned examples, the dimensional synthesis process is carried out in the four-bar mechanism to be able to use it in the specific application.

The four-bar mechanism can be synthesized to carry out tasks of generation function, motion, and path [5]. In this work, the synthesis of four-bar mechanisms for path

The associate editor coordinating the review of this manuscript and approving it for publication was Huaqing Li.

generation is of interest to determine the dimensions of the mechanism. Through a finite number of points, named desired or precision points, the representation of a discrete path can be followed by a position in the coupler link, named coupler point.

The dimensional synthesis problem of four-bar mechanisms has been tackled by different methods, including graphical methods through atlases or catalog of mechanisms [6], which can provide a quick solution with low accuracy. Another strategy is through analytical methods, such as the harmonic analysis of closed-loop equations, where the dimension of the search space is reduced by approximating the generated path in Fourier series [7], or through a combination of graphical and analytical methods such as for the synthesis of mechanisms with and without prescribed timing [8].

The graphical and analytical methods for the mechanism synthesis have limitations in the accuracy and complexity of the path to be followed. In the case of the analytical method, it is limited to five precision points [9]. When there are a large number of precision points, the path generation problem becomes over-constrained, and to find mechanisms that produce the desired path is a difficult task [10].

The exponential growth in the information processing capacity of computers has promoted the optimization method [11] in the dimensional synthesis of mechanisms. In this method, an optimization problem must be formulated and solved through numerical procedures. In recent years, metaheuristic algorithms (MA) have been used to provide the solution to the mechanism synthesis problem [12]–[14]. The MA can converge to promising solutions within a complex search space (non-linear, discontinuous, etc.); likewise, they are not dependent on the problem characteristics to be solved, and they can be endowed with different search approaches to improve their performance.

Research related to the optimal synthesis of four-link mechanisms with the optimization method has been divided into two large groups. In the first one are the investigations that focus on the development and formulation of the optimization problem, where the mathematical formalism is described to obtain the requirements of the problem, such as the design objective, the inherent constraints of the synthesis, and the design parameters [15]–[19]. Once the problem is stated, it is solved with some state-of-the-art optimization technique. The second group is related to the research that concentrate on achieving modifications to optimization techniques in order to find better design solutions in the mechanism synthesis problem [9], [20]–[28]. The main motivation of modifying optimization techniques is that the complexity of the optimization problems produces multiple optimal local solutions [20]. So, the optimization techniques could get stuck in local regions (therefore obtaining solutions that are not sufficiently precise).

Some contributions related to the first group are presented next. In [15], the concept of orientation structural error of the fixed link is incorporated in the synthesis of four-bar mechanisms with the crank-rocker configuration, and the problem is solved with the Genetic Algorithm (GA). In [16], the use of the finite element method is proposed to perform the mechanism synthesis. The GA is implemented to find a solution. In [17], the circular proximity function is incorporated for the synthesis of four-bar mechanisms. The optimization problem is solved with the Differential Evolution (DE) algorithm. Other investigations found in this group are those related to applications of the mechanism optimal synthesis. In [18], the dimensional synthesis of the crank-slide mechanism, the Ross-Yoke mechanism, and the Rhombic mechanism of a Stirling engine are considered in order to maximize the output work. A comparative study is carried out between the GA, the Particle Swarm Optimization (PSO) algorithm, and the Imperialist Competitive Algorithm (ICA). In [19], an eight-bar mechanism is proposed to be used as a bipedal limb. The necessary elements for the optimization problem are formalized to solve it with the DE algorithm.

In the second group, there are works such as the one presented in [20] where it uses a combined sequential scheme of the Ant Colony Optimization (ACO) algorithm with the gradient search algorithm to find solutions in the dimensional synthesis problem of the four-bar mechanism for hybrid tasks including path, motion and function generation. Other works related to modifications in the optimization technique for the four-bar mechanism synthesis are presented in [9]. In that work, an exploitation strategy is incorporated into the DE algorithm. The strategy involves the inclusion of a local search method based on the Lagrange interpolation in the neighborhood of the best individual. Likewise, combinations of PSO and GA have been used in [21], combinations of the Random Coordinate search Algorithm (RCA) with the Taguchi method is proposed in [22], hybridization of the DE with GA is presented in [23], as well as combinations of different DE variants is included in [24]. Other works incorporate modifications to the original algorithm, such as for the Krill Herd algorithm [25] and DE [26]. Furthermore, proposals for self-tuning of the optimizer parameters, such as for the differential evolution algorithm [27]. In addition, new optimization techniques such as the Teaching-Learning-Based Optimization (TLBO) algorithm [28] have been developed to solve the mechanism synthesis problem.

In the works related to the first group of investigations, they focus mainly on the way to carry out the synthesis of mechanisms. Once the synthesis problem is stated, the second group focuses on performing algorithm modifications to solve it and find better design solutions. The similarity in the works related to the second group is the use of loop closure-equations included in the optimal synthesis problem to determine the kinematic motion; and the link parameterization is set in terms of the magnitude and the direction of the vector formed by the link. For convenience, in the following, this synthesis problem is called in this paper as Optimum Dimensional Synthesis using Vector-Loop Equation and the Vector space link Parameterization (ODSVLE+VP). Nevertheless, the kinematic synthesis problem statement using the relative angle method for the kinematic motion and the link parameterization in Cartesian coordinates is not addressed in the previous works. In addition, to the best author's knowledge, the optimization problem formulation in the mechanism synthesis problem could significantly influence in the search for solutions.

For this reason, the main motivation of this work is to show with empirical results that the ODSRA+CP for the path generation of four-bar mechanisms can promote an efficient search in the optimizer without requiring algorithm modifications. Five study cases of kinematic synthesis reported in the literature are considered with the purpose of comparing the performance of the ODSRA+CP with the ODSVLE+VP. The comparative results are presented and analyzed according to the result reliability, the obtained solution, and the similarity with those solutions obtained in previous works. Also, the advantages and disadvantages of the ODSRA+CP are discussed for showing the usefulness of the kinematic synthesis based on the relative angle method and the link parameterization in Cartesian coordinates.

The organization of this work is as follows. The kinematic analysis using the relative angle method is presented in Section II. In Section III, the ODSRA+CP approach is formally established as an optimization problem. In addition, the differential evolution algorithm is also described. The analysis and discussion of comparative results for the ODSRA+CP approach with respect to the ODSVLE+VP approach are presented in section IV. Finally, the conclusions are drawn in Section V.

II. KINEMATIC ANALYSIS THROUGH RELATIVE JOINT ROTATION ANGLES

In the optimal synthesis of the four-bar mechanism, the motion kinematic analysis is required. In this case, this analysis incorporates the use of the relative joint rotation angle method [29].

The four-bar mechanism is presented in Fig. 1. The vectors $a_0 = [a_{0x}, a_{0y}]^T$, $\boldsymbol{b}_0 = [b_{0x}, b_{0y}]^T$, $f = [f_x, f_y]^T$ and $\boldsymbol{s} = [s_x, s_y]^T$ are the coordinates or points of the joints in the Cartesian space with respect to the coordinate system $x - y$. The coupler point of the mechanism is defined by $\boldsymbol{p}_0 = [p_{0x}, p_{0y}]^T$. The length of the four-bar mechanism's links can be seen in Fig. 1, where r_1 represents the ground link's length, r_2 the input link or crank, r_3 the coupler link and r_4 the output link. On the other hand, r_5 represents the length of the link that joins the coupler point $\boldsymbol{p}_0$ with the joint coordinate $\boldsymbol{a}_0$ of the coupler link, therefore the lengths r_3 and r_5 belong to a rigid body. The links are defined by vectors at their ends, e.g., the vector related to the length r_2 of the input link (crank) is defined as $(\boldsymbol{a}_0 - \boldsymbol{f})$, and therefore its magnitude (length) is $r_2 = \|\boldsymbol{a}_0 - \boldsymbol{f}\|$. It is considered that the initial position of the four-bar mechanism is in the joint coordinates $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{f}$, $\boldsymbol{s}$ and $\boldsymbol{p}_0$. From this initial position, θ_j represents the $j - th$ angular displacement of the crank relative to the initial crank position.

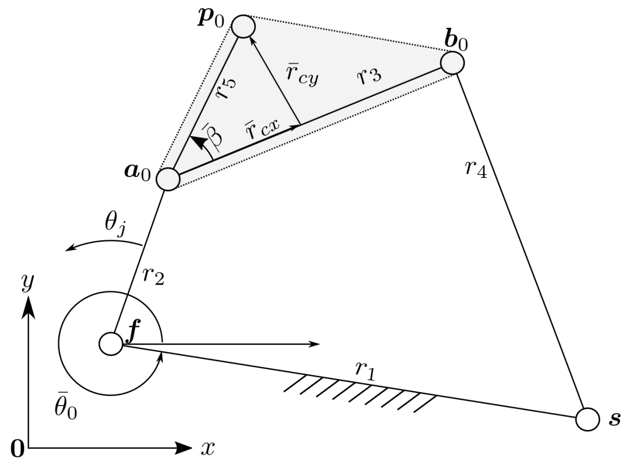


FIGURE 1. Initial position of the four-bar mechanism.

Let $\boldsymbol{a}_j$ be the $j - th$ next position from $\boldsymbol{a}_0$, that results from a rotation of the vector $(\boldsymbol{a}_0 - \boldsymbol{f})$ around the point $\boldsymbol{f}$ with the relative angle θ_j rad, as shown in Fig. 2(a). This displacement is mathematically described in (1), where $[R_{\theta_j}] = [R_{\theta_j,\hat{\mathrm{u}}}]$ is a rotation matrix in two-dimensional space of the unit vector û of the rotation axis by an angle θ_j.

$$\boldsymbol{a}_j = [R_{\theta_j}](\boldsymbol{a}_0 - \boldsymbol{f}) + \boldsymbol{f} \tag{1}$$

For the particular case, this matrix is similar to the rotation matrix around the z-axis considering $\hat{\boldsymbol{u}} = [0, 0, 1]^T$. In the case of spatial four-bar mechanism, the rotation matrix in three-dimensional space [30] results,

$$[R_{\theta,\hat{\boldsymbol{u}}}] = [I - Q_{\hat{\boldsymbol{u}}}]\cos(\theta) + [P_{\hat{\boldsymbol{u}}}]\sin(\theta) \tag{2}$$

where

$$[Q_{\hat{\boldsymbol{u}}}] = \begin{bmatrix} \hat{u}_x & \hat{u}_x\hat{u}_y & \hat{u}_x\hat{u}_z \\ \hat{u}_x\hat{u}_y & \hat{u}_y & \hat{u}_y\hat{u}_z \\ \hat{u}_x\hat{u}_z & \hat{u}_y\hat{u}_z & \hat{u}_z \end{bmatrix}$$
$$[P_{\hat{\boldsymbol{u}}}] = \begin{bmatrix} 0 & -\hat{u}_z & \hat{u}_y \\ \hat{u}_z & 0 & -\hat{u}_x \\ -\hat{u}_y & \hat{u}_x & 0 \end{bmatrix} \tag{3}$$

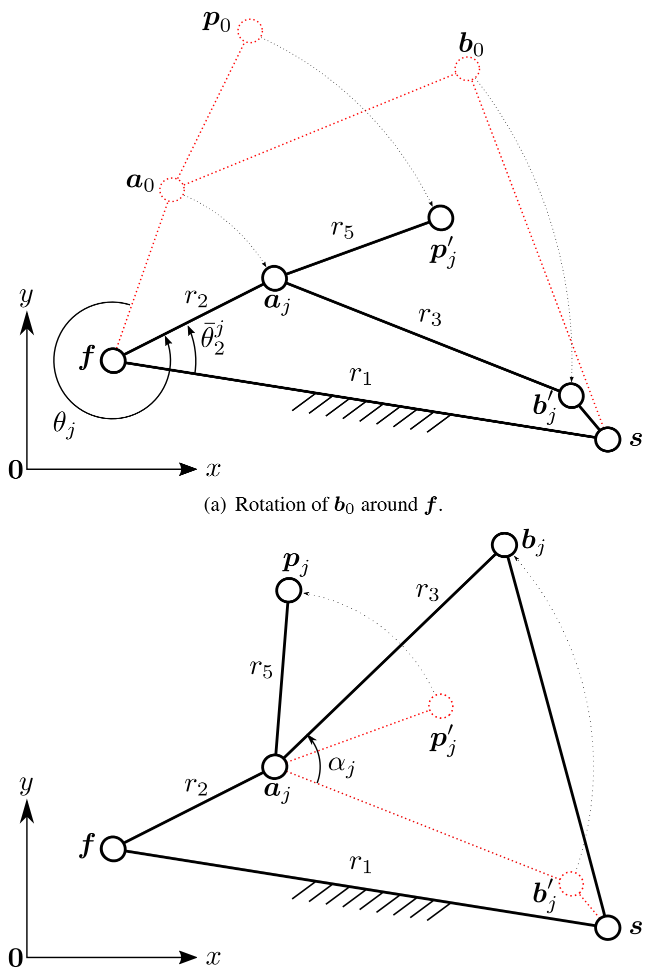
Also, let $\boldsymbol{b}_j$ be the $j - th$ next position of $\boldsymbol{b}_0$, as it can be observed in Fig. 2. Then, the position $\boldsymbol{b}_j$ is obtained as follows: the initial vector $(\boldsymbol{b}_0 - \boldsymbol{f})$ is rotated θ_j rad around the point $\boldsymbol{f}$ from this relative position, resulting in a new vector $\boldsymbol{b}'_j$. This new vector can be obtained in (4), and the graphical representation of the relative angular displacement is shown in Fig. 2(a). The mechanism with dotted red lines represents the initial position in that figure, while the mechanism with black lines is the new position with the aforementioned rotation.

$$\boldsymbol{b}'_j = [R_{\theta_j}](\boldsymbol{b}_0 - \boldsymbol{f}) + \boldsymbol{f} \tag{4}$$

Once $\boldsymbol{b}'_j$ has been obtained, it is possible to know the next position of $\boldsymbol{b}_0$, named $\boldsymbol{b}_j$, by making a rotation of the vector

$(\boldsymbol{b}'_j - \boldsymbol{a}_j)$ around the point $\boldsymbol{a}_j$ with an angle of α_j rad. This relative displacement from the reference vector $(\boldsymbol{b}'_j - \boldsymbol{a}_j)$ is shown in Fig. 2(b), and the equation that relates this movement is given in (5).

$$\boldsymbol{b}_j = [R_{\alpha_j}](\boldsymbol{b}'_j - \boldsymbol{a}_j) + \boldsymbol{a}_j \tag{5}$$



(a) Rotation of $\boldsymbol{b}_0$ around $\boldsymbol{f}$.

(b) Rotation of $\boldsymbol{b}'_j$ around $\boldsymbol{a}_j$.

FIGURE 2. **Kinematic movement of the four-bar mechanism.**

On the other hand, the following procedure must be carried out to find the angle α_j. In the first instance, the length of the output link r_4 must ensure to be a rigid body through the movement of θ_j. Then, it is necessary to guarantee that the vector distance in $(\boldsymbol{b}_0 - \boldsymbol{s})$ is the same as the vector $(\boldsymbol{b}_j - \boldsymbol{s})$, i.e., the following equation must be satisfied,

$$(\boldsymbol{b}_j - \boldsymbol{s})^T(\boldsymbol{b}_j - \boldsymbol{s}) = (\boldsymbol{b}_0 - \boldsymbol{s})^T(\boldsymbol{b}_0 - \boldsymbol{s}) \tag{6}$$

In the second instance, the expression in (6) must be expressed in terms of the angle α_j through the Freudenstein's equation [31]. So, substituting (5) in (6) results,

$$\begin{aligned}(\boldsymbol{b}'_j - \boldsymbol{a}_j)^T(\boldsymbol{b}'_j - \boldsymbol{a}_j) + 2(\boldsymbol{a}_j - \boldsymbol{s})^T[R_{\alpha_j}](\boldsymbol{b}'_j - \boldsymbol{a}_j)&\\ + (\boldsymbol{a}_j - \boldsymbol{s})^T(\boldsymbol{a}_j - \boldsymbol{s}) = (\boldsymbol{b}_0 - \boldsymbol{s})^T(\boldsymbol{b}_0 - \boldsymbol{s})&\end{aligned} \tag{7}$$

Considering

$$\begin{aligned}(\boldsymbol{a}_j - \boldsymbol{s})^T[R_{\alpha_j}](\boldsymbol{b}'_j - \boldsymbol{a}_j) = {}& (a_{jx} - s_x)[\cos(\alpha_j)(b'_{jx}\\ & - a_{jx}) - \sin(\alpha_j)(b'_{jy} - a_{jy})]\\ & + (a_{jy} - s_y)[\sin(\alpha_j)(b'_{jx}\\ & - a_{jx}) + \cos(\alpha_j)(b'_{jy} - a_{jy})]\end{aligned} \tag{8}$$

and substituting (8) in (7), it is possible to obtain (9).

$$\begin{aligned}&\cos(\alpha_j)((a_{jx} - s_x)(b'_{jx} - a_{jx}) + (a_{jy} - s_y)(b'_{jy} - a_{jy}))\\ &\quad + \sin(\alpha_j)((a_{jy} - s_y)(b'_{jx} - a_{jx})\text{-}(a_{jx} - s_x)(b'_{jy} - a_{jy}))\\ &\quad - \frac{1}{2}((\boldsymbol{b}_0 - \boldsymbol{s})^T(\boldsymbol{b}_0 - \boldsymbol{s})\text{-}(\boldsymbol{b}'_j - \boldsymbol{a}_j)^T(\boldsymbol{b}'_j - \boldsymbol{a}_j)\\ &\quad - (\boldsymbol{a}_j - \boldsymbol{s})^T(\boldsymbol{a}_j - \boldsymbol{s})) = 0\end{aligned} \tag{9}$$

Since (9) is in terms of angles α_j, it can be expressed as the Freudenstein's equation in (10).

$$E\cos(\alpha_j) + F\sin(\alpha_j) + G = 0 \tag{10}$$

where:

$$\begin{aligned}E &= (a_{jx} - s_x)(b'_{jx} - a_{jx}) + (a_{jy} - s_y)(b'_{jy} - a_{jy})\\ F &= (a_{jy} - s_y)(b'_{jx} - a_{jx})\text{-}(a_{jx} - s_x)(b'_{jy} - a_{jy})\\ G &= -\frac{1}{2}((\boldsymbol{b}_0 - \boldsymbol{s})^T(\boldsymbol{b}_0 - \boldsymbol{s})\text{-}(\boldsymbol{b}'_j - \boldsymbol{a}_j)^T(\boldsymbol{b}'_j - \boldsymbol{a}_j)\\ &\quad - (\boldsymbol{a}_j - \boldsymbol{s})^T(\boldsymbol{a}_j - \boldsymbol{s}))\end{aligned} \tag{11}$$

It is important to note that the solution of the Freudenstein's equation in (10) can produce imaginary solutions when $E^2 + F^2 - G^2 < 0$, resulting in mechanisms that cannot be assembled for the angle θ_j. Otherwise, two solutions are obtained for the angle α_j (α_j^1 and α_j^2) when $E^2 + F^2 - G^2 \geq 0$, and these solutions are expressed in (12) $\forall\, i = 1, 2$.

$$\alpha_j^i = 2atan2\left(-F + (-1)^i\sqrt{E^2 + F^2 - G^2}, (G - E)\right) \tag{12}$$

Once the two possible solutions for the angle α_j have been obtained, the one closer to the previous position α_{j-1} of the mechanism remains. Therefore, this condition can be stated as follows: whether $\left|\alpha_j^1 - \alpha_{j-1}\right| < \left|\alpha_j^2 - \alpha_{j-1}\right|$, then $\alpha_j = \alpha_j^1$, otherwise $\alpha_j = \alpha_j^2$. It is important to point out, that the computation of the relative angles α_{j-1}, α_j^1 and α_j^2 considers the reference (initial) position $(\boldsymbol{b}_0 - \boldsymbol{a}_0)$ of the coupler link r_3, as it can be seen in red dot line in Fig. 3. Hence, the initial angle $\alpha_0 = 0$ is set. The main issue in the above condition is to find the angle α_j of $(\boldsymbol{b}_j - \boldsymbol{a}_j)$ from α_j^i (12) closer to the previous position $(\boldsymbol{b}_{j-1} - \boldsymbol{a}_{j-1})$ of the coupler link r_3 with the angle α_{j-1}. So, the following procedure is done for setting the previous condition.

- Let the unit vectors of $(\boldsymbol{b}_{j-1} - \boldsymbol{a}_{j-1})$ and $(\boldsymbol{b}_j^i - \boldsymbol{a}_j^i)$ be described by $(\cos(\alpha_{j-1}), \sin(\alpha_{j-1}))$ and $(\cos(\alpha_j^i), \sin(\alpha_j^i))$ relative to the initial position $(\boldsymbol{b}_0 - \boldsymbol{a}_0)$ (coordinate system $x_{r_3} - y_{r_3}$ in Fig. 3), respectively.
- Using the al-Kashi's theorem [32] in such unit vectors, the minimum angle $\bar{\alpha}^i$ $\forall\, i = 1, 2$ between the vectors

$(\boldsymbol{b}_{j-1} - \boldsymbol{a}_{j-1})$ and $(\boldsymbol{b}_j^i - \boldsymbol{a}_j^i)$ can be computed as (13), shown at the bottom of the page.
- So, the next condition is included to select the angle α_j: If $\bar{\alpha}^1 < \bar{\alpha}^2$, then $\alpha_j = \alpha_j^1$, otherwise $\alpha_j = \alpha_j^2$.

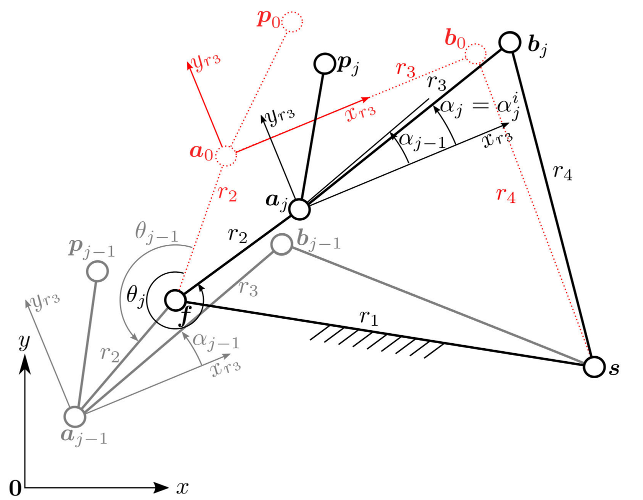


FIGURE 3. Schematic diagram of two movements (θ_{j-1} and θ_j) in the four-bar mechanism.

In order to know the next position of the coupler point $\boldsymbol{p}_j$ due to the $j-th$ angular displacement of the crank θ_j, a similar procedure to that described for $\boldsymbol{b}_j$, is required. Due to $\boldsymbol{p}_0$ and $\boldsymbol{b}_0$ are part of the same rigid body, the movement equations will affect the vectors $\boldsymbol{p}_j$ and $\boldsymbol{b}_j$ in the same way, as it can be observed in Fig. 2. As a result, the position $\boldsymbol{p}_j$ is defined in (14).

$$\boldsymbol{p}_j = [R_{\alpha_j}](\boldsymbol{p}_j' - \boldsymbol{a}_j) + \boldsymbol{a}_j \tag{14}$$

where:

$$\boldsymbol{p}_j' = [R_{\theta_j}](\boldsymbol{p}_0 - \boldsymbol{f}) + \boldsymbol{f} \tag{15}$$

Thus, the relative angles method focuses in determining the angle α_j to know the mechanisms' position with respect to the input relative angle θ_j. To know the complete path, described by the coupler point $\boldsymbol{p}_j$ $\forall\ j = 1, \ldots, n$, it is necessary to calculate (14) for all input angles $\theta_j \in [0, 2\pi]$.

III. DIMENSIONAL SYNTHESIS OF FOUR-BAR MECHANISMS FOR PATH GENERATION

The dimensional synthesis of the four-bar mechanisms for path generation is tackled in this work through the Optimum Dimensional Synthesis using Relative Angles and the Cartesian space link Parameterization (ODSRA+CP). In this approach, a non-linear constrained optimization problem is formulated with the use of the relative angle method for the motion analysis and the parameterization of the design variable vector in the Cartesian coordinates of the links. Section III-A formally establishes the optimization problem for the ODSRA+CP approach, and Section III-B describes the optimization technique to solve it.

A. OPTIMIZATION PROBLEM STATEMENT

The ODSRA+CP for the four-bar mechanism is established as a constrained optimization problem. This consists of finding the length of the links expressed in Cartesian coordinates which allow to follow n precision points $\boldsymbol{q}_j$ (desired points) by the coupler point $\boldsymbol{p}_j$ $\forall\, j = 1, 2, \ldots, n$, subject to inherent design constraints. The formal formulation of the optimization problem is presented below.

$$\min_{\boldsymbol{x}} J(\boldsymbol{x}) \tag{16}$$

$$\text{Subject to}: \quad g_i(\boldsymbol{x}) \leq 0, \quad \forall \quad i = 1, 2, \ldots, m$$
$$h_i(\boldsymbol{x}) = 0, \quad \forall \quad i = 1, 2, \ldots, k \tag{17}$$

with the design variable bounds:

$$\boldsymbol{x}_{min} \leq \boldsymbol{x} \leq \boldsymbol{x}_{max} \tag{18}$$

In the next sections, the elements of the optimization problem are detailed.

1) OBJECTIVE FUNCTION

The design objective shown in (19), is related to the quadratic error of the distance between the precision point $\boldsymbol{q}_j$ and the coupler point $\boldsymbol{p}_j$ $\forall\, j = 1, 2, \ldots, n$.

$$J = \sum_{j=1}^{n} \left[(\boldsymbol{q}_j - \boldsymbol{p}_j)^T (\boldsymbol{q}_j - \boldsymbol{p}_j) \right] \tag{19}$$

2) DESIGN VARIABLES

The links of the four-bar mechanism are parameterized in the Cartesian coordinates of the joints in the kinematic analysis using the relative angle method. So, the Cartesian coordinates of the link joints in the initial position given by $\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{f}, \boldsymbol{s}$ and $\boldsymbol{p}_0$ are included as the design variable vector. In addition, the crank relative angles θ_j $\forall\, j = 1, 2, \ldots, n$ are incorporated, where n is the number of precision points in the trajectory.

The design variables are grouped in the vector $\boldsymbol{x} \in \mathbb{R}^{10+n}$ (20) when the kinematic synthesis problems are without prescribed timing.

$$\boldsymbol{x} = \left[a_{0x}, a_{0y}, b_{0x}, b_{0y}, f_x, f_y, s_x, s_y, p_{0x}, p_{0y}, \right.$$
$$\left. \times\ \theta_1, \theta_2, \ldots, \theta_n \right] \tag{20}$$

Otherwise, whether the kinematic synthesis problem is with prescribed timing, the input crank relative angles are

$$\bar{\alpha}^i = \cos^{-1}\left(\frac{2 - \|(\cos(\alpha_{j-1}) - \cos(\alpha_j^i), \sin(\alpha_{j-1}) - \sin(\alpha_j^i))\|^2}{2} \right) \tag{13}$$

already introduced, such that those angles are removed from the vector $\boldsymbol{x}$. Then, the design variable vector results as $\boldsymbol{x} \in \mathbb{R}^{10}$ (21).

$$\boldsymbol{x} = \left[a_{0x}, a_{0y}, b_{0x}, b_{0y}, f_x, f_y, s_x, s_y, p_{0x}, p_{0y}\right] \tag{21}$$

3) CONSTRAINTS

The kinematic motion of the four-bar mechanism requires the computation of the angle α_j expressed in (12). In order to avoid complex numbers in the solution of α_j, the argument of the square root rewritten in (12) must be a positive real number. So, the constraint given in (22) is included to guarantee the kinematic motion. This constraint is considered a hard one [33], and this must be evaluated before the computation of the objective function. Whether the constraint is not feasible, the objective function will take a very high value without continuing its evaluation.

$$g_1(\boldsymbol{x}) : -(E^2 + F^2 - G^2) \leq 0 \tag{22}$$

Another constraint involves the fulfillment of the Grashof criterion [34] that allows a complete movement of one link of the four-bar mechanism. In this case, the link r_2 was selected as the crank one, resulting in a crank-rocker four-bar mechanism. Therefore, the Grashof criterion (23)-(25) is included as inequality constraints in the optimization problem. The larger and shorter lengths in such a criterion are established as the links r_1 and r_2.

$$g_2(\boldsymbol{x}) : \quad r_2 + r_1 - r_3 - r_4 \leq 0 \tag{23}$$
$$g_3(\boldsymbol{x}) : \quad r_2 + r_3 - r_1 - r_4 \leq 0 \tag{24}$$
$$g_4(\boldsymbol{x}) : \quad r_2 + r_4 - r_1 - r_3 \leq 0 \tag{25}$$

On the other hand, it is necessary for the crank link angles $\theta_j \; \forall \, j = 1, 2, \ldots, n$ to satisfy the movement in a counterclockwise direction. This is achieved by fulfilling the inequality equations expressed in (26). These constraints are only included when the kinematic synthesis problems are without prescribed timing.

$$g_{4+k}(\boldsymbol{x}) : \theta_k - \theta_{k+1} \leq 0 \quad \forall \, k = 1, 2, \ldots, n-1 \tag{26}$$

In addition, the link lengths of the mechanism are bounded according to the application. They are stated as inequality constraints in (27)-(34), where the $i-th$ minimum and maximum link lengths are represented as $r_{i_{min}}$ and $r_{i_{max}}$, respectively.

$$g_{n+4}(\boldsymbol{x}) : \quad \|\boldsymbol{s} - \boldsymbol{f}\| - r_{1_{max}} \leq 0 \tag{27}$$
$$g_{n+5}(\boldsymbol{x}) : \quad -\|\boldsymbol{s} - \boldsymbol{f}\| + r_{1_{min}} \leq 0 \tag{28}$$
$$g_{n+6}(\boldsymbol{x}) : \quad \|\boldsymbol{a}_0 - \boldsymbol{f}\| - r_{2_{max}} \leq 0 \tag{29}$$
$$g_{n+7}(\boldsymbol{x}) : \quad -\|\boldsymbol{a}_0 - \boldsymbol{f}\| + r_{2_{min}} \leq 0 \tag{30}$$
$$g_{n+8}(\boldsymbol{x}) : \quad \|\boldsymbol{b}_0 - \boldsymbol{a}_0\| - r_{3_{max}} \leq 0 \tag{31}$$
$$g_{n+9}(\boldsymbol{x}) : \quad -\|\boldsymbol{b}_0 - \boldsymbol{a}_0\| + r_{3_{min}} \leq 0 \tag{32}$$
$$g_{n+10}(\boldsymbol{x}) : \quad \|\boldsymbol{b}_0 - \boldsymbol{s}\| - r_{4_{max}} \leq 0 \tag{33}$$
$$g_{n+11}(\boldsymbol{x}) : \quad -\|\boldsymbol{b}_0 - \boldsymbol{s}\| + r_{4_{min}} \leq 0 \tag{34}$$

The last constraint, named box ones, involves that the design variables are inside an interval settled down by the designer or by the application. These constraints are shown in (35), where $\boldsymbol{x}_{min}$ and $\boldsymbol{x}_{max}$ represent the inferior and superior limits of the design variable vector.

$$\boldsymbol{x}_{min} \leq \boldsymbol{x} \leq \boldsymbol{x}_{max} \tag{35}$$

B. EVOLUTIONARY OPTIMIZER

The Differential Evolution algorithm (DE) [35] is used to solve the ODSRA+CP approach. The DE algorithm is an evolutionary algorithm that can be considered as a stochastic search method. DE has been used to solve a great variety of optimization problems in engineering because it presents an excellent approximation to a good solution in reasonable time [36]. Likewise, it is easy to implement; it presents great adaptability to different types of optimization problems (nonlinear, discontinuous), it has few parameters to tune, among other advantages.

The DE algorithm consists of four stages: initialization, mutation, crossover, and selection. In the literature, many variants of DE exist. In the current work, the variant DE/best/1/bin is implemented because this is the most competitive approach in a diverse set of benchmark problems with different characteristics [37]. This DE variant is explained below.

In the initialization, a set of NP possible solutions is randomly generated $x^{i,G} \; \forall \; i = 1, 2, \ldots, NP$ to the synthesis problem in the interval $[\boldsymbol{x}_{min}, \boldsymbol{x}_{max}]$. Each solution is called parent individual and the set of them is called parent population. The parent vector consists of D elements associated to the design variables $\boldsymbol{x}$. Thus, each design variable described in Section III-A2 is represented by $\boldsymbol{x}_j^{i,G} \; \forall \, j = 1, 2, \ldots, D$.

Once the initial population of parents is obtained, the mutation and recombination operators are applied, with the purpose of producing a new population of NP vectors called offspring. In particular, the differential mutation is used, as shown in (36). The vector difference in (36) is randomly perturbed through the scale factor $F \in [0, 1]$, which controls the population's change rate. The superscripts $\bar{r}_0$, $\bar{r}_1$, $\bar{r}_2$ indicate different individuals. The individual $\boldsymbol{x}^{\bar{r}_0,G}$ is obtained from the best (most apt) individual in the population. The indexes $\bar{r}_1$ and $\bar{r}_2$ are randomly determined from the population and different from the $i-th$ individual.

$$\boldsymbol{v}^{i,G} = \boldsymbol{x}^{\bar{r}_0,G} + F(\boldsymbol{x}^{\bar{r}_1,G} - \boldsymbol{x}^{\bar{r}_2,G}) \tag{36}$$

Likewise, the uniform crossover (37) is used to recombine the information of the vectors generated in the mutation (mutant vector) with the parent vectors. The crossover creates new solutions (offspring vectors $u^{i,G}$), and the crossover parameter $CR \in [0, 1]$ influences the probability of information transfer from the mutant vector or the parent one. In this work, the crossover parameter is constant through the algorithm execution.

$$u_j^{i,G} = \begin{cases} v_j^{i,G} & \textbf{if } (rand(0,1) < CR \textbf{ or } j = j_{rand}) \\ x_j^{i,G} & \textbf{otherwise} \end{cases} \tag{37}$$

Once the offspring vector $u^{i,G}$ is created, this competes with its parent $x^{i,G}$ with the purpose of preserving the information of the best individual (a better solution in the synthesis problem) to the next generation. This is summarized in (38).

$$\boldsymbol{x}^{i,G+1} = \begin{cases} \boldsymbol{u}^{i,G} & \textbf{if } J\left(\boldsymbol{u}^{i,G}\right) \leq J\left(\boldsymbol{x}^{i,G}\right) \\ \boldsymbol{x}^{i,G} & \textbf{otherwise} \end{cases} \tag{38}$$

Because the original algorithm of DE is used to solve unconstrained optimization problems, several strategies have been proposed for solving constrained optimization problems. One of the most frequent strategies used in mechanism synthesis is the transformation of the constrained optimization problem into an unconstrained optimization problem by penalty functions and then solving through the indirect methods [38], [39]. In the case presented in this work, the optimization problem is not transformed, and the constraints are directly handled through the Deb' feasibility rules [40]. These rules define the individual (solution in the synthesis problem) that best adapt to the problem (the solution that solves the problem better). The criterion for removing infeasible solutions is through Deb's feasibility rules, and it is established as follows:

1) Any feasible solution is preferred over an infeasible one.
2) Among two feasible solutions, the one with the best objective function is preferred.
3) Between two infeasible solutions, the one with the least distance ϕ (39) of violated constraints is preferred.

$$\phi = \sum_{j=1}^{n+11} max\left(0, g_j\left(\boldsymbol{x}\right)\right) \tag{39}$$

Finally, the pseudocode of DE/best/1/bin is presented in the Algorithm 1 for the dimensional synthesis problem of mechanisms.

IV. RESULTS

In this section, the ODSRA+CP approach is applied to five study cases of the dimensional synthesis of four-bar mechanisms for path generation. These study cases are commonly studied in the literature using different methodologies and algorithms. The obtained results are compared with the most common approach given in the state of art. The common approach is called in this paper as Optimal Dimensional Synthesis using Vector-Loop Equation and the Vector space link Parameterization (ODSVLE+VP). The main difference of the ODSVLE+VP approach is the use of the loop closure-equations to determine the kinematic motion in the optimal synthesis problem, as well as the parameterization of the design variable space from the magnitude and direction of the link vectors.

Algorithm 1 DE/Best/1/Bin

Begin
    $G \leftarrow 1$
    Create a random initial population $x^{i,G}$ $\forall$ $i = 1, \ldots, NP$.
    Evaluate $J(x^{i,G})$ $\forall$ $i = 1, \ldots, NP$ (Algorithm 2).
    Evaluate constraints $\phi(x^{i,G})$ $\forall$ $i = 1, \ldots, NP$ (39).
    while $G \leq G_{max}$ **do**
        for $i \leftarrow 1 to NP$ **do**
            Create vector $u^{i,G}$ (37).
            Evaluate $J(u^{i,G})$ (Algorithm 2).
            Evaluate constraints $\phi(u^{i,G})$ (39).
            Select between $x^{i,G}$ and $u^{i,G}$ using Deb's rules.
        end for
        $G \leftarrow G + 1$
    end while
End

Algorithm 2 Evaluation of the Objective Function

function [J]=Objective_function($\boldsymbol{a}$)
Input: $\boldsymbol{a}$
Output: J
Begin
    Calculate r_1, r_2, r_3, r_4.
    $\alpha_0 \leftarrow 0$
    $J \leftarrow 0$
    for $j \leftarrow 1$ **to** n **do**
        Calculate a_j (1).
        Calculate b_j' (4).
        Calculate p_j' (15).
        Calculate hard constraint $g_1(\mathrm{x})$ (22).
        if $g_1(\mathrm{x}) > 0$ **then**
            $J \leftarrow 10000$
            Break
        end if
        Calculate α_j^i from (12) and $\bar{\alpha}^i$ from (13).
        if $\bar{\alpha}^1 < \bar{\alpha}^2$ **then**
            $\alpha_j \leftarrow \alpha_j^1$
        else
            $\alpha_j \leftarrow \alpha_j^2$
        end if
        Calculate p_j (14).
    end for
    Calculate J (19).
End

In the first part of each subsection below, the variant DE/best/1/bin with other three DE variants solve the ODSRA+CP approach presented in this work. The performance results of the DE variants for solving the ODSRA+CP approach are analyzed based on statistics to evaluate the effectiveness of the variant DE/best/1/bin and to find the best result per each study case. The comparative variants consider binomial and exponential crossover as well as the use of random and the best individuals in the mutation process, as in the variants DE/rand/1/bin, DE/rand/1/exp and DE/best/1/exp. The algorithm tuning of the DE variants for the ODSRA+CP approach is set through a trial and error process. This process consists of systematically applying different configurations of *CR* and *F* values at each study case considering ten percent of changes. In the case of the scale factor *F*, the selected current configuration is decreased and increased at 10% of the maximum value of *F* to set its minimum and maximum values of the scale factor interval. The configuration that provides the best (minimum) objective function is contemplated in the optimization process. In order to make fair comparisons among DE variants, the same number of evaluations of the objective function is taken into account for the optimizers. The corresponding evaluation number is obtained from the reported works of the study cases in the four-bar mechanism dimensional synthesis for path generation. Besides, the limits in the parameters associated with the study cases are considered from those reported works. The optimization processes for the study cases are performed in a PC with 1.6GHz Intel Core(TM) i5 processor and 12GB of RAM, and programmed in Matlab© software.

In the second part of each subsection below, the best solution previously obtained is compared with the solution obtained by the ODSVLE+VP approach. The main goal is to show that with changes in the optimization problem through the ODSRA+CP approach applied to the four-bar mechanisms, it is possible to improve the obtained results in the state of the art (ODSVLE+VP) without modifying the optimizer. In the case of the comparative ODSVLE+VP approach, the optimization problems in the kinematic synthesis were regularly solved by proposing modifications in the optimization algorithms, and those results were compared with other works in the state of the art. The algorithm changes in the comparative ODSVLE+VP approach improve their search capabilities, and so, these achieve better results than those obtained from studies in which they made comparisons. In this work, in the study cases presented below, the best-reported algorithm for each particular study case is used in the ODSVLE+VP approach for making the comparative analysis. As it is mentioned in the first part, both approaches include the same in both the number of evaluations of the objective function and the limits in the parameters associated with the study cases to make fair comparisons and at the same time to describe the advantages and drawbacks of the ODSRA+CP approach with respect to the ODSVLE+VP approach. Finally, for comparative purposes, the design variables in the ODSRA+CP approach (Cartesian space link parameterization) are mapped into the corresponding in ODSVLE+VP approach (vector space link parameterization) to show their similarities and differences.

In the next subsections, each study case is described, and the comparative results are discussed.

A. STUDY CASE 1: SIX PRECISION POINTS WITHOUT PRESCRIBED TIMING

The first study case is reported in [13]. In this case, the coupler point of the four-bar mechanism must pass through six precision points aligned with a vertical straight line and without prescribed timing. In (40), the precision points are presented.

$$\boldsymbol{q}=[(20,20),(20,25),(20,30),(20,35),(20,40),(20,45)]\ [\text{mm}] \tag{40}$$

The design variables are shown in (41), and their limits are displayed in Table 1. Also, the limits on the link lengths proposed in [13] for the inequality constraints (27)-(34), are presented in the same Table.

$$\boldsymbol{x} = [a_{0x}, a_{0y}, b_{0x}, b_{0y}, f_x, f_y, s_x, s_y, p_{0x}, p_{0y}, \times \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6] \in \mathbb{R}^{16} \tag{41}$$

TABLE 1. Limits of design variables and the minimum/maximum link dimensions for the study case 1.

Parameter	a_{0x} [mm]	a_{0y} [mm]	b_{0x} [mm]	b_{0y} [mm]	f_x [mm]	f_y [mm]	s_x [mm]
$\boldsymbol{x}_{min}$	-100	-100	-100	-100	-60	-60	-100
$\boldsymbol{x}_{max}$	100	100	100	100	60	60	100
Parameter	s_y [mm]	p_{0x} [mm]	p_{0y} [mm]	θ_1 [rad]	θ_2 [rad]	θ_3 [rad]	θ_4 [rad]
$\boldsymbol{x}_{min}$	-100	-100	-100	0	0	0	0
$\boldsymbol{x}_{max}$	100	100	100	2π	2π	2π	2π
Parameter	θ_5 [rad]	θ_6 [rad]					
$\boldsymbol{x}_{min}$	0	0					
$\boldsymbol{x}_{max}$	2π	2π					
Length	r_1 $i=1$ [mm]	r_2 $i=2$ [mm]	r_3 $i=3$ [mm]	r_4 $i=4$ [mm]			
$r_{i_{min}}$	5	5	5	5			
$r_{i_{max}}$	60	60	60	60			

For the solution of the optimization problem in the ODSRA+CP approach with the DE variants, the following algorithm parameters are considered: A population size of 100 individuals and a generation number of 1000. The crossover factor *CR* is set per execution and the random scale factor *F* is fixed per generation randomly chosen in an interval. Those factors are displayed in Table 2.

TABLE 2. Crossover and scale factors for DE variants of the study case 1.

	DE/best/1/bin	DE/best/1/exp	DE/rand/1/bin	DE/rand/1/exp
CR	0.8	0.8	0.4	0.4
F	$[0.4, 0.6]$	$[0.85, 1]$	$[0.8, 1]$	$[0.8, 1]$

Thirty executions of the algorithm are performed and the best values of the objective functions per each DE variant are stored. Those data are the sample for making the descriptive and inferential statistics. The boxplots in logarithmic scale of those data are shown in Fig. 4 and the numerical results are presented in Table 3. The columns of such table represents the mean (Mean(J)), the standard deviation (σ(J)), the best solution (Best(J)) and the worse one (Worst(J)). The best solution is obtained by DE/best/1/bin with a performance function value of $J = 8.6622e - 06$. The standard deviation indicates that a diversity of solutions exists at the end of the executions per each DE variant and so, the algorithms converge to different local optimum solutions. For the particular best solution given by the variant DE/best/1/bin, this implies that the worst solution (100) has a percentage difference of 99.99% in the objective function value with regards to the best value ($8.6622e - 06$).

TABLE 3. Descriptive statistics of the obtained solutions by using DE variants in the study case 1.

Algorithm	Mean(J)	σ(J)	Best(J)	Worst(J)
ED/best/1/bin	10.174	21.193	$\mathbf{8.6622e - 6}$	100.00
ED/best/1/exp	10.952	38.887	$4.0826e - 3$	209.68
ED/rand/1/bin	**6.6811**	**8.5302**	$7.5350e - 4$	**25.065**
ED/rand/1/exp	54.913	35.272	5.2963	139.61

In order to know the general behavior (general conclusion) of the comparative results of DE variants, the non-parametric inferential statistical test of Friedman for multiple comparisons [41] are presented in Table 4 to carry out an accurate pairwise comparisons. Boldface indicates the winner with two-tailed hypothesis test and a 5% of significance level. It is observed that DE/best/1/bin, DE/best/1/exp and DE/rand/1/bin win with respect to the DE/rand/1/exp. On the contrary, when those former three algorithms compete, there are not enough evidence to guarantee a better performance. So, according to the number of wins, the DE/best/1/bin, DE/best/1/exp and DE/rand/1/bin are the most promising optimizers for the study case 1.

TABLE 4. Friedman test for the multiple comparison tests among all DE variants for the study case 1.

Hypotesis	p-value
DE/best/1/bin vs DE/best/1/exp	5.7433E-02
DE/best/1/bin vs DE/rand/1/bin	6.8916E-01
DE/best/1/bin vs DE/rand/1/exp	**1.1981E-08**
DE/best/1/exp vs DE/rand/1/bin	1.3361E-01
DE/best/1/exp vs DE/rand/1/exp	**1.4470E-04**
DE/rand/1/bin vs DE/rand/1/exp	**1.1580E-07**

On the other hand, the evolution of the objective function value J in logarithmic scale through generations is shown in Fig. 5 for the particular result of the best solution given by DE/best/1/bin. It is observed that in around 600 generations (60% of the maximum generation number), the algorithm converges to a solution. This indicates that the proposed generation maximum number in [13] is useful in the DE/best/1/bin algorithm because it can search for a suitable solution in the synthesis problem.

In Table 5, the design variables of the best solution found in the thirty executions among DE variants are shown.

The mechanism described by the best solution is shown in Fig. 6.

With the purpose of comparing the performance of the mechanism obtained through the ODSRA+CP approach,

TABLE 5. Design variables of the best solution given by DE/best/1/bin for the study case 1.

Design variables	
a_{0x}[mm]	-40.3810
a_{0y}[mm]	70.4760
b_{0x}[mm]	-11.0760
b_{0y}[mm]	-1.9011
f_x[mm]	-27.7229
f_y[mm]	1.3008
s_x[mm]	-31.9141
s_y[mm]	25.6686
p_{0x}[mm]	18.0487
p_{0y}[mm]	-4.3324
θ_1[rad]	1.3716
θ_2[rad]	1.6530
θ_3[rad]	1.9388
θ_4[rad]	2.2294
θ_5[rad]	2.5293
θ_6[rad]	2.8521

the mechanism obtained by the ODSVLE+VP approach given in [13] is used. In the ODSVLE+VP approach, the Imperialist Competitive Algorithm (ICA) was used to solve the associated optimization problem for the study case 1, and the results indicated that ICA shows an outstanding result with regards to the Genetic Algorithm(GA), Particle Swarm Optimization (PSO), Parallel simulated annealing and Differential Evolution (DE) in its variant DE/rand/1/exp.

In Table 6, the Cartesian coordinate design variables of links found in this work are transformed into to link lengths r_i and angles $\bar{\beta}$, $\bar{\theta}_0$, $\bar{\theta}_j^2$ (see Fig. 1 and Fig. 2) to carry out a correlation analysis with respect to the obtained mechanism in the ODSVLE+VP approach, as well as, to evaluate its performance. It is observed that the objective function with the mechanism obtained by the ODSRA+CP approach is significantly reduced in a 99.56% with reference to the one in the ODSVLE+VP approach, using the same number of objective function evaluations. Also, the design obtained in this work presents a Pearson correlation coefficient of 0.3672, indicating that the mechanism has significant differences between them. It is important to clarify that the Pearson correlation coefficient is applied to the parameters associated with the dimension of the mechanism in Table 6 without considering the crank angles $\bar{\theta}_2^j$.

B. STUDY CASE 2: FIVE PRECISION POINTS WITH PRESCRIBED TIMING

This study case is taken from [26]. Here, the dimensional synthesis of the four-bar mechanism requires that the coupler point generates a curve path that passes through five precision points with prescribed timing. In this case, the origin of the inertial coordinate system is located at the point $\boldsymbol{f}$ with the x axis collinear to the direction of link r_1, as a consequence, the parameters $f_x = 0, f_y = 0$ and $s_y = 0$ are set. Furthermore, the link r_2 is collinear to the link r_1 at the beginning, as a consequence $a_{0y} = 0$. The precision points are described in (42) with the crank angles (43).

$$\begin{aligned} \boldsymbol{q} = [&(3.000, 3.000), (2.759, 3.363), (2.372, 3.663), \\ &\times (1.890, 3.862), (1.355, 3.943)][\text{mm}] \end{aligned} \quad (42)$$

TABLE 6. Objective function and design parameters for the study case 1 with both approaches.

Parámetros	ODSVLE+VP ICA [13]	ODSRA+CP DE/best/1/bin
r_1[mm]	60	24.7256
r_2[mm]	17.51102	12.6722
r_3[mm]	60	29.4207
r_4[mm]	33.2268	34.5590
r_5[mm]	60	58.6465
β[rad]	4.2153	0.0027
f_x[mm]	60	-27.7229
f_y[mm]	-1.45505	1.3009
θ_0[rad]	0.93032	1.7411
$\bar{\theta}_2^1$[rad]	5.54638	2.8192
$\bar{\theta}_2^2$[rad]	5.70544	3.1006
$\bar{\theta}_2^3$[rad]	5.85061	3.3863
$\bar{\theta}_2^4$[rad]	5.99041	3.6770
$\bar{\theta}_2^5$[rad]	6.13185	3.9769
$\bar{\theta}_2^6$[rad]	6.28318	4.2996
J	0.002	8.6622e-06

$$[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5] = \left[\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{\pi}{2}\right] \tag{43}$$

The design variable vector is stated in (44).

$$\boldsymbol{x} = \left[a_{0x}, b_{0x}, b_{0y}, s_x, p_{0x}, p_{0y}\right] \in \mathbb{R}^6 \tag{44}$$

The design variable bounds and the minimum and maximum link dimensions are presented in Table 7.

TABLE 7. Limits of design variables and the minimum/maximum link dimensions for the study case 2.

Parameter	a_{0x} [mm]	b_{0x} [mm]	b_{0y} [mm]	s_x [mm]	p_{0x} [mm]	p_{0y} [mm]
$\boldsymbol{x}_{min}$	-50	-50	-50	-50	-50	-50
$\boldsymbol{x}_{max}$	50	50	50	50	50	50
Length	r_1 $i=1$ [mm]	r_2 $i=2$ [mm]	r_3 $i=3$ [mm]	r_4 $i=4$ [mm]		
$r_{i_{min}}$	0	0	0	0		
$r_{i_{max}}$	50	50	50	50		

The DE variant parameters for the solution of the optimization problem in the ODSRA+CP approach is set as: a population size of 50 individuals with a generation number of 100 and so, the maximum number of objective function evaluations is 5000. The crossover CR and scale F factors are chosen accordingly to Table 8.

TABLE 8. Crossover and scale factors for DE variants of the study case 2.

	DE/best/1/bin	DE/best/1/exp	DE/rand/1/bin	DE/rand/1/exp
CR	0.9	0.8	0.2	0.2
F	[0.4, 0.6]	[0.3, 0.4]	[0.8, 1.0]	[0.7, 0.9]

Thirty runs of the algorithms are executed, and the descriptive and inferential statistics are carried out by using the thirty best results. The boxplot in logarithmic scale of the objective functions for all DE variants is shown in Fig. 4 and the numerical results are presented in Table 9. The best solution is provided by DE/best/1/bin, which has a performance of $J = 7.6675e - 07$. It is also observed that the obtained solutions with DE variants in the ODSRA+CP approach have several local optimum ones due to the high value of the standard deviation. For the particular best solution given by variant DE/best/1/bin, the worst solution has a performance of $J = 2.4404$, which represents a high difference between the best and the worst values of 99.99%.

TABLE 9. Descriptive statistics of the obtained solutions by using DE variants in the study case 2.

Algorithm	Mean(J)	σ(J)	Best(J)	Worst(J)
ED/best/1/bin	$8.9875e-2$	0.44409	$\mathbf{7.667e-7}$	2.4404
ED/best/1/exp	$\mathbf{5.238e-3}$	$\mathbf{6.851e-3}$	$8.1786e-7$	$\mathbf{2.274e-2}$
ED/rand/1/bin	59.068	77.344	$2.2221e-3$	216.79
ED/rand/1/exp	2.8382	13.692	$2.5010e-4$	75.139

In order to know the general conclusion of the DE variant comparison, the non-parametric Friedman test for the multiple comparisons are presented in Table 10. Boldface indicates the winner with two-tailed hypothesis test and a 5% of significance level. It is observed that DE/best/1/bin and DE/best/1/exp win two times, DE/rand/1/exp wins one times. So, according to the number of wins, the DE/best/1/bin and DE/best/1/exp are the most promising optimizers for the study case 2, followed by DE/rand/1/exp.

TABLE 10. Friedman test for the multiple comparison tests among all DE variants for the study case 2.

Hypotesis	p-value
DE/best/1/bin vs DE/best/1/exp	4.8393E-01
DE/best/1/bin vs DE/rand/1/bin	**6.6641E-08**
DE/best/1/bin vs DE/rand/1/exp	**1.2419E-02**
DE/best/1/exp vs DE/rand/1/bin	**1.0607E-09**
DE/best/1/exp vs DE/rand/1/exp	**1.3743E-03**
DE/rand/1/bin vs **DE/rand/1/exp**	**3.7316E-03**

On the other hand, the behavior of the objective function value in logarithmic scale for the best solution through generations of the particular result given by DE/best/1/bin is shown in Fig. 5(b). In the first eleven generations, the obtained solution is not feasible. As a consequence, and according to Deb's feasibility rules, the best solution is obtained by comparing the violated constraints. So, in the first eleven generations, the objective function value presents oscillations. Also, note that the algorithm converges to a solution in 90 generations (90% of the maximum generation number), indicating that the proposed generation number in [26] is adequate to search for promising solutions, i.e., in the last generations, the variation in the obtained solution is almost not carried out.

The design variables of the best found individual in the thirty executions are shown in Table 11.

The mechanism described by the parameters given in Table 11 is shown in Fig. 6(b).

TABLE 11. Design variables of the best solution given by DE/best/1/bin for the study case 2.

Design variables	
a_{0x}[mm]	1.9977
b_{0x}[mm]	5.4028
b_{0y}[mm]	2.0568
s_x[mm]	3.6537
p_{0x}[mm]	2.5594
p_{0y}[mm]	2.3029

In the comparative analysis presented next, the Malaga University Mechanism Synthesis Algorithm (MUMSA) [26] was used to solve the ODSVLE+VP approach for the study case 2. The MUMSA showed in [26] outstanding results with regards to the evolutionary algorithms DE/best/1/bin and GA for the same optimization problem described in study case 2.

In order to compare the results of the current work with the results of [26], the best link design in the ODSRA+CP approach expressed Cartesian coordinates are transformed into link lengths r_i, the angle $\bar{\theta}_j^2$ and the coupler point coordinates $\bar{r}_{cx}$, $\bar{r}_{cy}$ (see Fig. 1). The obtained results for both approaches are shown in Table 12. It is important to note that the obtained design with the ODSRA+CP approach improves around 56.62% with respect to the one obtained with the ODSVLE+VP approach. The correlation coefficient of the associated mechanism lengths in both approaches is 0.9999, implying that the solution obtained by the ODSVLE+VP approach is closer than the one in the ODSRA+CP approach. As a consequence, the ODSVLE+VP approach's solution may tend to the same results given in the ODSRA+CP approach whether an increment of objective function evaluations in MUMSA is set.

TABLE 12. Objective function and design parameters for the study case 2 with both approaches.

Parameters	ODSVLE+VP MUMSA [26]	ODSRA+CP DE/best/1/bin
r_1[mm]	3.77326856	3.6537
r_2[mm]	2.00000403	1.9977
r_3[mm]	4.11697104	3.9780
r_4[mm]	2.74615671	2.7000
r_{cx}[mm]	1.67848787	1.6677
r_{cy}[mm]	1.67098007	1.6844
J	1.7678e-06	7.6675e-07

C. STUDY CASE 3: TEN PRECISION POINTS WITHOUT PRESCRIBED TIMING

The third study case is reported in [27]. This presents the dimensional synthesis of the four-bar mechanism with ten precision points (45) which forms an elliptical closed path without prescribed timing.

$$\begin{aligned} \boldsymbol{q} = &[(20, 10), (17.66, 15.142), (11.736, 17.878), \\ &\times (5, 16.928), (0.603, 12.736), (0.603, 7.263), \\ &\times (5, 3.071), (11.736, 2.121), (17.660, 4.857), \\ &\times (20, 10)][\text{mm}] \end{aligned} \tag{45}$$

The design variables vector for this study case is shown in (46).

$$\begin{aligned} \boldsymbol{x} = &[a_{0x}, a_{0y}, b_{0x}, b_{0y}, f_x, f_y, s_x, s_y, p_{0x}, p_{0y}, \\ &\times \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10}] \in R^{20} \end{aligned} \tag{46}$$

The superior and inferior limits of the design variables vector are presented in Table 13. The link length bounds are also shown at the end of the same table.

TABLE 13. Limits of design variables and the minimum/maximum link dimensions for the study case 3.

Parameter	a_{0x} [mm]	a_{0y} [mm]	b_{0x} [mm]	b_{0y} [mm]	f_x [mm]	f_y [mm]	s_x [mm]
$\boldsymbol{x}_{min}$	-300	-300	-300	-300	-80	-80	-300
$\boldsymbol{x}_{max}$	300	300	300	300	80	80	300
Parameter	s_y [mm]	p_{0x} [mm]	p_{0y} [mm]	θ_1 [rad]	θ_2 [rad]	θ_3 [rad]	θ_4 [rad]
$\boldsymbol{x}_{min}$	-300	-80	-80	0	0	0	0
$\boldsymbol{x}_{max}$	300	80	80	2π	2π	2π	2π
Parameter	θ_5 [rad]	θ_6 [rad]	θ_7 [rad]	θ_8 [rad]	θ_9 [rad]	θ_{10} [rad]	
$\boldsymbol{x}_{min}$	0	0	0	0	0	0	
$\boldsymbol{x}_{max}$	2π	2π	2π	2π	2π	2π	
Length	r_1 $i=1$ [mm]	r_2 $i=2$ [mm]	r_3 $i=3$ [mm]	r_4 $i=4$ [mm]			
$r_{i_{min}}$	0	0	0	0			
$r_{i_{max}}$	80	80	80	80			

The parameters of the DE variants, which solves the ODSRA+CP approach, are described as follows: population size of 50 individuals and a maximum number of generations of 2000. This generation number is obtained based on the number of objective function evaluations described in [27] multiplied by the two trial populations. The crossover and scale factors are selected according to the Table 14.

TABLE 14. Crossover and scale factors for DE variants of the study case 3.

	DE/best/1/bin	DE/best/1/exp	DE/rand/1/bin	DE/rand/1/exp
CR	0.85	0.9	0.8	0.8
F	$[0.7, 0.8]$	$[0.5, 0.7]$	$[0.3, 0.5]$	$[0.1, 0.3]$

Thirty executions of the algorithm are done, and the box-plots of the descriptive statistics from the best solutions of the DE variants are presented in Fig. 4. The numerical results of the statistics are included in Table 15. The best solution is obtained by DE/best/1/bin with a performance function value of $5.7279e-4$. A high diversity of solutions exists at the end of the executions per each DE variant because the standard deviation presents a large value indicating that the algorithms converge to different local optimum solutions. For the particular best solution given by variant DE/best/1/bin, the worst solution has a value of 272.01 which results in a percentage difference between the best and the worst ones of 99.99%.

The non-parametric inferential statistical test of Friedman [41] are presented in Table 16 to confirm the performance in the multiple comparisons. Boldface indicates the winner with two-tailed hypothesis test and a 5% of significance level. It is observed that DE/best/1/bin and

TABLE 15. Descriptive statistics of the obtained solutions by using DE variants in the study case 3.

Algorithm	Mean(J)	σ(J)	Best(J)	Worst(J)
ED/best/1/bin	9.4233	49.622	$\mathbf{5.7279e-4}$	272.01
ED/best/1/exp	**1.8324**	**5.7592**	$5.8201e-4$	28.229
ED/rand/1/bin	3.0619	6.4400	$1.1201e-3$	**24.472**
ED/rand/1/exp	11.377	13.368	$2.3757e-2$	49.671

DE/best/1/exp wins two times, followed by DE/rand/1/bin with one win. So, according to the number of wins, DE/best/1/bin and DE/best/1/exp are the most promising optimizer for the study case 3, also there are not enough evidence to guarantee the best performance when they compete.

TABLE 16. Friedman test for the multiple comparison tests among all DE variants for the study case 3.

Hypotesis	p-value
DE/best/1/bin vs DE/best/1/exp	4.2371E-01
DE/best/1/bin vs DE/rand/1/bin	**3.1822E-04**
DE/best/1/bin vs DE/rand/1/exp	**2.1435E-08**
DE/best/1/exp vs DE/rand/1/bin	**5.1103E-03**
DE/best/1/exp vs DE/rand/1/exp	**1.5867E-06**
DE/rand/1/bin vs DE/rand/1/exp	**4.5500E-02**

On the other hand, the evolution of the objective function J in logarithmic scale for the best solution in the DE/best/1/bin is shown in Fig. 5(c). It is observed that in the first 286 generations, the obtained solution is not feasible. Consequently, the objective function value is oscillated due to Deb's feasibility rules prioritizing the reduction of the constraint violation. It is also noticed that the algorithm tends to continue descending at the end of the maximum number of generations (2000), indicating that the DE/best/1/bin algorithm may improve even more the obtained solution by having more generations.

The best mechanism design obtained by the ODSRA+CP approach with the use of DE/best/1/bin is presented in Table 17, and its graphical representation is given in Fig. 6(c).

The best mechanism found with the ODSRA+CP approach is compared with the one reported in [27], which uses the ODSVLE+VP approach. The Ingeniería Mecánica Málaga (IMMa) Optimization Algorithm gives the solution of the ODSVLE+VP approach with Self-Adaptive Technique (IOA^{s-at}). For this third study case, the IOA^{s-at} shows outstanding results with respect to the algorithms GA, PSO and DE.

The link design in the Cartesian coordinate space is converted to link lengths, and those are summarized in Table 18. Moreover, the objective function J values are presented in it. It is important to note that the performance function value (19) reported in [27] presents a slight difference between the calculated performance function in this work by using the same optimal values of [27]. The Pearson correlation coefficient between both designs is 0.9575. Thus, the correlation coefficient indicates a high relation between both mechanisms. Nevertheless, the ODSRA+CP approach improves at 97.43% the performance function with respect to the ODSVLE+VP approach.

TABLE 17. Design variables of the best solution given by DE/best/1/bin for the study case 3.

Design variables	
a_{0x}[mm]	17.2764
a_{0y}[mm]	14.5929
b_{0x}[mm]	-20.5521
b_{0y}[mm]	47.0719
f_x[mm]	11.5875
f_y[mm]	20.3173
s_x[mm]	-63.4658
s_y[mm]	47.7211
p_{0x}[mm]	16.8019
p_{0y}[mm]	4.14279
θ_1[rad]	0.8056
θ_2[rad]	1.5012
θ_3[rad]	2.2122
θ_4[rad]	2.9251
θ_5[rad]	3.6334
θ_6[rad]	4.3410
θ_7[rad]	5.0354
θ_8[rad]	5.7248
θ_9[rad]	0.1200
θ_{10}[rad]	0.8055

TABLE 18. Objective function and design parameters for the study case 3 with both approaches.

Parameters	ODSVLE+VP IOA^{s-at} [27]	ODSRA+CP DE/best/1/bin
r_1[mm]	65.4287	79.8997
r_2[mm]	8.0163	8.0705
r_3[mm]	47.2216	49.8586
r_4[mm]	44.1365	42.9186
r_{cx}[mm]	-11.5708	-6.4474
r_{cy}[mm]	-1.9049	8.2377
$\bar{\theta}_0$[rad]	3.5860	2.7915
f_x[mm]	10.6354	11.5875
f_y[mm]	-1.6754	20.3173
$\bar{\theta}_2^1$[rad]	2.4199	3.5088
$\bar{\theta}_2^2$[rad]	3.1092	4.2043
$\bar{\theta}_2^3$[rad]	3.8129	4.9153
$\bar{\theta}_2^4$[rad]	4.5064	5.6282
$\bar{\theta}_2^5$[rad]	5.1811	0.0534
$\bar{\theta}_2^6$[rad]	5.8834	0.7610
$\bar{\theta}_2^7$[rad]	0.2962	1.4554
$\bar{\theta}_2^8$[rad]	0.9911	2.1448
$\bar{\theta}_2^9$[rad]	1.7077	2.8232
$\bar{\theta}_2^{10}$[rad]	2.4188	3.5087
J	$2.2289e-2$	$5.7279e-4$

D. STUDY CASE 4: EIGHTEEN PRECISION POINTS

The fourth study case is presented in [23]. It involves the dimensional synthesis of the four-bar mechanism for a closed trajectory with 18 precision points (47) with prescribed timing. In the prescribed timing, the input angle θ_j for each precision point is separated 20° among two adjacent precision points, and the input angle is displayed in (48), where θ_1 is a design variable.

$$\boldsymbol{q} = [(0.5, 1.1), (0.4, 1.1), (0.3, 1.1),$$
$$\times (0.2, 1), (0.1, 0.9), (0.005, 0.75),$$

$$\times (0.02, 0.6), (0.0, 0.5), (0.0, 0.4),$$
$$\times (0.03, 0.3), (0.1, 0.25), (0.15, 0.2),$$
$$\times (0.2, 0.3), (0.3, 0.4), (0.4, 0.5),$$
$$\times (0.5, 0.7), (0.6, 0.9), (0.6, 1.)][\text{mm}] \quad (47)$$
$$\theta_j = [\theta_1 + 20^\circ(j-1)] \quad \forall\, j = 1, 2, \ldots, 18 \quad (48)$$

The design variables are presented in (49), while their limits are shown in Table 19. At the end of such a table, the link lengths are shown.

$$\boldsymbol{x} = [a_{0x}, a_{0y}, b_{0x}, b_{0y}, f_x, f_y, s_x, s_y, p_{0x}, p_{0y}, \theta_1] \in \mathbb{R}^{11} \quad (49)$$

TABLE 19. Limits of design variables and the minimum/maximum link dimensions for the study case 4.

Parameter	a_{0x} [mm]	a_{0y} [mm]	b_{0x} [mm]	b_{0y} [mm]	f_x [mm]	f_y [mm]	s_x [mm]
$\boldsymbol{x}_{min}$	-100	-100	-100	-100	-50	-50	-100
$\boldsymbol{x}_{max}$	100	100	100	100	50	50	100
Parameter	s_y [mm]	p_{0x} [mm]	p_{0y} [mm]	θ_1 [rad]			
$\boldsymbol{x}_{min}$	-100	-50	-50	0			
$\boldsymbol{x}_{max}$	100	50	50	2π			
Length	r_1 $i=1$ [mm]	r_2 $i=2$ [mm]	r_3 $i=3$ [mm]	r_4 $i=4$ [mm]			
$r_{i_{min}}$	0	0	0	0			
$r_{i_{max}}$	50	50	50	50			

The optimization problem of the ODSRA+CP approach is solved by DE variants. The algorithm parameters are stated as follows: population size of 400 individuals with 1000 generations giving a total of 400000 objective function evaluations. The crossover and scale parameters are presented in Table 20.

TABLE 20. Crossover and scale factors for DE variants of the study case 4.

	DE/best/1/bin	DE/best/1/exp	DE/rand/1/bin	DE/rand/1/exp
CR	0.9	0.85	0.95	0.9
F	$[0.5, 0.7]$	$[.5, .7]$	$[.3, .5]$	$[.3, .5]$

Thirty executions are made per each DE variants from which the best solutions are stored. The boxplots in logarithmic scale of the objective function value obtained from those executions are shown in Fig. 4 and the numerical results are presented in Table 21. Based on the standard deviation, the DE variants converge to different local optimum solutions through different executions. The best solution among DE variants has a performance value of $J = 9.9119e-3$ and it is given by DE/best/1/bin and DE/best/1/exp. The percentage difference between the best and the worst solutions provided by the DE/best/1/bin and DE/best/1/exp is 62.3%.

The non-parametric inferential statistical test of Friedman for the multiple comparisons [41] is presented in Table 22 to carry out an accurate pairwise comparisons. Boldface indicates the winner with two-tailed hypothesis test and a 5% of significance level. It is observed that DE/rand/1/bin wins one times with respect to DE/rand/1/exp. Also, there are not enough evidence to guarantee a better performance with the rest of the comparisons. So, according to the number of wins, the DE/rand/1/bin is the most promising optimizer for the study case 4. Nevertheless, this does not provide the minimum objective function.

TABLE 21. Descriptive statistics of the obtained solutions by using DE variants in the study case 4.

Algorithm	Mean(J)	σ(J)	Best(J)	Worst(J)
ED/best/1/bin	$1.7007e-2$	$6.130e-3$	$\mathbf{9.911e-3}$	$\mathbf{2.629e-2}$
ED/best/1/exp	$1.6847e-2$	$\mathbf{5.870e-3}$	$\mathbf{9.911e-3}$	$\mathbf{2.629e-2}$
ED/rand/1/bin	$\mathbf{1.590e-2}$	$5.9594e-3$	$1.0585e-2$	$\mathbf{2.629e-2}$
ED/rand/1/exp	$2.0755e-2$	$1.0202e-2$	$1.0587e-2$	$4.0373e-2$

TABLE 22. Friedman test for the multiple comparison tests among all DE variants for the study case 4.

Hypotesis	p-value
DE/best/1/bin vs DE/best/1/exp	1
DE/best/1/bin vs DE/rand/1/bin	3.6812E-01
DE/best/1/bin vs DE/rand/1/exp	1.9360E-01
DE/best/1/exp vs DE/rand/1/bin	3.6812E-01
DE/best/1/exp vs DE/rand/1/exp	1.9360E-01
DE/rand/1/bin vs DE/rand/1/exp	**2.7807E-02**

On the other hand, the behavior of the objective function J in logarithmic scale through generations for the optimizer that provides the best results (DE/best/1/bin) is shown in Fig. 5(d). This is obtained from the best solutions of the thirty executions. It is observed that in around 760 generations (76% of the maximum generation number), the algorithm converges to a solution. This indicates that the proposed generation maximum number in [23] is useful in the DE/best/1/bin algorithm because it can search for a suitable solution in the synthesis problem.

The design variables of the best solution found in thirty executions are shown in Table 23. The graphical solution of the mechanism is presented in Fig. 6(d).

TABLE 23. Design variables of the best solution given by DE/best/1/bin for the study case 4.

Design variables	
a_{0x}[mm]	0.6373
a_{0y}[mm]	-0.0574
b_{0x}[mm]	0.8222
b_{0y}[mm]	0.8689
f_x[mm]	0.2624
f_y[mm]	0.1439
s_x[mm]	1.2617
s_y[mm]	0.4936
p_{0x}[mm]	0.3263
p_{0y}[mm]	0.4009
θ_1[rad]	1.6673

The results of the study case 4 are compared with the one reported in [23]. In [23], a hybrid algorithm between the DE and the GA was proposed to provide the solution of the optimization problem in the ODSVLE+VP approach. The obtained results of the GA-DE algorithm outperform the results gave in the GA and the Exact Gradient for the study case 4.

In Table 24, the Cartesian coordinate design variables of links found in this work are transformed into to link lengths r_i, angle $\bar{\theta}_0$, and $\bar{\theta}_2^1$ (see Fig. 1 and Fig. 2) to carry out a correlation analysis with respect to the obtained mechanism in the ODSVLE+VP approach, as well as, to evaluate its performance. It is observed that the objective function with the mechanism obtained in the ODSRA+CP approach reduces at 8.74% with respect to the one in the ODSVLE+VP approach, using the same number of objective function evaluations. Also, the design obtained in this work presents a Pearson correlation coefficient of 0.5650, indicating that the mechanism has differences between them despite slightly reducing the objective function.

TABLE 24. Objective function and design parameters for the study case 4 with both approaches.

Parameters	ODSVLE+VP GA-DE [23]	ODSRA+CP DE/best/1/bin
r_1[mm]	47.4379	1.0587
r_2[mm]	0.32477	0.42567
r_3[mm]	0.472857	0.9447
r_4[mm]	47.3093	0.5779
r_{cx}[mm]	0.1187	0.3887
r_{cy}[mm]	-0.3199	0.3947
$\bar{\theta}_0$[rad]	0.526988	0.3366
f_x[mm]	0.72393	0.2624
f_y[mm]	3.32029	0.1440
$\bar{\theta}_2^1$[rad]	3.51233	0.8377
J	$1.08613e-2$	$9.9119e-3$

E. STUDY CASE 5: TWENTY PRECISION POINTS

The fifth study case is given in [24]. In this case, the coupler point of the four-bar mechanism must pass through twenty precision points, forming a ∞ shaped curve trajectory without prescribed timing. The precision points are presented in (50).

$$\begin{aligned} \boldsymbol{q} = \; & [(-24, 40), (-30, 41), (-34, 40), (-38, 36), \\ & \times (-36, 30), (-28, 29), (-21, 31), (-17, 32), \\ & \times (-8, 34), (3, 37), (10, 41), (17, 41), \\ & \times (26, 39), (28, 33), (29, 26), (26, 23), \\ & \times (17, 23), (11, 24), (6, 27), (0, 31)][\text{mm}] \end{aligned} \tag{50}$$

The design variables vector is shown in (51), and its limits are shown in Table 25. Besides, the bounds of the link lengths are shown at the end of the previously mentioned table.

$$\begin{aligned} \boldsymbol{x} = [& a_{0x}, a_{0y}, b_{0x}, b_{0y}, f_x, f_y, s_x, s_y, p_{0x}, p_{0y}, \\ & \times \theta_1, \ldots, \theta_{20}] \in R^{30} \end{aligned} \tag{51}$$

In order to solve the optimization problem in the ODSRA+CP approach with the DE variants, the following conditions are set: population size of 200 individuals with a maximum generation number of 1000 to establish the same evaluation number of the objective function as in [24]. The crossover and scale factors are set as in Table 26. In this case, the percentage of the selected current configuration is changed at 30% to find the interval of the scale factor F.

TABLE 25. Limits of design variables and the minimum/maximum link dimensions for the study case 5.

Parameter	a_{0x} [mm]	a_{0y} [mm]	b_{0x} [mm]	b_{0y} [mm]	f_x [mm]	f_y [mm]	s_x [mm]
$\boldsymbol{x}_{min}$	-130	-130	-130	-130	-120	-120	-130
$\boldsymbol{x}_{max}$	130	130	130	130	120	120	130
Parameter	s_y [mm]	p_{0x} [mm]	p_{0y} [mm]	θ_1 [rad]	θ_2 [rad]	θ_3 [rad]	θ_4 [rad]
$\boldsymbol{x}_{min}$	-130	-120	-120	0	0	0	0
$\boldsymbol{x}_{max}$	130	120	120	2π	2π	2π	2π
Parameter	θ_5 [rad]	θ_6 [rad]	θ_7 [rad]	θ_8 [rad]	θ_9 [rad]	θ_{10} [rad]	θ_{11} [rad]
$\boldsymbol{x}_{min}$	0	0	0	0	0	0	0
$\boldsymbol{x}_{max}$	2π	2π	2π	2π	2π	2π	2π
Parameter	θ_{12} [rad]	θ_{13} [rad]	θ_{14} [rad]	θ_{15} [rad]	θ_{16} [rad]	θ_{17} [rad]	θ_{18} [rad]
$\boldsymbol{x}_{min}$	0	0	0	0	0	0	0
$\boldsymbol{x}_{max}$	2π	2π	2π	2π	2π	2π	2π
Parameter	θ_{19} [rad]	θ_{20} [rad]					
$\boldsymbol{x}_{min}$	0	0					
$\boldsymbol{x}_{max}$	2π	2π					
Length	r_1 $i=1$ [mm]	r_2 $i=2$ [mm]	r_3 $i=3$ [mm]	r_4 $i=4$ [mm]			
$r_{i_{min}}$	5	5	5	5			
$r_{i_{max}}$	120	120	120	120			

TABLE 26. Crossover and scale factors for DE variants of the study case 5.

	DE/best/1/bin	DE/best/1/exp	DE/rand/1/bin	DE/rand/1/exp
CR	0.8	0.95	0.9	0.85
F	$[0.3, 0.9]$	$[0.4, 0.6]$	$[0.3, 0.5]$	$[0.5, 0.7]$

Thirty executions of the algorithms are accomplished. The set of data, consisting of the best values of the objective functions from the thirty executions of the DE variants, are used to make the descriptive and inferential statistics. The boxplots in logarithmic scale of the descriptive statistics are shown in Fig. 4, and the numerical results are presented in Table 27. The best solution has a performance function value of $J = 7.8123$ and it is obtained by DE/best/1/bin. The standard deviation indicates that a diversity of solutions exists at the end of the executions in the DE variants. For the particular best solution given by variant DE/best/1/bin, this implies that the worst solution ($J = 2666.2$) has a percentage difference of 99.7% in the objective function value with regards to the best value ($J = 7.8123$). Thus, the DE/best/1/bin algorithm becomes stagnant for different local optimum solutions.

TABLE 27. Descriptive statistics of the obtained solutions by using DE variants in the study case 5.

Algorithm	Mean(J)	σ(J)	Best(J)	Worst(J)
ED/best/1/bin	$\mathbf{3.9234e2}$	$6.4338e2$	$\mathbf{7.8123}$	$2.6662e3$
ED/best/1/exp	$6.1470e2$	$\mathbf{4.7512e2}$	$1.0356e1$	$\mathbf{1.5305e3}$
ED/rand/1/bin	$1.4282e3$	$1.1649e3$	$1.5547e1$	$4.7301e3$
ED/rand/1/exp	$6.9324e4$	$6.3335e4$	$7.6866e3$	$2.3681e5$

The non-parametric inferential statistical test of Friedman for the multiple comparisons [41] are presented in Table 28. Boldface indicates the winner with two-tailed hypothesis test and a 5% of significance level. It is observed that DE/best/1/bin wins two times, followed by DE/best/1/exp and

DE/rand/1/bin with one win. So, according to the number of wins, the DE/best/1/bin is the most promising optimizer for the study case 5.

TABLE 28. Friedman test for the multiple comparison tests among all DE variants for the study case 5.

Hypotesis	p-value
DE/best/1/bin vs DE/best/1/exp	5.7433E-02
DE/best/1/bin vs DE/rand/1/bin	**4.6526E-04**
DE/best/1/bin vs DE/rand/1/exp	**6.2172E-15**
DE/best/1/exp vs DE/rand/1/bin	1.0960E-01
DE/best/1/exp vs DE/rand/1/exp	**3.6350E-09**
DE/rand/1/bin vs DE/rand/1/exp	**1.7080E-05**

On the other hand, the evolution of the objective function value J in logarithmic scale through generations for the DE/best/1/bin algorithm is shown in Fig. 5(e). This is obtained from the best solution of the thirty executions. It is observed that in around 400 generations (40% of the maximum generation number), the algorithm converges to a solution. This indicates that the proposed generation maximum number in [24] is useful in the DE/best/1/bin algorithm because it can search for a suitable solution in the synthesis problem.

The design variables of the best-found solution through thirty executions are shown in Table 29.

TABLE 29. Design variables of the best solution given by DE/best/1/bin for the study case 5.

Design variables	
a_{0x}	-67.0585
a_{0y}	30.9648
b_{0x}	-54.7665
b_{0y}	-63.7984
f_x	-67.9772
f_y	1.3481
s_x	-6.3825
s_y	45.9694
p_{0x}	5.4024
p_{0y}	27.7306
θ_1	1.0256
θ_2	1.2816
θ_3	1.4909
θ_4	1.9153
θ_5	2.5976
θ_6	3.0206
θ_7	3.3079
θ_8	3.4535
θ_9	3.7444
θ_{10}	4.0800
θ_{11}	4.3126
θ_{12}	4.5295
θ_{13}	4.8457
θ_{14}	5.0752
θ_{15}	5.3458
θ_{16}	5.5098
θ_{17}	5.8800
θ_{18}	6.0791
θ_{19}	6.2570
θ_{20}	0.1862

The mechanism described in Table 29 is shown in Fig. 6(e).

With the purpose of comparing the performance of the mechanism obtained through the ODSRA+CP approach, the mechanism obtained by the ODSVLE+VP approach given in [24] is used. In the ODSVLE+VP approach, the Combined-Mutation Differential Evolution (CMDE) algorithm was used to solve the associated optimization problem for the study case 5, and the results indicated that CMDE shows an outstanding result with regards to the DE variant DE/best/I and other approach related to changes in the optimization problem formulation (coupler-angle function curve method).

TABLE 30. Objective function and design parameters for the study case 5 with both approaches.

Parameters	ODSVLE+VP DE/combined/1/ 10%–10%–80% [24]	ODSRA+CP DE/best/1/bin
r_1[mm]	83.6638	76.0590
r_2[mm]	33.5251	29.6309
r_3[mm]	120	95.5571
r_4[mm]	96.8159	119.9583
r_{cx}[mm]	46.751	12.5283
r_{cy}[mm]	-74.6397	71.4429
f_x[mm]	-88.7106	-67.9772
f_y[mm]	57.9453	1.3481
$\bar{\theta}_0$[rad]	6.10457	0.6269
$\bar{\theta}_2^1$[rad]	1.9764	1.9385
$\bar{\theta}_2^2$[rad]	2.233	2.1945
$\bar{\theta}_2^3$[rad]	2.4417	2.4038
$\bar{\theta}_2^4$[rad]	2.8701	2.8282
$\bar{\theta}_2^5$[rad]	3.5359	3.5105
$\bar{\theta}_2^6$[rad]	3.9573	3.9335
$\bar{\theta}_2^7$[rad]	4.2441	4.2208
$\bar{\theta}_2^8$[rad]	4.3895	4.3664
$\bar{\theta}_2^9$[rad]	4.6819	4.6573
$\bar{\theta}_2^{10}$[rad]	5.0187	4.9929
$\bar{\theta}_2^{11}$[rad]	5.2527	5.2255
$\bar{\theta}_2^{12}$[rad]	5.4707	5.4424
$\bar{\theta}_2^{13}$[rad]	5.7875	5.7586
$\bar{\theta}_2^{14}$[rad]	6.0159	5.9881
$\bar{\theta}_2^{15}$[rad]	0.0013	6.2587
$\bar{\theta}_2^{16}$[rad]	0.1648	1.1395
$\bar{\theta}_2^{17}$[rad]	0.5388	0.5097
$\bar{\theta}_2^{18}$[rad]	0.7398	0.7089
$\bar{\theta}_2^{19}$[rad]	0.9189	0.8868
$\bar{\theta}_2^{20}$[rad]	1.1331	1.0991
J	7.82071	7.81234

TABLE 31. Summary of the number of wins in the inferential statistics obtained per each study case with different optimizers in the ODSRA+CP approach.

Study case	Number of wins in the inferential statistics DE/best/1/bin	DE/best/1/exp	DE/rand/1/bin	DE/rand/1/exp
1	1	1	1	0
2	2	2	0	1
3	2	2	1	0
4	0	0	1	0
5	2	1	1	0
Total:	7	6	4	1

In Table 30, the Cartesian coordinate design variables of links found in this work are transformed into to link lengths r_i, the angle $\bar{\theta}_2^j$ and the coupler point coordinates given by $\bar{r}_{cx}$ and $\bar{r}_{cy}$ (see Fig. 1 and Fig. 2) to carry out a correlation analysis with respect to the obtained mechanism in the ODSVLE+VP approach, as well as, to evaluate

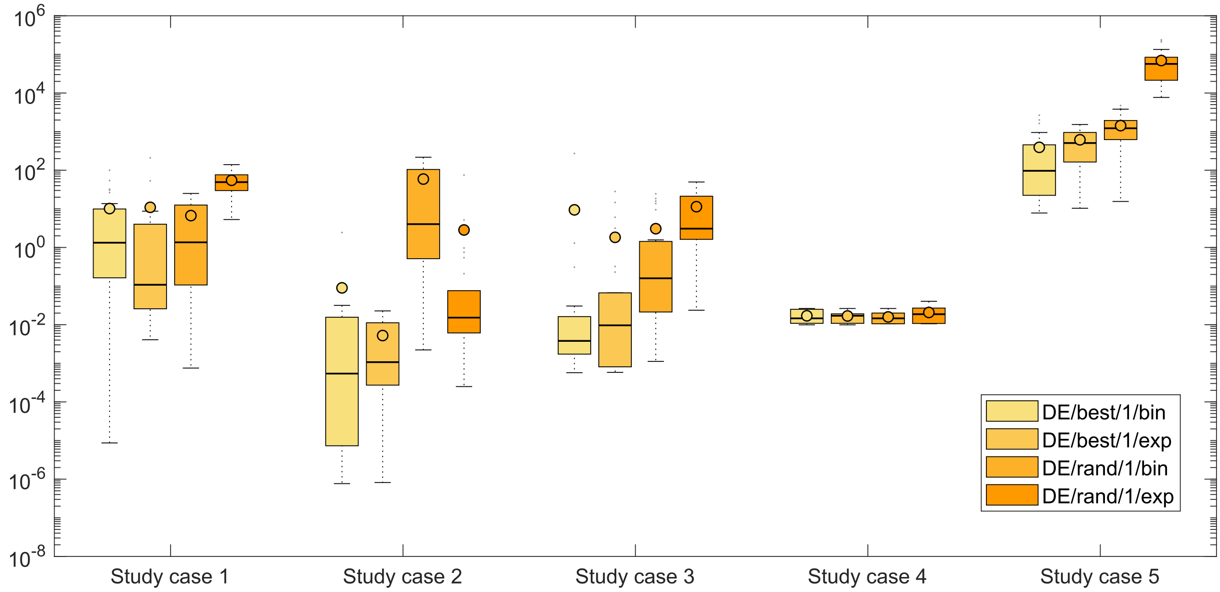


FIGURE 4. Boxplots of the thirty executions of the DE variants in the OSDRA+CP approach for each study case.

TABLE 32. Summary of the obtained mechanism and the algorithm performance in the ODSRA+CP approach compared with the ODSVLE+VP approach.

	Study case				
Obtained design	**1**	**2**	**3**	**4**	**5**
Pearson coefficient	0.3672	0.9999	0.9575	0.5650	0.6253
Improvement	99.56%	56.62%	97.43%	8.74%	0.1068%
Algorithm DE/best/1/bin					
Generation number for convergence	60%	90%	100%	76%	40%
Differences among executions	99.99%	99.99%	99.99%	62.3%	99.7%

its performance. It is observed that the objective function with the mechanism obtained in the ODSRA+CP approach is reduced in 0.1068% with respect to the one in the ODSVLE+VP approach, using the same number of objective function evaluations. Also, the design obtained in this work presents a Pearson correlation coefficient of 0.6253, indicating that the mechanism has differences between them.

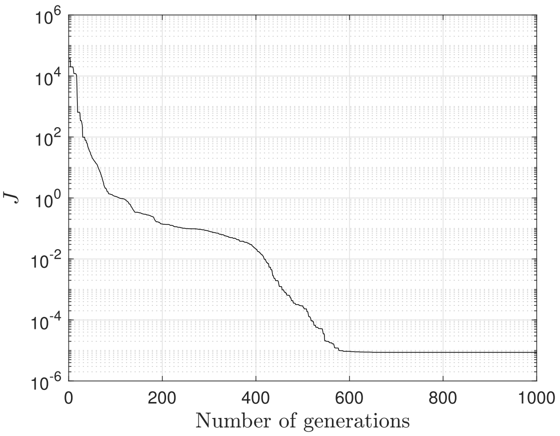
F. DISCUSSION

A summary of the number of wins for the DE variants applied to the solution of the ODSRA+CP approach is shown in Table 31. It is observed that the most promising optimizers include elitism strategies (the best solution in the mutation process), and the best of them is related to the DE/best/1/bin.

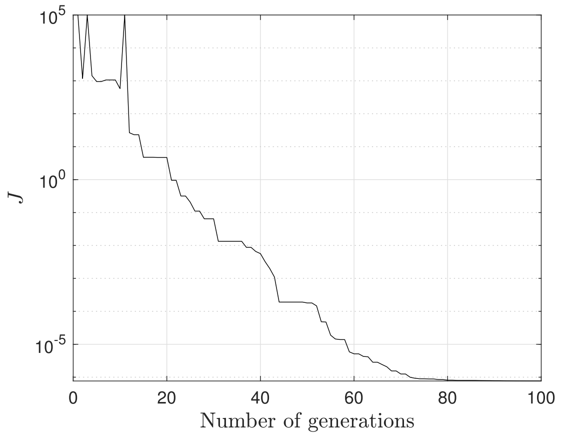
The similarity of the mechanisms and their corresponding improvements obtained by the ODSRA+CP approach with respect to the results reported in the ODSVLE+VP approach is summarized in Table 32. Furthermore, the percentage of the maximum number of generations to fulfill the convergence to a solution in the algorithm DE/best/1/bin is included in such table. Also, the percentage difference between the best and the worse performance function values obtained by the thirty executions per each study case is displayed. The graphical representation of the path traced by using the best four-bar mechanisms obtained by the ODSRA+CP and ODSVLE+VP approaches is shown in Fig. 7.

Through this empirical study, it is confirmed that the ODSRA+CP approach can improve, in all study cases, the obtained results reported in the literature (based on the ODSVLE+VP approach). In two cases (study cases 2 and 3), the found mechanisms are similar with respect to the ODSVLE+VP approach. Nevertheless, they reach the precision points with more accuracy than the ODSVLE+VP approach with an improvement of 56.62% and 97.31%, respectively. On the other hand, there is one case (study case 5), that in spite of confirming differences in the obtained mechanism, the improvement is low in around 0.1068%. In the rest of the cases (study cases 1 and 4), the obtained mechanisms are different, and the improvements are around 99.56% and 8.74%. This indicates that the way of stating the optimization problem in the kinematic synthesis of mechanisms positively influences the search for better solutions in the optimizer.

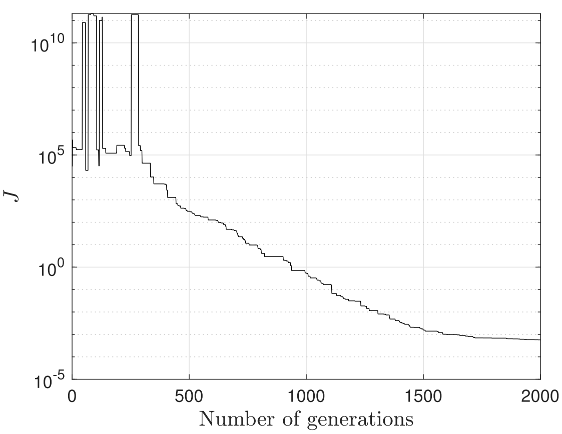
The confidence of finding the best solution in the optimizer (DE/best/1/bin) is given by setting different (thirty) executions to the same problem. It is confirmed that through executions, there is a significant difference of around $62\% - 99\%$ between the best and the worse solution at each



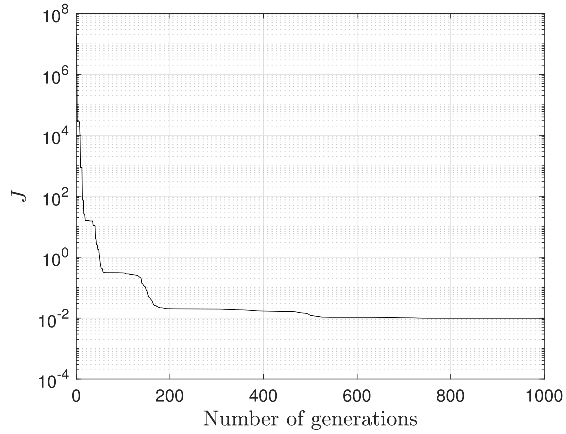
(a) Study case 1.



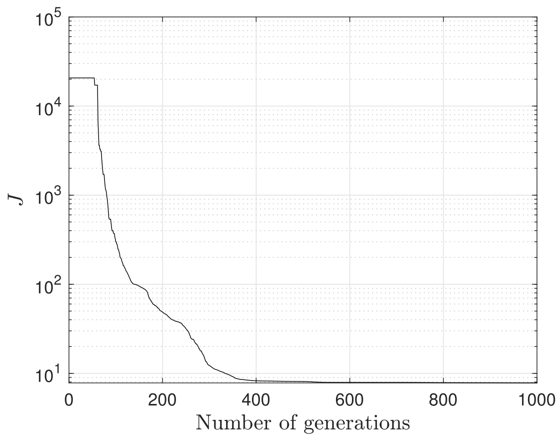
(b) Study case 2.



(c) Study case 3.



(d) Study case 4.

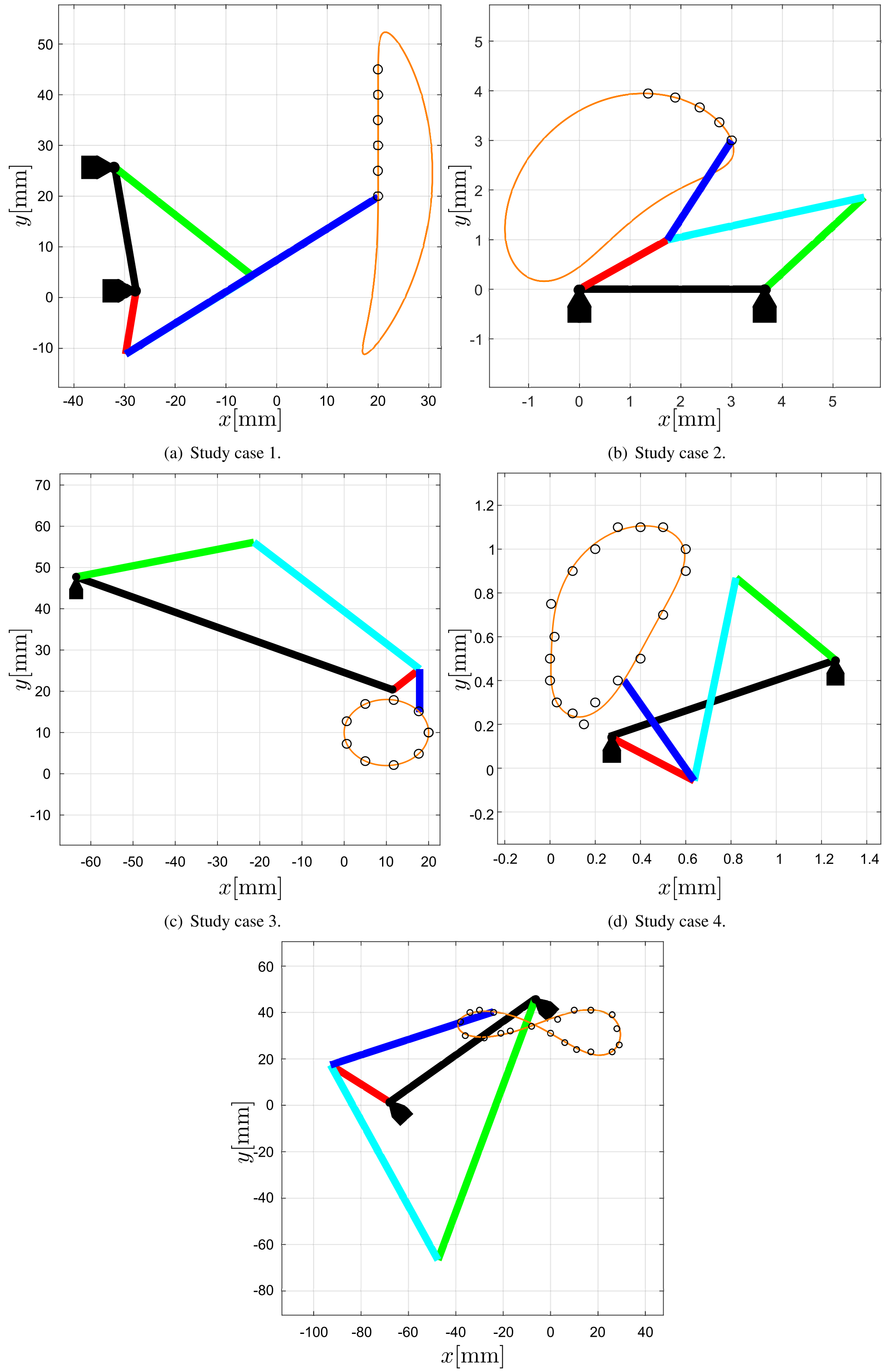


(e) Study case 5.

FIGURE 5. Objective function behavior in search of solutions to the ODSRA+CP approach in study cases using the best execution of DE/best/1/bin.

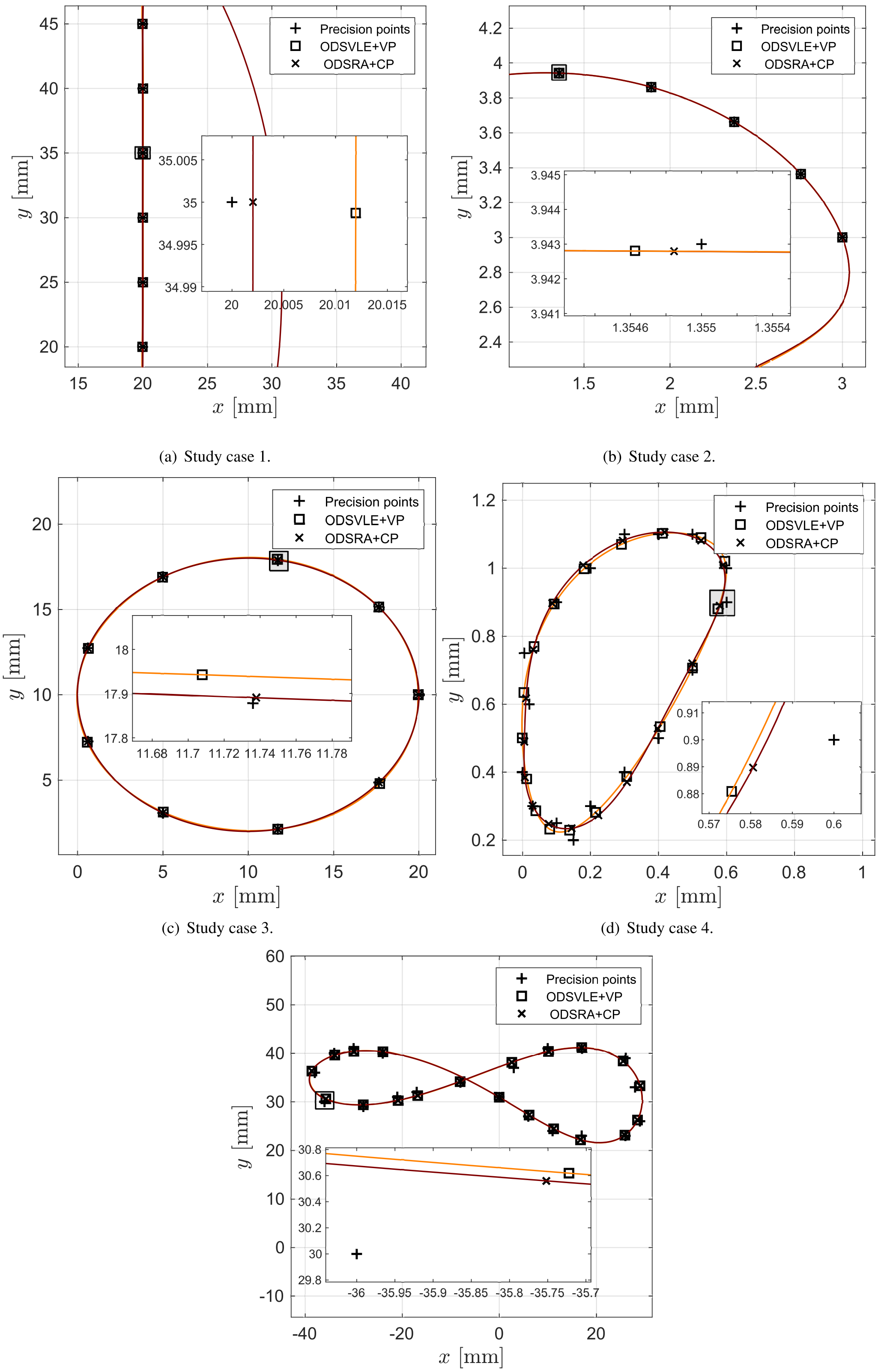
study case. This issue in the optimizer is sometimes reported in the literature, such as in [27], where fifty executions of the corresponding algorithm were established, resulting in a diverse set of solutions.

In addition, in the majority of the study cases (study cases 1, 2, 4 and 5), there exist the chance in the ODSRA+CP approach for further improving the obtained solution with the incorporation of different search strategies in the optimizer, because the algorithm DE/best/1/bin converges to a solution before the maximum generation number. It is important to note that in spite of not using an improved optimizer, the obtained results in this work can outperform the results



(a) Study case 1. (b) Study case 2. (c) Study case 3. (d) Study case 4. (e) Study case 5.

FIGURE 6. Mechanism obtained with the ODSRA+CP approach by solving with the DE/best/1/bin for each study case. The ground link length r_1 is in the black color line, the crank link length r_2 is in the red color line, the coupler link length r_3 is in the cyan color line, the output link length r_4 is in the green color line, and the distance r_5 is in the blue color line.



(a) Study case 1.

(b) Study case 2.

(c) Study case 3.

(d) Study case 4.

(e) Study case 5.

FIGURE 7. The path traced by the coupler point of the best mechanism obtained by two approaches.

in the state of the art that involves modifications in the algorithm.

Then, the formulation of the optimization problem in the ODSRA+CP approach indirectly promotes a better search space exploration by the optimizer, so that a great diversity of solutions and better ones can be obtained to the dimensional synthesis problem in the four-bar mechanism without needing to resort to the use of specialized algorithms.

V. CONCLUSION

In this work, the dimensional synthesis of the planar four-bar mechanism is stated using the relative angles method and the parameterization of the links in Cartesian coordinates. The synthesis performance has been compared in five study cases reported in the specialized literature. Furthermore, the solution to the optimization problem of the ODSRA+CP approach has been obtained through four DE variants, DE/best/1/bin, DE/best/1/exp, DE/rand/1/bin, DE/rand/1/exp.

The comparative statistical analysis reveals that DE/best/1/bin is the most promising optimizer because this presents the higher number of wins among the comparisons. This also obtains the best solution in all study cases. It is observed that elitism strategies with the use of best individuals in the mutation process given by DE/best/1/bin and DE/best/1/exp aid in the search of improved solution in the synthesis of four-bar mechanism.

The empirical comparative results show that the ODSRA+CP increases the exploration of the search space in a basic algorithm DE/best/1/bin. Thus, the solution of the ODSRA+CP provides mechanisms with better performance in the criterion by which it was optimized, even better than those reported in the specialized literature where modifications in the algorithms were considered. This confirms that the way of stating the optimization problem in the ODSRA+CP significantly improves the search process for finding promising solutions in the optimizer without requiring algorithm modifications.

Other advantage of the ODSRA+CP approach is that it is possible to extend the dimensional synthesis to spatial four-bar mechanism.

One of the disadvantages of the DE/best/1/bin algorithm in the ODSRA+CP approach is the rapid convergence towards a solution, which will require several (thirty) executions to know the best solution. Therefore, future work will involve the fusion of different operators in the optimizer, and the use of decentralized and self-organization strategies to increase the algorithm reliability in the optimal synthesis problem of four-bar mechanisms through the ODSRA+CP approach. Furthermore, it is expected that such specialized algorithms (memetic and hybrid algorithms) and also swarm intelligence algorithms would result in mechanisms with better performances than the obtained in the current study.

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LUIS ERNESTO VALENCIA-SEGURA received the B.S. degree in electronics and the M.Sc. degree in electronics sciences option in automation from the Benemérita Universidad Autónoma de Puebla (BUAP), Puebla City, Puebla, in 2015 and 2016, respectively. He is currently pursuing the Ph.D. degree in robotic and mechatronic systems engineering with the Instituto Politécnico Nacional (IPN), Unidad Profesional Interdisciplinaria en Ingeniería y Tecnologías Avanzadas (UPIITA), Mexico City, Mexico. His current research interests include the design and implementation of bio-inspired metaheuristics for optimization and their application to engineering problems.

MIGUEL GABRIEL VILLARREAL-CERVANTES (Member, IEEE) received the B.S. degree in electronics engineering from Veracruz Technologic Institute, Veracruz, Mexico, in 2003, and the M.Sc. and Ph.D. degrees in electrical engineering from the Center for Research and Advanced Studies, Cinvestav, Mexico City, Mexico, in 2005 and 2010, respectively. He is currently a Full Professor with the Postgraduate Department, Centro de Innovación y Desarrollo Tecnológico en Cómputo del, Instituto Politécnico Nacional (CIDETEC-IPN), Mexico City. He is a member of the National System of Researchers and the Expert Network on Robotics and Mechatronics, IPN (ENRM-IPN). He was the Head of both the Mechatronic Section with CIDETEC-IPN, from 2013 to 2016, and with ENRM-IPN, from 2012 to 2017. His current research interests include mechatronic design based on optimization (mono and multiobjective optimization), bio-inspired meta-heuristics in the optimal mechatronic design, optimal tuning for the mechatronic system control based on meta-heuristic algorithms (offline and online strategies), and robotics.

LEONEL GERMÁN CORONA-RAMÍREZ received the bachelor's degree in electromechanical engineering from the Technological Institute of Apizaco, the master's degree in electrical engineering from the Center for Research and Advanced Studies, National Polytechnic Institute (CINVESTAV-IPN), and the Ph.D. degree in mechanical engineering in the robotics line from the Higher School of Engineering and Mechanics, National Polytechnic Institute (ESIME-IPN). He is currently working as a Research Professor with the Department of Advanced Technologies in the area of Mechatronics in the Interdisciplinary Professional Unit in Engineering and Advanced Technologies of the National Polytechnic Institute (UPIITA-IPN). Between his developed works, it is possible to emphasize the following: author of books within the area of robotics, digital electronics, and sensors and actuators; publication of articles in various magazines and congresses; the Founder of the UPIITA-IPN Robotics Association; and the Director of Thesis in undergraduate and postgraduate studies.

FRANCISCO CUENCA-JIMÉNEZ received the B.S. degree from the Technological Institute of Oaxaca, in 1993, and the M.S. and Ph.D. degrees in mechanical engineering from the National Autonomous University of Mexico (UNAM), in 1996 and 2008, respectively. He has been a Professor in mechanical engineering with the Faculty of Engineering, Universidad Nacional Autónoma de México (UNAM), since 1995. His research interests include kinematic and dynamic multibody modeling and mechanism synthesis.

ROBERTO CASTRO-MEDINA received the B.S. degree in electronics engineering from Metropolitan Autonomous University (UAM), in 2005, and the M.Sc. degree from the Centro de Innovación y Desarrollo Tecnológico en Cómputo del, Instituto Politécnico Nacional (CIDETEC-IPN), in 2018, where he is currently pursuing the Ph.D. degree in robotic and mechatronic systems engineering. His research interests include modeling for the control of dynamic systems, optimization in mechatronic systems, and topics related to soft computing.