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Maximum Likelihood Estimation of Stochastic Fractional Singular Models

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ABSTRACT Using the non-causal nature of a fractional-order singular (FOS) model, this paper deals with the modification of an estimation algorithm developed by Nosrati and Shafiee, and demonstrates how the derived estimation procedure can be adjusted by additional information related to the future dynamics. The procedure adopts the maximum likelihood (ML) method leading to a 3-block fractional singular Kalman filter (FSKF). In addition to some conditions on existence and uniqueness of solutions for discrete-time linear stochastic FOS models, the estimability analyses are given and an optimal filter is presented. Finally, the performance of the derived filter is verified and validated via numerical simulation on a three machine infinite bus system.

INDEX TERMS Filtering and estimation, maximum likelihood approach, fractional-order singular model, non-causality, machine infinite bus system.

I. INTRODUCTION

Fractional or non-integer calculus steers us to a more general form called fractional-order singular (FOS) models, which are utilized to model various physical systems and scientific processes [2]–[4], and at the same time, share characteristics of both non-integer theories and singular systems. Although there have been some notable studies in stability [5], normalization and stabilization [6], estimator and observer design problems [1], [7], [8], issues have been reported for deterministic FOS models such as control designs for nonlinear and rectangular FOS systems, admissibility conditions and stabilization problems for convex intervals $1 < \alpha < 2$ of fractional order or based on the complex domain that has broader descriptions and more complex behaviors than linear square FOS systems with fractional order $0 < \alpha < 1$. However, there are many remaining challenges and few studies have considered the stochastic terms for applications in estimation and filter design.

In the state estimator problem, the Kalman filter (KF) is well established for optimal state observers [9]. The extension of this algorithm to the discrete time non-integer order model was elaborated in [10], which has been called the fractional KF (FKF). On the other hand, recursive state estimations for discrete-time singular models have been studied, in which different algorithms based on existing methods were derived [11]. That study transformed a singular model to a normal form to apply the classic KF algorithms to estimate the states of the system [12]. Moreover, some normal approaches such as the least-squares (LS), ML and deterministic approaches [13]–[15] were applied to solve the estimation problem without the need for transformation, yielding a singular KF (SKF).

Recently, the problem of filtering for FOS cases has been taken into account to a limited extent. In [12], in an indirect method, a FOS model in its continuous form was first decomposed into normal sub-systems with several transformations, and then, existing KF algorithms were used. Decomposition of the system may result in an important loss of relevant information, and in many cases, with inaccurate estimation of state variables. In order to overcome this problem and to decrease the estimation error, a direct approach is required. Using the data-fitting problem approach, a filtering algorithm directly from the original TI FOS model was derived [1]. That study aimed to formulate the deterministic estimation

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algorithm of such a model and introduced the fractional singular KF (FSKF) algorithm. In both these two studies, some issued such as solvability theorems, future dynamics, singular measurement noise remained open. The research motivation here addresses the noncausal FOS system filtering issue of using ML estimation concepts, which motivate us to derive a 3-block FSKF algorithm and Riccati equation that not only considers the future dynamics, but also performs when the measurement noise is singular, in which the derived algorithm requires only standard matrix inverses instead of pseudo inverses. The main contributions and outcomes of the present work are outlined as follows,

1) Solvability Theorem: The necessary and sufficient conditions on the solvability of discrete FOS models are established in this article. The model is considered to be stochastic with zero-mean white Gaussian measurement and process noise, in the form of Grunwald-Letnikov (GL) difference equations with consistent initial conditions.
2) ML-Based Estimation Algorithm: This article presents a review of the fundamental theories and aspects related to the ML method focused on deducing a valid form based on perfect measurements. This motivates us to derive an estimation algorithm for the considered system. The results demonstrate that the derived filter and its corresponding Riccati equation coincide with the results reported in [1], which cover all classical KFs.
3) Adjusted Estimation Algorithm: This work studies the impact of future information on the current states resulted from the non-causal dynamics of the singular part of the model, and modifies the derived FSKF to an adjusted estimate.
4) Modeling and Simulation (A three-machine infinite bus system): The estimation performance of the obtained filter is validated using a numerical simulation on a new FOS model of a three-machine infinite bus power system.

In the following, and in Section 2, we aim to bring some preliminaries on the discrete FOS model in its linear stochastic form, followed by necessary and sufficient conditions on an unique solution to this model. Then in Section 3, we present the ML estimation method that motivates us to take steps towards the filtering issue of the model and a modified version of the filter with respect to the information resulting from the constraints, which include future dynamics. Finally in Section 4, a FOS model of a three-machine infinite bus system is considered as a case study to verify our hypothesis.

II. PRELIMINARIES

Consider the following discrete stochastic FOS model

$$E\, {}_{0}^{GL}\triangle_{k+1}^{\alpha} x_{k+1} = Ax_k + w_k, \tag{1a}$$
$$y_k = Cx_k + v_k. \tag{1b}$$

where $E \in \mathbb{R}^{n\times n}$ with rank$E < n$, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$ are real constant matrices, $x_k \in \mathbb{R}^n$ is the state vector and $y_k \in \mathbb{R}^p$ is the output vector, and the sequences $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^p$ are zero mean white vectors. Also, ${}_{0}^{GL}\triangle_{k+1}^{\alpha} x_{k+1} \triangleq \left[{}_{0}^{GL}\triangle_{k+1}^{\alpha_1} x_{1,k+1} \;\cdots\; {}_{0}^{GL}\triangle_{k+1}^{\alpha_n} x_{n,k+1}\right]$, where $\alpha_i \in \mathbb{R}^n$, $i = 1, \ldots, n$ are the fractional orders assigned to n equations. Furthermore, ${}_{0}^{GL}\triangle_{k}^{\alpha_i}$ denotes the fractional GL difference defined as $h^{\alpha_i}\, {}_{0}^{GL}\triangle_{k}^{\alpha_i} x_k = \sum_{j=0}^{k}(-1)^j \Upsilon_j x_{k-j}$, in which h is the value of duration or the interval of each two samples out of k samples, where the derivative is computed, and $\Upsilon_j = \begin{bmatrix}\alpha_i\\ j\end{bmatrix}$.

Lemma 1 [1]: System (1) is called a regular FOS system if and only if ∃Q, P in which $QEP = diag(I_{n_1}, N)$ and $QAP = diag(A_1, I_{n_2})$, where Q and P are two invertible matrices of the appropriate size. Also, $A_1 \in \mathbb{R}^{n_1\times n_1}$ and $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix of the index μ, and $n_1 + n_2 = n$.

According to this lemma, a necessary and sufficient condition for the FOS model (1) to be regular is $\exists z_0 \in \mathbb{C}$ in which $\det(z_0^{\alpha} - A) \neq 0$.

Lemma 2: For matrix N if $\mu > 1$, the transfer function of the regular FOS model (1), i.e. $G(z) = C(z^{\alpha}E - A)^{-1}B$, for an input matrix B, might be improper.

Theorem 1: Let initial conditions of the FOS model (1) are consistent. Then, this model has a unique solution if and only if it is regular.

Proof: Sufficiency: Suppose that the FOS model (1) is regular, i.e., the determinant of matrix $z_0^{\alpha}E - A$ differs from zero for some complex number z_0. Then, according to Lemma 1, system (1a) can be decomposed into

$${}_{0}^{GL}\triangle_{k+1}^{\alpha} x_{1,k+1} = A_1 x_{1,k} + P_1 w_k, \tag{2a}$$
$$N\, {}_{0}^{GL}\triangle_{k+1}^{\alpha} x_{2,k+1} = x_{2,k} + P_2 w_k, \tag{2b}$$

where $x_{1,k} \in \mathbb{R}^{n_1}$, $x_{2,k} \in \mathbb{R}^{n_2}$, and $\begin{bmatrix}P_1^T & P_2^T\end{bmatrix} = P^T$ with $P_1 \in \mathbb{R}^{n_1\times n}$ and $P_2 \in \mathbb{R}^{n_2\times n}$. From (2b),

$$(N - I_{n_2}q^{-1})x_{2,k+1} = -N\sum_{j=1}^{k+1}(-1)^j \Upsilon_j x_{k+1-j} + P_2 w_k \tag{3}$$

where q^{-1} is the shift operator in which $q^{-1}x_k = x_{k-1}$. Now, applying the Taylor series expansion for the inverse of the matrix $(N - I_{n_2}q^{-1})$, one has the following equality:

$$\begin{aligned}(N - I_{n_2}q^{-1})^{-1} &= (-q^{-1}(I_{n_2} - Nq))^{-1}\\ &= -(I_{n_2} - Nq)^{-1}q = -(\sum_{i=0}^{\infty} N^i q^i)q\\ &= -\sum_{i=0}^{\infty} N^i q^{i+1}.\end{aligned}$$

In view of the relation $N^i = 0$ for $i \geq \mu$ (μ is the nilpotency index of the nilpotent matrix N), one obtains $(N - I_{n_2}q^{-1})^{-1} = -\sum_{i=0}^{\mu-1} N^i q^{i+1}$. Substituting this equality into (3) yields

$$x_{2,k} = -\sum_{i=0}^{\mu-1} N^i q^{i+1}(-N\sum_{j=1}^{k}(-1)^j \Upsilon_j x_{k-j} + P_2 w_{k-1})$$
$$= \sum_{i=0}^{\mu-1} N^{i+1}\sum_{j=1}^{k}(-1)^j \Upsilon_j x_{k+1+i-j} - \sum_{i=0}^{\mu-1} N^i P_2 w_{k+i}. \quad (4)$$

Also, for (2a) one has the solution $x_{1,k} = \phi_k x_{1,0} + \sum_{j=0}^{k-1}\phi_{k-j-1}P_1 w_{1,j}$ using iterative methods, where $x_{k+1} = \phi_k A_{1\alpha} + \sum_{i=2}^{k+1}(-1)^i \Upsilon_j \phi_{k-i+1}$ with $\phi_0 = I_{n_1}$ and $A_{1\alpha} = A_1 + \alpha I_{n_2}$. Therefore, from these two solutions, one can conclude that there is a solution for the FOS model (1). To show its uniqueness, in equation (2b), let $N = \text{diag}\left(N_1, N_2, \cdots, N_q\right)$ and $P_2^T = \left[\tilde{P}_1^T \; \tilde{P}_2^T \; \cdots \; \tilde{P}_q^T\right]$, where $\tilde{P}_i \in \mathbb{R}^{g_i\times n_2}$, $\sum_i g_i = n_2$ and $N_i \in \mathbb{R}^{g_i\times g_i}$, $i = 1, 2, \cdots, q$ are shift matrices with ones only on the super-diagonal, and zeroes elsewhere. Therefore, the ith relation of the equation (2b) has the form of $N_i \, {}_0^{GL}\triangle_{k+1}^{\alpha} x_{2i,k+1} = x_{2i,k} + \tilde{P}_i w_k$, where $x_{2i,k} = \left[x_{2i,k}^{(1)} \; x_{2i,k}^{(2)} \; \cdots \; x_{2i,k}^{(g_i)}\right]^T$ and $\tilde{P}_i = \left[\tilde{P}_i^{(1)} \; \tilde{P}_i^{(2)} \; \cdots \; \tilde{P}_i^{(g_i)}\right]$, which can be decomposed into the following equations:

$$\begin{aligned}
{}_0^{GL}\triangle_{k+1}^{\alpha} x_{2i,k+1}^{(2)} &= x_{2i,k}^{(1)} + \tilde{P}_i^{(1)} w_k\\
{}_0^{GL}\triangle_{k+1}^{\alpha} x_{2i,k+1}^{(3)} &= x_{2i,k}^{(2)} + \tilde{P}_i^{(2)} w_k\\
&\vdots\\
{}_0^{GL}\triangle_{k+1}^{\alpha} x_{2i,k+1}^{(g_i)} &= x_{2i,k}^{(g_i-1)} + \tilde{P}_i^{(g_i-1)} w_k\\
0 &= x_{2i,k}^{(g_i)} + \tilde{P}_i^{(g_i)} w_k. \quad (5)
\end{aligned}$$

It is obvious that (5) is equivalent to two equations $0 = x_{2i,k}^{(g_i)} + \tilde{P}_i^{(g_i)} w_k$ and $x_{2i,k}^{(g_i'-1)} = {}_0^{GL}\triangle_{k+1}^{\alpha} x_{2i,k+1}^{(g_i')} - \tilde{P}_i^{(g_i'-1)} w_k$, $g_i' = g_i, g_i - 1, \cdots, 2$. Suppose that equation (2) has two solutions defined by $x_{2,k}'$ and $x_{2,k}''$. From the first equation, one has that system (5) has a unique solution $x_{2,k}$, $i = 1, 2, \cdots, q$, and as a result, one can conclude that $x_{2,k}$ is a unique solution of equation (2b). Due to the uniqueness of solution for system (2a), the first part of proof is completed.

Necessity: Let us assume the solution of the FOS model (1) is unique. According to Kronecker's theorem [16], for any two matrices E and A, there exist the invertible matrices Q and P such that $QAP = \tilde{A}$ and $QEP = \tilde{E}$ with the matrices $\tilde{E}$ and $\tilde{A}$ given by $\tilde{E} = \text{diag}\left(0_{n_0\times n_0}, L_1, L_2, \cdots, L_p, L_1^{\infty}, L_2^{\infty}, \cdots, L_q^{\infty}, I, N\right)$ and $\tilde{A} = \text{diag}\left(0_{n_0\times n_0}, J_1, J_2, \cdots, J_p, J_1^{\infty}, J_2^{\infty}, \cdots, J_q^{\infty}, A_1, I\right)$, where $A_1 \in \mathbb{R}^{h\times h}$, $L_i, J_i \in \mathbb{R}^{n_i\times(n_i+1)}$ and $L_j^{\infty}, J_j^{\infty} \in \mathbb{R}^{(n_j+1)\times n_j}$ with

$$L_i = \begin{bmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix}, \quad J_i = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \end{bmatrix}$$

$$L_j^{\infty} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}, \quad J_j^{\infty} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 \\ & & & 1 \end{bmatrix}$$

for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$, and $N = \text{diag}\left(N_1, N_2, \cdots, N_l\right) \in \mathbb{R}^{g\times g}$, where $N_l \in \mathbb{R}^{k_l\times k_l}$, $l = 1, 2, \cdots, l$ possess the special forms described before. The dimensions of these matrices satisfy the relations $\sum_{i=1}^{p} k_i = g$, $\eta_0 + \sum_{i=1}^{p}\eta_i + \sum_{j=1}^{q}(n_j+1) = n$, $\eta_0 + \sum_{j=1}^{q}\eta_j + \sum_{i=1}^{p}(\eta_i + 1) = n$, where $\eta_0 = (n_0 + h)$ and $\eta_i = (n_i + k_i)$. Therefore, the FOS model (1) is equivalent to the following form

$$\tilde{E}\, {}_0^{GL}\triangle_{k+1}^{\alpha}\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{w}_k, \quad (6)$$

where $\tilde{x}_k = P^{-1}x_k$ and $\tilde{w}_k = Qw_k$. It is not difficult to show that the systems (1) and (6) have the same solution property. When a solution to (6) is derived, immediately, a solution to FOS model (1) can be obtained. According to the structures of the matrices $\tilde{E}$ and $\tilde{A}$, the vectors $\tilde{x}$ and $\tilde{w}$ can be partitioned as $\tilde{x}_k^T = \left[x_{n_0}^T \; x_p^T \; x_q^T \; x_h^T \; x_l^T\right]$ and $\tilde{w}_k^T = \left[w_{n_0}^T \; w_p^T \; w_q^T \; w_h^T \; w_l^T\right]$, where $\left[x_l \; w_l\right]^T = \text{diag}\left(\left[x_{k_1}^T \cdots x_{k_l}^T\right], \left[w_{k_1}^T \cdots w_{k_l}^T\right]\right)$ and $\left[x_\omega \; w_\omega\right]^T = \text{diag}\left(\left[x_{n_1}^T \cdots x_{n_\omega}^T\right], \left[w_{n_1}^T \cdots w_{n_\omega}^T\right]\right)$ for $\omega = p, q$. Then, system (6) can be decomposed into the following equations:

$$0_{n_0\times n_0}\, {}_0^{GL}\triangle_{k+1}^{\alpha} x_{n_0,k+1} = w_{n_0,k} \quad (7)$$
$$L_i\, {}_0^{GL}\triangle_{k+1}^{\alpha} x_{n_i,k+1} = J_i x_{n_i,k} + w_{n_i,k} \quad (8)$$
$$L_j^{\infty}\, {}_0^{GL}\triangle_{k+1}^{\alpha} x_{n_j,k+1} = J_j^{\infty} x_{n_j,k} + w_{n_j,k} \quad (9)$$
$$N_{k_m}\, {}_0^{GL}\triangle_{k+1}^{\alpha} x_{k_m,k+1} = x_{k_m,k} + w_{k_m,k} \quad (10)$$
$${}_0^{GL}\triangle_{k+1}^{\alpha} x_{h,k+1} = A_{f_1} x_{h,k} + w_{h,k} \quad (11)$$

for $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$ and $m = 1, 2, \cdots, l$. Thus, the solution of (6) is equivalent to the solutions of equations (7) to (11) which are elaborated as follows.

(a) For the identical Equation (7), there either does not exist a solution or does an infinite number of solutions for any differentiable function $x_{n_0,k}$.

(b) In Equation (8), suppose that the system has the dimension of $(r-1)\times r$. Let us denote the vectors x_{n_i} and w_{n_i} by z_i and w_i, respectively. Then one has ${}_0^{GL}\triangle_{k+1}^{\alpha} z_{j,k+1} = z_{j+1,k} + w_{j,k}$, for $j = 1, 2, \cdots, r-1$, which can be converted into the $z_{j+1,k} = {}_0^{GL}\triangle_{k+1}^{\alpha} z_{j,k+1} - w_{j,k}$ forms, for $j = 1, 2, \cdots, r-1$. Clearly, for any given function $z_{1,k}$, which is sufficiently differentiable, the other variables $z_{2,k}, \cdots, z_{r,k}$ can be obtained easily in turn. As a result, there exist an infinite number of solutions for this type of equations.

(c) In Equation (9), suppose that the system has the dimension of $(r+1)\times r$. Again denote the vectors x_{n_j} and w_{n_j} by z_i and w_i, respectively. Therefore, the system is equivalent to the equations $z_{r,k} + w_{r+1,k} = 0$,

${}_{0}^{GL}\triangle_{k+1}^{\alpha} z_{1,k+1} = w_{1,k}$, and ${}_{0}^{GL}\triangle_{k+1}^{\alpha} z_{j,k+1} = z_{j-1,k} + w_{j,k}$ for $j = 2, 3, \cdots, r$. Clearly, these set of equations can be written into a normal state space system. Therefore, it has a unique solution for any known inputs $w_{i,k}$ that is differentiable from sufficient order. However, the derived variable $z_{r,k}$ does not satisfy the equation $z_{r,k} + w_{r+1,k} = 0$ as a consistent condition. As a result, there is no solution for this form of equations. To be more specific, only when a special set of initial values and inputs satisfy the condition $z_{r,k} + w_{r+1,k} = 0$, there might exist a solution for this type of equation.

(d) Equation (10) may be rewritten as ${}_{0}^{GL}\triangle_{k+1}^{\alpha} z_{j,k+1} = z_{j-1,k} + w_{j-1,k}$ for $j = 2, 3, \cdots, r$, and $0 = z_{r,k} + w_{r,k}$. One has the equivalent equations $z_{j-1,k} = {}_{0}^{GL}\triangle_{k+1}^{\alpha} z_{j,k+1} - w_{j-1,k}, j = 2, 3, \cdots, r$, and $0 = z_{r,k} + w_{r,k}$. Therefore, when $z_{r,k} + w_{r,k} = 0$ has a unique solution $z_{r,k}$, the rest of the variables $z_{1,k}, \cdots, z_{r-1,k}$ can be determined in turn. Hence (10) has a unique solution for any $w_{r,k}$, and the initial value z_0 with $z_{r,0} = w_{r,0}$ and $z_{j,0}$, $j = 1, 2, \cdots, r-1$ arbitrary.

(e) Finally, Equation (11) is an ordinary differential equation with a unique solution for any piecewise continuous function w_k.

Following from the above points, one can conclude that the necessary and sufficient condition for the solution of system (6) to be unique is the fading of the equations (7) to (9). Accordingly, the linear FOS system (6) has a unique solution if and only if the matrices Q and P in the equivalent model (6) of the system (1) take the forms as $\tilde{E} = QEP = \operatorname{diag}(I, N)$ and $\tilde{A} = QAP = \operatorname{diag}(A_1, I_{n_2})$. Based on Lemma 1, this fact immediately gives that the system (1) is regular. □

Remark 1: Under regularity condition, the FOS system (1) is of index one and, accordingly, causal, if $\mu = 1$. On the contrary, for an index greater than one, we have a non-causal FOS system.

Now, we can take steps to obtain an appropriate dynamical model which interprets the system observations based on the following assumptions:

Assumption 1: The discrete stochastic FOS system (1) is regular.

Assumption 2: The initial state x_0 is a random variable as $x_0 \sim \mathcal{N}(\bar{x}_0, \bar{P}_0)$, where $\bar{x}_0$ is mean value and the positive definite (PD) matrix $\bar{P}_0$ is covariance of prior information.

Assumption 3: The independent vectors $w_k \in \mathbb{R}_n$ and $v_k \in \mathbb{R}_p$ of x_0 are zero mean white sequences with PD covariance as $\mathbb{E}\left\{\begin{bmatrix} w_k & v_k \end{bmatrix}^T \begin{bmatrix} w_l & v_l \end{bmatrix}\right\} = \operatorname{diag}(Q_k, R_k)\,\delta_{kl}$, where $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation and δ_{kl} is the discrete delta function.

Assumption 4: The set of observed signals $Y_l = \{y_i\}_{i=1}^{l}$ from discrete stochastic FOS regular model (1) are given.

The following section is devoted to the KF derivation for the causal and non-causal system (1) based on ML approach.

III. ML-BASED FSKF

This section aims to investigate the estimation issue of the FOS model (1) according to ML conception, with respect to the system dynamics and prior information on x_0 as additional piece of observations. Before deriving an algorithm of estimation based on ML approach, let us first discuss some features of this linear method of estimation.

A. ML ESTIMATION TECHNIQUE

Let $x \in \mathbb{R}^n$ be an unknown vector based on the following measurement vector

$$y = C_1 x + n_1 \tag{12}$$

where $y \in \mathbb{R}^p$, $n_1 = \mathcal{N}(0, N_1)$, $N_1 \in \mathbb{R}^{p\times p}$ and $C_1 \in \mathbb{R}^{p\times n}$ is a constant matrix. With respect to the estimation of this problem, we aim to bring here some aspects of ML estimation technique.

Lemma 3: The recent problem actually includes LS estimation method of Gaussian vectors. In other words, the issue of LS estimate for a vector $x = \mathcal{N}(m, p)$, $P \in \mathbb{R}^{n\times n}$ rely on the measurement $z = C_2 x + n_2$, where $n_2 \sim \mathcal{N}(0, N_2)$, $N_2 \in \mathbb{R}^{p\times p}$ and $C_2 \in \mathbb{R}^{p\times n}$ is a constant matrix, yields the same estimate as the ML problem with

$$y_{new} = \begin{bmatrix} m \\ z \end{bmatrix}, \quad C_{1,new} = \begin{bmatrix} I_n \\ C_2 \end{bmatrix}, \; N_{1,new} = \begin{bmatrix} P & 0 \\ 0 & N_2 \end{bmatrix} \tag{13}$$

where $y_{new} \in \mathbb{R}^{(n+p)\times 1}$, $C_{1,new} \in \mathbb{R}^{(n+p)\times n}$ and $N_{1,new} \in \mathbb{R}^{(n+p)\times(n+p)}$.

Proof: Let $p(y|x)$ denote a probability density function of y parameterized by x. Since n_1 is Gaussian, so is y, and as a result, $p(y|x) = \xi e^{\left(-\frac{1}{2}\left\{(y-C_1x)^T N_1^{-1}(y-C_1x)\right\}\right)}$, where ξ is a normalization constant. The ML estimate $\hat{x}^{\mathrm{ML}}$ based on observation (12) satisfies $p\left(y|\hat{x}^{\mathrm{ML}}\right) \geq p(y|x)$ for all x. Then, $\hat{x}^{\mathrm{ML}}$ can be obtained by noting that $\frac{\partial}{\partial x}\left(\ln p(y|x)|_{x=\hat{x}^{\mathrm{ML}}}\right) = 0$. Accordingly if C_1 and N_1 have full-rank, for the ML problem (12), one can obtain the following solution

$$\hat{x}^{\mathrm{ML}} = \left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1} y, \tag{14a}$$

with the associated error variance

$$\begin{aligned} P^{\mathrm{ML}} &= \mathbb{E}\left\{\left(x - \hat{x}^{\mathrm{ML}}\right)\left(x - \hat{x}^{\mathrm{ML}}\right)^T\right\} \\ &= \mathbb{E}\left\{\left(x - \left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1}(C_1 x + n_1)\right)\right. \\ &\quad \left.\left(x - \left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1}(C_1 x + n_1)\right)^T\right\} \\ &= \left(\left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1}\right)\mathbb{E}\left\{n_1 n_1^T\right\} \\ &\quad \left(\left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1}\right)^T \\ &= \left(C_1^T N_1^{-1} C_1\right)^{-1}. \end{aligned} \tag{14b}$$

When an *a priori* estimate of x exists, we can consider this statistics as an extra observation which takes the form $m = x + r$, where $r \sim \mathcal{N}(0, P)$, $P \in \mathbb{R}^{n\times n}$ is an independent Gaussian vector. By defining a new independent zero-mean Gaussian vector as $n_{1,\text{new}}^T = \left[r^T \; n_2^T\right]$, a new ML estimation problem can be defined as (12). Applying the ML estimation technique ((14a) and (14b)) to this problem, one can rewrite the ML estimate $\hat{x}^{\text{ML}}$ as

$$\begin{aligned}\hat{x}^{\text{ML}} &= \left(\begin{bmatrix} I_n \\ C_2\end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & N_2\end{bmatrix}^{-1} \begin{bmatrix} I_n \\ C_2\end{bmatrix}\right)^{-1} \begin{bmatrix} I_n \\ C_2\end{bmatrix}^T \\ &\quad\times \begin{bmatrix} P & 0 \\ 0 & N_2\end{bmatrix}^{-1} \begin{bmatrix} m \\ z\end{bmatrix} \\ &= \left(P^{-1} + C_2^T N_2^{-1} C_2\right)^{-1} \left(P^{-1}m + C_2^T N_2^{-1} z\right) \\ &= \left(P - PC_2^T\left(N_2 + C_2PC_2^T\right)^{-1} C_2P\right) \\ &\quad\times \left(P^{-1}m + C_2^T N_2^{-1} z\right) \\ &= \left(I_n - PC_2^T\left(N_2 + C_2PC_2^T\right)^{-1} C_2\right) m \\ &\quad+ \left(P - PC_2^T\left(N_2 + C_2PC_2^T\right)^{-1} C_2P\right) \\ &\quad\times C_2^T N_2^{-1} z \\ &= \left(I_n - PC_2^T\left(N_2 + C_2PC_2^T\right)^{-1} C_2\right) m \\ &\quad+ PC_2^T\left(N_2 + C_2PC_2^T\right)^{-1} z.\end{aligned}$$

Denoting $K = PC_2^T\left(N_2 + C_2PC_2^T\right)^{-1}$ as the gain vector, one can obtain the filter equation as $\hat{x}^{\text{ML}} = m + K\left(z - C_2m\right)$ with the associated error variance

$$\begin{aligned}\hat{P}^{\text{ML}} &= \left(\begin{bmatrix} I_n \\ C_2\end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & N_2\end{bmatrix}^{-1} \begin{bmatrix} I_n \\ C_2\end{bmatrix}\right)^{-1} \\ &= \left(P^{-1} + C_2^T N_2^{-1} C_2\right)^{-1} = \left(I_n - KC_2\right)P,\end{aligned}$$

which are exactly the Bayesian estimate algorithm (generating a posterior density $p(x|z)$ from the prior density and current measurement, and then updating this density to be the prior density for the next time step) reported in [17]. □

Remark 2: According to Lemma 3, any linear Gaussian Bayesian estimation problem can be transformed into an ML problem by considering a priori statistics of x as an observation.

Remark 3: To estimate the components of x completely, the measurement (12) should provide us adequate constraints. To guarantee this condition, the matrix C_1 must assure the relation $\text{rank}(C_1) = \dim(x)$ which is always satisfied in the Bayesian problems.

In spite of the explicit recursive formulation for sequential estimation issues, Lemma 3 cannot be feasible for a singular matrix N_1. To overcome this problem when N_1 is a singular matrix, it is sufficient to consider the ML estimation issue as the quadratic minimization approach [14]. Using the Lagrange multipliers, one can derive a set of equations as $\Pi\left[\lambda^T \; x^T\right]^T = \left[y^T \; 0_{1\times n}\right]^T$ of dimension $(p+n)$, where

$$\Pi = \begin{bmatrix} N_1 & C_1 \\ C_1^T & 0_{n\times n}\end{bmatrix}. \tag{15}$$

To calculate recent equations, the obvious question is about the conditions on invertibility of the matrix Π. According to [14], for a positive semi-definite matrix and full column rank matrix C_1, if $\left[N_1 \; C_1\right]$ has full row rank, then Π is invertible. Now, the solution to the aforementioned ML problem can be expressed by the following lemma.

Lemma 4: Suppose that Π is invertible. The ML estimate of and its error covariance based on the measurement vector (12) is given by

$$\begin{bmatrix} \hat{x}^{ML} & P^{ML}\end{bmatrix} = \begin{bmatrix} 0_{n\times p} & I_n\end{bmatrix} \Pi^{-1} \begin{bmatrix} y & 0_{p\times n} \\ 0_{n\times 1} & -I_n\end{bmatrix}. \tag{16}$$

Proof: By solving the equation $\Pi\left[\lambda^T \; x^T\right]^T = \left[y^T \; 0_{1\times n}\right]^T$ of dimension $(p+n)$ in terms of x, one has the ML estimate of x as $\hat{x}^{\text{ML}} = \left[0_{n\times p} \; I_n\right]\Pi^{-1}\left[I_P \; 0_{p\times n}\right]y$. Also, By substituting $\hat{x}^{\text{ML}}$ into the equation $P^{\text{ML}} = \mathbb{E}\left\{\left(x - \hat{x}^{\text{ML}}\right)\left(x - \hat{x}^{\text{ML}}\right)^T\right\}$, the ML error covariance is given by

$$\begin{aligned}\hat{P}^{\text{ML}} = \mathbb{E}\Bigg\{&\left(x - \begin{bmatrix} 0_{p\times n} \\ I_n\end{bmatrix}^T \Pi^{-1}\left(\begin{bmatrix} C_1 \\ 0_n\end{bmatrix} x + \begin{bmatrix} n_1 \\ 0_{n\times 1}\end{bmatrix}\right)\right) \\ &\left(x - \begin{bmatrix} 0_{p\times n} \\ I_n\end{bmatrix}^T \Pi^{-1}\left(\begin{bmatrix} C_1 \\ 0_n\end{bmatrix} x + \begin{bmatrix} n_1 \\ 0_{n\times 1}\end{bmatrix}\right)\right)^T\Bigg\}.\end{aligned}$$

Let

$$\Pi^{-1} = \begin{bmatrix} W & U \\ U^T & T\end{bmatrix},$$

where $W \in \mathbb{R}^{p\times p}$, $U \in \mathbb{R}^{p\times n}$ and $T \in \mathbb{R}^{n\times n}$. From $\Pi^{-1}\Pi = I_{n+p}$, one has $U^T N_1 + TC_1^T = 0_n$ and $C_1^T U = I_n$. Post multiplying the former equation by U and substituting the latter one into it, we have $U^T N_1 U + T = 0_n$. Then

$$\begin{aligned}\hat{P}^{\text{ML}} &= \begin{bmatrix} 0_{n\times p} & I_n\end{bmatrix} \Pi^{-1} \begin{bmatrix} C_1 \\ 0_n\end{bmatrix} \\ &= \begin{bmatrix} 0_{n\times p} & I_n\end{bmatrix} \begin{bmatrix} W & U \\ U^T & T\end{bmatrix} \begin{bmatrix} C_1 \\ 0_n\end{bmatrix} \\ &= U^T C_1 = I_n.\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{P}^{\text{ML}} &= \mathbb{E}\Bigg\{\left(\begin{bmatrix} 0_{p\times n} \\ I_n\end{bmatrix}^T \Pi^{-1} \begin{bmatrix} n_1 \\ 0_{n\times 1}\end{bmatrix}\right) \\ &\qquad\left(\begin{bmatrix} 0_{p\times n} \\ I_n\end{bmatrix}^T \Pi^{-1} \begin{bmatrix} n_1 \\ 0_{n\times 1}\end{bmatrix}\right)^T\Bigg\} \\ &= \begin{bmatrix} 0_{p\times n} \\ I_n\end{bmatrix}^T \Pi^{-1} \begin{bmatrix} N_1 & 0_{p\times n} \\ 0_{n\times p} & 0_n\end{bmatrix} \Pi^{-1} \begin{bmatrix} 0_{p\times n} \\ I_n\end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}^T \left(\Pi^{-1} \begin{bmatrix} N_1 & 0_{p\times n} \\ 0_{n\times p} & 0_n \end{bmatrix} \Pi^{-1} + \Pi^{-1} - \Pi^{-1} \right) \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}$$

$$= \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}^T \begin{bmatrix} W + WN_1W & U + WN_1U \\ U^T + U^T N_1 W & T + U^T N_1 U \end{bmatrix} \times \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix} - \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}^T \Pi^{-1} \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}$$

$$= T + U^T N_1 U - \begin{bmatrix} 0_{n\times p} & I_n \end{bmatrix} \Pi^{-1} \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}$$

$$= -\begin{bmatrix} 0_{n\times p} & I_n \end{bmatrix} \Pi^{-1} \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}.$$

□

Remark 4: Let N_1 be a PD matrix. The derived ML-based estimate of state x and its error covariance can be stated as follows by the inverse partitioned matrix lemma [18]

$$\begin{bmatrix} \hat{x}^{ML} & P^{ML} \end{bmatrix} = \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}^T \Pi^{-1} \begin{bmatrix} y & 0_{p\times n} \\ 0_{n\times 1} & -I_n \end{bmatrix}$$
$$= \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix}^T \begin{bmatrix} N_1^{-1} - \Delta^T \Lambda \Delta & \Delta^T \Lambda \\ \Lambda \Delta & -\Lambda \end{bmatrix} \times \begin{bmatrix} y & 0_{p\times n} \\ 0_{n\times 1} & -I_n \end{bmatrix} = \begin{bmatrix} \Lambda\Delta & \Lambda \end{bmatrix}$$

where $\Lambda = \left(C_1^T N_1^{-1} C_1 \right)^{-1}$ and $\Delta = C_1^T N_1^{-1}$.

It should be noted that if the matrix Π is singular, it is mandatory to apply pseudo-inverse algorithms. The derived outcomes here will cause to our describing constraints of dynamic as in (1a), which can be considered as additional pieces of observations in the upcoming section.

B. FILTER DESIGN

Using the dynamics (1a), the observations presented by Y_k and the given information on x_0, an ML-based estimation algorithm can be adopted here. Applying the GL definition and known observations, one can take the following three equations to design the algorithm for the FOS model (1)

$$y_k = Cx_k + v_k \tag{17a}$$
$$Ex_k = Ax_{k-1} - \sum_{j=1}^{k} E(-1)^j \upsilon_j x_{k-j} + w_{k-1} \tag{17b}$$
$$\bar{x}_0 = x_0 + \bar{r}_0 \tag{17c}$$

where $\bar{r}_0 \sim \mathcal{N}\left(0, \bar{P}_0\right)$ is a Gaussian independent vector of w_k and v_k. Equations (17) provide a set measurements related to the states vector $X_k = \{x_i\}_{i=1}^{k}$. By investigating this set of measurements according to the problem (12), one can deduce that just the terms Ex_k and Cx_k include the variable x_k, which yield the block matrix $C_1^T = \begin{bmatrix} E^T & C^T \end{bmatrix}$.

Remark 5: According to Remark 3, if the matrix is full column rank, then is estimable. This obtained result coincides with the estimability theorem of state variable of a discrete stochastic FOS model reported in [1].

TABLE 1. **ML-based FSKF recursive algorithm.**

Step 0	Initial values	$P_0^{\mathrm{ML}} = \begin{bmatrix} 0_{n\times n} \\ 0_{p\times n} \\ I_n \end{bmatrix}^T \begin{bmatrix} \bar{P}_0 & 0_{n\times p} & I_n \\ 0_{p\times n} & R_0 & C \\ I_n & C^T & 0_n \end{bmatrix}^{-1} \begin{bmatrix} 0_{n\times n} \\ 0_{p\times n} \\ -I_n \end{bmatrix}$ $\hat{x}_0^{\mathrm{ML}} = \begin{bmatrix} 0_{n\times n} \\ 0_{p\times n} \\ I_n \end{bmatrix}^T \begin{bmatrix} \bar{P}_0 & 0_{n\times p} & I_n \\ 0_{p\times n} & R_0 & C \\ I_n & C^T & 0_n \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}_0 \\ y_0 \\ 0_{n\times 1} \end{bmatrix}$
Step k	- Prediction	$\tilde{P}_k = Q_{k-1} + (A + E\Upsilon_1) P_{k-1} (A + E\Upsilon_1)^T + \sum_{j=2}^{k} (E\Upsilon_j) P_{k-j} (E\Upsilon_j)^T$ $\tilde{x}_k = (A + E\Upsilon_1) \hat{x}_{k-1} - E \sum_{j=2}^{k} (-1)^j \Upsilon_j \hat{x}_{k-1}$
	- Update	$P_k^{\mathrm{ML}} = \begin{bmatrix} 0_{n\times n} \\ 0_{p\times n} \\ I_n \end{bmatrix}^T \begin{bmatrix} \tilde{P}_k & 0_{n\times p} & E \\ 0_{p\times n} & R_k & C \\ E^T & C^T & 0_n \end{bmatrix}^{-1} \begin{bmatrix} 0_{n\times n} \\ 0_{p\times n} \\ -I_n \end{bmatrix}$ $\hat{x}_k^{\mathrm{ML}} = \begin{bmatrix} 0_{n\times n} \\ 0_{p\times n} \\ I_n \end{bmatrix}^T \begin{bmatrix} \tilde{P}_k & 0_{n\times p} & E \\ 0_{p\times n} & R_k & C \\ E^T & C^T & 0_n \end{bmatrix}^{-1} \begin{bmatrix} \tilde{x}_k \\ y_k \\ 0_{n\times 1} \end{bmatrix}$

By the principle of mathematical induction and applying ML method, we can establish the following theorem to derive our desirable FSKF recursive algorithm.

Theorem 2: Consider the full column rank matrix $\begin{bmatrix} E^T & C^T \end{bmatrix}^T$, and suppose that the sequence Y_k together with the prior information about x_0 are given. The ML-based optimal estimate $\hat{x}_k^{ML}$ can be successively derived by use of the algorithm outlined in the Table 1.

Proof: From Remark 4, the full column rank matrix $\begin{bmatrix} E^T & C^T \end{bmatrix}^T$ ensure that (15) is a full column rank matrix, and then, the state vector x_k is estimable. For $k = 0$, the given observations are as $y_0 = Cx_0 + v_0$ and (17c). By using the fact that x_0 is estimable and defining the matrices (13) as

$$y_{\text{new}} = \begin{bmatrix} \bar{x}_0 \\ y_0 \end{bmatrix}, \quad C_{1,\text{new}} = \begin{bmatrix} I_n \\ C \end{bmatrix}, \; N_{1,\text{new}} = \begin{bmatrix} \bar{P}_0 & 0 \\ 0 & R_0, \end{bmatrix}$$

one needs to take Lemma 4 into (17a) and (17c), when $i = 0$, to derive $\hat{x}_0^{\text{ML}}$ and P_0^{ML} as follows,

$$\begin{bmatrix} \hat{x}_0^{\text{ML}} & P_0^{\text{ML}} \end{bmatrix} = \begin{bmatrix} 0_{n\times(n+p)} & I_n \end{bmatrix} \begin{bmatrix} N_{1,\text{new}} & C_{1,\text{new}} \\ C_{1,\text{new}}^T & 0_n \end{bmatrix}^{-1} \times \begin{bmatrix} y_{\text{new}} & 0_{(n+p)\times n} \\ 0_{n\times 1} & -I_n. \end{bmatrix}$$
$$= \begin{bmatrix} 0_{n\times n} & 0_{n\times p} & I_n \end{bmatrix} \begin{bmatrix} \bar{P}_0 & 0_{n\times p} & I_n \\ 0_{p\times n} & R_0 & C \\ I_n & C^T & 0_n \end{bmatrix}^{-1} \times \begin{bmatrix} \bar{x}_0 & 0_{n\times n} \\ y_0 & 0_{p\times n} \\ 0_{n\times 1} & -I_n \end{bmatrix}.$$

In an analogous manner, the observations can be given as $y_1 = Cx_1 + v_1$ and $Ex_1 = (A + E\upsilon_1) x_0 + w_0$ for $k = 1$. By substituting the derived estimation at the prior step as $\hat{x}_0^{\text{ML}} = x_0 + r_0$, where $r_0 \sim \mathcal{N}\left(0, \bar{P}_0^{\text{ML}}\right)$ is an independent Gaussian vector, in these observations, one has the following equations

$$y_1 = Cx_1 + v_1 \tag{18a}$$
$$(A + E\Upsilon_1) \hat{x}_0^{\text{ML}} = Ex_1 + (A + E\Upsilon_1) r_0 - w_0. \tag{18b}$$

To derive $\hat{x}_1^{\text{ML}}$ and P_1^{ML}, it is sufficient to apply Lemma 4 to matrices (13) as

$$y_{\text{new}} = \begin{bmatrix} \tilde{x}_1 \\ y_1 \end{bmatrix}, \quad C_{1,\text{new}} = \begin{bmatrix} E \\ C \end{bmatrix}, \; N_{1,\text{new}} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & R_1 \end{bmatrix},$$

where $\tilde{x}_1 = (A + E\Upsilon_1)\hat{x}_0^{\text{ML}}$ and $\tilde{P}_1 = \mathbb{E}\left\{w_{0,\text{new}} w_{0,\text{new}}^T\right\} = Q_0 + (A + E\Upsilon_1) P_0^{\text{ML}} (A + E\Upsilon_1)^T$ with $w_{0,\text{new}} = (A + E\Upsilon_1)$. Then, the 3-bolck FSKF in the second step can be given as

$$\begin{aligned}\begin{bmatrix} \hat{x}_1^{\text{ML}} & P_1^{\text{ML}} \end{bmatrix} &= \begin{bmatrix} 0_{n\times(n+p)} & I_n \end{bmatrix} \begin{bmatrix} N_{1,\text{new}} & C_{1,\text{new}} \\ C_{1,\text{new}}^T & 0_n \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} y_{\text{new}} & 0_{(n+p)\times n} \\ 0_{n\times 1} & -I_n \end{bmatrix} \\ &= \begin{bmatrix} 0_{n\times n} & 0_{n\times p} & I_n \end{bmatrix} \begin{bmatrix} \tilde{P}_1 & 0_{n\times p} & E \\ 0_{p\times n} & R_1 & C \\ E^T & C^T & 0_n \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} \tilde{x}_1 & 0_{n\times n} \\ y_1 & 0_{p\times n} \\ 0_{n\times 1} & -I_n \end{bmatrix}.\end{aligned}$$

Likewise for $k > 1$, if we denote $P_{k-1}^{\text{ML}}, \cdots, P_2^{\text{ML}}$ and P_1^{ML} as the error covariance matrices associated with the filtered estimates $\hat{x}_{k-1}^{\text{ML}}, \cdots, \hat{x}_2^{\text{ML}}$ and $\hat{x}_1^{\text{ML}}$, with initial conditions $\hat{x}_0^{\text{ML}}$ and P_0^{ML} as the prior distribution for x_0. By defining the prior observations as $\hat{x}_i^{\text{ML}} = x_i + r_i$, $0 \leq i \leq k-1$, where $r_i \sim \mathcal{N}\left(0, \bar{P}_i^{\text{ML}}\right)$ are Gaussian random independent vectors of w_i. By substituting these observations into (17b), one can obtain the following observations.

$$y_k = Cx_k + v_k \quad \text{(19a)}$$

$$(A + E\Upsilon_1)\hat{x}_{k-1}^{\text{ML}} - \sum_{j=2}^{k} E(-1)^j \Upsilon_j \hat{x}_{k-j}^{\text{ML}} = Ex_k + w_{k-1,\text{new}}, \quad \text{(19b)}$$

where $w_{k-1,\text{new}} = (A + E\Upsilon_1) r_{k-1} - \sum_{j=1}^{k} E(-1)^j \Upsilon_j r_{k-j} - w_{k-1}$. Therefore, in order to derive the recursive equations of estimation at step k, one can apply Lemma 4 to matrices (13) as

$$y_{\text{new}} = \begin{bmatrix} \tilde{x}_k \\ y_k \end{bmatrix}, \quad C_{1,\text{new}} = \begin{bmatrix} E \\ C \end{bmatrix}, \; N_{1,\text{new}} = \begin{bmatrix} \tilde{P}_k & 0 \\ 0 & R_k \end{bmatrix},$$

where $\tilde{x}_k = (A + E\Upsilon_1)\hat{x}_{k-1}^{\text{ML}} - E\sum_{j=2}^{k}(-1)^j \Upsilon_j \hat{x}_{k-j}^{\text{ML}}$ and $\tilde{P}_k = \mathbb{E}\left\{w_{k-1,\text{new}} w_{k-1,\text{new}}^T\right\} = Q_{k-1} + (A + E\Upsilon_1) P_{k-1}^{\text{ML}} (A + E\Upsilon_1)^T + \sum_{j=2}^{k} (E\Upsilon_j) P_{k-j}^{\text{ML}} (E\Upsilon_j)^T$. Finally, it is easy to see that the 3-bolck FSKF $\hat{x}_k^{\text{ML}}$ and P_k^{ML} can be respectively derived as

$$\begin{aligned}\begin{bmatrix} \hat{x}_k^{\text{ML}} & P_k^{\text{ML}} \end{bmatrix} &= \begin{bmatrix} 0_{n\times(n+p)} & I_n \end{bmatrix} \begin{bmatrix} N_{1,\text{new}} & C_{1,\text{new}} \\ C_{1,\text{new}}^T & 0_n \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} y_{\text{new}} & 0_{(n+p)\times n} \\ 0_{n\times 1} & -I_n \end{bmatrix} \\ &= \begin{bmatrix} 0_{n\times n} & 0_{n\times p} & I_n \end{bmatrix} \begin{bmatrix} \tilde{P}_k & 0_{n\times p} & E \\ 0_{p\times n} & R_k & C \\ E^T & C^T & 0_n \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} \tilde{x}_k & 0_{n\times n} \\ y_k & 0_{p\times n} \\ 0_{n\times 1} & -I_n \end{bmatrix}.\end{aligned}$$

□

Theorem 2 organizes a recursive FSKF algorithm begins from x_0 and P_0 as *a priori* information. Then, at the following steps, the estimate of $\hat{x}_k^{\text{ML}}$ can be achieved by using the prediction and the update equations.

Corollary 1: Suppose that system (1) has an input term defined by $B_k u_k$ with $B_k \in \mathbb{R}^{n\times m}$ and $u_k \in \mathbb{R}^m$ in right-hand side of its first equation. Then, Theorem 2 can be stated with a slight modification as follows: Consider the full column rank matrix $\left[E^T \; C^T\right]^T$, and let $U_k = \{u_i\}_{i=0}^k$ and Y_k be the known input and measurement data, respectively. Now, for the modified FOS model (1), the FSKF algorithm presented in Table 1 remains unchanged, except for $\tilde{x}_k$ that can be rewritten as $\tilde{x}_k = (A + E\Upsilon_1)\hat{x}_{k-1}^{ML} + Bu_{k-1} - E\sum_{j=2}^{k}(-1)^j \Upsilon_j \hat{x}_{k-j}^{ML}$.

Proof: In the same analogous as the proof of Theorem, one needs to take the observation (19b) as □

$$(A + E\Upsilon_1)\hat{x}_{k-1}^{\text{ML}} - \sum_{j=2}^{k} E(-1)^j \Upsilon_j \hat{x}_{k-j}^{\text{ML}} + Bu_{k-1} = Ex_k + w_{k-1,\text{new}}$$

for $k > 0$, and apply Lemma 4 to matrices (13) as

$$y_{\text{new}} = \begin{bmatrix} \tilde{x}_k \\ y_k \end{bmatrix}, \quad C_{1,\text{new}} = \begin{bmatrix} E \\ C \end{bmatrix}, \; N_{1,\text{new}} = \begin{bmatrix} \tilde{P}_k & 0 \\ 0 & R_k \end{bmatrix},$$

with $\tilde{x}_k = (A + E\Upsilon_1)\hat{x}_{k-1}^{\text{ML}} + Bu_{k-1} - E\sum_{j=2}^{k}(-1)^j \Upsilon_j \hat{x}_{k-j}^{\text{ML}}$ and $\tilde{P}_k$ same as before.

Remark 6: It is easy to see that the derived 3-block FSKF recursive algorithm based on ML method coincides to that of LS-based approach introduced in [1].

C. ADJUSTED FSKF ALGORITHM

According to Remark 1, the FOS model (1) with $\mu > 1$ possess non-causal behavior in its dynamics. This phenomenon is completely alien to normal models. This causes the state vector x_i to be subject to constraints due to future, which makes the situation even more complicated. It can be seen that the information provided by this dynamics is not dependent of i. The impact of these dynamics can be considered as one observation. By incorporating this observation as additional information, an adjusted estimation algorithm can be derived by modification of the FSKF algorithm derived from Theorem 2. In the following, we apply the effect of future dynamics in 17b and replace it with an augmented observation given as

$$Fx_k = \eta_k, \quad (20)$$

where $\eta_i \sim \mathcal{N}(0, \Gamma_i)$. Therefore, the our issue is to find the matrices F and Γ_k in terms of the matrices E, A and Q_k. Manipulating 17b, one can derive the following matrix equation

$$\Theta \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \\ \vdots \end{bmatrix} = \begin{bmatrix} W_k \\ W_{k+1} \\ W_{k+2} \\ \vdots \end{bmatrix}, \tag{21}$$

where $W_{k+n} = w_{k+n} - \sum_{j=n+2}^{k+n+1}(-1)^j E\Upsilon_j x_{k+n+1-j}$, $n = 0, 1, \cdots$ and

$$\Theta = \begin{bmatrix} -(A+E\Upsilon_1) & E & 0_n & \cdots \\ (-1)^2 E\Upsilon_2 & -(A+E\Upsilon_1) & E & \cdots \\ (-1)^3 E\Upsilon_3 & (-1)^2 E\Upsilon_2 & -(A+E\Upsilon_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Equation (21) provides some additional data about the state x_k together with the state vectors x_i for $i > k$, which can be indicated as exogenous variables. Using the elementary row operations, the state vectors x_i, $i > k$ can be dropped out to eliminate the rest of the measurement. To be specific, we suppose

$$\left(\Theta \begin{bmatrix} T_0^T \\ T_1^T \\ T_2^T \\ \vdots \end{bmatrix}\right)^T = \begin{bmatrix} (-T_0(A+E\Upsilon_1)+T_1E\Upsilon_2+\cdots)^T \\ 0 \\ 0 \\ \vdots \end{bmatrix}^T \tag{22}$$

where $T_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots$ are the elementary matrices. Now, right multiplying (21) by $\begin{bmatrix} T_0 & T_1 & T_2 & \cdots \end{bmatrix}$ and applying (22), one has

$$\left(-T_0(A+E\Upsilon_1) + \sum_m (-1)^{m+1} T_m E\Upsilon_{m+1}\right) x_k$$
$$= \sum_m T_m w_{k+m} - \sum_m T_m \sum_{i=2+m}^{k+m+1} (-1)^i E\Upsilon_i x_{k+m+1-i} \tag{23}$$

with $m = 0, 1, 2, \cdots$. Equation (23) is of the form (20) with $\eta_i = \sum_m T_m W_{i+m}$ and $F = -T_0(A+E\Upsilon_1) + \sum_m (-1)^{m+1} T_m E\Upsilon_{m+1}$. So the problem is finding a highest row rank matrix satisfying (22) which can be rewritten as

$$T(z)\left(zE - (A+E\Upsilon_1)\, z(-1)^2 E\Upsilon_2 z^{-1} + \cdots\right)$$
$$= -T_0(A+E\Upsilon_1) + T_1(-1)^2 E\Upsilon_2 + \cdots, \tag{24}$$

where $T(z) = T_0 + zT_1 + z^2 T_2 + \cdots$ is a polynomial in terms of z. In (24), its left hand-side can be rewritten as

$$\begin{aligned} & T(z)\left(zE - (A+E\Upsilon_1) + (-1)^2 E\Upsilon_2 z^{-1} + \cdots\right) \\ &= T(z)\left(zE\left(I - \Upsilon_1 z^{-1} + \Upsilon_2 z^{-2} - \Upsilon_3 z^{-3} + \cdots\right) - A\right) \\ &= T(z)\left(zE\left(\sum_{k=0}^{\infty} \binom{\alpha}{k} 1^{\alpha-k}(-z^{-1})^k\right) - A\right) \\ &= T(z)\left(z(1-z^{-1})^\alpha E - A\right). \end{aligned}$$

Thus, one needs to solve the following equation to derive the matrix $T(z)$ with largest rank.

$$T(z)\left(z(1-z^{-1})^\alpha E - A\right) = \text{constant matrix}. \tag{25}$$

Finally, from (23) one has $Fx_k = \sum_m T_m w_{k+m} - \sum_m T_m \sum_{i=2+m}^{k+m+1}(-1)^i E\Upsilon_i x_{k+m+1-i}$, where $\sum_{m=1} T_m z^m = T(z)$. Thus, one obtains (20) with

$$\Gamma_k = \sum_m T_m Q_{k+m} T_m^T + \sum_m \sum_{i=2+m}^{k+m+1} T_m E\Upsilon_i P_{k+m+1-i}^{\mathrm{ML}} (T_m E\Upsilon_i)^T.$$

Now, applying the presented method in the previous subsection, we manipulate the optimal estimate $\hat{x}_k^{\mathrm{ML}}$ of x_k relied on the past dynamics and the information $\hat{x}_k^{\mathrm{ML}} = x_k + h_k$, where $h_k \sim \mathcal{N}\left(0, P_k^{\mathrm{ML}}\right)$ is a Gaussian vector. Applying the future dynamics (20), we can derive the estimate of the state vector x_k by using the information (12) given as

$$\begin{bmatrix} \hat{x}_k^{\mathrm{ML}} \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ F \end{bmatrix} x_k + \begin{bmatrix} h_k \\ \eta_k \end{bmatrix}.$$

According to Lemma 4 and modified matrices (13) as

$$y_{\mathrm{new}} = \begin{bmatrix} \hat{x}_k^{\mathrm{ML}} \\ 0 \end{bmatrix}, \quad C_{1,\mathrm{new}} = \begin{bmatrix} I \\ F \end{bmatrix}, \quad N_{1,\mathrm{new}} = \begin{bmatrix} P_k^{\mathrm{ML}} & 0 \\ 0 & \Gamma_k \end{bmatrix},$$

it is easy to see that the new and adjusted 3-block FSKF $\hat{x}_i^{\mathrm{ML_{adjusted}}}$ and the corresponding error covariance $P_i^{\mathrm{ML_{adjusted}}}$ can be derived as

$$\begin{aligned} & \begin{bmatrix} \hat{x}_i^{\mathrm{ML_{adjusted}}} & P_i^{\mathrm{ML_{adjusted}}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} N_{1,\mathrm{new}} & C_{1,\mathrm{new}} \\ C_{1,\mathrm{new}}^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} y_{\mathrm{new}} & 0 \\ 0 & -I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_k^{\mathrm{ML}} & 0 & I \\ 0 & \Gamma_k & F \\ I & F^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{x}_k^{\mathrm{ML}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix}. \end{aligned}$$

As seen, the effect of future dynamics is interpreted as *a posteriori* modification to the estimate that relies on past dynamics and observations. The obstacle of this method is the use of polynomial matrix manipulations, which makes it numerically intractable. A solution to alleviate this problem can be to convert the estimation problem to where future information does not have any effect on present state estimates, which can be considered as a future research direction.

IV. SIMULATION RESULTS: A THREE MACHINE INFINITE BUS POWER SYSTEM

Here, a three machine infinite bus power system is used to verify the estimation performances of the derived filter in subsection III. As shown in Fig. 1, the dynamic behavior of model is controlled by the swing equations of the machines G1, G2 and G3. The spinning of the generators has considerable inertia where it makes the dynamic behaviors that are dependent on the history. According to the features of

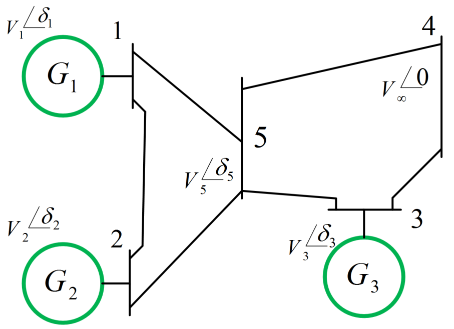

FIGURE 1. A three machine infinite bus system.

fractional calculus, a fractional order model of the generator system can be considered based on [19]. Also, the fifth node introduces the algebraic behavior [20]. Owing to the conception of fractional calculus and singular theory, the corresponding linear FOS model of the three machines infinite bus system is described as follows,

$$
\begin{aligned}
D^{\alpha}\delta_1 &= \omega_1 \quad D^{\alpha}\delta_2 = \omega_2 \quad D^{\alpha}\delta_3 = \omega_3\\
D^{\alpha}\omega_1 &= \frac{1}{M_1}\left(P_1 - Y_{12}V_1V_2\left(\delta_1-\delta_2\right) - Y_{15}V_1V_5\left(\delta_1-\delta_5\right)\right.\\
&\quad \left.-D_1\omega_1\right)\\
D^{\alpha}\omega_2 &= \frac{1}{M_2}\left(P_2 - Y_{21}V_2V_1\left(\delta_2-\delta_1\right) - Y_{25}V_2V_5\left(\delta_2-\delta_5\right)\right.\\
&\quad \left.-D_2\omega_2\right)\\
D^{\alpha}\omega_3 &= \frac{1}{M_3}\left(P_3 - Y_{34}V_3V_{\infty}\delta_3 - Y_{35}V_3V_5\left(\delta_3-\delta_5\right)\right.\\
&\quad \left.-D_3\omega_3\right)\\
0 &= -Y_{51}V_5V_1\left(\delta_5-\delta_1\right) - Y_{52}V_5V_2\left(\delta_5-\delta_2\right)\\
&\quad - Y_{53}V_5V_3\left(\delta_5-\delta_3\right) - Y_{54}V_5V_{\infty}\delta_5. \qquad (26)
\end{aligned}
$$

The parameter α is the fractional order and D^{α} refers to GL operator with $0 < \alpha < 2$. Also, δ_1, δ_2 and δ_3 are the generator angles, δ_5 is the bus angle, and ω_1, ω_2 and ω_3 denotes the variation of the generator angles. In addition, P_1, P_2 and P_3 are the mechanical power inputs, M_1, M_2 and M_3 are the angular momenta, Y_{12}, Y_{15}, Y_{25}, Y_{34}, Y_{35} and Y_{45} are the admittances, D_1, D_2 and D_3 are the damping factors, and V_3, V_5 and V_{∞} are the potentials. The nominal values of parameters are chosen as follows,

$$
\begin{array}{lll}
M_1 = 14 & M_2 = 26 & M_3 = 20\\
D_1 = 0.057 & D_2 = 0.15 & D_3 = 0.11\\
Y_{12} = 1 & Y_{15} = 0.5 & Y_{25} = 1.2\\
Y_{34} = 0.7 & Y_{35} = 0.8 & Y_{45} = 1\\
V_i = 1 & i = 1, 2, 3, 5, \infty & \alpha = 1.
\end{array}
$$

Assuming that the only available measurements are the generator angles and denoting δ_1, δ_2, δ_3, ω_1, ω_2, ω_3 and δ_5 by the state variables $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7, respectively, and also the parameters P_1, P_2 and P_3 by the inputs u_1, u_2 and u_3, respectively, the FOS model of the three machines infinite

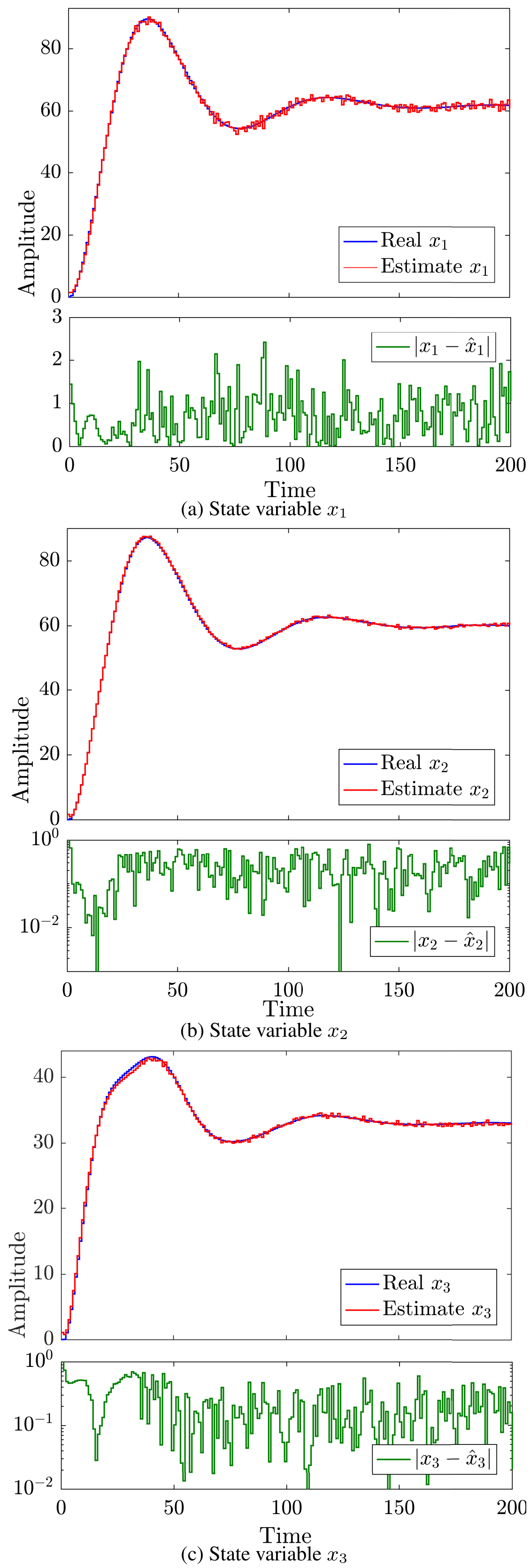

FIGURE 2. Trajectories of the generator angles, their estimated and absolute error values of system (26).

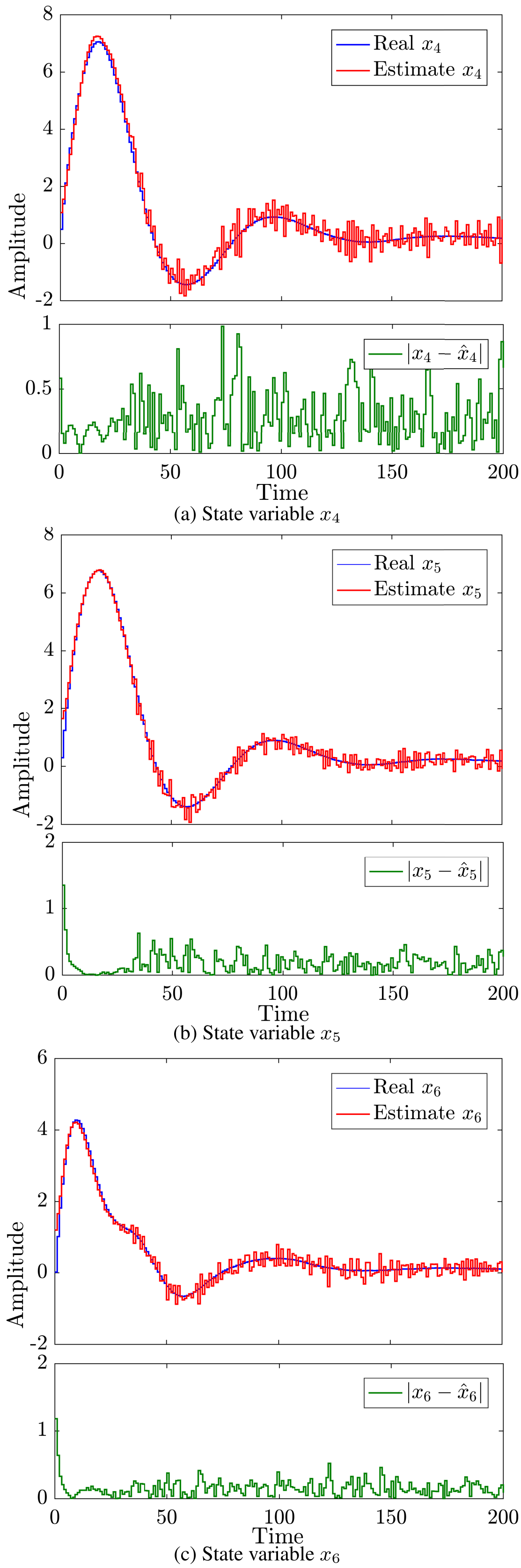

FIGURE 3. Trajectories of the variation of the generator angles, their estimated and absolute error values of system (26).

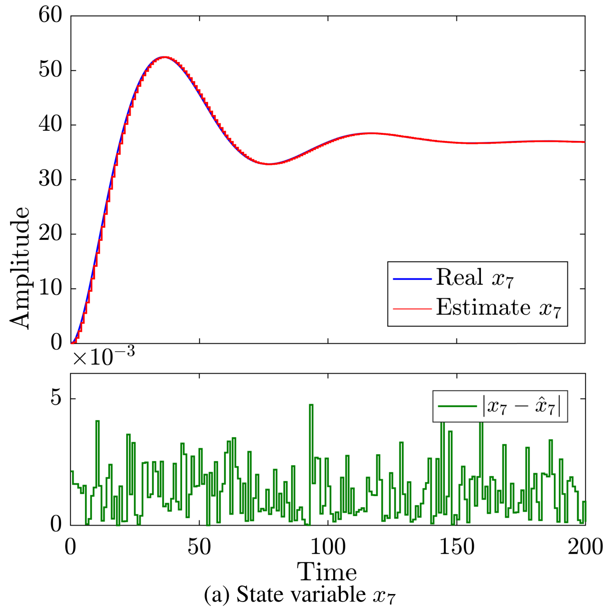

FIGURE 4. Trajectory of the bus angle, its estimated and absolute error value of system (26).

bus system (26) can be described in the form of (1) with an input term by the following matrices:

$$E = \text{diag}\{I_6, 0_{1\times 6}\}, \ A = \begin{bmatrix} 0_3 & I_3 & 0_{3\times 1} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0_{1\times 3} & -3.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0_{3\times 3} \\ b_{21} \\ 0_{1\times 3} \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $a_{22} = -\text{diag}\{0.0041, 0.0058, 0.0055\}$, $b_{21} = \text{diag}\{0.07, 0.038, 0.05\}$, $a_{31} = \begin{bmatrix} 0.5 & 1.2 & 0.8 \end{bmatrix}$ and

$$a_{21} = \begin{bmatrix} -0.107 & 0.071 & 0 \\ 0.0038 & -0.0085 & 0 \\ 0 & 0 & -0.075 \end{bmatrix}, \ a_{23} = \begin{bmatrix} 0.0036 \\ 0.0046 \\ 0.004 \end{bmatrix}.$$

With respect to Definition 2 in [1], the continuous FOS infinite bus system can be discretized. First, the block matrix $\begin{bmatrix} E^T & C^T \end{bmatrix}^T$ is full column rank and accordingly (26) is estimable. Then, after discretization and applying the derived ML-based FSKF algorithm, the results including actual value of states, their estimated signals and absolute error values are depicted in Figs. (2), (3) and (4). One can see that the results make sense as the states of (26) are estimated with reasonable accuracy at the right time. Also, the absolute error of each state is depicted beneath the time graph of actual value and its estimation to provide an improved clarity of estimation performance. We observe that these errors between actual values and their estimations are desirable. Note that electric power systems are very complicated dynamical systems due to their intrinsic property of high nonlinearity. In three machine infinite bus system, the variation of generator angles

TABLE 2. MSEs of ML-based FSKF algorithm dealing with state estimation for the three machine infinite bus system.

	States						
	x_1	x_2	x_3	x_4	x_5	x_6	x_7
MSE	0.9092	0.3391	0.3313	0.33	0.2497	0.2088	0.0016

has complex dynamics with more coupling effects and behavioral interactions than the generator and bus angle. Also, the only available measurements are the generator angles, which have been picked off by the observation matrix C that converts the system state estimate from the state space to the measurement space and outputs. Here, the matrix C just select certain states in the form of a linear transformation and projects the three first states and the last one to the measurement unit. That is why the estimated plots for the states x_i, $i = 4, 5, 6$ show slightly different behavior than the other four states where they show constant steps during the estimation process. Moreover, the mean square errors (MSEs) of the FSKF algorithm for the state estimation of the three machine infinite bus system are presented in Table 2. Again, we observe that the estimation of each state has enough accuracy.

V. CONCLUSION

In this study, we have considered an estimation problem using stochastic FOS discrete linear models. First, we showed that regularity is a necessary and sufficient condition on the solvability of these systems. Second, the ML-based optimal filter algorithm and its corresponding error covariance have been derived in a 3-block structure. This possesses a number of advantages in comparison to the existing filtering algorithms. It works when the measurement noise is singular, in which the derived algorithm requires only standard matrix inverses instead of pseudo inverses. In addition, we have demonstrated that how the derived estimation procedure can be adjusted by some additional information related to the future dynamics. Finally, we verified the estimation performance parameters of the proposed filter by a numerical example on a new FOS model of a three-machine infinite bus system, where the estimation algorithm showed a desirable result with enough accuracy. It may be noted that a drawback of the adjusted filter is its implementation feasibility. Due to increasing the matrix dimension, computational complexity may limit some practical implementations. For some classes of problems with large matrices, it has been shown that use of geometric algebra can result in improvement in terms of tractabilty [21]. This problem together with the investigation of filter stability, continuous-time case studies, etc. can be considered as possible extensions of this work.

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