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# Lyapunov Spectrum Local Assignability of Linear Discrete Time-Varying Systems by Static Output Feedback

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**ABSTRACT** We consider a linear discrete time-varying input-output system. Our goal is to study the problem of local assignability of the Lyapunov spectrum by static output feedback control. To this end we introduce the notion of uniform consistency for discrete-time linear systems which is the extension of the notion of uniform complete controllability to input-output systems. The property of uniform consistency is investigated, some necessary and sufficient conditions for this property are obtained. The notion of uniform local attainability is introduced for the closed-loop system. We prove that uniform consistency implies uniform local attainability of the closed-loop system. The property of local Lyapunov reducibility is introduced for the closed-loop system. We prove that uniform local attainability implies local Lyapunov reducibility. We prove that, for a locally Lyapunov reducible system, the Lyapunov spectrum is locally assignable, if the free system is diagonalizable or regular (in the Lyapunov sense) or has the stable Lyapunov spectrum.

**INDEX TERMS** Linear discrete time-varying input-output systems, local assignability, Lyapunov spectrum, pole assignment problem, static output feedback, uniform consistency.

## I. INTRODUCTION

It is well known that a well-designed feedback controller is expected not only to produce the required output, but also to ensure the satisfactory quality for the transition process, e.g., provide the required overall decay rate of the solutions or some appropriate oscillatory properties. In many cases, these properties are determined by the asymptotic behavior of some linear system, which usually arises as a system in variations for the original system and most often turns out to be non-stationary. Mathematical problems arising here are diverse and often difficult.

The simplest case of such a problem and, at the same time, its classic example is the stabilization problem, where the

deviations of the process parameters from the target values are to be suppressed in the shortest possible time. The state-of-the-art for the stabilization problem of continuous-time linear systems is described in [1] (see also [2]), and for discrete time systems in [3]. Another classic examples are the pole assignment problem for a stationary system and the problem of assigning the multiplier spectrum for a periodic system. Here the ultimate goal is not only to influence the decay rate of the solutions, but also on other characteristics of the transient process. It is well known that a necessary and, in the stationary case, a sufficient condition for the solvability of these problems is the complete controllability of the open system [4]–[7].

The spectrum of eigenvalues of a stationary system determines almost all features of this system. That is why for stationary systems it is possible to fine-tune their asymptotic

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properties even with the help of stationary linear feedback. In the non-stationary case, it is difficult to provide something like this, and we are to seek for some alternative approaches.

One of the useful way to handle the asymptotic properties of time-varying linear systems is the use of the Lyapunov spectrum and some related characteristics of these systems such as the Bohl exponents, the dichotomy spectrum, the properties of stability, reducibility, regularity in the Lyapunov sense, etc. For example, the stabilization problem as a rule can be reduced to the problem of assigning the higher Lyapunov exponent to be negative. In turn, to ensure uniform stability, the upper Bohl exponent is to be assigned. It should be stressed that by assigning some asymptotic characteristics of a linear system, we can influence various properties of this system that are not reduced only to the overall decay rate of solutions. In particular, by the simultaneous assignment of all Lyapunov exponents of a system (i.e., the assignment of the Lyapunov spectrum), it is possible to influence the conditional stability of its solutions. By assigning a zero value to the irregularity coefficient, it is possible to ensure a more reliable preservation of stability under the action of nonlinear perturbations.

All the above characteristics and properties are studied within the framework of the theory of Lyapunov exponents, the foundations of which were laid by A.M. Lyapunov in his doctoral thesis of 1892 [8]. Since then, the exponents theory for both differential and discrete case has been intensively developed in many directions and is now a well-established mathematical theory having many applications. The current state-of-the-art and basic definitions can be found in [9]–[14]. Some necessary definitions are also given below.

Thus, we may assert that the problem of ensuring the required quality of the transition process leads to the problem of assigning some prescribed asymptotic properties of a given linear control system by introducing some appropriate linear feedback into it. These problems have been intensively investigated for the continuous-time case, and the monograph [15] contains a summary and history of this research before 2012. Recently, some substantially new results have been obtained in this direction. In particular, necessary and sufficient conditions for assignability of the dichotomy spectrum for continuous time-varying linear systems are obtained in [16].

There exist few alternative approaches to the problem of assigning asymptotic properties of a linear system. An approach based on reducing of a periodic system to a stationary form using special feedback were considered, for example, in [17]–[19]. Starting from [20], a number of authors have tried to solve the problem of assigning asymptotic properties for a system with smooth coefficients by reducing it to the second canonical Luenberger form with subsequent transformation into stationary one by means of suitable feedback, see e.g. [21]–[23]. These results are quite advanced and provide a well-developed computational technique for practical applications. However, they have

significant limitations of the scope due to the requirements for the original control system.

Much less is currently known for discrete systems. Results related to the canonical Luenberger form are presented in [24]–[26]. An approach similar to the approach of [17] for discrete systems is developed in [27]. Sufficient conditions for assignability of the dichotomy spectra for discrete time-varying linear control systems were obtained in [28]. Necessary and sufficient conditions for assignability of the dichotomy spectrum for one-sided discrete time-varying linear systems are obtained in [29].

A series of papers [30]–[33] discussed investigations of the problem of Lyapunov exponents placement for discrete-time systems. In these works, sufficient conditions are obtained for the solvability of the problem of assigning the Lyapunov spectrum of discrete non-stationary systems in various formulations. The main one among these conditions is, as in the continuous case, the uniform complete controllability of the original (open-loop) control system.

More precisely, in [32] it was established that if a linear discrete-time system

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad t \in \mathbb{Z}, \quad x \in \mathbb{R}^n, \quad (1)$$

with time-varying coefficients is uniformly completely controllable and the free system

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{Z}, \quad x \in \mathbb{R}^n, \quad (2)$$

is diagonalizable or regular (in the sense of Lyapunov) or has the stable Lyapunov spectrum, then the Lyapunov spectrum of the closed-system (by linear state feedback)

$$x(t+1) = (A(t) + B(t)U(t))x(t), \quad t \in \mathbb{Z} \quad x \in \mathbb{R}^n, \quad (3)$$

is proportionally locally assignable. Here proportional local assignability means that for an arbitrary set of numbers lying in a small neighborhood of the Lyapunov spectrum of the free system (2), we can construct a small-norm control  $(U(t))_{t \in \mathbb{Z}}$  such that the Lyapunov spectrum of the closed-loop system (3) coincides with the given set. Moreover, we can choose the control  $(U(t))_{t \in \mathbb{Z}}$  so that the value of  $\|U(t)\|$  satisfies some Lipschitz-type estimate with respect to the required exponents shift.

An essential feature of the above result is the use of static state feedback. Such a restriction significantly narrows the scope of the result, but makes it easier to obtain. Our main goal in this paper is to overcome this deficiency. Here we consider the problem of assignment of the Lyapunov spectrum for a linear input-output discrete-time system with time-varying coefficients

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad (4)$$

$$y(t) = C^*(t)x(t), \\ t \in \mathbb{Z}, \quad (x, u, y) \in \mathbb{K}^n \times \mathbb{K}^m \times \mathbb{K}^k, \quad (5)$$

where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ , by means of linear static output feedback

$$u(t) = U(t)y(t), \quad t \in \mathbb{Z}, \quad (6)$$

that is for the closed-loop system of the form

$$x(t + 1) = (A(t) + B(t)U(t)C^*(t))x(t), \quad t \in \mathbb{Z}. \quad (7)$$

The problem is considered in a local setting, i.e., for an arbitrary set of numbers lying in a small neighborhood of the Lyapunov spectrum of the free system (2) one needs to construct a small-norm control  $(U(t))_{t \in \mathbb{Z}}$  such that the Lyapunov spectrum of the closed-loop system (7) coincides with the given set. Note that we do not assume the control  $(U(t))_{t \in \mathbb{Z}}$  to have any Lipschitz-type estimate with respect to the required exponents shift.

To extend the results obtained in [32] to system (7), we use the concept of uniform consistency of system (4), (5), which is a generalization of the concept of uniform complete controllability of system (4). The definition of uniform consistency was given in [34] for continuous-time systems, and in [35] for discrete-time systems. This new notion allows us to obtain sufficient condition for the above formulated problem. However, unlike the case of state feedback, we failed to obtain proportional local controllability. In [32] we construct the control explicitly and due to that we easily obtain the desired estimate. To construct a control in the case of output feedback, we have to use some sophisticated technique that severely restricts our options.

The paper is organized as follows. In Section II, we introduce the definition of uniform consistency for a linear discrete-time input-output system. The properties of uniformly consistent systems are investigated. Necessary conditions and sufficient conditions for uniform consistency are established, in terms of the coefficients of the original system, as well as in terms of the coefficients of the big system. In Section III, we recall the concept of dynamical equivalence and its sufficient condition. In Section IV, criteria for uniform consistency are obtained for time-invariant systems. In Section V, we introduce and investigate the definition of uniform local attainability for a closed-loop system by static output feedback. In Section VI, we establish an interrelation between the properties of the uniform consistency and uniform local attainability. It is proved that uniform consistency of the open-loop system implies uniform local attainability of the closed-loop system but the converse is not true. In Section VII, we introduce the definition of the property of local Lyapunov reducibility for the closed-loop system and prove that the uniform local attainability is a sufficient condition for local Lyapunov reducibility. In Section VIII, we introduce the definition of local assignability of the Lyapunov spectrum for a closed-loop system by static output feedback. We prove that, under some additional assumptions on the matrix of the free system, uniform local attainability implies local assignability of the Lyapunov spectrum. Corollaries are obtained on local assignability of the Lyapunov spectrum for uniformly consistent systems. In Section IX, an example is presented to illustrate the results obtained. In Section X, conclusion comments are given.

*Notation.* Relations  $\alpha := \beta$  and  $\beta =: \alpha$  mean that  $\alpha$  is assumed, by definition, equal to  $\beta$ . Let  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{K} = \mathbb{R}$ ;  $\mathbb{K}^n = \{x = \text{col}(x_1, \dots, x_n) : x_i \in \mathbb{K}\}$  is the linear space of vectors over  $\mathbb{K}$ ;  $M_{m,n}(\mathbb{K})$  is the space of  $m \times n$ -matrices over  $\mathbb{K}$ ;  $M_n(\mathbb{K}) := M_{n,n}(\mathbb{K})$ ;  $I \in M_n(\mathbb{K})$  is the identity matrix;  $[e_1, \dots, e_n] := I$ ; set  $A^0 := I$  for any  $A \in M_n(\mathbb{K})$ ;  $\bar{A}$  is the complex conjugation of a matrix  $A$ ;  $T$  is the transposition and  $*$  is the Hermitian conjugation of a matrix or a vector treated as a matrix;  $|x| = \sqrt{x^*x}$  is the norm in  $\mathbb{K}^n$ ;  $\|A\| = \max_{|x|=1} |Ax|$  is the norm in  $M_{m,n}(\mathbb{K})$ ;  $\mathbf{B}_\varepsilon(H) := \{G \in M_{m,n}(\mathbb{K}) : \|G - H\| \leq \varepsilon\}$ ;  $\mathbb{N}$  and  $\mathbb{Z}$  are the sets of natural numbers and integers, respectively; an interval  $[t_0, t_1]$ , where  $t_0, t_1 \in \mathbb{Z}$ ,  $t_0 < t_1$ , is understood as the set of integer points  $t_0, t_0 + 1, \dots, t_1 - 1$  (respectively  $[t_0, +\infty) = \{t_0, t_0 + 1, \dots\}$ );  $[t_0, t_1] := \{t_0, t_0 + 1, \dots, t_1\}$ ;  $A \otimes B$  denotes the right Kronecker (tensor) product of matrices  $A$  and  $B$  [36, Sect. 12.1]. By  $\text{vec} : M_{p,q}(\mathbb{K}) \rightarrow \mathbb{K}^{pq}$  denote the mapping, which “unrolls” a matrix  $H = \{h_{ij}\}$ ,  $i = \overline{1, p}$ ,  $j = \overline{1, q}$ , by rows into the column vector  $\text{vec } H = \text{col}(h_{11}, \dots, h_{1q}, \dots, h_{p1}, \dots, h_{pq}) \in \mathbb{K}^{pq}$ . For any matrices  $A \in M_{n,m}(\mathbb{K})$ ,  $B \in M_{m,k}(\mathbb{K})$ ,  $C \in M_{k,l}(\mathbb{K})$ , we have

$$\text{vec}(ABC) = (A \otimes C^T)\text{vec } B. \quad (8)$$

A quadratic form  $V_P(y) := y^*Py$  is identified with its Hermitian matrix  $P = P^*$ ; the inequalities  $P > Q$  and  $P \geq Q$  for Hermitian matrices  $P, Q$  are understood in the sense of quadratic forms, i.e.,  $P > Q$  iff  $V_P(y) > V_Q(y)$  for all  $y \in \mathbb{K}^n \setminus \{0\}$ , and  $P \geq Q$  iff  $V_P(y) \geq V_Q(y)$  for all  $y \in \mathbb{K}^n$ .

Let

$$\mathcal{N}_n(\alpha) := \{H \in M_n : \det H \neq 0, \|H\| \leq \alpha, \|H^{-1}\| \leq \alpha\}.$$

For every matrix  $H \in \mathcal{N}_n(\alpha)$  the inequality

$$2\alpha \geq \|H\| + \|H^{-1}\| \geq \|H\| + \|H\|^{-1} \geq 2$$

holds; therefore, the set  $\mathcal{N}_n(\alpha)$  is nonempty only for  $\alpha \geq 1$ . For this reason, below we consider the set  $\mathcal{N}_n(\alpha)$  only for  $\alpha \geq 1$ .

We say that a sequence  $L(\cdot) = (L(t))_{t \in \mathbb{Z}} \subset M_n$  forms a *Lyapunov sequence* if there exists  $\alpha \geq 1$  such that  $L(t) \in \mathcal{N}_n(\alpha)$  for all  $t \in \mathbb{Z}$ .

For any sequence  $F(\cdot) = (F(t))_{t \in \mathbb{Z}} \subset M_{m,n}$  we define  $\|F\|_\infty = \sup_{t \in \mathbb{Z}} \|F(t)\|$ .

## II. UNIFORM CONSISTENCY FOR LINEAR DISCRETE-TIME SYSTEMS

### A. DEFINITIONS

Consider a linear discrete time-varying control system (4), (5) with a Lyapunov sequence  $(A(t))_{t \in \mathbb{Z}} \subset M_n(\mathbb{K})$  and bounded sequences  $(B(t))_{t \in \mathbb{Z}} \subset M_{n,m}(\mathbb{K})$ ,  $(C(t))_{t \in \mathbb{Z}} \subset M_{n,k}(\mathbb{K})$ . We assume that the inclusion

$$A(\cdot) = (A(t))_{t \in \mathbb{Z}} \subset \mathcal{N}_n(a) \quad (9)$$

and the inequalities

$$\|B\|_\infty \leq b, \quad \|C\|_\infty \leq c \quad (10)$$

hold for some finite  $a \geq 1, b > 0, c > 0$ .

By  $X_A(t, \tau), t, \tau \in \mathbb{Z}$ , denote the transition matrix of the corresponding free system (2), that is

$$X_A(t, \tau) = \begin{cases} A(t-1) \cdots A(\tau), & t > \tau, \\ I, & t = \tau, \\ A^{-1}(t) \cdots A^{-1}(\tau-1), & t < \tau. \end{cases}$$

From (9) it follows that, for any  $t \in \mathbb{Z}$ ,

$$\|A(t)\| \leq a, \quad \|A^{-1}(t)\| \leq a,$$

hence, by definition of  $X_A(t, \tau)$ , for every  $t, \tau \in \mathbb{Z}$  the estimation

$$\|X_A(t, \tau)\| \leq a^{|t-\tau|} \quad (11)$$

holds.

Recall that system (4) is said to be *completely reachable on* (*completely controllable on*)  $[t_0, t_1]$  if for any  $\hat{x} \in \mathbb{K}^n$  there exists a control function  $\hat{u}(t), t \in [t_0, t_1]$ , steering the solution of system (4) from the state  $x(t_0) = 0$  ( $x(t_0) = \hat{x}$ ) into the state  $x(t_1) = \hat{x}$  ( $x(t_1) = 0$ ).

Let us construct *the reachability gramian*

$$W_1(t_1, t_0) = \sum_{s=t_0}^{t_1-1} X_A(t_1, s+1)B(s)B^*(s)X_A^*(t_1, s+1)$$

and *the controllability gramian*

$$W_2(t_1, t_0) = \sum_{s=t_0}^{t_1-1} X_A(t_0, s+1)B(s)B^*(s)X_A^*(t_0, s+1)$$

( $t_1 > t_0$ ). Note that  $W_j(t_1, t_0) = W_j^*(t_1, t_0), W_j(t_1, t_0) \geq 0, j = 1, 2$ , and

$$W_1(t_1, t_0) = X_A(t_1, t_0)W_2(t_1, t_0)X_A^*(t_1, t_0).$$

It is known the following proposition.

*Proposition 1:* System (4) is completely reachable (controllable) on  $[t_0, t_1]$  iff  $W_1(t_1, t_0) > 0$  ( $W_2(t_1, t_0) > 0$ ).

*Remark 1:* If condition (9) is satisfied, then the properties of complete reachability and complete controllability on  $[t_0, t_1]$  are equivalent, and relations  $W_1(t_1, t_0) > 0$  and  $W_2(t_1, t_0) > 0$  are equivalent.

*Definition 1* (see [37, Definition 1]): System (4), (5) is said to be *consistent on the interval*  $[\tau, \tau + \vartheta]$  if for any matrix  $G \in M_n(\mathbb{K})$  there exists a control function  $\hat{U}(t) \in M_{m,k}(\mathbb{K}), t \in [\tau, \tau + \vartheta]$ , steering the solution of the matrix system

$$Z(t+1) = A(t)Z(t) + B(t)\hat{U}(t)C^*(t)X_A(t, \tau) \quad (12)$$

from the state  $Z(\tau) = 0$  into the state  $Z(\tau + \vartheta) = G$ .

*Remark 2:* The property of consistency is, in a sense, a generalization of the notion of complete reachability (controllability) from systems with the complete output ( $C(t) \equiv I, y = x$ ) to systems with the incomplete output

( $\text{rank } C(t) < n$ ). In particular: (a) if system (4), (5) is consistent on the interval  $[\tau, \tau + \vartheta]$ , then system (4), (5) is completely reachable (controllable) and completely observable on  $[\tau, \tau + \vartheta]$  [37, Proposition 3]; (b) If  $C(t) \equiv I, t \in [\tau, \tau + \vartheta]$ , then system (4), (5) is consistent on  $[\tau, \tau + \vartheta]$  if and only if it is completely reachable (controllable) on  $[\tau, \tau + \vartheta]$  [37, Proposition 7].

Let  $\vartheta \in \mathbb{N}$  be fixed. Let us construct the following matrices for  $t \in [\tau, \tau + \vartheta]$ :

$$\tilde{B}(t, \tau) := X_A(\tau + \vartheta, t + 1)B(t) \in M_{n,m}(\mathbb{K}),$$

$$\tilde{C}(t, \tau) := X_A^*(t, \tau)C(t) \in M_{n,k}(\mathbb{K}),$$

$$Q_1(t, \tau) := \tilde{B}(t, \tau) \otimes (\tilde{C}^*(t, \tau))^T \in M_{n^2, mk}(\mathbb{K}),$$

$$T_1(\tau, \vartheta) := [Q_1(\tau, \tau), \dots, Q_1(\tau + \vartheta - 1, \tau)] \in M_{n^2, mk\vartheta}(\mathbb{K}),$$

$$S_1(\tau, \vartheta) := T_1(\tau, \vartheta)T_1^*(\tau, \vartheta) \in M_{n^2}(\mathbb{K}). \quad (13)$$

Then  $S_1^*(\tau, \vartheta) = S_1(\tau, \vartheta), S_1(\tau, \vartheta) \geq 0$ , and the matrix  $S_1(\tau, \vartheta)$  has the form (see [37])

$$S_1(\tau, \vartheta) = \sum_{s=\tau}^{\tau+\vartheta-1} (\tilde{B}(s, \tau)\tilde{B}^*(s, \tau)) \otimes (\tilde{C}(s, \tau)\tilde{C}^*(s, \tau))^T.$$

The matrix  $S_1(\tau, \vartheta)$  for system (4), (5) is an analogue of the matrix  $W_1(\tau + \vartheta, \tau)$  for system (4). The following proposition is similar to Proposition 1.

*Proposition 2* (see [37, Proposition 2]): The following assertions are equivalent.

1. System (4), (5) is consistent on  $[\tau, \tau + \vartheta]$ .
2.  $\text{rank } T_1(\tau, \vartheta) = n^2$ .
3.  $S_1(\tau, \vartheta) > 0$ .

*Definition 2:* System (4) is said to be  $\vartheta$ -uniformly completely controllable ( $\vartheta \in \mathbb{N}$ ) if there exist  $\alpha_i = \alpha_i(\vartheta) > 0, i = 1, 2, 3, 4$ , such that for all  $\tau \in \mathbb{Z}$  the following inequalities hold:

$$W_1(\tau + \vartheta, \tau) > 0,$$

$$0 < \alpha_1 I \leq W_1^{-1}(\tau + \vartheta, \tau) \leq \alpha_2 I,$$

$$0 < \alpha_3 I \leq X_A^*(\tau + \vartheta, \tau)W_1^{-1}(\tau + \vartheta, \tau)X_A(\tau + \vartheta, \tau) \leq \alpha_4 I.$$

System (4) is said to be *uniformly completely controllable* if there exists  $\vartheta \in \mathbb{N}$  such that system (4) is  $\vartheta$ -uniformly completely controllable (see [38, Definition 6.3]).

Definition 2 repeats Kalman's definition [39] for continuous-time systems adapted for discrete-time systems. Note that Definition 2 does not require invertibility of  $A(t)$ . Nevertheless, invertibility of  $A(t)$  follows necessarily from Definition 2 (see, e.g., [40, Proposition 1]). Hence,  $W_2(\tau + \vartheta, \tau)$  is well-defined, and, in fact, one can prove the following proposition (see, e.g., [40, Theorem 2]).

*Proposition 3:* System (4) is  $\vartheta$ -uniformly completely controllable iff there exist  $\alpha_i = \alpha_i(\vartheta) > 0, i = 5, 6, 7, 8$ , such that for all  $\tau \in \mathbb{Z}$  the following inequalities hold:

$$0 < \alpha_5 I \leq W_1(\tau + \vartheta, \tau) \leq \alpha_6 I,$$

$$0 < \alpha_7 I \leq W_2(\tau + \vartheta, \tau) \leq \alpha_8 I.$$



In addition, the following proposition was proved (see [40, Theorem 4]).

**Proposition 4:** System (4) is  $\vartheta$ -uniformly completely controllable if and only if there exists a number  $\kappa_1 > 0$  such that for any  $\tau \in \mathbb{Z}$  and for any  $x_1 \in \mathbb{K}^n$  there exists a control function  $\widehat{u}(t)$ ,  $t \in [\tau, \tau + \vartheta)$ , steering the solution of system (4) from the state  $x(\tau) = 0$  into the state  $x(\tau + \vartheta) = x_1$ , and the inequality  $|\widehat{u}(t)| \leq \kappa_1 |x_1|$  holds for all  $t \in [\tau, \tau + \vartheta)$ .

Let us introduce the following definition by analogy with Definition 2.

**Definition 3:** System (4), (5) is said to be  $\vartheta$ -uniformly consistent ( $\vartheta \in \mathbb{N}$ ) if there exist  $\alpha_i = \alpha_i(\vartheta) > 0$ ,  $i = 1, 2, 3, 4$ , such that for all  $\tau \in \mathbb{Z}$  the following inequalities hold:

$$S_1(\tau, \vartheta) > 0, \tag{14}$$

$$0 < \alpha_1 I \leq S_1^{-1}(\tau, \vartheta) \leq \alpha_2 I, \tag{15}$$

$$0 < \alpha_3 I \leq (X_A^*(\tau + \vartheta, \tau) \otimes I) S_1^{-1}(\tau, \vartheta) \times (X_A(\tau + \vartheta, \tau) \otimes I) \leq \alpha_4 I. \tag{16}$$

System (4), (5) is said to be *uniformly consistent* if there exists  $\vartheta \in \mathbb{N}$  such that system (4), (5) is  $\vartheta$ -uniformly consistent.

The following criterion for uniform consistency holds (see [35, Definition 7, Theorem 10]).

**Theorem 1:** System (4), (5) is  $\vartheta$ -uniformly consistent if and only if there exists  $\kappa_2 > 0$  such that for any  $\tau \in \mathbb{Z}$  and for any  $G \in M_n(\mathbb{K})$  there exists a control function  $\widehat{U}(t) \in M_{m,k}(\mathbb{K})$ ,  $t \in [\tau, \tau + \vartheta)$ , steering the solution of system (12) from the state  $Z(\tau) = 0$  into the state  $Z(\tau + \vartheta) = G$ , and the inequality  $\|\widehat{U}(t)\| \leq \kappa_2 \|G\|$  holds for all  $t \in [\tau, \tau + \vartheta)$ .

The aim of this section is studying the property of uniform consistency.

### B. AUXILIARY STATEMENTS

Let us give some auxiliary statements.

**Lemma 1:** Let  $W \in M_p(\mathbb{K})$  be a Hermitian, positive definite matrix, and  $0 < \mu_1 I \leq W \leq \mu_2 I$ . Then the matrix  $W^{-1}$  is also Hermitian, positive definite and

$$0 < \mu_2^{-1} I \leq W^{-1} \leq \mu_1^{-1} I.$$

**Lemma 2:** Let  $W \in M_p(\mathbb{K})$  be a Hermitian, positive semidefinite matrix. Then inequality  $\|W\| \leq \alpha$  is equivalent to  $W \leq \alpha I$ .

Lemmas 1 and 2 are clear. The proofs are given, e.g., in [35, Lemma 8 and Lemma 9].

**Lemma 3:** For any  $Y \in M_{p,q}(\mathbb{K})$  and  $Z \in M_{r,s}(\mathbb{K})$ ,

$$\|Y \otimes Z\| = \|Y\| \|Z\|.$$

*Proof:* It is known that: (a) by properties of the spectral norm [41, Example 5.6.6],  $\|Y\|$  coincides with the square root of the largest eigenvalue of  $Y^*Y$ ; (b) if  $\lambda_1, \dots, \lambda_q$  are the eigenvalues of  $F \in M_q(\mathbb{K})$  and  $\mu_1, \dots, \mu_s$  are the eigenvalues of  $G \in M_s(\mathbb{K})$ , then  $\{\lambda_i \mu_j, i = 1, 2, \dots, q, j = 1, 2, \dots, s\}$  are the eigenvalues of  $F \otimes G$  [36, Sect. 12.2]. We have

$$(Y \otimes Z)^*(Y \otimes Z) = (Y^*Y) \otimes (Z^*Z). \tag{17}$$

Set  $F := Y^*Y$  and  $G := Z^*Z$ . Then all eigenvalues of  $F$  and  $G$  are real and nonnegative:  $0 \leq \lambda_1 \leq \dots \leq \lambda_q$  and  $0 \leq \mu_1 \leq \dots \leq \mu_s$ . By (a),  $\|Y\| = \sqrt{\lambda_q}$  and  $\|Z\| = \sqrt{\mu_s}$ . By (a), (17), and (b), we get  $\|Y \otimes Z\| = \sqrt{\lambda_q \mu_s}$ . Q.E.D. ■

**Remark 3:** Note that, by properties of the spectral norm, for any  $Y \in M_{p,q}(\mathbb{K})$ ,  $\|Y\| = \|Y^*\| = \|Y^T\|$ .

### C. PROPERTIES OF UNIFORMLY CONSISTENT SYSTEMS

#### 1) NECESSARY CONDITIONS AND SUFFICIENT CONDITIONS

**Proposition 5:** If system (4), (5) is  $\vartheta$ -uniformly consistent, then for any  $\tau \in \mathbb{Z}$  it is consistent on  $[\tau, \tau + \vartheta)$ .

Proposition 5 follows from inequality (14) of Definition 3 and Proposition 2. The converse is not true, in general, by the following example.

**Example 1:** Suppose that  $n = m = k = 1$ ,

$$A(t) = B(t) \equiv 1, \quad C(t) = \begin{cases} 1/t, & t \geq 1, \\ 1, & t \leq 0. \end{cases} \tag{18}$$

For  $\vartheta = 1$ , inequality (14) holds for any  $\tau \in \mathbb{Z}$ , hence, system (4), (5), (18) is consistent on  $[\tau, \tau + \vartheta)$ . But the last inequality  $S_1^{-1}(\tau, \vartheta) \leq \alpha_2 I$  of (15) does not hold, hence, (4), (5), (18) is not  $\vartheta$ -uniformly consistent.

Suppose that system (4), (5) is  $\vartheta$ -uniformly consistent. Let us construct the following matrices for  $t \geq \tau$ :

$$\begin{aligned} \widehat{B}(t, \tau) &:= X_A(\tau, t + 1)B(t) \in M_{n,m}(\mathbb{K}), \\ Q_2(t, \tau) &:= \widehat{B}(t, \tau) \otimes (\widetilde{C}^*(t, \tau))^T \in M_{n^2, mk}(\mathbb{K}), \\ T_2(\tau, \vartheta) &:= [Q_2(\tau, \tau), \dots, Q_2(\tau + \vartheta - 1, \tau)] \in M_{n^2, mk\vartheta}(\mathbb{K}), \\ S_2(\tau, \vartheta) &:= T_2(\tau, \vartheta)T_2^*(\tau, \vartheta) \in M_{n^2}(\mathbb{K}). \end{aligned} \tag{19}$$

Then  $S_2^*(\tau, \vartheta) = S_2(\tau, \vartheta)$ ,  $S_2(\tau, \vartheta) \geq 0$ , and the matrix  $S_2(\tau, \vartheta)$  has the form

$$S_2(\tau, \vartheta) = \sum_{s=\tau}^{\tau+\vartheta-1} (\widehat{B}(s, \tau)\widehat{B}^*(s, \tau)) \otimes (\widetilde{C}(s, \tau)\widetilde{C}^*(s, \tau))^T.$$

Matrices (13) and (19) are related by the equality

$$S_1(\tau, \vartheta) = (X_A(\tau + \vartheta, \tau) \otimes I) \times S_2(\tau, \vartheta)(X_A(\tau + \vartheta, \tau) \otimes I)^* \tag{20}$$

for all  $\tau \in \mathbb{Z}$ ,  $\vartheta \in \mathbb{N}$ . Thus, inequality (15) is equivalent to

$$0 < \alpha_3 I \leq S_2^{-1}(\tau, \vartheta) \leq \alpha_4 I. \tag{21}$$

**Theorem 2:** System (4), (5) is  $\vartheta$ -uniformly consistent if and only if there exist  $\alpha_i = \alpha_i(\vartheta) > 0$ ,  $i = 5, 6, 7, 8$ , such that for all  $\tau \in \mathbb{Z}$  the following inequalities hold:

$$0 < \alpha_5 I \leq S_1(\tau, \vartheta) \leq \alpha_6 I, \tag{22}$$

$$0 < \alpha_7 I \leq S_2(\tau, \vartheta) \leq \alpha_8 I. \tag{23}$$

The proof follows from Lemma 1 and inequalities (15) and (21) if we set  $\alpha_5 := \alpha_2^{-1}$ ,  $\alpha_6 := \alpha_1^{-1}$ ,  $\alpha_7 := \alpha_4^{-1}$ ,  $\alpha_8 := \alpha_3^{-1}$ .

**Remark 4:** Theorem 2 is an analogue of Proposition 3.

2) THE BIG SYSTEMS

Let us construct the control system (so-called *the big system*)

$$\begin{aligned} \omega(t+1) &= F(t)\omega(t) + G(t)v(t), \\ F(t) &= A(t) \otimes (A^{-1}(t))^T \in M_{n^2}(\mathbb{K}), \end{aligned} \quad (24)$$

$$\begin{aligned} G(t) &= B(t) \otimes \left( (A^{-1}(t))^T \bar{C}(t) \right) \in M_{n^2, mk}(\mathbb{K}), \\ t \in \mathbb{Z}, \quad \omega &\in \mathbb{K}^{n^2}, \quad v \in \mathbb{K}^{mk}. \end{aligned} \quad (25)$$

By  $\Omega(t, \tau)$  denote the transition matrix of the free system

$$\omega(t+1) = F(t)\omega(t).$$

Then, by properties of the Kronecker product, we get

$$\Omega(t, \tau) = X_A(t, \tau) \otimes X_A^T(\tau, t).$$

*Theorem 3:* System (4), (5) is  $\vartheta$ -uniformly consistent iff system (24), (25) is  $\vartheta$ -uniformly completely controllable.

*Proof:* By  $V_1(t_1, t_0)$  and  $V_2(t_1, t_0)$  denote the reachability and controllability gramian respectively for system (24), (25). Then

$$V_2(\tau + \vartheta, \tau) = \sum_{s=\tau}^{\tau+\vartheta-1} P(s, \tau)P^*(s, \tau) \quad (26)$$

where  $P(s, \tau) = \Omega(\tau, s+1)G(s)$ , and

$$V_1(\tau + \vartheta, \tau) = \Omega(\tau + \vartheta, \tau)V_2(\tau + \vartheta, \tau)\Omega^*(\tau + \vartheta, \tau). \quad (27)$$

We have

$$\begin{aligned} P(s, \tau) &= \Omega(\tau, s+1)G(s) \\ &= \left( X_A(\tau, s+1) \otimes X_A^T(s+1, \tau) \right) \\ &\quad \times \left( B(s) \otimes \left( (A^{-1}(s))^T \bar{C}(s) \right) \right) \\ &= \left( X_A(\tau, s+1)B(s) \right) \otimes \left( X_A^T(s, \tau)\bar{C}(s) \right) \\ &= \widehat{B}(s, \tau) \otimes \left( \widetilde{C}^*(s, \tau) \right)^T = Q_2(s, \tau). \end{aligned} \quad (28)$$

So, it follows from (19), (28), and (26) that

$$V_2(\tau + \vartheta, \tau) = S_2(\tau, \vartheta). \quad (29)$$

Consequently, by (20) and (27), the equalities

$$\begin{aligned} S_1(\tau, \vartheta) &= \left( X_A(\tau + \vartheta, \tau) \otimes I \right) \\ &\quad \times V_2(\tau + \vartheta, \tau) \left( X_A(\tau + \vartheta, \tau) \otimes I \right)^* \\ &= \left( X_A(\tau + \vartheta, \tau) \otimes I \right) \Omega(\tau, \tau + \vartheta) V_1(\tau + \vartheta, \tau) \\ &\quad \times \Omega^*(\tau, \tau + \vartheta) \left( X_A(\tau + \vartheta, \tau) \otimes I \right)^* \end{aligned} \quad (30)$$

hold.

Let system (4), (5) be  $\vartheta$ -uniformly consistent. Then inequalities (22) and (23) hold. From (23) and (29), it follows that, for all  $\tau \in \mathbb{Z}$ ,

$$\alpha_7 I \leq V_2(\tau + \vartheta, \tau) \leq \alpha_8 I. \quad (31)$$

From (9) and Lemma 3, it follows that

$$\|\Omega^{\pm 1}(\tau + \vartheta, \tau)\| \leq a^{2\vartheta}. \quad (32)$$

Here the notation  $\Omega^{\pm 1}(\tau + \vartheta, \tau)$  means the matrix  $\Omega(\tau + \vartheta, \tau)$  and its inverse  $\Omega^{-1}(\tau + \vartheta, \tau)$ . From (27), (31), and (32), it follows that there exist  $\alpha'_5 = \alpha'_5(\vartheta) > 0$  and  $\alpha'_6 = \alpha'_6(\vartheta) > 0$  such that

$$\alpha'_5 I \leq V_1(\tau + \vartheta, \tau) \leq \alpha'_6 I. \quad (33)$$

So, from (31) and (33), by Proposition 3, it follows that system (24), (25) is  $\vartheta$ -uniformly completely controllable.

Conversely, let system (24), (25) be  $\vartheta$ -uniformly completely controllable. Then the matrices  $V_1(\tau + \vartheta, \tau)$  and  $V_2(\tau + \vartheta, \tau)$  satisfy the inequalities (33) and (31), respectively, where the positive numbers  $\alpha'_5(\vartheta)$ ,  $\alpha'_6(\vartheta)$ ,  $\alpha_7(\vartheta)$ ,  $\alpha_8(\vartheta)$  do not depend on  $\tau \in \mathbb{Z}$ . From (31) and (29), we get the inequality (23). It follows from (30), (32) and (33) that there exist positive  $\alpha_5(\vartheta)$ ,  $\alpha_6(\vartheta)$  such that (22) holds for every  $\tau \in \mathbb{Z}$ . Therefore, by virtue of Theorem 2, system (4), (5) is  $\vartheta$ -uniformly consistent. ■

*Theorem 4:* If system (4), (5) is  $\vartheta$ -uniformly consistent, then this system is  $\vartheta_1$ -uniformly consistent for any  $\vartheta_1 \geq \vartheta$ .

The proof follows from Theorem 3 and [40, Corollary 1].

Let us construct the other *big system*

$$\begin{aligned} \xi(t+1) &= K(t)\xi(t) + N(t)v(t), \\ K(t) &= A(t) \otimes (A^{-1}(t-1))^T \in M_{n^2}(\mathbb{K}), \end{aligned} \quad (34)$$

$$\begin{aligned} N(t) &= B(t) \otimes \bar{C}(t) \in M_{n^2, mk}(\mathbb{K}), \\ t \in \mathbb{Z}, \quad \xi &\in \mathbb{K}^{n^2}, \quad v \in \mathbb{K}^{mk}. \end{aligned} \quad (35)$$

By  $\mathcal{E}(t, \tau)$  denote the transition matrix of the free system

$$\xi(t+1) = K(t)\xi(t).$$

Then  $\mathcal{E}(t, \tau) = X_A(t, \tau) \otimes X_A^T(\tau-1, t-1)$ . By  $\mathcal{V}_1(t_1, t_0)$  and  $\mathcal{V}_2(t_1, t_0)$  denote the reachability and controllability gramian respectively for system (34), (35). Then

$$\mathcal{V}_1(\tau + \vartheta, \tau) = \sum_{s=\tau}^{\tau+\vartheta-1} \mathcal{P}(s, \tau) \mathcal{P}^*(s, \tau) \quad (36)$$

where  $\mathcal{P}(s, \tau) = \mathcal{E}(\tau + \vartheta, s+1)N(s)$ , and

$$\mathcal{V}_2(\tau + \vartheta, \tau) = \mathcal{E}(\tau, \tau + \vartheta) \mathcal{V}_1(\tau + \vartheta, \tau) \mathcal{E}^*(\tau, \tau + \vartheta). \quad (37)$$

We have

$$\begin{aligned} \mathcal{P}(s, \tau) &= \left( X_A(\tau + \vartheta, s+1) \otimes X_A^T(s, \tau + \vartheta - 1) \right) \left( B(s) \otimes \bar{C}(s) \right) \\ &= \left( X_A(\tau + \vartheta, s+1)B(s) \right) \otimes \left( X_A^T(s, \tau + \vartheta - 1)\bar{C}(s) \right) \\ &= \left( I \otimes X_A^T(\tau, \tau + \vartheta - 1) \right) \\ &\quad \times \left( X_A(\tau + \vartheta, s+1)B(s) \right) \otimes \left( X_A^T(s, \tau)\bar{C}(s) \right) \\ &= \left( I \otimes X_A^T(\tau, \tau + \vartheta - 1) \right) Q_1(s, \tau). \end{aligned} \quad (38)$$

Therefore, it follows from (36), (38), and (13) that

$$\begin{aligned} \mathcal{V}_1(\tau + \vartheta, \tau) &= \left( I \otimes X_A^T(\tau, \tau + \vartheta - 1) \right) S_1(\tau, \vartheta) \\ &\quad \times \left( I \otimes X_A^T(\tau, \tau + \vartheta - 1) \right)^*. \end{aligned} \quad (39)$$

By (9) and (39), inequalities (22) are equivalent to

$$\alpha'_5 I \leq \mathcal{V}_1(\tau + \vartheta, \tau) \leq \alpha'_6 I, \quad (40)$$

for some  $\alpha'_i = \alpha'_i(\vartheta) > 0, i = 5, 6$ , for all  $\tau \in \mathbb{Z}$ . By (9), estimations (22) and (23) are equivalent. By (9),  $K(\cdot)$  is a Lyapunov sequence. Therefore, due to (37), it follows that inequalities (40) are equivalent to

$$\alpha'_7 I \leq \mathcal{V}_2(\tau + \vartheta, \tau) \leq \alpha'_8 I,$$

for some  $\alpha'_i = \alpha'_i(\vartheta) > 0, i = 7, 8$ , for all  $\tau \in \mathbb{Z}$ . Thus, the following theorem is true.

**Theorem 5:** System (4), (5) is  $\vartheta$ -uniformly consistent iff system (34), (35) is  $\vartheta$ -uniformly completely controllable.

### 3) INTERCONNECTION BETWEEN UNIFORM COMPLETE CONTROLLABILITY AND UNIFORM CONSISTENCY

The following statements establish an interconnection between the properties of uniform complete controllability of system (4) and uniform consistency of system (4), (5).

**Theorem 6:** Suppose that system (4), (5) is  $\vartheta$ -uniformly consistent. Then system (4) is  $\vartheta$ -uniformly completely controllable.

*Proof:* For the proof, we use Theorem 1. Let  $\tau \in \mathbb{Z}$  and  $x_1 \in \mathbb{K}^n$  be given. Construct  $G = [x_1, 0, \dots, 0] \in M_n(\mathbb{K})$ . Then  $\|G\| = |x_1|$ . By using Theorem 1, let us construct the control function  $\widehat{U}(t) \in M_{m,k}(\mathbb{K}), t \in [\tau, \tau + \vartheta]$ , steering the solution of system (12) from  $Z(\tau) = 0$  into  $Z(\tau + \vartheta) = G$  such that  $\|\widehat{U}(t)\| \leq \kappa_2 \|G\|$ . Then  $\widehat{u}(t) = \widehat{U}(t)C^*(t)X(t, \tau)e_1$  steers the solution of (4) from  $x(\tau) = 0$  into  $x(\tau + \vartheta) = x_1$ . We have

$$\begin{aligned} |\widehat{u}(t)| &\leq \|\widehat{U}(t)\| \|C^*(t)\| \|X(t, \tau)\| \\ &\leq ca^{\vartheta-1} \kappa_2 |x_1| =: \kappa_1 |x_1|. \end{aligned}$$

So, the required follows from Proposition 4.  $\blacksquare$

**Theorem 7:** Let  $k = n$  and  $C(t) \in M_n(\mathbb{K}), t \in \mathbb{Z}$ , be a Lyapunov sequence. Suppose that system (4) is  $\vartheta$ -uniformly completely controllable. Then system (4), (5) is  $\vartheta$ -uniformly consistent.

*Proof:* Suppose that  $C(\cdot) = (C(t))_{t \in \mathbb{Z}} \subset \mathcal{N}_n(c)$ , where  $c \geq 1$ . Let  $\tau \in \mathbb{Z}$  and  $G = [g_1, \dots, g_n] \in M_n(\mathbb{K})$  be given;  $g_i \in \mathbb{K}^n, i = \overline{1, n}$ . We have

$$|g_i| \leq \|G\|, \quad i = \overline{1, n}. \quad (41)$$

By Proposition 4, for every  $i = \overline{1, n}$  construct the functions  $v_i(t), t \in [\tau, \tau + \vartheta]$ , steering the solution of (4) from  $x(\tau) = 0$  into  $x(\tau + \vartheta) = g_i$  such that  $|v_i(t)| \leq \kappa_1 |g_i|$ . By (41), we have  $|v_i(t)| \leq \kappa_1 \|G\|, t \in [\tau, \tau + \vartheta]$ . Set

$$\begin{aligned} V(t) &:= [v_1(t), \dots, v_n(t)] \in M_{m,n}(\mathbb{K}), \\ \widehat{U}(t) &:= V(t)X_A(\tau, t)(C^*(t))^{-1}. \end{aligned}$$

Then the function  $\widehat{U}(t) \in M_{m,n}(\mathbb{K}), t \in [\tau, \tau + \vartheta]$ , steers the solution of system (12) from  $Z(\tau) = 0$  into  $Z(\tau + \vartheta) = G$ .

We have

$$\begin{aligned} \|\widehat{U}(t)\| &\leq \|V(t)\| \|X_A(\tau, t)\| \|(C^*(t))^{-1}\| \\ &\leq a^{\vartheta-1} c \sqrt{n} \max_{i=1, \dots, n} |v_i(t)| \leq a^{\vartheta-1} c \sqrt{n} \kappa_1 \|G\| =: \kappa_2 \|G\|. \end{aligned}$$

So, by Theorem 1, system (4), (5) is  $\vartheta$ -uniformly consistent.  $\blacksquare$

### III. DYNAMIC EQUIVALENCE

**Definition 4** (see [42, p. 15]): Let  $L: \mathbb{Z} \rightarrow M_n(\mathbb{K})$  be a Lyapunov sequence. A linear transformation

$$\theta = L(t)x, \quad t \in \mathbb{Z}, \quad (42)$$

of the space  $\mathbb{K}^n$  is called a *Lyapunov transformation*.

**Definition 5** (see [42, p. 15]): We say that system (2) is *dynamically equivalent* to the system

$$\theta(t + 1) = A(t)\theta(t), \quad t \in \mathbb{Z}, \quad \theta \in \mathbb{K}^n, \quad (43)$$

if there exists a Lyapunov transformation (42) which connects these systems, i.e., for every solution  $x(t)$  of system (2) the function  $\theta(t) = L(t)x(t)$  is a solution of system (43) and for every solution  $\theta(t)$  of system (43) the function  $x(t) = L^{-1}(t)\theta(t)$  is a solution of system (2).

Let us note that if a Lyapunov transformation (42) establishes the dynamic equivalence between systems (2) and (43), then

$$\begin{aligned} \theta(t + 1) &= L(t + 1)x(t + 1) = L(t + 1)A(t)x(t) \\ &= L(t + 1)A(t)L^{-1}(t)\theta(t), \quad t \in \mathbb{Z}; \end{aligned}$$

hence,

$$A(t) = L(t + 1)A(t)L^{-1}(t), \quad t \in \mathbb{Z}. \quad (44)$$

Thus, systems (2) and (43) are dynamically equivalent if and only if there exists a Lyapunov sequence  $L(\cdot) = (L(t))_{t \in \mathbb{Z}}$ , such that the equality (44) is satisfied.

Denote by  $\Theta(t, \tau)$  the transition matrix of the free system (43). There is the following criterion of dynamic equivalence [30, Lemma 4.5].

**Lemma 4:** Suppose that  $(A(t))_{t \in \mathbb{Z}}$  and  $(\mathcal{A}(t))_{t \in \mathbb{Z}}$  are Lyapunov sequences. Assume that  $\Theta(t_{i+1}, t_i) = X_{\mathcal{A}}(t_{i+1}, t_i)$  for all  $t_i \in \mathbb{Z}$ , where  $t_i, i \in \mathbb{Z}$ , is a sequence of integer numbers such that  $0 < t_{i+1} - t_i \leq c < \infty$  for all  $i \in \mathbb{Z}$ . Then systems (2) and (43) are dynamically equivalent.

**Remark 5:** Lemma 4 was proved in [30, Lemma 4.5] for positive semiaxis of integers but the proof remains the same for the whole axes of integers.

### IV. UNIFORM CONSISTENCY FOR TIME-INVARIANT SYSTEMS

Consider a linear discrete time-invariant control system

$$x(t + 1) = Ax(t) + Bu(t), \quad (45)$$

$$y(t) = C^*x(t),$$

$$t \in \mathbb{Z}, \quad (x, u, y) \in \mathbb{K}^n \times \mathbb{K}^m \times \mathbb{K}^k, \quad (46)$$

where  $\det A \neq 0$ . It is clear that if system (45),(46) is consistent on  $[\tau, \tau + \vartheta)$  for some  $\tau \in \mathbb{Z}$ , then it is consistent on  $[\tau, \tau + \vartheta)$  for any  $\tau \in \mathbb{Z}$ . In this case, we say that the system (45), (46) is  $\vartheta$ -consistent.

**Theorem 8:** System (45), (46) is  $\vartheta$ -uniformly consistent iff it is  $\vartheta$ -consistent.

*Proof:* Necessity follows from Proposition 5.

*Sufficiency.* Since  $\det A \neq 0$ , it follows (9). Moreover (10) holds as well. We have  $S_1(\tau, \vartheta) \equiv S_1(\vartheta)$ ,  $\tau \in \mathbb{Z}$ . By Proposition 2,  $S_1(\vartheta) > 0$ . Hence,  $S_1(\vartheta) \geq \alpha_5 I > 0$  where  $\alpha_5 = \alpha_5(\vartheta)$  is the least eigenvalue of  $S_1(\vartheta)$ . The estimation  $S_1(\vartheta) \leq \alpha_6 I > 0$  follows from (13). In turn, from (20) we obtain the inequalities (23). It follows from Theorem 2 that system (45), (46) is  $\vartheta$ -uniformly consistent. ■

The property of  $\vartheta$ -consistency of system (45), (46) was studied in detail in [26], [37].

For system (45),(46), let us construct the big system (34), (35):

$$\begin{aligned} \xi(t+1) &= K\xi(t) + Nv(t), \\ K &= A \otimes (A^{-1})^T \in M_{n^2}(\mathbb{K}), \end{aligned} \quad (47)$$

$$\begin{aligned} N &= B \otimes \bar{C} \in M_{n^2, mk}(\mathbb{K}), \\ t \in \mathbb{Z}, \quad \xi &\in \mathbb{K}^{n^2}, \quad v \in \mathbb{K}^{mk}. \end{aligned} \quad (48)$$

System (47), (48) is completely controllable iff

$$\text{rank} [N, KN, \dots, K^{n^2-1}N] = n^2. \quad (49)$$

Suppose that system (45),(46) is uniformly consistent. Hence, by Theorem 8, there exists  $\vartheta \in \mathbb{N}$  such that system (45), (46) is consistent on  $[0, \vartheta)$ . Then, by [37, Theorem 1], system (47), (48) is completely controllable on  $[0, \vartheta)$ . Therefore, condition (49) is satisfied.

Vice versa, let condition (49) be satisfied. Then system (47),(48) is completely controllable on  $[0, n^2)$ . By [37, Theorem 1], system (45), (46) is consistent on  $[0, n^2)$ . Then by Theorem 8, system (45), (46) is  $n^2$ -uniformly consistent, hence, it is uniformly consistent. Thus, the following theorem is proved.

**Theorem 9:** System (45),(46) is uniformly consistent if and only if condition (49) is satisfied.

## V. UNIFORM LOCAL ATTAINABILITY

Consider system (4), (5). Suppose that the control in this system is constructed as static output feedback (6). The closed-loop system has the form (7). By  $\Phi_U(t, \tau)$ ,  $t \geq \tau$ , denote the transition matrix of system (7). In particular,

$$\Phi_0(t, \tau) = X_A(t, \tau). \quad (50)$$

**Definition 6:** System (7) is said to be:

(a)  $\vartheta$ -uniformly locally attainable if there exists  $\gamma \geq 1$  such that, for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, for any matrix  $H \in \mathbf{B}_\delta(I) \subset M_n(\mathbb{K})$  and any  $t_0 \in \mathbb{Z}$ , there exists a control function  $U : [t_0, t_0 + \vartheta) \rightarrow \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$  ensuring the equality

$$\Phi_U(t_0 + \vartheta, t_0) = \Phi_0(t_0 + \vartheta, t_0)H, \quad (51)$$

and the following inclusion holds:

$$A(t) + B(t)U(t)C^*(t) \in \mathcal{N}_n(\gamma), \quad t \in [t_0, t_0 + \vartheta). \quad (52)$$

(b) uniformly locally attainable if there exists  $\vartheta > 0$  such that the system is  $\vartheta$ -uniformly locally attainable.

Let us prove some properties of uniformly locally attainable systems.

**Proposition 6:** Suppose that system (7) is  $\vartheta$ -uniformly locally attainable. Then system (7) is  $(\vartheta + 1)$ -uniformly locally attainable.

*Proof:* Let  $\gamma \geq 1$  be from Definition 6. Denote  $\gamma_1 := \max\{a, \gamma\}$ , where  $a \geq 1$  is from (9). Let  $U(t)$ ,  $t \in [t_0, t_0 + \vartheta)$ , by Definition 6, ensure equality (51) and inclusion (52). Construct

$$V(t) = \begin{cases} U(t), & t \in [t_0, t_0 + \vartheta), \\ 0, & t = t_0 + \vartheta. \end{cases}$$

Then, taking into account (51), we obtain

$$\begin{aligned} \Phi_V(t_0 + \vartheta + 1, t_0) &= \Phi_V(t_0 + \vartheta + 1, t_0 + \vartheta)\Phi_V(t_0 + \vartheta, t_0) \\ &= \Phi_0(t_0 + \vartheta + 1, t_0 + \vartheta)\Phi_U(t_0 + \vartheta, t_0) \\ &= \Phi_0(t_0 + \vartheta + 1, t_0)H. \end{aligned}$$

In addition, for all  $t \in [t_0, t_0 + \vartheta)$ , the estimations

$$\left\| (A(t) + B(t)V(t)C^*(t))^{\pm 1} \right\| \leq \gamma_1$$

hold due to (52). At  $t = t_0 + \vartheta$ , these estimations hold due to (9). ■

**Corollary 1:** If system (7) is  $\vartheta$ -uniformly locally attainable, then this system is  $\vartheta_1$ -uniformly locally attainable for any  $\vartheta_1 \geq \vartheta$ .

## VI. INTERCONNECTION BETWEEN UNIFORM CONSISTENCY AND UNIFORM LOCAL ATTAINABILITY

**Lemma 5:** Let inclusion (9) and inequality (23) hold. Then for any  $t_0 \in \mathbb{Z}$  and  $t \in [t_0, t_0 + \vartheta)$  the following estimations hold:

$$\|Q_2(t, t_0)\| \leq \sqrt{\alpha_8}, \quad (53)$$

$$\|B(t) \otimes \bar{C}(t)\| \leq a\sqrt{\alpha_8} \quad (54)$$

*Proof:* By definition (19) of the matrix  $S_2(t_0, \vartheta)$ , we have

$$Q_2(t, t_0)Q_2^*(t, t_0) \leq S_2(t_0, \vartheta). \quad (55)$$

From (23), it follows that the maximal eigenvalue  $\Lambda$  of the matrix  $Q_2(t, t_0)Q_2^*(t, t_0)$  satisfies the inequality  $\Lambda \leq \alpha_8$ . Hence,

$$\|Q_2(t, t_0)\| = \|Q_2^*(t, t_0)\| = \sqrt{\Lambda} \leq \sqrt{\alpha_8}.$$

By definition,

$$(X_A(t_0, t+1)B(t)) \otimes (X_A^T(t, t_0)\bar{C}(t)) = Q_2(t, t_0).$$



Substituting here  $t_0 = t$ , we obtain

$$\begin{aligned} Q_2(t, t) &= (X_A(t, t+1)B(t)) \otimes (X_A^T(t, t)\bar{C}(t)) \\ &= (A^{-1}(t)B(t)) \otimes \bar{C}(t) \\ &= (A^{-1}(t) \otimes I)(B(t) \otimes \bar{C}(t)). \end{aligned}$$

Hence,  $(B(t) \otimes \bar{C}(t)) = (A(t) \otimes I)Q_2(t, t)$ , so,

$$\|B(t) \otimes \bar{C}(t)\| \leq \|A(t)\| \|Q_2(t, t)\| \leq a\sqrt{\alpha_8}.$$

■

*Lemma 6:* If system (4),(5) is  $\vartheta$ -uniformly consistent, then there exists  $\varepsilon_0 = \varepsilon_0(\vartheta) > 0$  such that, for any  $\varepsilon \in (0, \varepsilon_0]$ , there exists  $\delta = \delta(\varepsilon) > 0$  ensuring the following property: for any  $G \in \mathbf{B}_\delta(0) \subset M_n(\mathbb{K})$  and any  $t_0 \in \mathbb{Z}$  there exists a function

$$[t_0, t_0 + \vartheta) \ni t \mapsto V(t) = V(t; t_0, G) \in \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$$

steering the solution  $Y(\cdot)$  of the system

$$\begin{aligned} Y(t+1) &= A(t)Y(t) + B(t)V(t)C^*(t)X_A(t, t_0) \\ &\quad + B(t)V(t)C^*(t)Y(t) \end{aligned} \quad (56)$$

from the state

$$Y(t_0) = 0 \quad (57)$$

into the state

$$Y(t_0 + \vartheta) = G. \quad (58)$$

*Proof:* By Theorem 2, inequalities (22), (23) hold. Without loss of generality, we can assume that

$$\alpha_7 \leq 1 \leq \alpha_8. \quad (59)$$

Let  $t_0 \in \mathbb{Z}$  be fixed. Let us construct the matrix

$$\mathcal{W}(t, \tau) = \sum_{s=\tau}^{t-1} Q_2(s, t_0)Q_2^*(s, t_0) \in M_{n^2}(\mathbb{K})$$

for  $t > \tau$ ; set  $\mathcal{W}(\tau, \tau) := 0 \in M_{n^2}(\mathbb{K})$ . Then  $\mathcal{W}(t_0 + \vartheta, t_0) = S_2(t_0, \vartheta)$ , hence, by (23),

$$0 < \alpha_7 I \leq \mathcal{W}(t_0 + \vartheta, t_0) \leq \alpha_8 I. \quad (60)$$

For  $t, \tau \in [t_0, t_0 + \vartheta]$ , let us construct the following matrices:

$$N(t, \tau) = \mathcal{W}^{-1}(t_0 + \vartheta, t_0)(X_A(t_0, \tau) \otimes I) \in M_{n^2}(\mathbb{K}),$$

$$K_1(t, \tau) = -(X_A(t, t_0) \otimes I)\mathcal{W}(t_0 + \vartheta, t)N(t_0, \tau) \in M_{n^2}(\mathbb{K}),$$

$$K_2(t, \tau) = (X_A(t, t_0) \otimes I)\mathcal{W}(t, t_0)N(t_0, \tau) \in M_{n^2}(\mathbb{K}),$$

$$K(t, \tau) = \begin{cases} K_1(t, \tau), & t_0 \leq \tau \leq t \leq t_0 + \vartheta, \\ K_2(t, \tau), & t_0 \leq t < \tau \leq t_0 + \vartheta, \end{cases}$$

$$L(t, \tau) = Q_2^*(t, t_0)N(t_0, \tau) \in M_{mk, n^2}(\mathbb{K}),$$

$$\Pi(t, \tau) = \begin{bmatrix} -K(t, \tau) \\ -L(t, \tau) \end{bmatrix} \in M_{n^2+mk, n^2}(\mathbb{K}).$$

Let us show that the following assertion (21) holds:

(a) the pair  $(\widehat{Y}(t), \widehat{V}(t))$ ,  $t \in [t_0, t_0 + \vartheta)$ , is a solution of the nonlinear matrix system of equations

$$\begin{aligned} \text{vec } Y(t) &= - \sum_{s=t_0}^{t_0+\vartheta-1} K(t, s+1) \text{vec } (B(s)V(s)C^*(s)Y(s)) \\ &\quad + K(t, t_0 + \vartheta) \text{vec } G, \end{aligned} \quad (61)$$

$$\begin{aligned} \text{vec } V(t) &= - \sum_{s=t_0}^{t_0+\vartheta-1} L(t, s+1) \text{vec } (B(s)V(s)C^*(s)Y(s)) \\ &\quad + L(t, t_0 + \vartheta) \text{vec } G, \end{aligned} \quad (62)$$

if and only if

(b)  $\widehat{Y}(t)$ ,  $t \in [t_0, t_0 + \vartheta]$ , is a solution of equation (56) (with  $V(t) = \widehat{V}(t)$ ) with the initial condition (57) satisfying condition (58).

Let (b) hold. From (56) and (57), it follows that

$$\begin{aligned} \widehat{Y}(t) &= \sum_{s=t_0}^{t-1} X_A(t, s+1)B(s)\widehat{V}(s)C^*(s) \\ &\quad \times (X_A(s, t_0) + \widehat{Y}(s)). \end{aligned} \quad (63)$$

From (58), it follows that

$$\begin{aligned} &\sum_{s=t_0}^{t_0+\vartheta-1} X_A(t_0 + \vartheta, s+1)B(s)\widehat{V}(s)C^*(s)X_A(s, t_0) \\ &+ \sum_{s=t_0}^{t_0+\vartheta-1} X_A(t_0 + \vartheta, s+1)B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s) = G. \end{aligned} \quad (64)$$

Multiplying (64) from the left by  $X_A(t_0, t_0 + \vartheta)$ , we obtain

$$\begin{aligned} &\sum_{s=t_0}^{t_0+\vartheta-1} X_A(t_0, s+1)B(s)\widehat{V}(s)C^*(s)X_A(s, t_0) \\ &= X_A(t_0, t_0 + \vartheta)G - \sum_{s=t_0}^{t_0+\vartheta-1} X_A(t_0, s+1)B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s). \end{aligned} \quad (65)$$

Applying  $\text{vec}$  to (65), we obtain

$$\begin{aligned} &\sum_{s=t_0}^{t_0+\vartheta-1} Q_2(s, t_0) \text{vec } \widehat{V}(s) \\ &= (X_A(t_0, t_0 + \vartheta) \otimes I) \text{vec } G \\ &\quad - \sum_{s=t_0}^{t_0+\vartheta-1} (X_A(t_0, s+1) \otimes I) \text{vec } (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)). \end{aligned} \quad (66)$$

Solving (66) with respect to  $\text{vec } \widehat{V}(s)$  in the left-hand side, we obtain

$$\begin{aligned} \text{vec } \widehat{V}(t) &= Q_2^*(t, t_0)\mathcal{W}^{-1}(t_0 + \vartheta, t_0) \\ &\quad \times ((X_A(t_0, t_0 + \vartheta) \otimes I) \text{vec } G \end{aligned} \quad (67)$$

$$\begin{aligned}
 & - \sum_{s=t_0}^{t_0+\vartheta-1} (X_A(t_0, s+1) \otimes I) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \\
 & = - \sum_{s=t_0}^{t_0+\vartheta-1} L(t, s+1) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \\
 & \quad + L(t, t_0 + \vartheta) \text{vec} G. \tag{68}
 \end{aligned}$$

Next, from (63), it follows that

$$\begin{aligned}
 \widehat{Y}(t) & = X_A(t, t_0) \\
 & \quad \times \left( \sum_{s=t_0}^{t-1} X_A(t_0, s+1)B(s)\widehat{V}(s)C^*(s)X_A(s, t_0) \right. \\
 & \quad \left. + \sum_{s=t_0}^{t-1} X_A(t_0, s+1)B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s) \right). \tag{69}
 \end{aligned}$$

Applying  $\text{vec}$  to (69), we get

$$\begin{aligned}
 \text{vec} \widehat{Y}(t) & = (X_A(t, t_0) \otimes I) \left( \sum_{s=t_0}^{t-1} Q_2(s, t_0) \text{vec} \widehat{V}(s) \right. \\
 & \quad \left. + \sum_{s=t_0}^{t-1} (X_A(t_0, s+1) \otimes I) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \right). \tag{70}
 \end{aligned}$$

Substituting (67) for  $\text{vec} \widehat{V}(s)$  in the first summand of (70), we get

$$\begin{aligned}
 \text{vec} \widehat{Y}(t) & = (X_A(t, t_0) \otimes I) \left( \mathcal{W}(t, t_0) \mathcal{W}^{-1}(t_0 + \vartheta, t_0) \right. \\
 & \quad \times (X_A(t_0, t_0 + \vartheta) \otimes I) \text{vec} H - \mathcal{W}(t, t_0) \mathcal{W}^{-1}(t_0 + \vartheta, t_0) \\
 & \quad \times \sum_{s=t_0}^{t_0+\vartheta-1} (X_A(t_0, s+1) \otimes I) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \\
 & \quad \left. + \sum_{s=t_0}^{t-1} (X_A(t_0, s+1) \otimes I) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \right). \tag{71}
 \end{aligned}$$

Dividing in (71) the summing over  $s$  from  $t_0$  to  $t_0 + \vartheta - 1$  into two parts — from  $t_0$  to  $t-1$  and from  $t$  to  $t_0 + \vartheta - 1$ , and using the equality  $\mathcal{W}(t, t_0) = \mathcal{W}(t_0 + \vartheta, t_0) - \mathcal{W}(t_0 + \vartheta, t)$ , from (71), we obtain that

$$\begin{aligned}
 \text{vec} \widehat{Y}(t) & = (X_A(t, t_0) \otimes I) \mathcal{W}(t_0 + \vartheta, t) \mathcal{W}^{-1}(t_0 + \vartheta, t_0) \\
 & \quad \times \sum_{s=t_0}^{t-1} (X_A(t_0, s+1) \otimes I) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \\
 & \quad - (X_A(t, t_0) \otimes I) \mathcal{W}(t, t_0) \mathcal{W}^{-1}(t_0 + \vartheta, t_0) \\
 & \quad \times \sum_{s=t}^{t_0+\vartheta-1} (X_A(t_0, s+1) \otimes I) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \\
 & \quad + (X_A(t, t_0) \otimes I) \mathcal{W}(t, t_0) \mathcal{W}^{-1}(t_0 + \vartheta, t_0) \\
 & \quad \times (X_A(t_0, t_0 + \vartheta) \otimes I) \text{vec} G
 \end{aligned}$$

$$\begin{aligned}
 & = - \sum_{s=t_0}^{t_0+\vartheta-1} K(t, s+1) \text{vec} (B(s)\widehat{V}(s)C^*(s)\widehat{Y}(s)) \\
 & \quad + K(t, t_0 + \vartheta) \text{vec} G. \tag{72}
 \end{aligned}$$

Thus, from (68) and (72), it follows (a). All above arguments are reversible, hence, (a)  $\Rightarrow$  (b). So, (2) is true.

Let us prove the existence of a solution to system (61), (62) by a fixed-point theorem. Let  $Z := \begin{bmatrix} \text{vec} Y \\ \text{vec} V \end{bmatrix} \in \mathbb{K}^{n^2+mk}$ . Denote

$$\rho := 2a^{2\vartheta} \alpha_8 \alpha_7^{-1}, \quad \beta := \vartheta \sqrt{\alpha_8},$$

where  $\alpha_7, \alpha_8$  are from (60), and  $a$  from (9). Let

$$\psi(t) := \frac{1 - \sqrt{1 - 4\rho^2 \beta t}}{2\rho\beta}.$$

The function  $t \mapsto \psi(t)$  is well-defined, continuous, and strictly increasing on  $[0, (4\rho^2\beta)^{-1}]$ ;  $\psi(0) = 0$ ,  $\psi((4\rho^2\beta)^{-1}) = (2\rho\beta)^{-1}$ . Hence, the function  $t \mapsto \psi(t)$  is a one-to-one mapping from  $[0, (4\rho^2\beta)^{-1}]$  to  $[0, (2\rho\beta)^{-1}]$ . In addition,  $\psi(t), t \in [0, (4\rho^2\beta)^{-1}]$ , satisfies

$$\rho(\beta\psi^2(t) + t) = \psi(t). \tag{73}$$

We set  $\varepsilon_0 := (2\rho\beta)^{-1}$ , and for any  $\varepsilon \in (0, \varepsilon_0]$  we set  $\delta := \psi^{-1}(\varepsilon)/\sqrt{n}$ . Let us fix an arbitrary matrix  $G \in \mathbf{B}_\delta(0) \subset M_n(\mathbb{K})$ . Let  $g = \text{vec} G$ , then

$$|g| \leq \sqrt{n} \|G\| \leq \sqrt{n} \delta = \psi^{-1}(\varepsilon) \leq (4\rho^2\beta)^{-1},$$

hence,

$$\psi(|g|) \leq \varepsilon. \tag{74}$$

Denote  $E_1 := [I \ 0] \in M_{n^2, n^2+mk}(\mathbb{K})$  where  $I \in M_{n^2}(\mathbb{K})$ ,  $0 \in M_{n^2, mk}(\mathbb{K})$ ;  $E_2 := [0 \ I] \in M_{mk, n^2+mk}(\mathbb{K})$  where  $I \in M_{mk}(\mathbb{K})$ ,  $0 \in M_{mk, n^2}(\mathbb{K})$ . Then  $E_1 Z = \text{vec} Y$ ,  $E_2 Z = \text{vec} V$ . Denote  $v_1: \mathbb{K}^{n^2} \rightarrow M_n(\mathbb{K})$ ,  $v_1(q) := \text{vec}^{-1}(q)$ ;  $v_2: \mathbb{K}^{mk} \rightarrow M_{m,k}(\mathbb{K})$ ,  $v_2(q) := \text{vec}^{-1}(q)$ . Then  $v_1$  and  $v_2$  are linear operators.

Consider the space  $\mathfrak{M} = \{Z: [t_0, t_0 + \vartheta) \rightarrow \mathbb{K}^{n^2+mk}\}$  with the norm  $\|Z(\cdot)\|_{\mathfrak{M}} = \max_{t \in [t_0, t_0+\vartheta)} |Z(t)|$ . The space  $\mathfrak{M}$  is finite-dimensional and isomorphic to  $M_{n^2+mk, \vartheta}(\mathbb{K})$  consisting of the elements  $[Z(t_0), Z(t_0+1), \dots, Z(t_0+\vartheta-1)]$ . Consider the operator  $F: \mathfrak{M} \rightarrow \mathfrak{M}$ :

$$\begin{aligned}
 (FZ)(t) & = -\Pi(t, t_0 + \vartheta) \text{vec} G \\
 & \quad + \sum_{s=t_0}^{t_0+\vartheta-1} \Pi(t, s+1) \text{vec} (B(s)v_2(E_2 Z(s))C^*(s)v_1(E_1 Z(s))).
 \end{aligned}$$

The operator  $F$  is continuous. In the space  $\mathfrak{M}$ , take the convex bounded closed set

$$\mathfrak{N} = \mathfrak{N}(G) = \{Z(\cdot) \in \mathfrak{M}: \|Z(\cdot)\|_{\mathfrak{M}} \leq \psi(|g|)\}.$$

Let us show that  $F$  maps the set  $\mathfrak{N}(G)$  to itself. Let  $Z_0 = \begin{bmatrix} \text{vec } Y_0 \\ \text{vec } V_0 \end{bmatrix}$ ,  $\|Z_0(\cdot)\|_{\mathfrak{N}} \leq \psi(|g|)$ . Then, for all  $t \in [t_0, t_0 + \vartheta)$ ,

$$\|Y_0(t)\| \leq |\text{vec } Y_0(t)| \leq \psi(|g|), \tag{75}$$

$$|\text{vec } V_0(t)| \leq \psi(|g|). \tag{76}$$

Denote  $FZ_0 =: Z_1 =: \begin{bmatrix} \text{vec } Y_1 \\ \text{vec } V_1 \end{bmatrix}$ . Then, for all  $t \in [t_0, t_0 + \vartheta)$ ,

$$\begin{aligned} \text{vec } Y_1(t) = & - \sum_{s=t_0}^{t_0+\vartheta-1} K(t, s+1) \text{vec} (B(s)V_0(s)C^*(s)Y_0(s)) \\ & + K(t, t_0 + \vartheta) \text{vec } G, \end{aligned} \tag{77}$$

$$\begin{aligned} \text{vec } V_1(t) = & - \sum_{s=t_0}^{t_0+\vartheta-1} L(t, s+1) \text{vec} (B(s)V_0(s)C^*(s)Y_0(s)) \\ & + L(t, t_0 + \vartheta) \text{vec } G, \end{aligned} \tag{78}$$

For all  $t \in [t_0, t_0 + \vartheta)$ , we have

$$\mathcal{W}(t_0 + \vartheta, t) \leq \mathcal{W}(t_0 + \vartheta, t_0) \leq \alpha_8 I,$$

$$\mathcal{W}(t, t_0) \leq \mathcal{W}(t_0 + \vartheta, t_0) \leq \alpha_8 I,$$

hence,

$$\|\mathcal{W}(t_0 + \vartheta, t)\| \leq \alpha_8, \quad \|\mathcal{W}(t, t_0)\| \leq \alpha_8. \tag{79}$$

From (60), Lemma 1, and Lemma 2, it follows that

$$\|\mathcal{W}^{-1}(t_0 + \vartheta, t_0)\| \leq \alpha_7^{-1}. \tag{80}$$

For  $t \in [t_0, t_0 + \vartheta)$ , we have

$$\|X_A(t, t_0) \otimes I\| = \|X_A(t, t_0)\| \leq a^{\vartheta-1}. \tag{81}$$

For  $s \in [t_0, t_0 + \vartheta)$ , we have

$$\|X_A(t_0, s+1) \otimes I\| \leq a^{\vartheta}. \tag{82}$$

We have

$$\begin{aligned} & \text{vec} (B(s)V_0(s)C^*(s)Y_0(s)) \\ & = (I \otimes Y_0^T(s))(B(s) \otimes \bar{C}(s)) \text{vec} (V_0(s)). \end{aligned} \tag{83}$$

By (75),

$$\|I \otimes Y_0^T(s)\| = \|Y_0^T(s)\| \leq \psi(|g|). \tag{84}$$

By using (81), (79), (80), (82), (83), (84), (54), (76), and (73), from (77), we obtain

$$\begin{aligned} |\text{vec } Y_1(t)| & \leq \vartheta a^{\vartheta-1} \alpha_8 \alpha_7^{-1} a^{\vartheta} \psi(|g|) a \sqrt{\alpha_8} \psi(|g|) \\ & \quad + a^{\vartheta-1} \alpha_8 \alpha_7^{-1} a^{\vartheta} |g| \leq a^{2\vartheta} \alpha_8 \alpha_7^{-1} (\vartheta \sqrt{\alpha_8} \psi^2(|g|) + |g|) \\ & = \rho(\beta \psi^2(|g|) + |g|)/2 = \psi(|g|)/2. \end{aligned} \tag{85}$$

By using (53), (80), (82), (83), (84), (54), (76), and (73), from (78), we obtain

$$\begin{aligned} |\text{vec } V_1(t)| & \leq \vartheta \sqrt{\alpha_8} \alpha_7^{-1} a^{\vartheta} \psi(|g|) a \sqrt{\alpha_8} \psi(|g|) \\ & \quad + \sqrt{\alpha_8} \alpha_7^{-1} a^{\vartheta} |g|. \end{aligned} \tag{86}$$

Since

$$a \geq 1, \quad \vartheta \geq 1, \quad \alpha_8 \geq 1, \tag{87}$$

from (86), we obtain

$$\begin{aligned} |\text{vec } V_1(t)| & \leq a^{2\vartheta} \alpha_8 \alpha_7^{-1} (\vartheta \sqrt{\alpha_8} \psi^2(|g|) + |g|) \\ & = \rho(\beta \psi^2(|g|) + |g|)/2 = \psi(|g|)/2. \end{aligned} \tag{88}$$

By (85) and (88),

$$|Z_1(t)| \leq |\text{vec } Y_1(t)| + |\text{vec } V_1(t)| \leq \psi(|g|).$$

Thus,  $F$  maps the set  $\mathfrak{N}(G)$  to itself. Therefore, due to the fixed-point theorem, the equation  $Z = FZ$  has at least one solution

$$\widehat{Z}(\cdot) = \widehat{Z}(\cdot, G) = \begin{bmatrix} \widehat{Y}(\cdot, G) \\ \widehat{V}(\cdot, G) \end{bmatrix}$$

in  $\mathfrak{N}(G)$ . By definition of the operator  $F$ , the pair  $(\widehat{Y}(\cdot, G), \widehat{V}(\cdot, G))$  is a solution of system (61), (62). By assertion (A),  $\widehat{Y}(\cdot, G)$  is a solution of (56) (with  $V(t) = \widehat{V}(t, G)$ ) satisfying conditions (57) and (58). In addition, due to (74), the following estimation holds:

$$\max_{t \in [t_0, t_0 + \vartheta)} \|\widehat{V}(t, G)\| \leq \max_{t \in [t_0, t_0 + \vartheta)} |\text{vec } \widehat{V}(t, G)| \leq \psi(|g|) \leq \varepsilon.$$

**Lemma 7:** Let  $Q, S \in M_n(\mathbb{K})$  be arbitrary matrices such that  $\|Q^{-1}\| \leq a$ ,  $\|S\| \leq 1/(4a)$  for some  $a > 0$ . Then the matrix  $Q + S$  is invertible and  $\|(Q + S)^{-1}\| \leq 4a/3$ .

*Proof:* We have  $\|Q^{-1}S\| \leq \|Q^{-1}\| \|S\| \leq 1/4$ . Hence (see [41, p. 351]), the matrix  $I + Q^{-1}S$  is invertible and

$$\|(I + Q^{-1}S)^{-1}\| \leq \frac{1}{1 - \|Q^{-1}S\|} \leq \frac{1}{1 - 1/4} = 4/3.$$

Since the matrices  $Q$  and  $I + Q^{-1}S$  are invertible and the equality  $Q + S = Q(I + Q^{-1}S)$  holds, it follows that the matrix  $Q + S$  is invertible and the estimates  $\|(Q + S)^{-1}\| = \|(I + Q^{-1}S)^{-1}Q^{-1}\| \leq \|(I + Q^{-1}S)^{-1}\| \|Q^{-1}\| \leq 4a/3$  are valid.

**Theorem 10:** Suppose that system (4), (5) is  $\vartheta$ -uniformly consistent. Then system (7) is  $\vartheta$ -uniformly locally attainable.

*Proof:* By Theorem 2, inequalities (22), (23) hold. Set  $a_1 := 4a/3$ . Let arbitrary  $\varepsilon > 0$  be given. Let us set  $\delta = \delta(\varepsilon) > 0$  in accordance to the proof of Lemma 6, namely:

$$\delta := \begin{cases} \psi^{-1}(\varepsilon)/\sqrt{n}, & \text{if } \varepsilon \in (0, \varepsilon_0], \\ \psi^{-1}(\varepsilon_0)/\sqrt{n}, & \text{if } \varepsilon > \varepsilon_0. \end{cases}$$

Then, for any  $G \in \mathbf{B}_\delta(0) \subset M_n(\mathbb{K})$ , (74) holds, where  $g = \text{vec } G$ . Set  $\delta_1 := \delta/a^\vartheta$ . Let  $H \in \mathbf{B}_{\delta_1}(I) \subset M_n(\mathbb{K})$ . Set

$$G := X_A(t_0 + \vartheta, t_0)H - X_A(t_0 + \vartheta, t_0).$$

Hence,

$$\begin{aligned} \|G\| & = \|X_A(t_0 + \vartheta, t_0)H - X_A(t_0 + \vartheta, t_0)\| \\ & \leq \|X_A(t_0 + \vartheta, t_0)\| \|H - I\| \leq a^\vartheta \delta_1 = \delta. \end{aligned}$$

So,  $G \in \mathbf{B}_\delta(0) \subset M_n(\mathbb{K})$ . By Lemma 6, there exists a function  $\widehat{V}: [t_0, t_0 + \vartheta) \rightarrow \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$  such that the

corresponding solution  $\widehat{Y}(t)$  of equation (56) (with  $V(t) = \widehat{V}(t)$ ) with the initial condition  $\widehat{Y}(t_0) = 0$  satisfies condition

$$\widehat{Y}(t_0 + \vartheta) = X_A(t_0 + \vartheta, t_0)H - X_A(t_0 + \vartheta, t_0). \quad (89)$$

Set

$$U(t) := \widehat{V}(t), \quad t \in [t_0, t_0 + \vartheta]. \quad (90)$$

Then  $\|U(t)\| \leq \varepsilon$ .

Consider the function  $Z: [t_0, t_0 + \vartheta] \rightarrow M_n(\mathbb{K})$  defined by the equality

$$Z(t) = X_A(t, t_0) + \widehat{Y}(t). \quad (91)$$

Then  $Z(t_0) = X_A(t_0, t_0) + \widehat{Y}(t_0) = I$ . Next, by (90), we have, for all  $t \in [t_0, t_0 + \vartheta]$ ,

$$\begin{aligned} Z(t+1) &= X_A(t+1, t_0) + \widehat{Y}(t+1) \\ &= A(t)X_A(t, t_0) + A(t)\widehat{Y}(t) \\ &\quad + B(t)U(t)C^*(t)X_A(t, t_0) + B(t)U(t)C^*(t)\widehat{Y}(t) \\ &= (A(t) + B(t)U(t)C^*(t))(X_A(t, t_0) + \widehat{Y}(t)) \\ &= (A(t) + B(t)U(t)C^*(t))Z(t). \end{aligned}$$

By the uniqueness of the solution, we obtain that, for all  $t \in [t_0, t_0 + \vartheta]$ ,

$$Z(t) = \Phi_U(t, t_0). \quad (92)$$

Substituting  $t = t_0 + \vartheta$  into (92), and taking into account (91), (89), and (50), we obtain

$$\Phi_U(t_0 + \vartheta, t_0) = \Phi_0(t_0 + \vartheta, t_0)H. \quad (93)$$

By Lemma 5, from (54), for all  $t \in [t_0, t_0 + \vartheta]$ , we have

$$\|B(t) \otimes \overline{C}(t)\| \leq a\sqrt{\alpha_8}.$$

If  $\varepsilon \leq \varepsilon_0$ , then

$$|\text{vec } U(t)| = |\text{vec } \widehat{V}(t)| \leq \varepsilon \leq \varepsilon_0 = \frac{1}{2\rho\beta} = \frac{\alpha_7}{4a^{2\vartheta}\alpha_8\vartheta\sqrt{\alpha_8}}.$$

If  $\varepsilon > \varepsilon_0$ , then, by construction of  $\delta$ , we have, for  $G \in \mathbf{B}_\delta(0)$ ,

$$|g| \leq \sqrt{n}\|G\| \leq \sqrt{n}\delta = \psi^{-1}(\varepsilon_0),$$

i.e.,  $\psi(|g|) \leq \varepsilon_0$ . Hence,

$$|\text{vec } U(t)| = |\text{vec } \widehat{V}(t)| \leq \psi(|g|) \leq \varepsilon_0,$$

as well. Thus,

$$\begin{aligned} \|B(t)U(t)C^*(t)\| &\leq |\text{vec } (B(t)U(t)C^*(t))| \\ &= |(B(t) \otimes \overline{C}(t))\text{vec } U(t)| \\ &\leq \|B(t) \otimes \overline{C}(t)\| |\text{vec } U(t)| \leq \frac{\alpha_7}{4a^{2\vartheta-1}\alpha_8\vartheta}. \end{aligned} \quad (94)$$

Taking into account (87) and (59), from (94), we obtain that

$$\|B(t)U(t)C^*(t)\| \leq 1/(4a). \quad (95)$$

For all  $[t_0, t_0 + \vartheta]$ , from (9) and (95), we obtain

$$\|A(t) + B(t)U(t)C^*(t)\| \leq a + \frac{1}{4a} \leq a + \frac{a}{4} < a_1. \quad (96)$$

For all  $[t_0, t_0 + \vartheta]$ , from (9), (95), and applying Lemma 7 to  $Q = A(t)$ ,  $S = B(t)U(t)C^*(t)$ , we obtain

$$\|(A(t) + B(t)U(t)C^*(t))^{-1}\| \leq 4a/3 = a_1. \quad (97)$$

It follows from (93), (96), and (97) that system (7) is  $\vartheta$ -uniformly locally attainable. ■

*Corollary 2:* Suppose that system (4), (5) is uniformly consistent. Then system (7) is uniformly locally attainable.

*Remark 6:* By Corollary 2, uniform consistency of system (4), (5) is a sufficient condition for uniform local attainability of system (7). But it is not a necessary condition. The following example 2 confirms this.

*Example 2:* Consider system (4), (5) with  $n = 2, m = 1, k = 1; A(t) \equiv I, t \in \mathbb{Z}; B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}, C(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}, B(t) = B(t+3), C(t) = C(t+3), t \in \mathbb{Z}$ ,

$$\begin{aligned} b_1(0) = c_2(0) = 1, \quad b_2(0) = c_1(0) = 0; \\ b_1(1) = c_2(1) = b_2(1) = 1, \quad c_1(1) = 0; \\ b_1(2) = c_2(2) = 0, \quad b_2(2) = c_1(2) = 1. \end{aligned}$$

Let us show that the system is not uniformly consistent. We have

$$\begin{aligned} \widetilde{B}(t, \tau) &= B(t), \quad \widetilde{C}(t, \tau) = C(t), \\ Q_1(t, \tau) &= Q_1(t), \quad Q_1(t) = Q_1(t+3), \end{aligned} \quad (98)$$

$$Q_1(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad Q_1(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad Q_1(2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (99)$$

From (98) and (99), it follows that, for any  $\tau \in \mathbb{Z}$  and  $\vartheta \in \mathbb{N}$ , the first row of the matrix  $T_1(\tau, \vartheta)$  is equal to zero. Hence,  $\text{rank } T_1(\tau, \vartheta) < n^2$ . Thus, the system is not consistent on  $[\tau, \tau + \vartheta]$ , hence, is not  $\vartheta$ -uniformly consistent. Nevertheless, the system is  $\vartheta$ -uniformly locally attainable for  $\vartheta = 12$ . The proof of this assertion is the same as the proof of the similar assertion for the corresponding continuous-time system. This proof was given in [43, Theorem 3] on the base of Lemmas 2 and 3 [43], and there was constructed the control  $U(\cdot)$  ensuring (51). The proof of Lemma 2 [43] remains the same with a simpler estimation for  $|u_2|$ . The proofs of Lemma 3 and Theorem 3 [43] remains the same. The constructed control  $U(\cdot)$  is  $\vartheta$ -periodic. This will imply estimations (52) for some  $\gamma \geq 1$ .

### VII. LOCAL LYAPUNOV REDUCIBILITY

Along with the free system (2) we consider the perturbed system

$$z(t+1) = A(t)R(t)z(t), \quad t \in \mathbb{Z}, \quad z \in \mathbb{K}^n. \quad (100)$$

The perturbation  $R(\cdot) = (R(t))_{t \in \mathbb{Z}} \subset M_n(\mathbb{K})$  will be called a multiplicative perturbation of system (2).



*Definition 7 (see [44]):* A multiplicative perturbation  $R(\cdot)$  is called *admissible* if the sequence  $(R(t))_{t \in \mathbb{Z}}$  is a Lyapunov sequence.

The set of all admissible multiplicative perturbations will be denoted by  $\mathcal{R}$  and the subset of  $\mathcal{R}$  consisting of perturbations satisfying the condition  $\|R - I\|_\infty < \delta$  will be denoted by  $\mathcal{R}_\delta$ .

Let  $X_{AR}(t, s)$  be the transition matrix of system (100). Below we will need the following lemma.

*Lemma 8 (see [32]):* For any integers  $t > s$  the following equality holds:

$$X_{AR}(t, s) = X_A(t, s) + \sum_{j=s}^{t-1} X_A(t, j)(R(j) - I)X_{AR}(j, s).$$

*Definition 8:* We say that system (7) has the property of *local Lyapunov reducibility* if for any  $\varepsilon > 0$  there exists  $\widehat{\delta} = \widehat{\delta}(\varepsilon) > 0$  such that, for any perturbation  $R(\cdot) \in \mathcal{R}_{\widehat{\delta}}$  there exists a control function  $U: \mathbb{Z} \rightarrow \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$  ensuring dynamical equivalence of systems (7) and (100).

*Theorem 11:* If system (7) is uniformly locally attainable, then this system has the property of local Lyapunov reducibility.

*Proof:* Let  $\vartheta \in \mathbb{N}$  be fixed such that system (7) is  $\vartheta$ -uniformly locally attainable. Let  $a \geq 1$  be from (9) and  $\gamma \geq 1$  be from Definition 6. Let  $\varepsilon > 0$  and  $s \in \mathbb{Z}$  be given. Denote  $J_s := [(s-1)\vartheta, s\vartheta]$ . Construct  $\delta = \delta(\varepsilon) > 0$  due to Definition 6. Let us define  $\widehat{\delta} = \widehat{\delta}(\varepsilon) > 0$  from the equality

$$\widehat{\delta} \vartheta a^{2\vartheta} (\widehat{\delta} + 1)^\vartheta = \delta.$$

Since the function  $t \mapsto t(t+1)^\vartheta$  is continuous and strictly increasing on  $[0, +\infty)$  from 0 to  $+\infty$ , it follows that there exists such  $\widehat{\delta} > 0$ .

Take any  $R(\cdot) \in \mathcal{R}_{\widehat{\delta}}$ . Consider the corresponding multiplicatively perturbed system (100). Then, by Lemma 8, we have

$$\begin{aligned} X_{AR}(s\vartheta, (s-1)\vartheta) &= X_A(s\vartheta, (s-1)\vartheta) \\ &+ \sum_{j=(s-1)\vartheta}^{s\vartheta-1} X_A(s\vartheta, j)(R(j) - I)X_{AR}(j, (s-1)\vartheta) \\ &= X_A(s\vartheta, (s-1)\vartheta)H_s, \end{aligned}$$

where

$$H_s = I + \sum_{j=(s-1)\vartheta}^{s\vartheta-1} X_A((s-1)\vartheta, j)(R(j) - I)X_{AR}(j, (s-1)\vartheta).$$

Note that

$$\begin{aligned} \|H_s - I\| &\leq \sum_{j=(s-1)\vartheta}^{s\vartheta-1} \|X_A((s-1)\vartheta, j)\| \cdot \|R(j) - I\| \\ &\times \|X_{AR}(j, (s-1)\vartheta)\| \leq \vartheta a^\vartheta \|R - I\|_\infty (a \|R\|_\infty)^\vartheta \\ &\leq \widehat{\delta} \vartheta a^{2\vartheta} (\widehat{\delta} + 1)^\vartheta = \delta. \end{aligned}$$

By Definition 6, for this matrix  $H_s$  there exists a control  $U_s: J_s \rightarrow \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$  ensuring the equality

$$\Phi_{U_s}(s\vartheta, (s-1)\vartheta) = \Phi_0(s\vartheta, (s-1)\vartheta)H_s$$

and inclusions (52) for  $t_0 = (s-1)\vartheta$ .

Set  $U(t) := U_s(t)$ ,  $t \in J_s$ . Then  $U: \mathbb{Z} \rightarrow \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$ ,  $(A(t) + B(t)U(t)C^*(t))_{t \in \mathbb{Z}} \subset \mathcal{N}_n(\gamma)$ , and for every integer  $s$  the following equalities hold:

$$\begin{aligned} \Phi_U(s\vartheta, (s-1)\vartheta) &= \Phi_0(s\vartheta, (s-1)\vartheta)H_s \\ &= X_A(s\vartheta, (s-1)\vartheta)H_s = X_{AR}(s\vartheta, (s-1)\vartheta). \end{aligned}$$

From this, it follows, due to Lemma 4, that systems (100) and (7) with the constructed control  $U(\cdot)$  are dynamically equivalent. ■

*Corollary 3:* If system (4), (5) is uniformly consistent then system (7) has the property of local Lyapunov reducibility.

Corollary 3 follows from Corollary 2 and Theorem 11.

### VIII. LOCAL ASSIGNABILITY OF THE LYAPUNOV SPECTRUM

By  $\mathbb{R}_\leq^n$  we denote the set of all nondecreasing sequences of  $n$  real numbers. For a fixed sequence  $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}_\leq^n$  and any  $\delta > 0$ , let us denote by  $\mathcal{O}_\delta(\nu)$  the set of all sequences  $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}_\leq^n$  such that  $\max_{j=1, \dots, n} |\nu_j - \mu_j| < \delta$ . In other words,  $\mathcal{O}_\delta(\nu)$  is a  $\delta$ -neighborhood of the sequence  $\nu \in \mathbb{R}_\leq^n$  with respect to the metric generated by the vector  $l_\infty$  norm of the space  $\mathbb{R}^n$  [41, p. 265] on its subset  $\mathbb{R}_\leq^n$ .

Suppose that (9) holds. Then the Lyapunov spectrum

$$\lambda(A) = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)) \in \mathbb{R}_\leq^n$$

of the free system (2) is well-defined (see [13], [30] for the definition of this concept).

Let the control in the system (4), (5) have the form of static output feedback (6). We identify (6) with the sequence  $U(\cdot) = (U(t))_{t \in \mathbb{Z}}$ .

*Definition 9:* A bounded function  $U: \mathbb{Z} \rightarrow M_{m,k}(\mathbb{K})$  is said to be an *admissible feedback control* for system (4), (5) if  $(A(t) + B(t)U(t)C^*(t))_{t \in \mathbb{Z}}$  is a Lyapunov sequence.

Let  $U(\cdot)$  be any admissible feedback control for system (4), (5). Then, for a closed-loop system (7), the Lyapunov spectrum  $\lambda(A + BUC^*)$  is well-defined.

*Definition 10:* The Lyapunov spectrum of system (7) is called:

1) *locally assignable* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any  $\mu \in \mathcal{O}_\delta(\lambda(A))$  there exists an admissible feedback control  $U: \mathbb{Z} \rightarrow \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$  for system (4), (5), ensuring the equality

$$\lambda(A + BUC^*) = \mu; \tag{101}$$

2) *proportionally locally assignable* if there exist  $\ell > 0$  and  $\delta > 0$  such that for any sequence  $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{O}_\delta(\lambda(A))$  there exists an admissible feedback control  $U(\cdot)$  for system (4), (5), satisfying the estimate

$$\|U\|_\infty \leq \ell \max_{j=1, \dots, n} |\lambda_j(A) - \mu_j|$$

and providing the validity of the relation (101).

It is clear that the property of proportional local assignability implies the property of local assignability, and the reverse implication is generally not true.

It turns out that the concepts of local and proportional local assignability of the Lyapunov spectrum of system (7) are closely related to the concept of proportional global assignability of the Lyapunov spectrum of the system (100), in which the multiplicative perturbation  $R(\cdot)$  is understood as a control.

*Definition 11:* The Lyapunov spectrum of system (100) is called *proportionally globally assignable* if for any  $\Delta > 0$  there exists  $\widehat{\ell} = \widehat{\ell}(\Delta) > 0$  such that for any sequence  $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{O}_\Delta(\lambda(A))$  there exists a perturbation  $R(\cdot) \in \mathcal{R}$  satisfying the estimation

$$\|R - I\|_\infty \leq \widehat{\ell} \max_{j=1, \dots, n} |\lambda_j(A) - \mu_j| \quad (102)$$

and providing the validity of the relation

$$\lambda(AR) = \mu. \quad (103)$$

*Theorem 12:* Suppose that system (7) has the property of local Lyapunov reducibility. If the Lyapunov spectrum of (100) is proportionally globally assignable, then the Lyapunov spectrum of (7) is locally assignable.

*Proof:* From the proportional global assignability of the spectrum of (100) it follows that for  $\Delta = 1$  there exists  $\widehat{\ell} = \widehat{\ell}(1) > 0$  such that for any  $\mu \in \mathcal{O}_1(\lambda(A))$  there exists a perturbation  $R(\cdot) \in \mathcal{R}$  satisfying the estimate (102) and providing the validity of relation (103). Since (7) has the property of local Lyapunov reducibility, then according to Definition 8, for any  $\varepsilon > 0$  there exists  $\widehat{\delta} = \widehat{\delta}(\varepsilon) > 0$  such that, for any sequence  $R(\cdot) \in \mathcal{R}_{\widehat{\delta}}$ , there exists a control function  $U: \mathbb{Z} \rightarrow \mathbf{B}_\varepsilon(0) \subset M_{m,k}(\mathbb{K})$  ensuring dynamical equivalence of systems (7) and (100). Let any  $\varepsilon > 0$  be given. Define  $\delta = \delta(\varepsilon) := \min\{1, \widehat{\delta}/\widehat{\ell}\}$  and consider any sequence  $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{O}_\delta(\lambda(A)) \subset \mathcal{O}_1(\lambda(A))$ . From the proportional global assignability of the spectrum of (100) it follows that there exists a sequence  $R \in \mathcal{R}$  such that

$$\|R - I\|_\infty \leq \widehat{\ell} \max_{j=1, \dots, n} |\lambda_j(A) - \mu_j| \leq \widehat{\ell}\delta \leq \widehat{\delta}$$

and (103) is satisfied. By the local Lyapunov reducibility of system (7) for this sequence  $R(\cdot)$  there exists an admissible feedback control  $U(\cdot)$  for system (4), (5) such that  $\|U\|_\infty < \varepsilon$  and such that the systems (7) and (100) are dynamically equivalent. Since equivalent systems have the same spectrum, it follows that

$$\lambda(A + BUC^*) = \lambda(AR) = \mu. \quad \blacksquare$$

*Corollary 4:* Suppose that system (4), (5) is uniformly consistent. If the Lyapunov spectrum of (100) is proportionally globally assignable, then the Lyapunov spectrum of (7) is locally assignable.

This corollary follows from Corollary 3 and Theorem 12.

Consider a linear control system (4). Let the control in this system have the form of static state feedback

$$u(t) = U(t)x(t), \quad t \in \mathbb{Z}.$$

We get the closed-loop system (3). This system has the form (7) with  $k = n$  and  $C(t) \equiv I, t \in \mathbb{Z}$ , therefore, for this system, one can introduce the concepts of local assignability and proportional local assignability of the Lyapunov spectrum.

*Corollary 5:* Suppose that system (4) is uniformly completely controllable. If the Lyapunov spectrum of (100) is globally proportionally assignable, then the Lyapunov spectrum of (3) is locally assignable.

This corollary follows from Theorem 7 and Corollary 4.

Note that in the paper [32] the more strong assertion was proved.

*Theorem 13 (see [32]):* If system (4) is uniformly completely controllable and the Lyapunov spectrum of (100) is globally proportionally assignable, then the Lyapunov spectrum of (3) is proportionally locally assignable.

The method of proving this theorem used in [32] is not applicable to the input-output system (4), (5).

Now we present results about local assignability of the Lyapunov spectrum of system (7). They are expressed in the forms of certain concepts from the asymptotic theory of linear systems, which are defined below.

*Definition 12:* System (2) is called *diagonalizable* if it is dynamically equivalent to system (43) with a diagonal matrix  $A(t), t \in \mathbb{Z}$ .

*Definition 13 (see [24, p. 63]):* System (2) is called *regular* (in the Lyapunov sense) if the following equality holds:

$$\sum_{i=1}^n \lambda_i(A) = \liminf_{t \rightarrow +\infty} \frac{1}{t} \sum_{j=1}^{t-1} \ln |\det A(j)|.$$

The notion of regularity of linear differential systems was introduced in the famous paper of Lyapunov [8]. Some facts about regularity of discrete equations may be found in the works [24], [45], [46]. Let us notice that all time-invariant or all periodic systems are regular.

*Definition 14 (see [44]):* The Lyapunov spectrum of system (2) is called *stable* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\lambda(\mathcal{R}_\delta) \subset \mathcal{O}_\varepsilon(\lambda(A))$ , where  $\lambda(\mathcal{R}_\delta) := \{\lambda(AR) | R \in \mathcal{R}_\delta\}$ .

The effect of instability of the Lyapunov spectrum under the influence of small coefficient perturbations for linear continuous-time systems was discovered by O. Perron [47]. Later the stability property of the Lyapunov spectrum for these systems was investigated in [48], [49]. The study of this property for discrete-time systems was started in [44].

In [32], sufficient conditions were obtained for proportional global assignability of the Lyapunov spectrum of system (100).

*Theorem 14 (see [32]):* Assume that at least one of the following conditions holds:

- (i) system (2) is regular;
- (ii) system (2) is diagonalizable;
- (iii) the Lyapunov spectrum of system (2) is stable.

Then the Lyapunov spectrum of system (100) is proportionally globally assignable.

*Theorem 15:* Suppose that system (7) has the property of local Lyapunov reducibility and at least one of the conditions (i), (ii), (iii) of Theorem 14 holds. Then the Lyapunov spectrum of system (7) is locally assignable.

This Theorem follows from Theorems 12 and 14.

*Corollary 6:* Suppose that system (4), (5) is uniformly consistent. If at least one of the conditions (i), (ii), (iii) of Theorem 14 holds, then the Lyapunov spectrum of (7) is locally assignable.

Corollary 6 follows from Corollary 4 and Theorem 15.

*Corollary 7:* Suppose that system (4) is uniformly completely controllable. If at least one of the conditions (i), (ii), (iii) of Theorem 14 holds, then the Lyapunov spectrum of (3) is locally assignable.

Corollary 7 follows from Theorem 7 and Corollary 6.

From Theorem 13, it follows a more strong assertion, which have been proved in [32].

*Theorem 16 (see [32]):* If system (4) is uniformly completely controllable and at least one of the conditions (i), (ii), (iii) of Theorem 14 holds, then the Lyapunov spectrum of (3) is proportionally locally assignable.

### IX. EXAMPLE

Let us illustrate the results obtained. Consider system (4), (5) where  $\mathbb{K} = \mathbb{R}$ ,  $m = k = 1$ ,  $n = 2$ ,

$$\begin{aligned} A(t) &\equiv A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; & B(t) &\equiv B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \\ C(t) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & t = 0, 1; & C(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & t = 2, 3; \\ C(t+4) &= C(t), & t \in \mathbb{Z}. \end{aligned} \tag{104}$$

System (4), (5), (104) is a periodic with the period  $\omega = 4$ . Set  $\vartheta := 4$ . Let us construct matrices (13), (19) for  $\tau = 0, 1, 2, 3$ . For any  $\tau \in \mathbb{Z}$ , we have

$$\begin{aligned} \tilde{B}(\tau, \tau) &= \tilde{B}(\tau + 2, \tau) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\ \tilde{B}(\tau + 1, \tau) &= \tilde{B}(\tau + 3, \tau) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

We have

$$\begin{aligned} \tilde{C}(0, 0) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \tilde{C}(1, 0) &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \tilde{C}(2, 0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{C}(3, 0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \tilde{C}(1, 1) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \tilde{C}(2, 1) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \tilde{C}(3, 1) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{C}(4, 1) &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \tilde{C}(2, 2) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{C}(3, 2) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \tilde{C}(4, 2) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \tilde{C}(5, 2) &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \tilde{C}(3, 3) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{C}(4, 3) &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \tilde{C}(5, 3) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \tilde{C}(6, 3) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

So,

$$\begin{aligned} T_1(0, 4) &= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix}, & T_1(1, 4) &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \\ T_1(2, 4) &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, & T_1(3, 4) &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Hence,

$$S_1(0, 4) = S_1(1, 4) = S_1(2, 4) = S_1(3, 4) = 2I.$$

By  $\vartheta$ -periodicity of the system, we have  $S_1(\tau, \vartheta) = 2I$  for any  $\tau \in \mathbb{Z}$ . Since  $X_A(\tau + \vartheta, \tau) = I$  for any  $\tau \in \mathbb{Z}$ , we have  $S_2(\tau, \vartheta) = 2I$ . Hence, the system is  $\vartheta$ -uniformly consistent. In accordance with the notation of the proof of Lemma 6, we have  $a = 1$ ,  $\alpha_8 = 2$ ,  $\alpha_7 = 1$ ,  $\rho = 4$ ,  $\beta = 4\sqrt{2}$ ,  $\psi(t) = (1 - \sqrt{1 - 256\sqrt{2}t})/(32\sqrt{2})$ ,  $\varepsilon_0 = (32\sqrt{2})^{-1}$ ,  $\psi^{-1}(\varepsilon_0) = (4\rho^2\beta)^{-1} = (256\sqrt{2})^{-1}$ . Let arbitrary  $\varepsilon > 0$  be given. Set

$$\delta := \begin{cases} \psi^{-1}(\varepsilon)/\sqrt{n}, & \text{if } \varepsilon \in (0, \varepsilon_0), \\ \psi^{-1}(\varepsilon_0)/\sqrt{n}, & \text{if } \varepsilon > \varepsilon_0. \end{cases}$$

Then  $\delta \leq 1/512$ . Let

$$G = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix},$$

$G \in \mathbf{B}_\delta(0)$ . Let  $t_0 = 0$ . Construct system (61), (62). Let

$$Y(t) = \begin{bmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{bmatrix}, \quad V(t) = [v(t)].$$

Then system (61), (62) represents the following nonlinear system of equations:

$$\begin{aligned} y_{11}(0) &= 0, y_{12}(0) = 0, y_{11}(1) = 0, y_{12}(1) = v(0), \\ y_{21}(0) &= 0, y_{22}(0) = 0, y_{21}(1) = 0, y_{22}(1) = v(0), \\ y_{11}(2) &= 0, y_{12}(2) = v(1)v(0) + v(0) - v(1), \\ y_{21}(2) &= 0, y_{22}(2) = v(1)v(0) - v(0) - v(1), \\ y_{11}(3) &= v(2), y_{12}(3) = v(2)y_{12}(2) + y_{12}(2), \\ y_{21}(3) &= v(2), y_{22}(3) = v(2)y_{12}(2) - y_{22}(2), \\ v(3)v(2) &+ v(2) + v(3) = g_1, v(3)y_{12}(3) + y_{12}(3) = g_2, \\ v(3)v(2) &- v(2) + v(3) = g_3, v(3)y_{12}(3) - y_{22}(3) = g_4. \end{aligned}$$

Since  $\delta > 0$  is sufficiently small, this system has a solution  $(\hat{Y}(t), \hat{V}(t))$ ,  $t = 0, 1, 2, 3$ . This solution can be found explicitly. Finding this solution, we obtain

$$\hat{v}(0) = \frac{g_2 - g_4 - g_1g_4 + g_2g_3}{2(1 + g_1)},$$

$$\begin{aligned} \widehat{v}(1) &= -\frac{g_2 + g_4 + g_1g_4 - g_2g_3}{2 + 2g_1 - g_2 + g_4 + g_1g_4 - g_2g_3}, \\ \widehat{v}(2) &= \frac{g_1 - g_3}{2}, \quad \widehat{v}(3) = \frac{g_1 + g_3}{2 + g_1 - g_3}, \\ \widehat{Y}(0) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \widehat{Y}(1) &= \begin{bmatrix} 0 & \frac{g_2 - g_4 - g_1g_4 + g_2g_3}{2(1 + g_1)} \\ 0 & \frac{g_2 - g_4 - g_1g_4 + g_2g_3}{2(1 + g_1)} \end{bmatrix}, \\ \widehat{Y}(2) &= \begin{bmatrix} 0 & \frac{g_2}{1 + g_1} \\ 0 & \frac{g_4 + g_1g_4 - g_2g_3}{1 + g_1} \end{bmatrix}, \\ \widehat{Y}(3) &= \begin{bmatrix} \frac{g_1 - g_3}{2} & \frac{g_2(2 + g_1 - g_3)}{2(1 + g_1)} \\ \frac{g_1 - g_3}{2} & \frac{g_1g_2 + g_2g_3 - 2g_4 - 2g_1g_4}{2(1 + g_1)} \end{bmatrix}. \end{aligned}$$

One can check that the matrix sequence  $\widehat{Y}(\cdot)$  is a solution of system (56) and boundary condition (58) holds. So, Lemma 6 is confirmed on  $[t_0, t_0 + \vartheta) = [0, 4)$ . For  $t_0 = 1, 2, 3$ , one can check the solvability of system (61), (62) similarly.

Construct the control  $U(t) := \widehat{V}(t)$ , and the function  $Z(t) = X_A(t, t_0) + \widehat{Y}(t)$ . One can check that  $Z(t)$  satisfies equation

$$Z(t + 1) = (A(t) + B(t)U(t)C^*(t))Z(t)$$

for  $t \in [0, 4)$ , and  $Z(0) = I$ . Hence, (92) holds and equality (93) holds. For  $t_0 = 1, 2, 3$ , one can also construct  $U(\cdot)$  ensuring (93). So, the closed-loop system is  $\vartheta$ -uniformly locally attainable. This confirms Theorem 10.

The free system is diagonal. By Corollary 6, the Lyapunov spectrum of the closed-loop system (7) is locally assignable. Let us check it. Note that the Lyapunov spectrum  $\lambda(A) = (\lambda_1(A), \lambda_2(A))$  of the free system (2) consists of numbers  $\lambda_1(A) = \lambda_2(A) = 0$ . The free system is Lyapunov stable but not asymptotically stable [24, p.33]. Let  $\delta_1 > 0$  be such that, if  $|v| < \delta_1$ , then  $|e^{4v} - 1| < \delta$ . Take any sequence  $\mu = (\mu_1, \mu_2) \in \mathcal{O}_{\delta_1}(\lambda(A))$ . Set  $H := \text{diag}(\exp(4\mu_1), \exp(4\mu_2)) \in M_2(\mathbb{R})$ . Let  $G = H - I$ . Then  $G \in \mathcal{B}_{\delta}(0)$ . Let us construct, for this matrix  $G$ , the control  $U(t) = \widehat{V}(t)$ ,  $t \in [0, 4)$ , as above. Then  $\Phi_U(4, 0) = \Phi_0(4, 0)H = H$  and  $\|U(t)\| \leq \varepsilon$ ,  $t \in [0, 4)$ . Let us extend  $U(t)$ ,  $t \in [0, 4)$ , periodically on  $\mathbb{Z}$  with the period  $\omega = 4$ . Then  $\Phi_U(4s, 0) = \text{diag}(\exp(4s\mu_1), \exp(4s\mu_2))$ ,  $s \in \mathbb{N}$ . Hence,  $\lambda_1(A + BUC^*) = \mu_1$ ,  $\lambda_2(A + BUC^*) = \mu_2$ . If the selected values  $\mu_1, \mu_2$  are negative, then this means that the closed-loop system is asymptotically stable, and each its solution tends exponentially to zero as  $t \rightarrow +\infty$ . If  $\mu_1 < 0 < \mu_2$ , then the closed-loop system is hyperbolic. If  $\mu_1, \mu_2$  are positive, then the closed-loop system is unstable, and the norm of each of its nontrivial solutions tends exponentially to infinity as  $t \rightarrow +\infty$ .

## X. CONCLUSION

In the paper, linear discrete time-varying input-output systems have been studied. The problem of local assignability of the Lyapunov spectrum by static output feedback control have been investigated. The notion of uniform consistency for discrete-time systems have been introduced. This property is, in some sense, the extension of the notion of uniform complete controllability for input-output systems. The property of uniform consistency have been developed in detail, the necessary and sufficient conditions for this property have been obtained. The notions of uniform local attainability and local Lyapunov reducibility have been introduced, which were previously introduced for continuous-time systems. We have proved that uniform consistency implies uniform local attainability of the closed-loop system. In turn, uniform local attainability implies local Lyapunov reducibility. We have proved that, for a locally Lyapunov reducible system, the Lyapunov spectrum is locally assignable, if the free system is diagonalizable, or regular (in the Lyapunov sense), or has the stable Lyapunov spectrum. This is an extension of the corresponding results proved earlier for continuous-time systems and for discrete-time systems with static state feedback.

Further development of the results of the paper could be as follows. We plan to study in more detail the properties of uniform consistency and uniform local attainability, including for the case when conditions (9) and (10) are not satisfied. In addition, the invariance of these properties under the Lyapunov transformations will be proved. We also plan to prove that, in Theorem 15, the conditions (i)–(iii) could be weakened. Further, we plan to extend the theory of uniformly consistent systems to more general systems, namely, to bilinear systems of the form

$$x(t + 1) = (A(t) + u_1(t)A_1(t) + \dots + u_r(t)A_r(t))x(t) \quad (105)$$

and to obtain the corresponding results on uniform local attainability, local Lyapunov reducibility and Lyapunov spectrum assignability for systems (105). Some results concerning global assignability of Lyapunov spectrum for *time-invariant* consistent systems of the form (105) were obtained in [26], [37].

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