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Distributed Event-Triggered Impulsive Control for Synchronization of Coupled Harmonic Oscillators

GUODONG MA[1], JIE REN[2], YANSEN LIU[1], AND GUOPING LU[1]
[1]School of Electrical Engineering, Nantong University, Nantong, Jiangsu 226019, China
[2]Xinglin College, Nantong University, Nantong, Jiangsu 226200, China
Corresponding author: Guoping Lu (lu.gp@ntu.edu.cn)

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ABSTRACT This paper addresses the synchronization problem of coupled harmonic oscillators via event-triggered impulsive control. Different from the existing results where both position and velocity measurements are required to achieve the synchronization, two kinds of event-triggered impulsive mechanisms, that is, the position-only and velocity-only approaches are proposed respectively, which effectively extends the event-triggered impulsive control strategy to the synchronization problem with limited information. Sufficient conditions are proposed to exhibit the convergence behavior of the coupled harmonic oscillators. Meanwhile, it is derived that Zeno behavior can be excluded by using the designed controller. Two simulation examples are illustrated to substantiate the performance of the proposed synchronization protocols.

INDEX TERMS Coupled harmonic oscillator, event-triggered impulsive control, synchronization, leader-following.

I. INTRODUCTION

Coordination control problems of multi-agent systems have been extensively discussed in recent decades, including synchronization control (or equivalently consensus control) [1]–[3], formation control [4]–[6], containment control [7]–[9] and so on (see literature review in [10]). Particularly, synchronization control, aiming at regulating the states or outputs of agents to converge to a common value, plays an important role in seeking collaborative policies such that collective behaviors among agents can emerge. The synchronization of coupled harmonic oscillators, which can be considered as a second-order system with a restoring-force-like term [11], has potential applications in various aspects, such as the coordination of mobile robots [12] and the design of electrical networks [13]. In this sense, it is of both theoretical and practical significance to investigate the synchronization problem of coupled harmonic oscillators.

According to the available measurements therein, the existing synchronization protocols for coupled harmonic oscillators can be classified into velocity-based [14]–[17] and position-based methods [18], [19]. Ren in [14] designed distributed synchronization law using velocity measurements, and analyze the convergence under directed fix and switching topologies. Su *et al.* developed a control law to achieve synchronization without any connectivity assumption on the agents [15]. It is worth noting that the control laws in [14] and [15] were continuous-time, which results in a high communication consumption, therefore authors in [16] and [17] proposed sampled-velocity-based control laws, and further considered the practical issues including communication and input delays. Nevertheless, velocity measurements are not always available due to some technological difficulties and external disturbances, and the position states are relatively more convenient to obtain [20]. In [18], synchronization control law was designed using the delayed position measurements. In [19], two sampled-position-based control algorithms were subsequently designed to reach synchronization using position information obtained at current and previous sampling time, respectively. Sampled-data-based control can effectively reduce the communication burdens, however, the periodic interactions and control still result in unnecessary energy and resource consumption. An alternative approach is called event-triggered control [21], wherein the sampling is activated if and only if the state-dependent error signals are beyond certain thresholds. Consequently, event-based control can reduce the computation and transmission costs compared with time-based sampling.

Recently, event-triggered control has achieved great progresses in synchronization of coupled harmonic oscillators.

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Authors in [22] proposed centralized and decentralized event-triggered control strategy by designing state-dependent threshold to achieve asymptotic synchronization, and further extended results to an event-triggered control with fixed threshold such that the synchronization errors approach a small bound around zero. In [23], an edge event-triggered control mechanism was developed, under which each oscillator only needed to update some of the relative states of its neighbors. This is different from the event-triggered rules in [22], where all relative states of the neighbors were required to update once an event happens. Dai and Xiao extended the results in [23] to edge-self-triggered control for coupled harmonic oscillators with quantization and time delays [24]. In [25], small gain technology and the integral quadratic constraints were employed to analyze the stability conditions under event-triggered control scheme. It is worth mentioning that a basic requirement of event-triggered control is that the events should not occur arbitrarily. In other words, the time interval between any two consecutive events should not be zero, which is also known as Zeno-behavior. Otherwise, the sampling frequency will be increased and the objective of reducing the resource consumption cannot be satisfied. However, if Zeno-behaviour is avoided, the synchronization rate of the oscillators will slow down because the events are triggered with low frequency [26]. Therefore, how to handle the contradiction between excluding Zeno-behavior and promoting convergence speed should be further considered.

To achieve the synchronization with faster speed, impulsive strategy has been found important application in coupled harmonic oscillators. In [27], distributed impulsive control strategy was proposed to achieve the synchronization using only position measurement, and demonstrated that the desired sampling period can be determined by the positive gain and the eigenvalues of Laplacian matrix associated with the network. Liu *et al.* studied bipartie consensus control problem of oscillators on the basis of impulsive control [28]. Song *et al.* further investigated the impulsive-controlled convergent behaviors of coupled harmonic oscillators using current and previous sampled positions [11]. Note that the impulsive frequency in [11], [27], and [28] were typically predefined and constant, which results in high computational and communication costs. To remove such constraint, researchers have proposed event-triggered impulsive control, under which the impulsive instants are determined by well-defined event conditions. Therefore, no signal transmission or computation is required between two consecutive triggering instants, and the impulsive control signal can promote the synchronization performance. Some related control approaches for multi-agent systems can be found in [26], [29], and [30]. Nevertheless, these methods cannot be always applied to coupled harmonic oscillators, because they required that both position and velocity should be measurable. In other words, event-triggered impulsive control for coupled harmonic oscillators using either position or velocity measurements is an issue worthy of study.

Motivated by the above considerations, this paper intends to address the synchronization problem of coupled harmonic oscillators via event-triggered impulsive control. We respectively design the position-only and velocity-only event-triggered impulsive mechanisms, and provide necessary conditions to ensure the synchronization in the absence of Zeno behavior. The main contribution of this paper is threefold, as follows:

1) We extend the event-based approaches in [22]–[25] and impulse-based approaches in 11], [27], and [28], such that a tradeoff between sampling frequency and convergence speed is achieved. By integrating event-triggered method into impulsive control, the control input is only activated at the triggered instants, i.e., no control input is required during any two consecutive triggered instants. In this sense, event-triggered impulsive control can reduce the resource consumption.
2) Compared with event-triggered impulsive control methods presented in [26], [29], and [30] where both position and velocity measurements are needed, our method only require either position or velocity measurement, which is less restricted and more flexible.
3) We present the global asymptotic and exponential stability criteria to achieve synchronization. In addition, the Zeno-behavior is guaranteed to be excluded.

The remainder of this paper is organized as follows. Section II provides some preliminaries and formulates the synchronization problem of coupled harmonic oscillators. Then we present position-only and velocity-only event-triggered impulsive control approaches in Section III. Numerical experiment results are given in Section IV to show the effectiveness of the proposed synchronization protocols. Conclusions are summarized in Section V.

Notation: $\mathbb{R}^n$ denotes the Euclidean space with dimension n. $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices. Let $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$ be the identity matrix. The symbol $\otimes$ is the Kronecker product. $\|\cdot\|$ denotes the L_2 norm. $\lambda_{\min}(\boldsymbol{P})$ and $\lambda_{\max}(\boldsymbol{P})$ are the smallest and largest eigenvalue of the matrix $\boldsymbol{P}$. $\exp(\cdot)$ stands for an exponential function.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. GRAPH THEORY

The interactions among oscillators can be characterized by directed or undirected graphs, and we particularly focus on directed case in this research. A directed graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is a vertex set and $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}\}$ is the edge set. A directed edge (j,i) means that vertex i can receive information from vertex j. The neighbor set of vertex i is denoted by $N_i = \{j \in \mathcal{V} | (j,i) \in \mathcal{E}\}$. The adjacency matrix $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $a_{ij} > 0$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Note that $a_{ii} = 0$ for all vertices. The Laplacian matrix $\boldsymbol{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $l_{ij} = -a_{ij}$ if $i \neq j$, otherwise $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$. A directed path from vertex i_0 to vertex i_k is a sequence of directed edges with the form of (i_l, i_{l+1}), where $i_l \in \mathcal{V}\,(l = 0, 1, \ldots, k-1)$. A directed graph contains a spanning tree if there exists at least one vertex connecting to all the other vertices through directed paths.

If there exists a leader, interactions from leader to followers can be characterized by adjacency matrix $\boldsymbol{B} = diag\{b_1, \ldots, b_n\}$, where $b_i = 1$ if follower i can receive information from the leader. Let the leader be represented by vertex 0, and we denote the augmented graph containing leader and followers by $\bar{\mathcal{G}} = \left(\bar{\mathcal{V}}, \bar{\mathcal{E}}\right)$, where $\bar{\mathcal{V}} = \mathcal{V} \cup \{0\}$ and $\bar{\mathcal{E}}$ includes $\mathcal{E}$ and edges from the leader to followers. Denote matrix $\boldsymbol{H} \in \mathbb{R}^{n\times n}$ as $\boldsymbol{H} = \boldsymbol{L} + \boldsymbol{B}$. From [25], matrix $\boldsymbol{H}$ is positive definite if $\bar{\mathcal{G}}$ contains a spanning tree with the leader being the root vertex.

B. PROBLEM FORMULATION

Consider a group of n coupled harmonic oscillators described by following dynamic equations [22]:

$$\begin{cases} \dot{r}_i(t) = v_i(t) \\ \dot{v}_i(t) = -\alpha r_i(t) + u_i(t), \end{cases} \quad i = 1, 2, \ldots, n \tag{1}$$

where $r_i(t), v_i(t) \in \mathbb{R}$ denote the position and velocity of the ith oscillator. $\sqrt{\alpha}$ is the frequency of the oscillator and $u_i(t)$ is the control input to be designed.

The leader oscillator can be described with the following dynamics:

$$\begin{cases} \dot{r}_0(t) &= v_0(t) \\ \dot{v}_0(t) &= -\alpha r_0(t) \end{cases} \tag{2}$$

where $r_0(t), v_0(t) \in \mathbb{R}$ are the position and velocity of the leader.

Definition 1: A group of coupled harmonic oscillators is said to achieve synchronization with the leader, if for any $r_i(0), v_i(0) \in \mathbb{R}$, the position and velocity of the follower converge to that of the leader, i.e.,

$$\begin{cases} \lim\limits_{t\to\infty} \|r_i(t) - r_0(t)\| = 0 \\ \lim\limits_{t\to\infty} \|v_i(t) - v_0(t)\| = 0, \end{cases} \quad i = 1, 2, \ldots, n \tag{3}$$

Assumption 1: The directed graph $\bar{\mathcal{G}}$ contains a spanning tree with the leader as the root.

The objective of this paper is to design the event-triggered impulsive control strategies such that a synchronization among oscillators is achieved. Fig. 1 shows the diagram of the event-triggered impulsive control. Each oscillator continuously computes the triggering function $f_i(t)$ via the states (position or velocity) of the neighboring oscillators and itself. Once $f_i(t)$ violates the well-designed condition, a triggering instant t_i^k is determined and oscillator is required to calculate the error signal $p_i\left(t_i^k\right)$ at the instant t_i^k. Subsequently, the impulse generator will be activated and generates the corresponding discrete-time control input $u_i\left(t_i^k\right)$. Therefore, the oscillator will update the states and continue monitoring $f_i(t)$ until the next event happens.

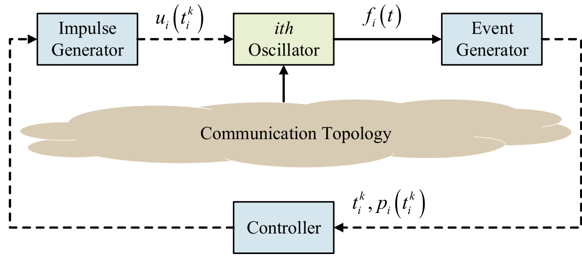


FIGURE 1. A diagram of event-triggered impulsive control scheme, wherein the continuous-time and discrete-time signals are denoted by solid arrow and dashed arrow, respectively.

Remark 1: From the above analysis, it can be observed that event-triggered impulsive control is discrete-time and non-periodic. This is different from conventional event-triggered control [22]–[25] where the control inputs are maintained constant and non-zero between any two consecutive event instants. In addition, most impulsive control such as [11], [27], [28] assumed that the impulsive frequency is typically predefined and constant. For the sake of fast convergence speed, the impulsive frequency must be designed high enough. In this sense, event-triggered impulsive control can effectively reduce the energy cost.

Before introducing the main results, we introduce a useful lemma, namely Gronwall inequality.

Lemma 1 [31]: Let $x(t)$ be a continuous function on $[t_0, T), T \leq +\infty$, which satisfies the inequality

$$x(t) < h + \int_{t_0}^{t} wx(s)\,ds, \quad t \in [t_0, T) \tag{4}$$

where h is a non-negative constant and w is a constant. Then

$$x(t) \leq he^{w(t-t_0)}, \quad t \in [t_0, T) \tag{5}$$

III. MAIN RESULTS

This section will present two event-triggered impulsive control strategies using respectively position and velocity measurements, and then theoretically analyze the convergence behavior and the exclusion of Zeno behavior.

A. POSITION-ONLY EVENT-TRIGGERED IMPULSIVE CONTROL STRATEGY

For i-th oscillator with dynamics described by Eq. (1), let $\boldsymbol{x}_i(t) = [r_i(t), v_i(t)]^T$. We then define the position error relative to the neighboring oscillators as

$$\begin{aligned} p_i(t) &= \sum_{j=1}^{n} a_{ij}\left(r_j(t) - r_i(t)\right) + b_i\left(r_0(t) - r_i(t)\right) \\ &= \boldsymbol{l}_p\left(\sum_{j=1}^{n} a_{ij}\left(\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)\right) + b_i\left(\boldsymbol{x}_0(t) - \boldsymbol{x}_i(t)\right)\right) \end{aligned} \tag{6}$$

where $\boldsymbol{l}_p = [1, 0]$.

We design the following distributed event-triggered impulsive control law:

$$u_i(t) = \gamma p_i\left(t_i^k\right)\delta\left(t - t_i^k\right) \tag{7}$$

where $\gamma > 0$ is the impulsive strength and $\delta(\cdot)$ is the Dirac delta function. $\left\{t_i^k | k = 0, 1, \ldots\right\}$ is the triggering sequence satisfying $t_0 = t_i^0 < t_i^1 < \ldots < t_i^k < \ldots$ and $\lim_{k\to\infty} t_i^k = +\infty$. Different from the impulsive control where the impulsive interval is predefined and constant, the triggering sequence is defined by

$$t_i^k = \inf\left\{t : t > t_i^{k-1}, f_i(t) \geq 0\right\}, k = 1, 2, \ldots, \tag{8}$$

where $f_i(t)$ is the triggering function defined by

$$f_i(t) = \left\|\boldsymbol{E}_i\left(t_i^k\right) - \boldsymbol{E}_i(t)\right\|^2 - \left(\beta\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2 + \frac{\eta}{(t-t_0)^2}\right) \tag{9}$$

where $\beta > 0$, $\eta > 0$ and $\boldsymbol{E}_i(t) = \left[-p_i(t), -\dot{p}_i(t)\right]^T$.

It is worth noting that impulsive control law (7) is only activated at the event instant t_i^k. In other words, $u_i(t) = 0, \forall t \notin \left\{t_i^k | k = 0, 1, \ldots\right\}$. Therefore, for these non-event instants, the system dynamics is updated as

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t), t \in \left[t_i^{k-1}, t_i^k\right) \tag{10}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} \tag{11}$$

At event instant t_i^k, using the property of Dirac function, apply impulse control law (7) to Eq. (1) yielding

$$\boldsymbol{x}_i\left(t_i^k\right) = \boldsymbol{x}_i\left(t_i^{k-}\right) - \gamma \boldsymbol{C}_p\left(\sum_{j=1}^{n} a_{ij}\left(\boldsymbol{x}_i\left(t_i^{k-}\right) - \boldsymbol{x}_j\left(t_i^{k-}\right)\right) + b_i\left(\boldsymbol{x}_i\left(t_i^{k-}\right) - \boldsymbol{x}_0\left(t_i^{k-}\right)\right)\right) \tag{12}$$

where $\boldsymbol{x}_i\left(t_i^{k-}\right) = \lim_{t\to t_i^{k-}} \boldsymbol{x}_i(t)$ and

$$\boldsymbol{C}_p = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} \tag{13}$$

Define the synchronization error as $\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{x}_0(t)$, using Eqs. (2), (10) and (12), the error dynamics is given by

$$\begin{cases} \dot{\boldsymbol{e}}_i(t) = \boldsymbol{A}\boldsymbol{e}_i(t), t \in \left[t_i^{k-1}, t_i^k\right) \\ \boldsymbol{e}_i\left(t_i^k\right) = \boldsymbol{e}_i\left(t_i^{k-}\right) - \boldsymbol{C}_p\left(\sum_{j=1}^{n} a_{ij}\left(\boldsymbol{e}_i\left(t_i^{k-}\right) - \boldsymbol{e}_j\left(t_i^{k-}\right)\right) \right. \\ \qquad \left. + b_i \boldsymbol{e}_i\left(t_i^{k-}\right)\right) \end{cases} \tag{14}$$

For convenience, we rearrange all the triggering sequences $\left\{t_i^k | k = 0, 1, \ldots, i \in \mathcal{V}\right\}$ with chronological order: $\{t_k | k = 0, 1, \ldots\} = \left\{t_i^s | s = 0, 1, \ldots, i \in \mathcal{V}\right\}$ satisfying $t_0 < t_1 < \ldots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = +\infty$.

Algorithm 1 Distributed Event-Triggered Impulsive Control Algorithm for Coupled Harmonic Oscillators Using Only Position Measurements

Input: Initial position $r_i(t_0)$ and velocity $v_i(t_0)$, $i = 0, 1, \ldots, n$;

Objective: $\lim_{t\to\infty} \|\boldsymbol{e}_i(t)\| = 0$

Initialize controller parameters γ, β and η;
$k \leftarrow 0$;
Initialize event instant $t_i^k \leftarrow t_0$;
while $t \leq t_{end}$ **do**
  Update leader's position $r_0(t)$ and velocity $v_0(t)$;
  for i from 1 to n **do**
    Receive position information $r_j(t)$ from neighboring oscillators, $j \in N_i$;
    Compute position error $p_i(t)$ using Eq. (6);
    Compute triggering function $f_i(t)$ using Eq. (9);
    if $f_i(t) \geq 0$ **then**
      $k \leftarrow k + 1$;
      Update event instant t_i^k and save $\boldsymbol{E}_i\left(t_i^k\right)$;
      Activate impulse input $u_i(t)$ using Eq. (7);
      Update $r_i\left(t_i^k\right)$ and $v_i\left(t_i^k\right)$ using Eq. (12);
    else
      Update $r_i(t)$ and $v_i(t)$ using Eq. (10);
    end if
  end for
end while

At triggering instant t_k, we define the activation function $\sigma_i(t_k)$ to indicate whether ith oscillator is triggered or not: $\sigma_i(t_k) = 1$ if $t_k \in \left\{t_i^s | s = 0, 1, \ldots\right\}$, and $\sigma_i(t_k) = 0$ otherwise. Let $\boldsymbol{e} = \left[\boldsymbol{e}_1^T(t), \ldots, \boldsymbol{e}_n^T(t)\right]^T$, we rewrite Eq. (14) in a compact form as

$$\begin{cases} \dot{\boldsymbol{e}}(t) = (\boldsymbol{I}_n \otimes \boldsymbol{A})\boldsymbol{e}(t), t \in [t_{k-1}, t_k) \\ \boldsymbol{e}(t_k) = (\boldsymbol{I}_n \otimes \boldsymbol{I}_2)\boldsymbol{e}\left(t_k^-\right) - \left(\boldsymbol{\sigma}(t_k)\boldsymbol{H} \otimes \boldsymbol{C}_p\right)\boldsymbol{e}\left(t_k^-\right) \end{cases} \tag{15}$$

where $\boldsymbol{\sigma}(t_k)$ is the event-triggered matrix defined as

$$\boldsymbol{\sigma}(t_k) = \begin{bmatrix} \sigma_1(t_k) & 0 & \cdots & 0 \\ 0 & \sigma_2(t_k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n(t_k) \end{bmatrix} \tag{16}$$

We summarize the event-triggered impulsive control algorithm in Algorithm 1. Then, several sufficient conditions will be given to converge the synchronization error signal $\boldsymbol{e}(t)$ to zero under our proposed event-triggered impulsive control scheme, as specified in the following theorem.

Theorem 1: Consider the coupled harmonic oscillators (1) with event-triggered impulsive control law (7), under Assumption 1, for any $\beta > 0$ and $\eta > 0$, if there exists positive constant μ, positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{2n\times 2n}$ and event-triggered matrix $\boldsymbol{\sigma}(k)$, such that

$$(\boldsymbol{I}_n \otimes \boldsymbol{A})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{I}_n \otimes \boldsymbol{A}) + \mu(\boldsymbol{I}_n \otimes \boldsymbol{I}_2) \leq 0 \tag{17}$$

and

$$\sum_{j=1}^{k}\left(\ln\left(\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\left\|Z_p(t_j)\right\|\right)-\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}\left(t_{j+1}-t_j\right)\right)\leq\varphi(t,t_0) \tag{18}$$

where $k=1,2,\ldots$, $\boldsymbol{Z}_p(t_k)=\boldsymbol{I}_n\otimes\boldsymbol{I}_2-\boldsymbol{\sigma}(t_k)\boldsymbol{H}\otimes\boldsymbol{C}_p$ and $\varphi(t,t_0)$ is a continuous function satisfying

$$\lim_{t\to\infty}\varphi(t,t_0)=-\infty \tag{19}$$

then synchronization error $\boldsymbol{e}(t)$ globally asymptotically converges to zero. Moreover, if there exists a positive constant $c>0$ such that

$$\varphi(t,t_0)\leq -c(t-t_0), \tag{20}$$

then synchronization error $\boldsymbol{e}(t)$ converges to zero globally exponentially. Additionally, Zeno-behavior can be excluded under triggering function (9).

Proof: We consider the Lyapunov function $V(\boldsymbol{e}(t))=\boldsymbol{e}^T(t)\boldsymbol{P}\boldsymbol{e}(t)$. For $t\in[t_{k-1},t_k)$, $k=1,2,\ldots$, the derivative of $V(\boldsymbol{e}(t))$ along Eq. (14) is given by

$$\begin{aligned}\dot{V}(\boldsymbol{e}(t))&=\boldsymbol{e}^T(t)\left[(\boldsymbol{I}_n\otimes\boldsymbol{A})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{I}_n\otimes\boldsymbol{A})\right]\boldsymbol{e}(t)\\&\leq-\mu\|\boldsymbol{e}(t)\|^2\end{aligned} \tag{21}$$

wherein we use the inequality (17). Moreover, for any $t\in[t_{k-1},t_k)$, one has

$$\begin{aligned}\lambda_{\min}(\boldsymbol{P})\|\boldsymbol{e}(t)\|^2&\leq V(\boldsymbol{e}(t))\\&\leq V(\boldsymbol{e}(t_{k-1}))+\int_{t_{k-1}}^{t}-\mu\|\boldsymbol{e}(s)\|^2ds\\&\leq\lambda_{\max}(\boldsymbol{P})\|\boldsymbol{e}(t_{k-1})\|^2\\&\quad+\int_{t_{k-1}}^{t}-\mu\|\boldsymbol{e}(s)\|^2ds\end{aligned} \tag{22}$$

Using the Gronwall inequality, one has

$$\|\boldsymbol{e}(t)\|^2\leq\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\|\boldsymbol{e}(t_{k-1})\|^2\times\exp\left(\int_{t_{k-1}}^{t}-\frac{\mu}{\lambda_{\min}(\boldsymbol{P})}ds\right) \tag{23}$$

Therefore, we have

$$\begin{aligned}\|\boldsymbol{e}(t)\|&\leq\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\times\|\boldsymbol{e}(t_{k-1})\|\\&\quad\times\exp\left(\int_{t_{k-1}}^{t}-\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}ds\right)\\&\leq\|\boldsymbol{Z}(k-1)\|\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\times\left\|\boldsymbol{e}\left(t_{k-1}^-\right)\right\|\\&\quad\times\exp\left(\int_{t_{k-1}}^{t}-\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}ds\right)\end{aligned} \tag{24}$$

where the last inequality uses $\boldsymbol{e}(t_k)=\boldsymbol{Z}_p(t_k)\boldsymbol{e}\left(t_k^-\right)$, given in Eq. (15).

Via a similar procedure as shown in Eq. (24), it can be derived that

$$\left\|\boldsymbol{e}\left(t_{k-1}^-\right)\right\|\leq\|\boldsymbol{Z}(k-2)\|\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\times\left\|\boldsymbol{e}\left(t_{k-2}^-\right)\right\|\exp\left(\int_{t_{k-2}}^{t_{k-1}}-\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}ds\right) \tag{25}$$

Introducing Eq. (25) into Eq. (24), we can obtain

$$\|\boldsymbol{e}(t)\|\leq\left\|\boldsymbol{e}\left(t_{k-2}^-\right)\right\|(\|\boldsymbol{Z}(k-1)\|\,\|\boldsymbol{Z}(k-2)\|)\times\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\exp\left(\int_{t_{k-2}}^{t}-\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}ds\right) \tag{26}$$

Consequently, we have

$$\begin{aligned}\|\boldsymbol{e}(t)\|&\leq\|\boldsymbol{e}(t_0)\|\left(\prod_{j=1}^{k-1}\|\boldsymbol{Z}(j)\|\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\right)\\&\quad\times\exp\left(-\sum_{j=1}^{k-1}\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}\xi\left(T_j(t_0,t)\right)\right)\\&\leq\|\boldsymbol{e}(t_0)\|\exp\left(\sum_{j=1}^{k-1}\left(\ln\left(\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\left\|Z_p(j)\right\|\right)-\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}\xi\left(T_j(t_0,t)\right)\right)\right)\\&\leq\|\boldsymbol{e}(t_0)\|\exp(\varphi(t,t_0))\end{aligned} \tag{27}$$

where the last inequality uses Eq. (18), and $\xi\left(T_j(t_0,t)\right)$ is the Lebesgue measure of the set $T_j(t_0,t)$ [26].

From Eq. (27), we can conclude that the synchronization error $\boldsymbol{e}(t)$ globally asymptotically converges to zero if Eq. (19) is satisfied, and $\boldsymbol{e}(t)$ globally exponentially converges to zero if Eq. (20) is satisfied.

It rest to demonstrate that Zeno-behavior can be excluded using the triggering function (9). Let $\boldsymbol{h}_i$ denote the i-th row of matrix $\boldsymbol{H}$, then Eq. (6) can be rewritten as $p_i(t)=-\left(h_i\otimes l_p\right)\boldsymbol{e}(t)$.

Referring to $\boldsymbol{E}_i(t)=[-p_i(t),-\dot{p}_i(t)]^T$, we obtain $\boldsymbol{E}_i(t)=(\boldsymbol{h}_i\otimes\boldsymbol{I}_2)\boldsymbol{e}(t)$. In addition, for any $t\in\left[t_i^k,t_i^{k+1}\right)$, the derivative of $\boldsymbol{E}_i(t)$ is given by $\dot{\boldsymbol{E}}_i(t)=(\boldsymbol{h}_i\otimes\boldsymbol{A})\boldsymbol{e}(t)$.

Let $\boldsymbol{Y}_i(t)=\boldsymbol{E}_i(t)-\boldsymbol{E}_i\left(t_i^k\right)$ and $\boldsymbol{\Gamma}_i(t)=\boldsymbol{Y}_i^T(t)\dot{\boldsymbol{Y}}_i(t)$, one has $\dot{\boldsymbol{\Gamma}}_i(t)=2\boldsymbol{Y}_i^T(t)\dot{\boldsymbol{Y}}_i(t)\leq\boldsymbol{Y}_i^T(t)\boldsymbol{Y}_i(t)+\dot{\boldsymbol{Y}}_i^T(t)\dot{\boldsymbol{Y}}_i(t)$. Furthermore, we have

$$\begin{aligned}\dot{\boldsymbol{\Gamma}}_i(t)&\leq\boldsymbol{\Gamma}_i(t)+\boldsymbol{e}^T(t)\left(\boldsymbol{h}_i^T\otimes\boldsymbol{A}^T\right)(\boldsymbol{h}_i\otimes\boldsymbol{A})\boldsymbol{e}(t)\\&\leq\boldsymbol{\Gamma}_i(t)+\boldsymbol{e}^T(t)\left(\boldsymbol{h}_i^T\boldsymbol{h}_i\otimes\boldsymbol{A}^T\boldsymbol{A}\right)\boldsymbol{e}(t)\\&\leq\boldsymbol{\Gamma}_i(t)+\|\boldsymbol{A}\|^2\|\boldsymbol{E}_i(t)\|^2\\&\leq\boldsymbol{\Gamma}_i(t)+\|\boldsymbol{A}\|^2\left\|\boldsymbol{Y}_i(t)+\boldsymbol{E}_i\left(t_i^k\right)\right\|^2\\&\leq\boldsymbol{\Gamma}_i(t)+\|\boldsymbol{A}\|^2\left(2\boldsymbol{\Gamma}_i(t)+2\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2\right)\\&\leq\left(1+2\|\boldsymbol{A}\|^2\right)\boldsymbol{\Gamma}_i(t)+2\|\boldsymbol{A}\|^2\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2\end{aligned} \tag{28}$$

Using the fact $\boldsymbol{Y}_i\left(t_i^k\right) = \mathbf{0}$, we obtain

$$\|\boldsymbol{Y}_i(t)\|^2 \leq \frac{2\|\boldsymbol{A}\|^2\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2}{1+2\|\boldsymbol{A}\|^2}\left(e^{\left(1+2\|\boldsymbol{A}\|^2\right)\left(t-t_i^k\right)}-1\right) \tag{29}$$

Referring to the triggering function (9), after instant t_i^k, the next triggered instant will arrive if

$$\begin{aligned}\|\boldsymbol{Y}_i(t)\|^2 &= \beta\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2+\frac{\eta}{(t-t_0)^2}\\ &\leq \frac{2\|\boldsymbol{A}\|^2\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2}{1+2\|\boldsymbol{A}\|^2}\left(e^{\left(1+2\|\boldsymbol{A}\|^2\right)\left(t-t_i^k\right)}-1\right)\end{aligned} \tag{30}$$

For any $t \in \left[t_i^k, t_i^{k+1}\right)$, one has

$$\frac{\eta}{(t-t_0)^2} \leq \frac{2\|\boldsymbol{A}\|^2\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2}{1+2\|\boldsymbol{A}\|^2}\left(e^{\left(1+2\|\boldsymbol{A}\|^2\right)\left(t-t_i^k\right)}-1\right) \tag{31}$$

Let $T_i^k = t_i^{k+1} - t_i^k$, we have

$$\frac{\eta}{\left(T_i^k+t_i^k-t_0\right)^2} \leq \frac{2\|\boldsymbol{A}\|^2\left\|\boldsymbol{E}_i\left(t_i^k\right)\right\|^2}{1+2\|\boldsymbol{A}\|^2}\left(e^{\left(1+2\|\boldsymbol{A}\|^2\right)T_i^k}-1\right) \tag{32}$$

Zeno behavior means that there exists an infinite number of events occurring in a finite time interval, in other words, T_i^k will be equal to zero if Zeno behavior happens. In this case, Eq. (32) will be rewritten as $\eta\Big/\left(T_i^k+t_i^k-t_0\right)^2 \leq 0$, which means $\eta \leq 0$. This is a contradiction with triggering function Eq. (9) where η is a positive constant. Therefore, T_i^k should be non-zero, and Zeno-behavior can be excluded. ■

Remark 2: From Eq. (30), it can be noticed that if the parameters β and η are increased, the next event-triggered instant will be delayed, i.e., the interval between two consecutive event instants will be enlarged, therefore the impulse frequency will be decreased.

B. VELOCITY-ONLY EVENT-TRIGGERED IMPULSIVE CONTROL

It is worth noting that position measurement is not always available in practice. To extend the application domain, in this section, we attempt to design event-triggered impulsive control law via only velocity measurements.

Similarly, for ith oscillator, we define the velocity error relative to the neighboring oscillators as

$$\begin{aligned}z_i(t) &= \sum_{j=1}^{n} a_{ij}\left(v_j(t)-v_i(t)\right)+b_i\left(v_0(t)-v_i(t)\right)\\ &= \boldsymbol{l}_z\left(\sum_{j=1}^{n} a_{ij}\left(\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t)\right)+b_i\left(\boldsymbol{x}_0(t)-\boldsymbol{x}_i(t)\right)\right)\end{aligned} \tag{33}$$

where $l_z = [0, 1]$. Subsequently, we propose the following control law

$$u_i(t) = \gamma z_i\left(t_i^k\right)\delta\left(t-t_i^k\right) \tag{34}$$

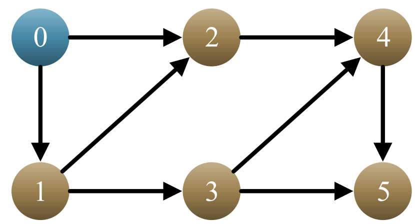


FIGURE 2. **The network topology among six harmonic oscillators in numerical simulations.**

The triggering function $f_i(t)$ is defined as

$$f_i(t) = \left\|\tilde{\boldsymbol{E}}_i\left(t_i^k\right)-\tilde{\boldsymbol{E}}_i(t)\right\|^2-\left(\beta\left\|\tilde{\boldsymbol{E}}_i\left(t_i^k\right)\right\|^2+\frac{\eta}{(t-t_0)^2}\right) \tag{35}$$

where $\tilde{\boldsymbol{E}}_i(t) = \left[\dot{z}_i(t)\big/\alpha, -z_i(t)\right]^T$.

Similar to the procedure established in Eq. (14), the error dynamics under control law (34) is given by

$$\begin{cases}\dot{\boldsymbol{e}}(t) = (\boldsymbol{I}_n \otimes \boldsymbol{A})\,\boldsymbol{e}(t), t \in [t_{k-1}, t_k)\\ \boldsymbol{e}(t_k) = (\boldsymbol{I}_n \otimes \boldsymbol{I}_2)\,\boldsymbol{e}\left(t_k^-\right)-(\boldsymbol{\sigma}(t_k)\boldsymbol{H} \otimes \boldsymbol{C}_v)\,\boldsymbol{e}\left(t_k^-\right)\end{cases} \tag{36}$$

where

$$\boldsymbol{C}_v = \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix} \tag{37}$$

Theorem 2: Consider the coupled harmonic oscillators (1) with event-triggered impulsive control law (34), under Assumption 1, for any $\beta > 0$ and $\eta > 0$, if there exists positive constant μ and positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{2n\times 2n}$ and event-triggered matrix $\boldsymbol{\sigma}(k)$, such that Eq. (17) and

$$\sum_{j=1}^{k}\left(\ln\left(\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\left\|\boldsymbol{Z}_v(t_j)\right\|\right)+\frac{\mu}{2\lambda_{\min}(\boldsymbol{P})}\left(t_j-t_{j+1}\right)\right) \leq \varphi(t, t_0) \tag{38}$$

where $\boldsymbol{Z}_v(t_k) = \boldsymbol{I}_n \otimes \boldsymbol{I}_2 - \boldsymbol{\sigma}(t_k)\boldsymbol{H} \otimes \boldsymbol{C}_v$, then synchronization error $\boldsymbol{e}(t)$ converges to zero globally asymptotically if Eq. (19) exists, and converges to zero globally exponentially if Eq. (20) exists.

Proof: The proof is similar to that of Theorem 1, and therefore we omit the proof due to page limitation. ■

IV. SIMULATION RESULTS

In this section, numerical simulation results are provided to validate the effectiveness of the proposed event-triggered impulsive control algorithms. We consider a group of six harmonic oscillators with $\alpha = 2$, and the interactions among oscillators are presented in Fig. 2. The initial conditions of oscillators are given by $\boldsymbol{x}_0(0) = [0, 1]^T$, $\boldsymbol{x}_1(0) = [0.5, 0]^T$, $\boldsymbol{x}_2(0) = [1, 0.2]^T$, $\boldsymbol{x}_3(0) = [1.5, 0.2]^T$, $\boldsymbol{x}_4(0) = [-0.5, 0]^T$, $\boldsymbol{x}_5(0) = [1.2, 0]^T$.

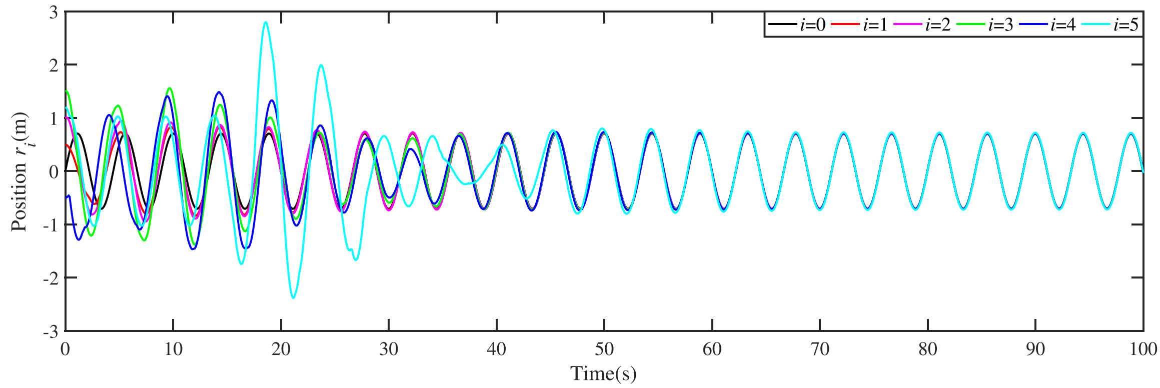


FIGURE 3. Positions of oscillators using position-only synchronization protocol (7).

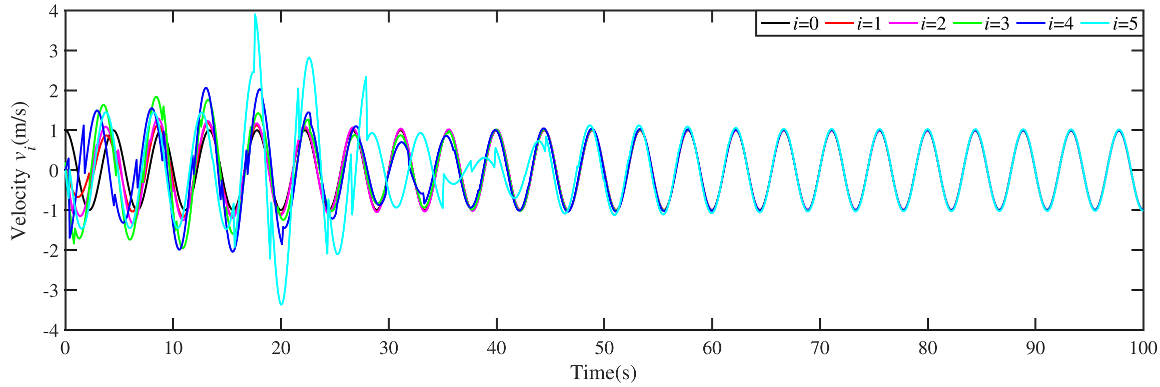


FIGURE 4. Velocities of oscillators using position-only synchronization protocol (7).

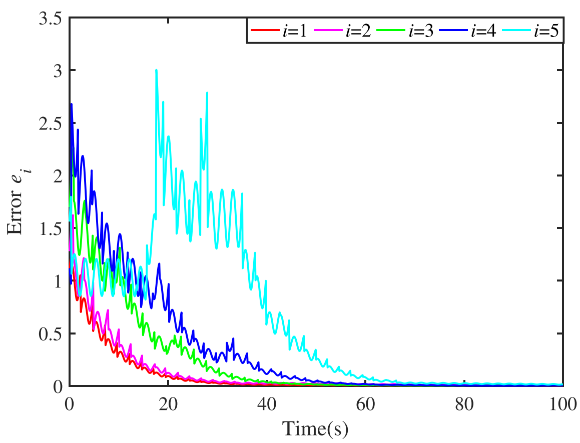


FIGURE 5. Synchronization errors under position-only synchronization protocol (7).

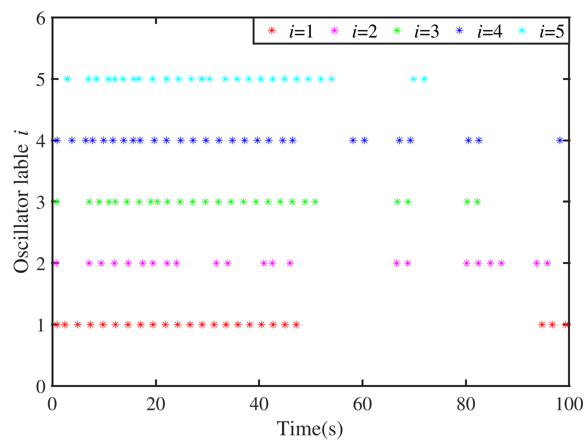


FIGURE 6. Event instants of the five oscillators under position-only synchronization protocol (7).

A. SYNCHRONIZATION OF OSCILLATORS UNDER POSITION-ONLY PROTOCOL

We firstly evaluate the synchronization behaviors of oscillators using position-only protocol (7). The parameters of triggering function (9) is given by $\beta = 6$ and $\eta = 2$, and the impulse strength parameter is $\gamma = 1$. Using Matlab LMI toolbox to solve inequality (17), we can obtain $\mu = 0.869$ and matrix $P \in \mathbb{R}^{10\times 10}$ (which is omitted due to page limitation).

The simulation results are plotted in Figs. 3-6. Fig. 3 and Fig. 4 present the position and velocity evolution of six oscillators, respectively. It can be seen that the

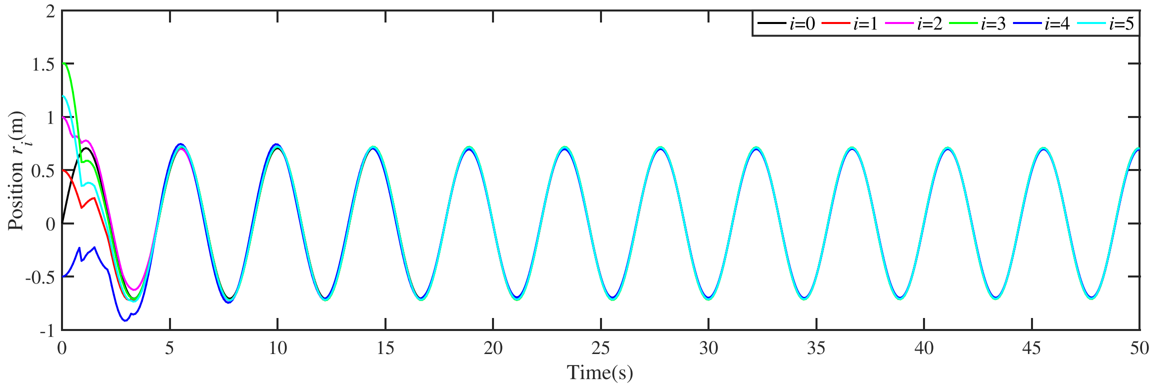


FIGURE 7. Positions of oscillators using velocity-only synchronization protocol (34).

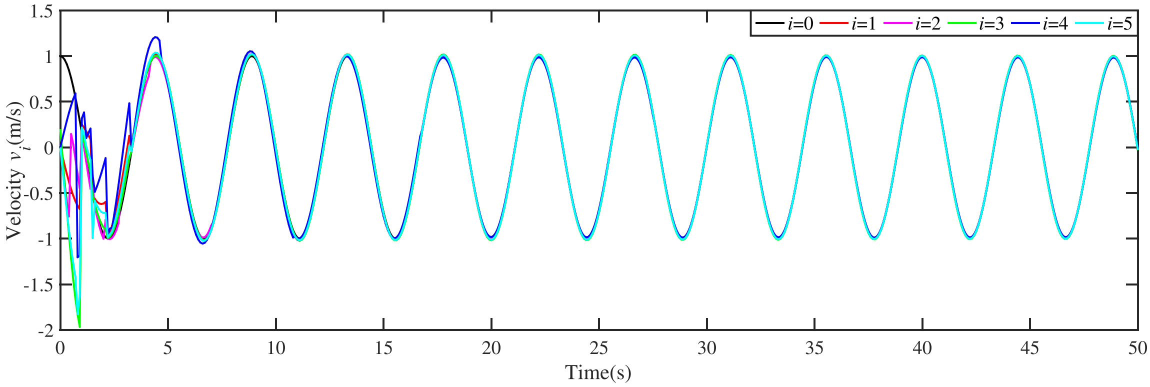


FIGURE 8. Velocities of oscillators using velocity-only synchronization protocol (34).

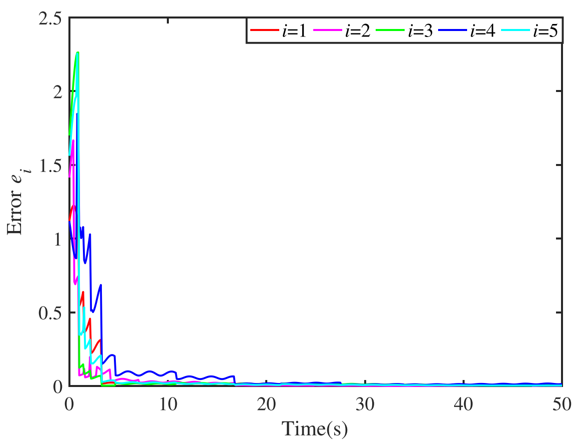


FIGURE 9. Synchronization errors under velocity-only synchronization protocol (34).

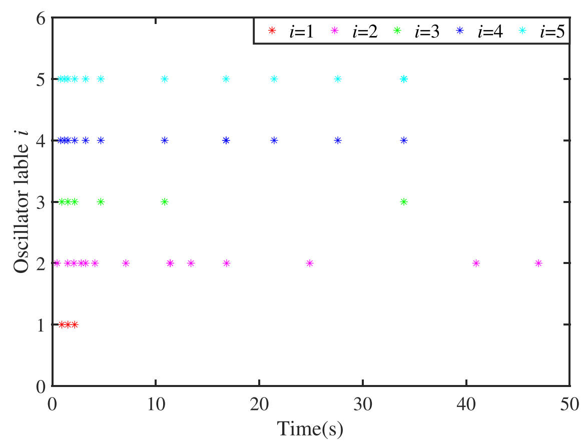


FIGURE 10. Event instants of the five oscillators under velocity-only synchronization protocol (34).

follower oscillators approaches the leader oscillator under the designed protocol. The evolution of synchronization errors is given in Fig. 5, from which we can notice that the synchronization errors converge to zero. The event instants of five followers are also presented in Fig. 6. From the above analysis, we can conclude that oscillators can achieve synchronization under our proposed position-only synchronization protocol.

B. SYNCHRONIZATION OF NETWORK UNDER VELOCITY-ONLY PROTOCOL

In this subsection, we will verify the performance of the velocity-only protocol (34). The parameters are given by

$\beta = 0.1$, $\eta = 1$ and $\gamma = 0.8$. The parameter $\mu = 0.869$ and matrix P is consistent with that in position-only protocol, this is because Eq. (17) is same for both methods.

The position and velocity trajectories of the oscillators are depicted in Figs. 7 and Fig. 8, respectively, which demonstrates that oscillator synchronization is achieved under the proposed velocity-only protocol. The synchronization errors are shown in Fig. 9, which converge to zero after nearly 20 seconds. Fig. 10 depicts the triggering instants of the oscillators. This simulation example validates the effectiveness of the velocity-only synchronization protocol.

V. CONCLUSION

This paper investigates the distributed synchronization problem of harmonic oscillators using event-triggered impulsive control. Specifically, we design position-only and velocity-only protocols to achieve synchronization in the absence of Zeno-behavior, respectively. It is worth noting that the impulsive control of each oscillator is activated if and only if the triggering function is violated, which can effectively reduce the resource consumption. Two simulation examples are provided to validate the proposed synchronization protocols. There are two shortcomings in this research: continuous communication between neighboring agents are required, and the effect of time delay on the synchronization is not considered. Therefore, it is of our great interest to study synchronization problem of harmonic oscillator systems with discrete communication and time delay in the future work.

REFERENCES

[1] J. Ren, Q. Song, and G. Lu, "Event-triggered bipartite leader-following consensus of second-order nonlinear multi-agent systems under signed digraph," *J. Franklin Inst.*, vol. 356, no. 12, pp. 6591–6609, Aug. 2019.

[2] J. Ren, Q. Song, Y. Gao, and G. Lu, "Leader-following bipartite consensus of second-order time-delay nonlinear multi-agent systems with event-triggered pinning control under signed digraph," *Neurocomputing*, vol. 385, pp. 186–196, Apr. 2020.

[3] J. Ren, Q. Song, Y. Gao, M. Zhao, and G. Lu, "Leader-following consensus of nonlinear singular multi-agent systems under signed digraph," *Int. J. Syst. Sci.*, vol. 52, no. 2, pp. 277–290, Jan. 2021.

[4] Z. Gao and G. Guo, "Fixed-time leader-follower formation control of autonomous underwater vehicles with event-triggered intermittent communications," *IEEE Access*, vol. 6, pp. 27902–27911, 2018.

[5] L. He, J. Zhang, Y. Hou, X. Liang, and P. Bai, "Time-varying formation control for second-order discrete-time multi-agent systems with directed topology and communication delay," *IEEE Access*, vol. 7, pp. 33517–33527, 2019.

[6] K. K. Oh, M. C. Park, and H. S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, Mar. 2015.

[7] L. Wang, J. Wu, X.-S. Zhan, T. Han, and H. Yan, "Fixed-time bipartite containment of multi-agent systems subject to disturbance," *IEEE Access*, vol. 8, pp. 77679–77688, 2020.

[8] W. Wang, H. Liang, Y. Pan, and T. Li, "Prescribed performance adaptive fuzzy containment control for nonlinear multiagent systems using disturbance observer," *IEEE Trans. Cybern.*, vol. 50, no. 9, pp. 3879–3891, Sep. 2020.

[9] D. Wang and W. Wang, "Necessary and sufficient conditions for containment control of multi-agent systems with time delay," *Automatica*, vol. 103, pp. 418–423, May 2019.

[10] A. Dorri, S. S. Kanhere, and R. Jurdak, "Multi-agent systems: A survey," *IEEE Access*, vol. 6, pp. 28573–28593, 2018.

[11] Q. Song, G. Wen, D. Meng, Z.-W. Liu, and F. Liu, "Distributed impulsive control for signed networks of coupled harmonic oscillators with sampled positions," *IEEE Trans. Control Netw. Syst.*, vol. 8, no. 1, pp. 111–122, Mar. 2021.

[12] L. Ballard, Y. Cao, and W. Ren, "Distributed discrete-time coupled harmonic oscillators with application to synchronised motion coordination," *IET Control Theory Appl.*, vol. 4, no. 5, pp. 806–816, 2010.

[13] S. E. Tuna, "Synchronization of harmonic oscillators under restorative coupling with applications in electrical networks," *Automatica*, vol. 75, pp. 236–243, Jan. 2017.

[14] W. Ren, "Synchronization of coupled harmonic oscillators with local interaction," *Automatica*, vol. 44, no. 2, pp. 3195–3200, 2008.

[15] H. Su, X. Wang, and Z. Lin, "Synchronization of coupled harmonic oscillators in a dynamic proximity network," *Automatica*, vol. 45, no. 10, pp. 2286–2291, 2009.

[16] W. Sun, J. Lü, S. Chen, and X. Yu, "Synchronisation of directed coupled harmonic oscillators with sampled-data," *IET Control Theory Appl.*, vol. 8, no. 11, pp. 937–947, Jul. 2014.

[17] L. Zhao, J. Zhou, and Q. Wu, "Sampled-data synchronisation of coupled harmonic oscillators with communication and input delays subject to controller failure," *Int. J. Syst. Sci.*, vol. 47, no. 1, pp. 235–248, 2016.

[18] Q. Song, W. Yu, J. Cao, and F. Liu, "Reaching synchronization in networked harmonic oscillators with outdated position data," *IEEE Trans. Cybern.*, vol. 46, no. 7, pp. 1566–1578, Jul. 2016.

[19] Q. Song, F. Liu, G. Wen, J. Cao, and Y. Tang, "Synchronization of coupled harmonic oscillators via sampled position data control," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 63, no. 7, pp. 1079–1088, Jul. 2016.

[20] Y. W. Wang and J. W. Yi, "Consensus in second-order multi-agent systems via impulsive control using position-only information with heterogeneous delays," *IET Control Theory Appl.*, vol. 9, no. 3, pp. 336–345, 2015.

[21] C. Peng and F. Li, "A survey on recent advances in event-triggered communication and control," *Inf. Sci.*, vols. 457–458, pp. 113–125, Aug. 2018.

[22] B. Wei and F. Xiao, "Event-triggered control for synchronization of coupled harmonic oscillators," *Syst. Control Lett.*, vol. 97, pp. 163–168, Nov. 2016.

[23] B. Wei, F. Xiao, and M.-Z. Dai, "Edge event-triggered synchronization in networks of coupled harmonic oscillators," *IEEE Trans. Cybern.*, vol. 47, no. 12, pp. 4162–4168, Dec. 2017.

[24] M.-Z. Dai and F. Xiao, "Edge-event- and self-triggered synchronization of coupled harmonic oscillators with quantization and time delays," *Neurocomputing*, vol. 310, pp. 172–182, Oct. 2018.

[25] Y. Xu, Z.-G. Wu, and Y.-J. Pan, "Synchronization of coupled harmonic oscillators with asynchronous intermittent communication," *IEEE Trans. Cybern.*, vol. 51, no. 1, pp. 258–266, Jan. 2019.

[26] X. Tan, J. Cao, and X. Li, "Consensus of leader-following multiagent systems: A distributed event-triggered impulsive control strategy," *IEEE Trans. Cybern.*, vol. 49, no. 3, pp. 792–801, Mar. 2019.

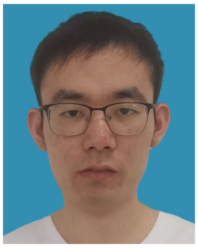
[27] H. Zhang and J. Ji, "Group synchronization of coupled harmonic oscillators without velocity measurements," *Nonlinear Dyn.*, vol. 91, no. 4, pp. 2773–2788, 2018.

[28] J. Liu, H. Li, and J. Luo, "Impulse bipartite consensus control for coupled harmonic oscillators under a coopetitive network topology using only position states," *IEEE Access*, vol. 7, pp. 20316–20324, 2019.

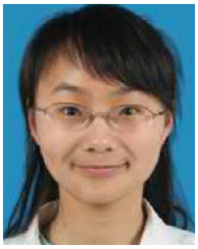
[29] W. Zhu, D. Wang, L. Liu, and G. Feng, "Event-based impulsive control of continuous-time dynamic systems and its application to synchronization of memristive neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 8, pp. 3599–3609, Aug. 2018.

[30] X. Li, D. Peng, and J. Cao, "Lyapunov stability for impulsive systems via event-triggered impulsive control," *IEEE Trans. Autom. Control*, vol. 65, no. 11, pp. 4908–4913, Nov. 2020.

[31] T. H. Gronwall, "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," *Ann. Math.*, pp. 292–296, 1919.



GUODONG MA was born in Xuzhou, China, in 1997. He received the B.Eng. degree in electrical engineering and automation from the School of Electrical Engineering, Nanhang Jincheng College, Nanjing, China, in 2019. He is currently pursuing the M.Sc. degree with Nantong University, Nantong, China. His current research interest includes consensus problem of multi-agent systems.



JIE REN received the B.Sc. degree in information and computer science and the M.Sc. degree in control theory and control engineering from Nantong University, Nantong, China, in 2007 and 2010, respectively, where she is currently pursuing the Ph.D. degree in information and communication engineering. She joined Xinglin College, Nantong University, Nantong, in 2010. Her current research interest includes bipartite consensus problems of multi-agent systems under signed digraph.

YANSEN LIU was born in Yangzhou, China, in 1998. He received the B.Eng. degree from the School of Electrical Engineering, Nantong University, Nantong, China, in 2020, where he is currently pursuing the M.Sc. degree. His current research interest includes consensus problem of multi-agent systems.

GUOPING LU received the B.S. degree from the Department of Applied Mathematics, Chengdu University of Science and Technology, Chengdu, China, in 1984, and the M.S. and Ph.D. degrees in applied mathematics from the Department of Mathematics, East China Normal University, Shanghai, China, in 1989 and 1998, respectively. He is currently a Professor with the School of Electrical Engineering, Nantong University, Jiangsu, China. His research interests include singular systems, multi-agent systems, networked control, and nonlinear signal processing.