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Novel Soliton Solutions of Two-Mode Sawada-Kotera Equation and Its Applications

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ABSTRACT The Sawada-Kotera equations illustrate the non-linear wave phenomena in shallow water, ion-acoustic waves in plasmas, fluid dynamics, etc. In this article, the two-mode Sawada-Kotera equation (tmSKE) occurring in fluid dynamics is considered which is important model equations for shallow water waves, the capillary waves, the waves of foam density, the electro-hydro-dynamical model. The improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion methods are utilized in this model and abundant of solitary wave solutions of different kinds such as bright and dark solitons, multi-peak soliton, breather type waves, periodic solutions, and other wave results are obtained. These achieved novel solitary and other wave results have significant applications in fluid dynamics, applied sciences and engineering. By granting appropriate values to parameters, the structures of few results are presented in which many structures are novel. The graphical moments of the results are provided to signify the impact of the parameters. To explain the novelty between the present results and the previously attained results, a comparative study has been carried out. The restricted conditions are also added on solutions to avoid singularities. Furthermore, the executed techniques can be employed for further studies to explain the realistic phenomena arising in fluid dynamics correlated with any physical and engineering problems.

INDEX TERMS Improve F-expansion method, generalized $\exp(-\phi(\zeta))$-expansion method, two-mode Sawada-Kotera equation, traveling and dual wave solutions, breather waves, periodic solitons.

I. INTRODUCTION

The dynamic complexity of physical phenomena in the real world can be expressed by the changes in temporal and spatial events. The temporal and spatial changes of physical phenomena are greatest articulated by partial differential equations (PDEs). The nonlinear PDEs are utilized for expressing various physical phenomena in the real world to get an insight through qualitative and quantitative features of many models that arise in diverse fields. Nonlinear wave phenomena emerge in plasma physics, fluid mechanics, solid-state physics, dynamics of chemical, non-linear optics, population model and other fields of science and engineering [1]–[19]. The analytical solutions of non-linear PDEs play a decisive part in non-linear science as they inform us deep imminent into the physical characteristics of the model and can provide further physical information to help in other applications. In recent years, the approximate and exact solutions of non-linear PDEs have attracted more and more attention, as they are utilized to illustrate the nonlinear complex phenomena in dissimilar scientific areas. Numerous real-world problems are altered into equations mathematically by differential equations. Thus, the finding wave results of all kinds of PDEs are a major problem, such as the present direction of non-linear science, which originated from the research of chemistry, physics, material science, biology, and many more, and has a burly practical backdrop. They have significant realistic applications and theoretical study in mathematics.

Lately, a novel family of nonlinear PDEs have been recognized in the name of ''dual-mode'' or ''two-mode'' about temporal and spatial derivatives. With regard to this curiosity,

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researchers have established some dual-mode nonlinear PDEs, namely two-mode (tm) mKdV equation [20], [21], tm KdV equation [10], [22], tm Sharma-Tasso-Olver equation [15], tm fifth order KdV (tmfKdV) equation [5], [23], two-mode Burger equation (tmBE) [24], tm Ostrovsky equation [25], tm perturbed Burger (tmPB) equation [25], tm KdV Burgers (tmKdVB) equation [26], tm Kadomtsev Petviashvili (tmKP) equation [27], [28], two-mode dispersive Fisher (tmdF) equation [29], tm Kuramoto-Sivashinsky (tmKS) equation [30], tm Boussinesq Burgers (tmBB) equation [31], two-mode coupled KdV and mKdV [32], [33], two-mode non-linear Schrödinger (tmNLS) [34], and tm Hirota Satsuma coupled KdV (tmHSKdV) [35] equations and the related dual-wave solutions are analyzed by different methods, such as Tanh expansion technique, (G'/G)-expansion technique, rational sine-cosine technique, Kudryshov technique, simplified Hirota technique, tanh-coth tachnque, sech-csch technique, Fourier spectral technique, B$\ddot{a}$cklund transformation scheme and trigonometric function technique [20]–[35]. As results, few solitons results in the form Kink, Kinks type of multiple soliton, periodic wave of singular kind, dark and bright solitons solutions have been conceded out for the aforementioned models.

The researcher Wazwaz [5] developed the tmSKE from the tmfKdV equation, and few multiple solitons results were determined by the simplified Hirota technique. Later on, the researchers in [23] investigated the tmfKdV model and established some Kink, bright and periodic solutions in singular form by using sine-cosine function and Kudryashov techniques. The authors in [18] were used modified Kudryashov and auxiliary equation methods, and dual wave solutions were constructed. It should be pointed out that the tmSKE is a special case of the tmfKdV equation. As far as the author is aware, although some two-mode PDEs have been extensively studied, the contributions to the above tmSKE are limited. It can be seen from the literature that there is room for further study of the tmSKE through the improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion methods, as well as the illustrating their physical explanations. The results executed by the projected methods are to be novel in the sense of methods application.

Several powerful and systematic methods (analytic, semi-analytic, and numerical methods) have been developed for studying non-linear PDEs [29]–[62], such as modified direct algebraic technique, Hirota bilinear technique, modified simple equation technique, B$\ddot{a}$cklund transformation scheme, F-expansion method, modified Kudryashov method, Darboux transform technique, (G'/G)-expansion technique, rational sine-cosine technique, inverse scattering scheme, auxiliary equation method, painlevé analysis method, trigonometric function technique, tanh/coth method, sine and sinh Gordon equation expansion methods, general symmetry technique, variational iteration technique, reduced differential transform method, Fourier spectral technique, finite difference technique, Adomian decomposition technique, finite element technique, the wavelet technique and other techniques.

This work aims to obtain solitons and other wave results of tmSKE. It is of interest to note here that the generalized $\exp(-\phi(\zeta))$-expansion method is an extended form of the $\exp(-\phi(\zeta))$-expansion method, and the improved F-expansion method is also an extended form of F-expansion method. Thus, motivated by the existing literature, a modest effort has been made in this study to construct some new dual-wave solutions to the TmSK equation via the project methods. The solutions attained by the improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion methods are to be new in the sense of methods application. The constructed results are novel and more general. To our best knowledge, these approaches are not utilized to address the early work on this equation.

This paper is structured as follows. In Section 1, specifies the introduction. In Section 2, a summary of the general form of tm standard and tm SK equations are summarized. In Section 3, the review of the improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion techniques are depicted. The constructed results from the investigation are given in Section 4. In Section 5, a general discussion and graphical illustrations of some acquired solutions are presented. Finally, the conclusion and future recommendations of the article are illustrated in Section 6.

II. FORMULATION OF MATHEMATICAL MODELS

A. GENERAL TYPE OF DUAL-MODE STANDARD MODEL

The general type of the two-mode or dual-mode model proposed by Korsonski [10] is as

$$\frac{\partial^2 u}{\partial t^2}-\nu\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial}{\partial t}-\beta\nu\frac{\partial}{\partial x}\right)G\left(u,u\frac{\partial u}{\partial x},\ldots\right)+\left(\frac{\partial}{\partial t}-\gamma\nu\frac{\partial}{\partial x}\right)N\left(\frac{\partial^2 u}{\partial r\partial x},r\geq 2\right)=0, \quad (1)$$

the above equation (1) is recognized from the equation of standard mode:

$$\frac{\partial u}{\partial t}+N\left(u,u\frac{\partial u}{\partial x},\ldots\right)+L\left(\frac{\partial^2 u}{\partial r\partial x},r\geq 2\right)=0.$$

In equation (1), the function $u(x,t)$ is an unknown with $(t,x)\in(-\infty,\infty)$, and $\nu>0$ is velocity of the phase, $\beta\leq 1,\gamma\leq 1$, β and γ symbolize nonlinearity and dispersion parameters respectively. The terms $L\left(\frac{\partial^2 u}{\partial r\partial x},r\geq 2\right)$ and $N\left(u,u\frac{\partial u}{\partial x},\ldots\right)$ signify the terms of linear and nonlinear respectively.

B. DUAL-MODE SAWADA-KOTERA MODEL

The SKE in standard form having two non-linear terms [5] has as

$$\frac{\partial u}{\partial t}+5\frac{\partial}{\partial x}\left(\frac{u^3}{3}+u\frac{\partial^2 u}{\partial x^2}\right)+\frac{\partial^5 u}{\partial x^5}=0, \quad (2)$$

in above equation, the terms $\frac{\partial^5 u}{\partial x^5}$ and $\frac{\partial}{\partial x}\left(\frac{u^3}{3}+u\frac{\partial^2 u}{\partial x^2}\right)$ are linear and nonlinear respectively.

Merging the sense of Korsunsky [10], and follow Wazwaz [5], the tmSKE of the standard SKE precises by equation (2) is presented as

$$\frac{\partial^2 u}{\partial t^2}-\nu\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial}{\partial t}-\beta\nu\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{5u^3}{3}+5u\frac{\partial^2 u}{\partial x^2}\right)+\left(\frac{\partial}{\partial t}-\gamma\nu\frac{\partial}{\partial x}\right)\frac{\partial^5 u}{\partial x^5}=0. \quad (3)$$

Obviously, for $\nu = 0$, the tmSKE specified through equation (3) after integrating the relevant time t has been simplified to the standard mode SKE given through equation (2).

The equation (3) illustrates the proliferation of two moving waves under the persuade of phase velocity ν, dispersion (γ), and non-linearity (β) factors.

III. PORTRAYAL OF PROPOSED METHODS

Here, we reveal the algorithms of suggested techniques namely as improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion methods for constructing the wave results of two-mode Sawada-Kotera model. The general nonlinear PDE has as

$$G\left(v, v_x, vv_x, v_t, v_{xx}, vv_{xx}, w_{tt}, \ldots\ldots\ldots\right)=0, \quad (4)$$

where the polynomial function G having unknown function $v(x,t)$ with respect to a few specific independent variables x and t, that also having derivative terms of linear and nonlinear. Assuming the transformation for changing independent variables into sole variable has as

$$v(x,t)=U(\zeta), \quad \zeta=kx-\omega t+\theta, \quad (5)$$

where the constant k and ω are wave length and frequency. Utilizing (5), the equation (4) is converting into ODE as

$$F\left(U, U', U'', UU'', \ldots\ldots\right)=0, \quad (6)$$

where $U'=\frac{dU}{d\zeta}$ and F is a polynomial of U and its derivatives.

A. IMPROVED F-EXPANTION METHOD

The main steps are as

1st Step: Consider the solution of Eq.(6) has as

$$U(\zeta)=\sum_{i=0}^{N}A_i\left(\mu+F(\zeta)\right)^i+\sum_{j=-1}^{-N}B_{-j}\left(\mu+F(\zeta)\right)^j, \quad (7)$$

where the constants A_i, B_{-j}, μ are real and the function $F(\zeta)$ in equation (7) pledges the below ODE

$$F'(\zeta)=\delta_0+\delta_1F(\zeta)+\delta_2F^2(\zeta)+\delta_3F^3(\zeta), \quad (8)$$

where δ_0, δ_1, δ_2 and δ_3 are real constants.

2nd Step: By utilizing homogeneous balance principle on Eq.(6), the positive integer N is obtained.

3rd Step: Deputizing Eq.(7) into Eq.(6) and taking the various coefficients of $\frac{F^i(\zeta)}{(\mu+F(\zeta))^j}$ to zero, capitulate a system of equation. By using Mathematica, this system is solved and constant values can be achieved. After substituting constant values and solutions of Eq.(6), the wave solutions of Eq.(7) are constructed.

B. GENERALIZED EXP($-\phi(\zeta)$)-EXPANSION METHOD

The main steps are as

1st Step: Assume the solution of Eq.(6) has the form as

$$U(\zeta)=\sum_{i=0}^{N}A_i\left(\exp(-\phi(\zeta))\right)^i, \quad (9)$$

where A_i $(0\le i\le N)$ are real constants such that $A_N\neq 0$ and $\phi=\phi(\zeta)$ pledges the ODE as

$$\phi'(\zeta)=a\exp(-\phi(\zeta))+b\exp(\phi(\zeta))+c, \quad (10)$$

where a, b, c are real constants.

2nd Step: Utilizing homogeneous balance principle on Eq.(6), the positive integer N is obtained.

3rd Step: By Deputizing equation (9) into (6) and polynomial obtained in $e^{(-\phi(\zeta))}$, and taking diverse powers of $(e^{(-\phi(\zeta))})^i$ to zero, capitulate a system of equation. By resolving this system and reverse substitution, we construct many exact solutions for Eq.(4).

IV. APPLICATIONS

In this part, we construct the solitons and other waves solutions of two-mode Sawada-Kotera equation by employing described methods. By employing the transformation described in Eq.(5), the Eq.(3) is converted into ODE as

$$\left(\omega^2-k^2\nu^2\right)U''-5k\left(\omega+\beta k\nu\right)\left(k^2UU^{(iv)}+2k^2U'U'''+k^2\left(U''\right)^2+U^2U''+2U\left(U'\right)^2\right)-k^5\left(\omega+\gamma k\nu\right)U^{(vi)}=0. \quad (11)$$

A. APPLICATION OF IMPROVED F-EXPANSION METHOD

Employing balancing principle on Eq.(11) and solution of equation (11) assumed as

$$U(\zeta)=A_0+A_1\left(\mu+F(\zeta)\right)+A_2\left(\mu+F(\zeta)\right)^2+\frac{B_1}{\mu+F(\zeta)}+\frac{B_2}{\left(\mu+F(\zeta)\right)^2}. \quad (12)$$

By substituting Eq.(12) into Eq.(11) and deputing the coefficients of $\frac{F^i(\zeta)}{(\mu+F(\zeta))^j}$ to zero, we attained a equations system $A_0, A_1, A_2, B_1, B_2, \delta_0, \delta_1, \delta_2, \delta_3, \beta, \gamma, k, \nu, \omega$ and θ. Mathematica 9 was utilized for solving this equation system. We attain the families of wave results as:

1st Family: Here assume $\delta_0=\delta_3=0$,

Set 1:

$$A_0=-\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(12\delta_2^2\mu^2-12\delta_2\delta_1\mu+\delta_1^2\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},$$

$$A_1=-\frac{12\delta_2\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_1-2\delta_2\mu\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},$$

$$A_2=-\frac{12\delta_2^2\sqrt{3\left(\gamma^2-1\right)\nu}}{\delta_1^2\sqrt{5(\beta-\gamma)}},$$

$$B_1 = B_2 = 0, \quad k = \mp\frac{\sqrt[4]{4\left(\gamma^2-1\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}},$$
$$\omega = \pm\frac{\gamma\nu\sqrt[4]{4\left(\gamma^2-1\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}}. \tag{13}$$

Set 2:

$$A_0 = -\frac{3k^2\left(12\delta_2^2\mu^2-12\delta_2\delta_1\mu+\delta_1^2\right)}{2},$$
$$A_1 = 18\delta_2k^2\left(2\delta_2\mu-\delta_1\right),$$
$$A_2 = -18\delta_2^2k^2, B_1 = 0,$$
$$B_2 = 0, \quad \nu = \frac{15\delta_1^4k^4(\beta-\gamma)}{4\left(\gamma^2-1\right)}, \ \omega = \frac{15\gamma\delta_1^4k^5(\gamma-\beta)}{4\left(\gamma^2-1\right)}. \tag{14}$$

Set 3:

$$A_0 = \frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(12\delta_2^2\mu^2-12\delta_2\delta_1\mu+\delta_1^2\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},$$
$$A_1 = \frac{12\delta_2\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_1-2\delta_2\mu\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},$$
$$B_1 = 0, \quad B_2 = 0, \ A_2 = \frac{12\delta_2^2\sqrt{3\left(\gamma^2-1\right)\nu}}{\delta_1^2\sqrt{5(\beta-\gamma)}},$$
$$k = \pm\frac{\sqrt[4]{4\left(1-\gamma^2\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}}, \quad \omega = \mp\frac{\gamma\nu\sqrt[4]{4\left(1-\gamma^2\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}}. \tag{15}$$

The soliton results of Eq.(3) from sets 1 and 2 are constructed in the form as

$$u_{1,2}(x,t) = -\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}+10\right)+1\right)}{\sqrt{5(\beta-\gamma)}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}-1\right)^2}, \quad \delta_1 > 0. \tag{16}$$

$$u_{3,4}(x,t) = -\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}-10\right)+1\right)}{\sqrt{5(\beta-\gamma)}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}+1\right)^2}, \quad \delta_1 < 0. \tag{17}$$

$$u_5(x,t) = -\frac{3\delta_1^2k^2\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}+10\right)+1\right)}{2\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}-1\right)^2}, \quad \delta_1 > 0. \tag{18}$$

$$u_6(x,t) = -\frac{3\delta_1^2k^2\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}-10\right)+1\right)}{2\left(\delta_2e^{\delta_1(\zeta+\zeta_0)}+1\right)^2}, \quad \delta_1 < 0. \tag{19}$$

Similar-way, one can construct more wave results of Eq.(3) from set 3.

In solution (16), the restricted conditions to evade singularities are $\beta \neq \gamma$ and $\delta \neq \frac{1}{e^{\delta_1(\zeta+\zeta_0)}}$. In solution (17), the restricted conditions to evade singularities are $\beta \neq \gamma$ and $\delta \neq -\frac{1}{e^{\delta_1(\zeta+\zeta_0)}}$.

In solutions (18) and (19), the restricted conditions to evade singularities are $\delta \neq \frac{1}{e^{\delta_1(\zeta+\zeta_0)}}$ and $\delta \neq -\frac{1}{e^{\delta_1(\zeta+\zeta_0)}}$.

2nd Family: In this family, we assume as $\delta_1 = \delta_3 = 0$,

Set 1:

$$A_0 = -\frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(3\delta_2\mu^2+2\delta_0\right)}{\delta_0\sqrt{5(\gamma-\beta)}},$$
$$A_1 = \frac{6\mu\delta_2\sqrt{3\left(1-\gamma^2\right)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}}, \ A_2 = -\frac{3\delta_2\sqrt{3\left(1-\gamma^2\right)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}}$$
$$B_1 = 0, \quad B_2 = 0, \ k = \mp\frac{\sqrt[4]{\left(1-\gamma^2\right)\nu}}{\sqrt[4]{60\delta_0^2\delta_2^2(\gamma-\beta)}},$$
$$\omega = \pm\frac{\gamma\nu\sqrt[4]{\left(1-\gamma^2\right)\nu}}{\sqrt[4]{60\delta_0^2\delta_2^2(\gamma-\beta)}}. \tag{20}$$

Set 2:

$$A_0 = \frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(3\delta_2\mu^2+2\delta_0\right)}{\delta_0\sqrt{5(\gamma-\beta)}},$$
$$A_1 = -\frac{6\mu\delta_2\sqrt{3\left(1-\gamma^2\right)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}}, \quad A_2 = \frac{3\delta_2\sqrt{3\left(1-\gamma^2\right)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}}$$
$$B_1 = 0, \quad B_2 = 0, \ k = \pm\frac{(-1)^{3/4}\sqrt[4]{\gamma^2-1}\sqrt[4]{\nu}}{\sqrt{2}\sqrt[4]{15}\sqrt[4]{\delta_0^2\delta_2^2(\gamma-\beta)}},$$
$$\omega = \mp\frac{(-1)^{3/4}\gamma\sqrt[4]{\gamma^2-1}\nu^{5/4}}{\sqrt{2}\sqrt[4]{15}\sqrt[4]{\delta_0^2\delta_2^2(\gamma-\beta)}}. \tag{21}$$

The wave solutions of Eq.(3) are constructed from solution sets 1 and 2 as

$$u_{7,8}(x,t) = -\frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(3\tan^2\left(\sqrt{\delta_0\delta_2}(\zeta+\zeta_0)\right)+2\right)}{\sqrt{5(\gamma-\beta)}}, \quad \delta_0\delta_2 > 0. \tag{22}$$

$$u_{9,10}(x,t) = \frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(3\tanh^2\left(\sqrt{-\delta_0\delta_2}(\zeta+\zeta_0)\right)-2\right)}{\sqrt{5(\gamma-\beta)}}, \quad \delta_0\delta_2 < 0. \tag{23}$$

$$u_{11,12}(x,t) = \frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(3\tan^2\left(\sqrt{\delta_0\delta_2}(\zeta+\zeta_0)\right)+2\right)}{\sqrt{5(\gamma-\beta)}}, \quad \delta_0\delta_2 > 0. \tag{24}$$

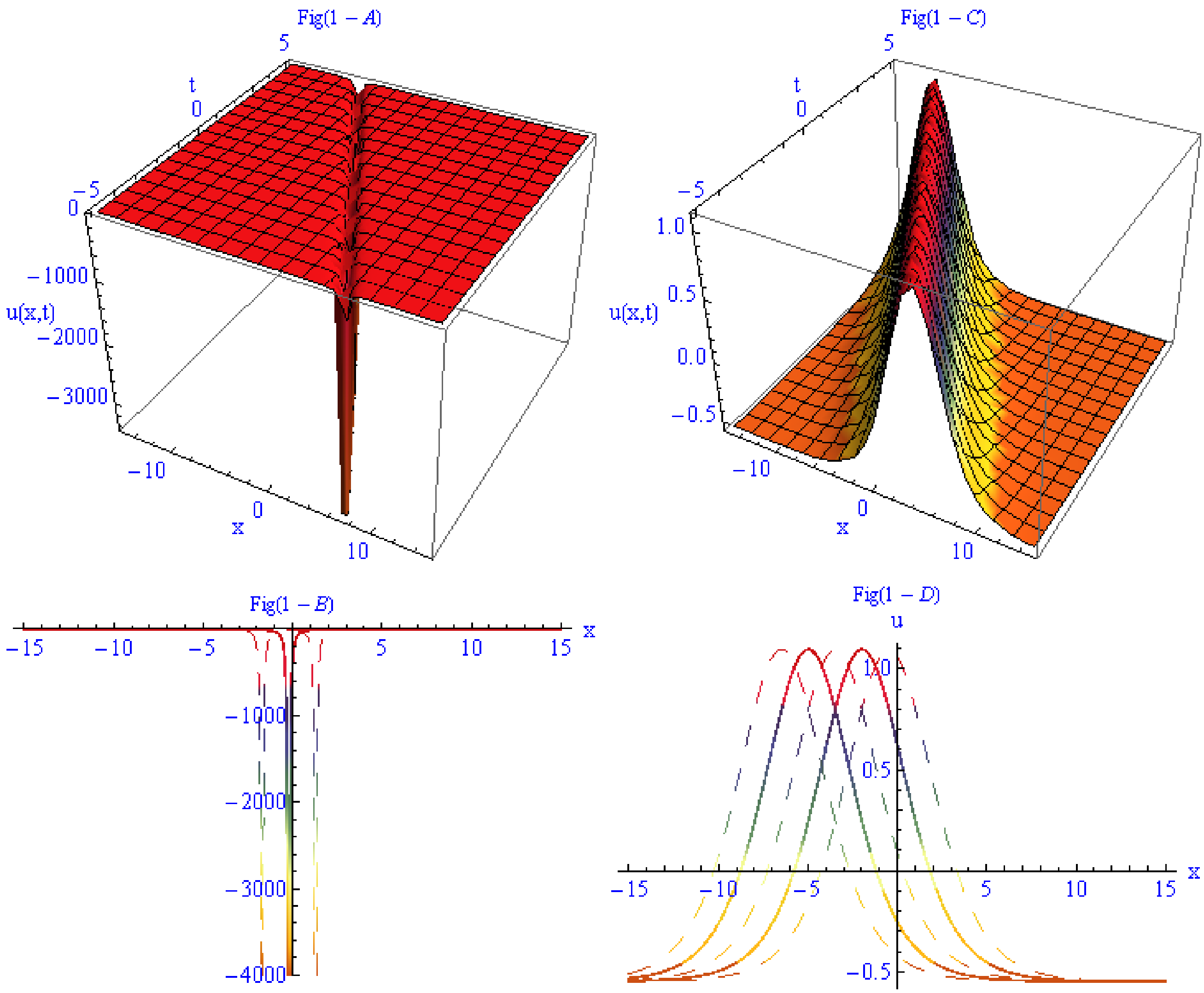

FIGURE 1. By granting appropriate values to parameters, the formation of solutions (16) and (17) are revealed as: Fig(1-A) Dark solitary wave and its 2-dimensional (2D) in Fig(1-B), Fig(1-C) bright soliton and its 2D in Fig(1-D).

$$u_{13,14}(x,t) = \frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(2-3\tanh^2\left(\sqrt{-\delta_0\delta_2}(\zeta+\zeta_0)\right)\right)}{\sqrt{5(\gamma-\beta)}}, \quad \delta_0\delta_2<0. \tag{25}$$

In solutions (22) and (23), the restricted condition to evade singularity is $\gamma \neq \beta$.

3rd Family: In this family, we assume as $\delta_3 = 0$,

Set 1: See (26), as shown at the bottom of the page.

Set 2:

$$A_0 = -\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(12\delta_2^2\mu^2+4\delta_2\left(2\delta_0-3\delta_1\mu\right)+\delta_1^2\right)}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}},$$

$$A_1 = -\frac{12\delta_2\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_1-2\delta_2\mu\right)}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}},$$

$$\begin{aligned} A_0 &= -\frac{3k^2}{2}\left(12\delta_2^2\mu^2+4\delta_2\left(2\delta_0-3\delta_1\mu\right)+\delta_1^2\right), \quad A_1 = -18\delta_2k^2\left(\delta_1-2\delta_2\mu\right), \; A_2 = -18\delta_2^2k^2, \; B_1 = 0, \\ B_2 &= 0, \quad \omega = \frac{15k^5\left(\delta_1^2-4\delta_0\delta_2\right)^2 \mp \sqrt{225\left(\delta_1^2-4\delta_0\delta_2\right)^4k^{10}+16k^2\nu\left(15\beta\left(\delta_1^2-4\delta_0\delta_2\right)^2k^4+4\nu\right)}}{8}, \\ \gamma &= -\frac{15k^5\left(\delta_1^2-4\delta_0\delta_2\right)^2 \mp \sqrt{225\left(\delta_1^2-4\delta_0\delta_2\right)^4k^{10}+16k^2\nu\left(15\beta\left(\delta_1^2-4\delta_0\delta_2\right)^2k^4+4\nu\right)}}{8k\nu}. \end{aligned} \tag{26}$$

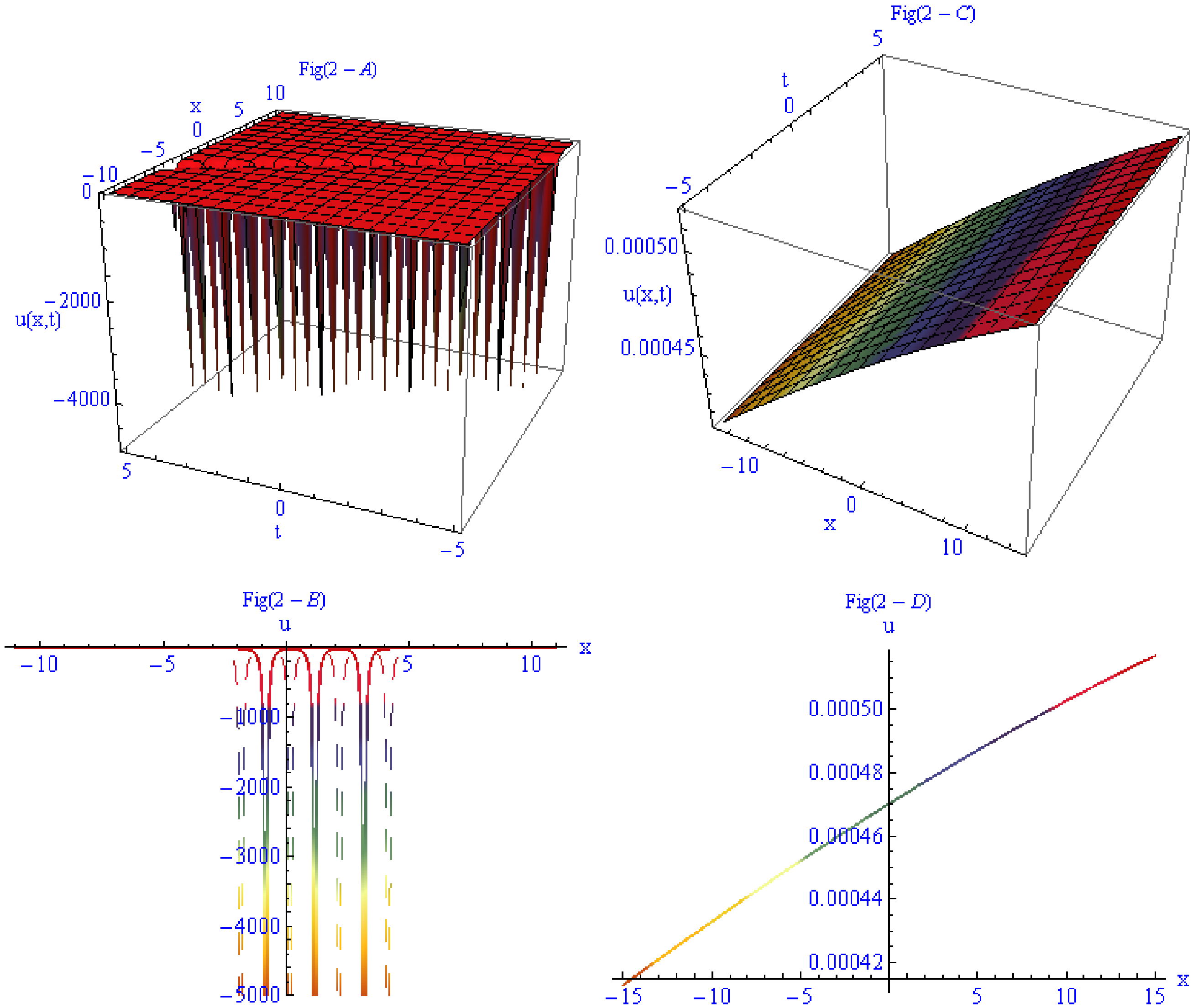

FIGURE 2. By granting appropriate values to parameters, the formation of solutions (18) and (19) are revealed as: Fig(2-A) is Multi-peak solitons and its 2D in Fig(2-B), Fig(2-C) is solitary wave of anti-Kink type and its 2D in Fig(2-D).

$$A_2 = -\frac{12\delta_2^2\sqrt{3\left(\gamma^2-1\right)\nu}}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}}, \quad B_1 = 0,\ B_2 = 0,$$
$$k = \mp\frac{\sqrt{2}\sqrt[4]{\left(\gamma^2-1\right)\nu}}{\sqrt[4]{15\left(\delta_1^2-4\delta_0\delta_2\right)^2(\beta-\gamma)}},$$
$$\omega = \pm\frac{\gamma\nu\sqrt[4]{4\left(\gamma^2-1\right)\nu}}{\sqrt[4]{15\left(\delta_1^2-4\delta_0\delta_2\right)^2(\beta-\gamma)}}. \tag{27}$$

The wave results of Eq.(3) from sets 1 and 2 are constructed as follows

$$u_{15,16}(x,t) = \frac{3k^2}{2}\left(\delta_1^2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)-10\right) - 4\delta_0\delta_2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)+2\right) + 12\sqrt{4\delta_0\delta_2-\delta_1^2}\,\delta_1\tan\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)\right),$$
$$4\delta_0\delta_2 > \delta_1^2; \tag{28}$$

$$u_{17,18}(x,t) = \frac{\sqrt{3\left(\gamma^2-1\right)\nu}}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}} \times \left(\delta_1^2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)-10\right)\right.$$

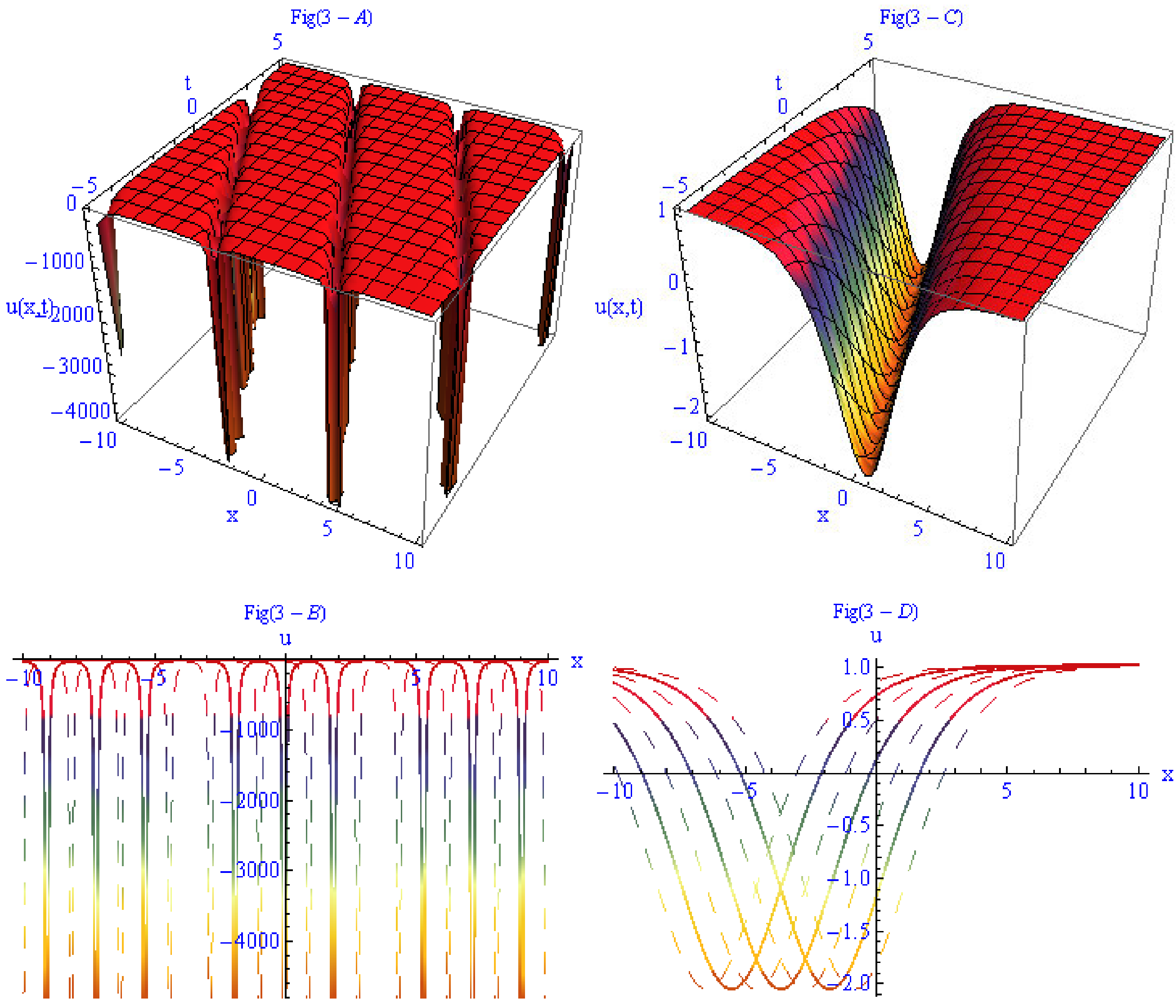

FIGURE 3. By granting appropriate values to parameters, the shape of solutions (22) and (23) are shown as: Fig(3-A) dark periodic solitary wave and its 2D in Fig(3-B), Fig(3-C) is dark soliton and its 2D in Fig(3-D).

$$
\begin{aligned}
&+12\sqrt{4\delta_0\delta_2-\delta_1^2}\,\delta_1 \tan\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)\\
&-4\delta_0\delta_2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)+2\right)\Bigg),\\
&4\delta_0\delta_2>\delta_1^2; \qquad (29)
\end{aligned}
$$

where ζ_0 is constant.

In solutions (24) and (29), the restricted conditions to evade singularities are $\beta \neq \gamma$ and $\gamma \neq \beta$ & $\delta_1^2 \neq 4\delta_0\delta_2$.

B. APPLICATION OF GENERALIZED EXP($-\phi(\zeta)$)-EXPANSION METHOD

In this part, we employ generalized exp($-\phi(\zeta)$)-expansion method on two-mode Sawada-Kotera for constructing the solitons and more waves solutions. Employing balancing principle of homogeneous on Eq.(11) and assume the wave solution as

$$U(\zeta) = A_0 + A_1\exp(-\phi(\zeta)) + A_2\left(\exp(-\phi(\zeta))\right)^2. \quad (30)$$

By substituting Eq.(30) into Eq.(11) and deputing the coefficients of $\left(e^{(-\phi(\zeta))}\right)^i$ to zero, we achieved a equations system $A_0, A_1, A_2, a, b, c, k, \nu, \omega, \eta, \beta$. Mathematica 9 was utilized to resolve the equations set. We attained below families as:

1st Family:

$$
\begin{aligned}
&A_0=-\frac{2\left(8abk^2+c^2k^2\right)}{3}, \quad A_1=-8ack^2,\\
&A_2=-8a^2k^2, \quad \omega=\mp k\nu, \quad \gamma=\frac{(10\beta\mp 1)}{9}. \quad (31)
\end{aligned}
$$

2nd Family:

$$A_2=0, \quad \omega=\mp k\nu, \ \gamma=\pm 1, \ \beta=\pm 1. \quad (32)$$

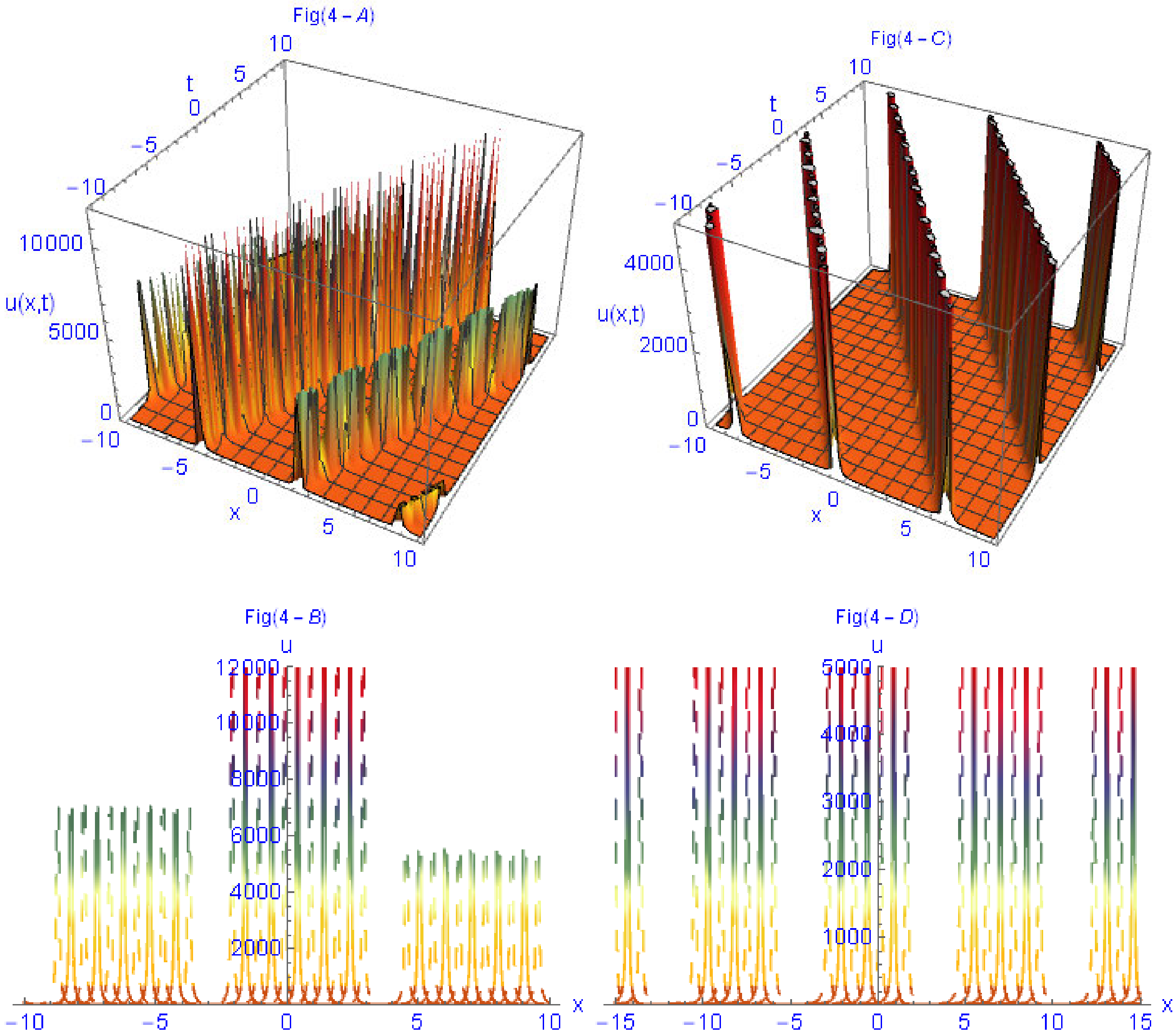

FIGURE 4. By granting appropriate values to parameters, the shape of solutions (24) and (29) are shown as: Fig(4-A) Multi peak soliton of different amplitude and its 2D in Fig(4-B), Fig(4-C) periodic solitary wave and its 2D in Fig(4-D).

3rd Family: See (33), as shown at the bottom of the page.

4th Family:

$$A_1 = -8ack^2, \quad A_2 = -8a^2k^2, \ \omega = \pm k\nu, \ \beta = \mp 1, \ \gamma = \mp 1. \tag{34}$$

From 1st family, the different forms of solitons and other solutions of Eq.(3) are obtained as

Type I: For $a = 1, b \neq 0, c^2 - 4b > 0$, (35), as shown at the bottom of the next page.

Type II: For $a = 1, \ b \neq 0, \ c^2 - 4b < 0$, (36), as shown at the bottom of the next page.

Type III: For $a = 1, \ b = 0, \ c \neq 0, \ c^2 - 4b > 0$,

$$u_{5,6}(\zeta) = -\frac{2k^2}{3}\left(c^2 + 8ab + \frac{12ac^2}{e^{c(\zeta+\zeta_0)} - 1} + \frac{12a^2c^2}{\left(e^{c(\zeta+\zeta_0)} - 1\right)^2}\right). \tag{37}$$

$$A_0 = -\left(\sqrt{5k(\beta k\nu + \omega)\left(16a^2b^2k^5(\beta k\nu + \omega) - 8abc^2k^5(\beta k\nu + \omega) + c^4k^5(\beta k\nu + \omega) - 4k^2\nu^2 + 4\omega^2\right)} + 40abk^3(\beta k\nu + \omega) + 5\beta c^2k^4\nu + 5c^2k^3\omega\right) / (10k(\beta k\nu + \omega)), \quad A_1 = -6ack^2, \ A_2 = -6a^2k^2, \ \gamma = \beta. \tag{33}$$

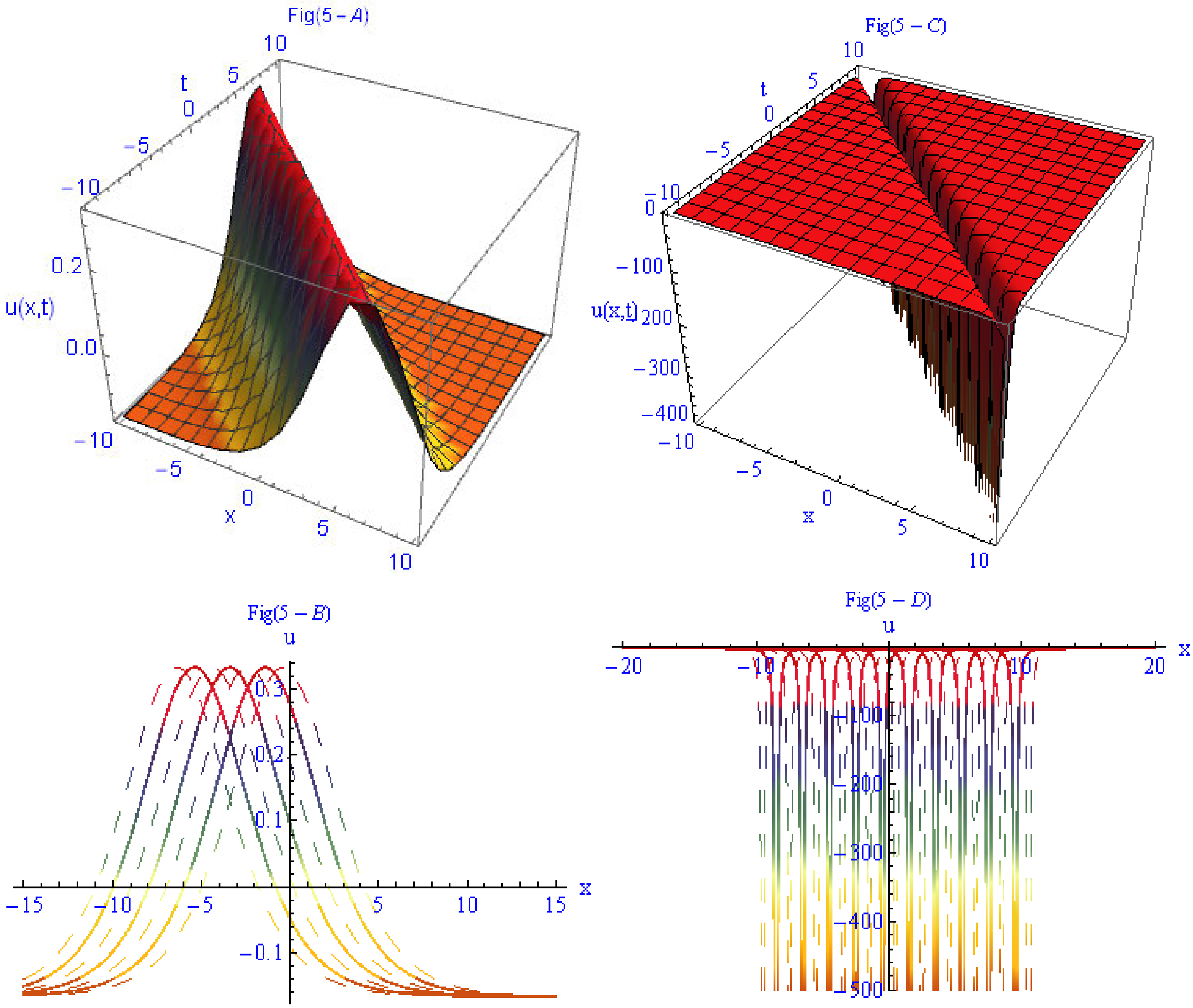

FIGURE 5. By granting appropriate values to parameters, the shape of solutions (35) and (37) are shown as: Fig(5-A) is bright soliton wave and its 2D in Fig(5-B), Fig(5-C) is dark solitary wave and its 2D in Fig(5-D).

Type IV: For $a = 1,\ b \neq 0,\ c \neq 0,\ c^2 - 4b = 0,$

$$u_{7,8}(\zeta) = \frac{2k^2}{3} \times \left(\frac{6ac^3(\zeta + \zeta_0)}{c(\zeta + \zeta_0) + 1} - \frac{3a^2c^4(\zeta + \zeta_0)^2}{(c(\zeta + \zeta_0) + 1)^2} - 8ab - c^2 \right). \quad (38)$$

Type V: For $c = 0,\ a > 0,\ b > 0,$

$$u_{9,10}(\zeta) = -\frac{2k^2}{3} \left(12b \cot\left(\sqrt{ab}(\zeta + \zeta_0)\right) \times \left(c\sqrt{\frac{a}{b}} + a \cot\left(\sqrt{ab}(\zeta + \zeta_0)\right)\right) + 8ab + c^2 \right). \quad (39)$$

$$u_{1,2}(\zeta) = \frac{2k^2}{3} \left(8ab \left(\frac{3\left(c^2 - 2ab + c\sqrt{c^2 - 4b}\tanh\left(\frac{\sqrt{c^2-4b}}{2}(\zeta + \zeta_0)\right)\right)}{\left(\sqrt{c^2 - 4b}\tanh\left(\frac{\sqrt{c^2-4b}}{2}(\zeta + \zeta_0)\right) + c\right)^2} - 1 \right) - c^2 \right). \quad (35)$$

$$u_{3,4}(\zeta) = \frac{2k^2}{3} \left(8ab \left(\frac{3\left(c^2 - 2ab - c\sqrt{4b - c^2}\tan\left(\frac{\sqrt{4b-c^2}}{2}(\zeta + \zeta_0)\right)\right)}{\left(c - \sqrt{4b - c^2}\tan\left(\frac{\sqrt{4b-c^2}}{2}(\zeta + \zeta_0)\right)\right)^2} - 1 \right) - c^2 \right). \quad (36)$$

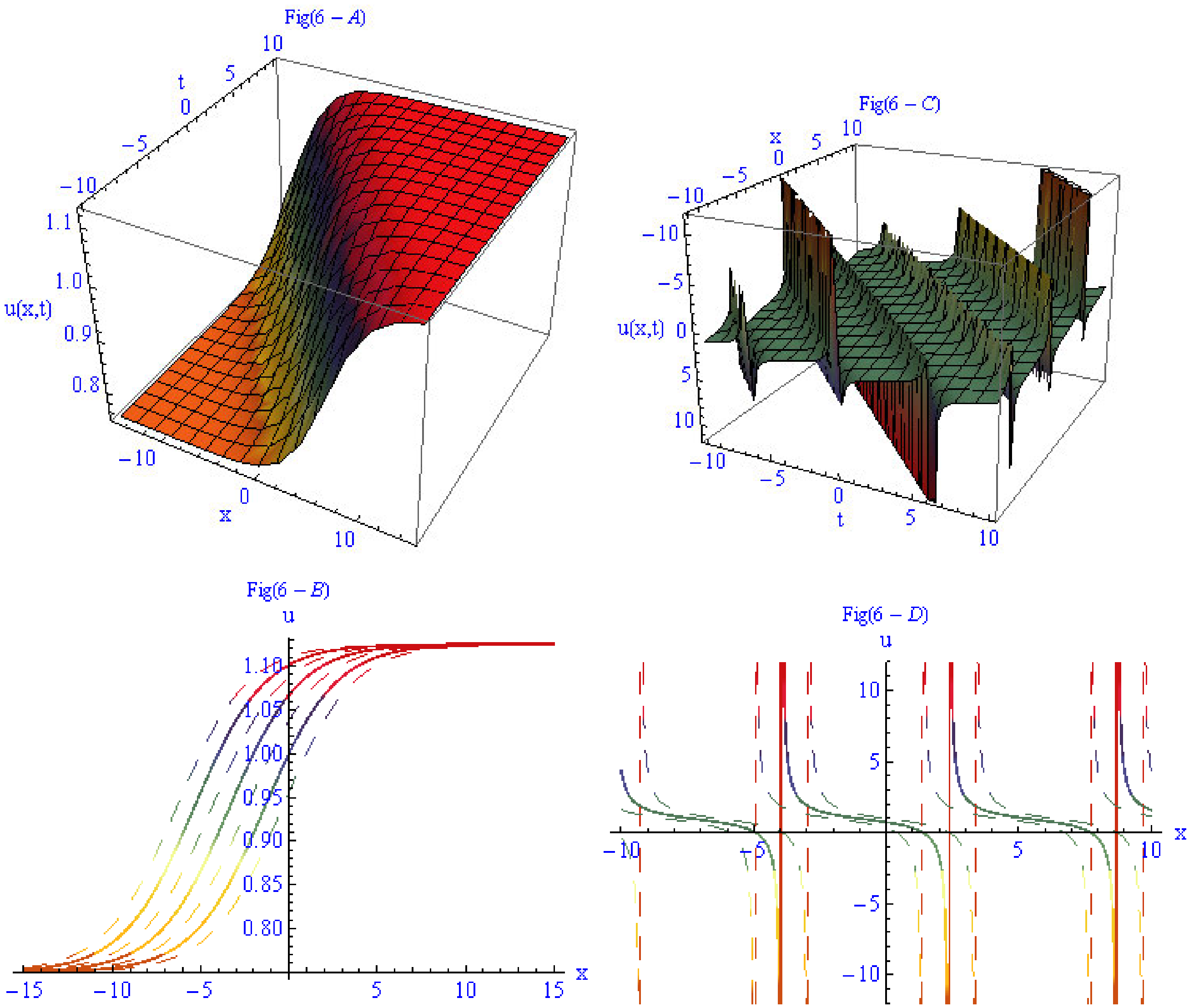

FIGURE 6. By granting appropriate values to parameters, the shape of solutions (44) and (45) are shown as: Fig(6-A) is Kink soliton wave and its 2D in Fig(6-B), Fig(6-C) is Breather wave of strange shape and its 2D in Fig(6-D).

Type VI: For $c = 0,\ a < 0,\ b < 0,$

$$u_{11,12}(\zeta) = -\frac{2k^2}{3}\left(8ab + c^2 - 12bc\sqrt{\frac{a}{b}}\cot\left(\sqrt{ab}(\zeta - \zeta_0)\right) + 12ab\cot^2\left(\sqrt{ab}(\zeta - \zeta_0)\right)\right). \tag{40}$$

Type VII: For $c = 0,\ a > 0,\ b < 0,$

$$u_{13,14}(\zeta) = \frac{2k^2}{3}\left(12bc\sqrt{-\frac{a}{b}}\coth\left(\sqrt{-ab}(\zeta - \zeta_0)\right) + 12ab\coth^2\left(\sqrt{-ab}(\zeta - \zeta_0)\right) - 8ab - c^2\right). \tag{41}$$

Type VIII: For $c = 0,\ a < 0,\ b > 0,$

$$u_{15,16}(\zeta) = \frac{2k^2}{3}\left(12ab\coth^2\left(\sqrt{-ab}(\zeta + \zeta_0)\right) - 12bc\sqrt{-\frac{a}{b}}\coth\left(\sqrt{-ab}(\zeta + \zeta_0)\right) - 8ab - c^2\right). \tag{42}$$

Type IX: For $b = 0,\ c = 0,$

$$u_{17,18}(\zeta) = -\frac{2k^2}{3}\left(8ab + c^2 + \frac{12c}{\zeta + \zeta_0} + \frac{12}{(\zeta + \zeta_0)^2}\right). \tag{43}$$

In solutions (35) and (37), the restricted conditions to evade singularities are $\sqrt{c^2 - 4\,b}\tanh\left(\frac{\sqrt{c^2-4\,b}}{2}(\zeta + \zeta_0)\right) \neq -c$ and $1 \neq e^{c(\zeta+\zeta_0)}$.

From 2nd family, the more solitons and other wave solutions of Eq.(3) are obtained as

Type I: For $a = 1, b \neq 0, c^2 - 4b > 0$,

$$u_{19,20}(\zeta) = A_0 - \frac{2A_1 b}{\sqrt{c^2 - 4b}\tanh\left(\frac{\sqrt{c^2-4b}}{2}(\zeta + \zeta_0)\right) + c}. \tag{44}$$

Type II: For $a = 1,\ b \neq 0,\ c^2 - 4b < 0$,

$$u_{21,22}(\zeta) = A_0 - \frac{2A_1 b}{c - \sqrt{4b - c^2}\tan\left(\frac{\sqrt{4b-c^2}}{2}(\zeta + \zeta_0)\right)}. \tag{45}$$

Type III: For $a = 1,\ b = 0,\ c \neq 0,\ c^2 - 4b > 0$,

$$u_{23,24}(\zeta) = A_0 - \frac{A_1 c}{1 - e^{c(\zeta+\zeta)}}. \tag{46}$$

Type IV: For $a = 1,\ b \neq 0,\ c \neq 0,\ c^2 - 4b = 0$,

$$u_{25,26}(\zeta) = A_0 - \frac{A_1 c^2(\zeta + \zeta_0)}{2c(\zeta + \zeta_0) + 2}. \tag{47}$$

Type V: For $c = 0,\ a > 0,\ b > 0$,

$$u_{27,28}(\zeta) = A_0 + \frac{A_1\sqrt{b}\cot\left(\sqrt{ab}(\zeta + \zeta_0)\right)}{\sqrt{a}}. \tag{48}$$

Type VI: For $c = 0,\ a < 0,\ b < 0$,

$$u_{29,30}(\zeta) = A_0 - \frac{A_1\sqrt{b}\cot\left(\sqrt{ab}(\zeta - \zeta_0)\right)}{\sqrt{a}}. \tag{49}$$

Type VII: For $c = 0,\ a > 0,\ b < 0$,

$$u_{31,32}(\zeta) = A_0 + A_1\sqrt{-\frac{b}{a}}\coth\left(\sqrt{-ab}(\zeta - \zeta_0)\right). \tag{50}$$

Type VIII: For $c = 0,\ a < 0,\ b > 0$,

$$u_{33,34}(\zeta) = A_0 - A_1\sqrt{-\frac{b}{a}}\coth\left(\sqrt{-ab}(\zeta + \zeta_0)\right). \tag{51}$$

Type IX: For $b = 0,\ c = 0$,

$$u_{35,36}(\zeta) = A_0 + \frac{A_1}{a(\zeta + \zeta_0)}. \tag{52}$$

Similarly, more general soliton results can construct of equation (3) from families 3rd and 4th.

In solutions (44) and (45), the restricted conditions to evade singularities are $c \neq -\sqrt{c^2 - 4\,b}\tanh\left(\frac{\sqrt{c^2-4\,b}}{2}(\zeta + \zeta_0)\right)$ and $c \neq -\sqrt{4\,b - c^2}\tan\left(\frac{\sqrt{4b-c^2}}{2}(\zeta + \zeta_0)\right)$.

TABLE 1. Comparisons between the outcomes of reported work and our work.

Sr. No.	Reported Work	Our study
1.	a) The authors in [23] used Kudryashov Kudryashov method and sine-cosine function techniques on new developed two-mode fifth KdV equation and stationary wave solutions are constructed. b) The author in [5] used simplified Hirota technique and soliton solutions are constructed. c) The authors in [18] used the modified Kudryashov and auxiliary equation methods and dual wave solutions are obtained.	In this work, we used Improved F-expansion and Generalized $\exp(-\phi(\zeta))$-expansion methods and solitons solutions in different form are obtained.
2.	The solution (2.10) in [23]	is similar to solution (17) if $\mu = 0$.
3.	The solution (2.12) in [23]	is similar to solution (46) if $A_1 = -A_1$.
4.	The solution (25) in [5]	is similar to solution (16) if $A_1 = 0$ in solution set 1 [(13)] .
5.	The solution (43) in [18]	is similar to solution (28) if $\mu = 0$.
6.	The solution (45) in [18]	is similar to solution (35).
7.	The solutions (56) and (57) in [18]	are similar to solutions (50) and (51).
8.	Only one or two families of analytic results have been achieved in [5], [18], [23] by using different approaches.	More than two families of analytic solutions have been accomplished in this technique.
9.	The results in literature obtained by other techniques contain no or less than 2 unknown parameters.	The attained results concerned with some unknown parameters which are used to obtained the different kinds of structures of solutions.
10.	The results in literatures are not more general.	Our results are more general and comprehensive.
11.	The literature techniques give limited types of soliton solutions.	The current techniques give many types of novel soliton solutions to explore many physical phenomena in nonlinear physical sciences and other related field.
12.	It is so complicated to study, when the balance number is high.	Our techniques are very useful easy to study, when the balance number is high.

V. DISCUSSION OF RESULTS AND GRAPHICAL REPRESENTATION

The accomplished solutions are dissimilar from the results obtained by other researchers in the previous methods. The equations (8) and (10) present numerous dissimilar kinds of solutions by giving different values of parameters. It was announced earlier that the tmSKE was studied by some authors is given in Table 1.

Pedestal on the applications of these methods, the authors report some bright, dark, multi-solitons, singular periodic and kink structured results with the restricted conditions $\beta = \gamma = 1$. However in this article, eighteen wave solutions are constructed through the improved F-expansion method and thirty-six wave solutions are constructed through the generalized $\exp(-\phi(\zeta))$-expansion technique. The explored solutions demonstrate the dual-mode bright, dark, periodic, Kink, multi soliton and singular wave behaviors that are being classified as waves of right/left mode. Evaluated with published results [5], [18], [23], it is worth revealed that the constructed dual-wave solutions are new for the interests of applied methods. As a result, we have constructed several original results, which have not been explained before.

The Figures 1 to 4 indicate the solitons and other waves in dissimilar structures are described. In the Figure 1, by granting appropriate values to parameters, the formation of solutions (16) and (17) are revealed as: Fig(1-A) Dark solitary wave and its 2-dimensional (2D) in Fig(1-B), Fig(1-C) bright soliton and its 2D in Fig(1-D). By granting appropriate values to parameters, the formation of solutions (18) and (19) in Figure 2 are revealed as: Fig(2-A) is Multi-peak solitons and its 2D in Fig(2-B), Fig(2-C) is solitary wave of anti-Kink type and its 2D in Fig(2-D). In Figure 3, by granting appropriate values to parameters, the shape of solutions (22) and (23) are shown as: Fig(3-A) dark periodic solitary wave and its 2D in Fig(3-B), Fig(3-C) is dark soliton and its 2D in Fig(3-D). By granting appropriate values to parameters, the shape of solutions (24) and (29) in Figure 4 are shown as: Fig(4-A) Multi peak soliton of different amplitude and its 2D in Fig(4-B), Fig(4-C) periodic solitary wave and its 2D in Fig(4-D).

The Figures 5 and 6 illustrate the solitary waves in dissimilar structures are described. In the Figure 5, By granting appropriate values to parameters, the shape of solutions (35) and (37) are shown as: Fig(5-A) is bright soliton wave and its 2D in Fig(5-B), Fig(5-C) is dark solitary wave and its 2D in Fig(5-D). By granting appropriate values to parameters, the shape of solutions (44) and (45) in Figure 6 are shown as: Fig(6-A) is Kink soliton wave and its 2D in Fig(6-B), Fig(6-C) is Breather wave of strange shape and its 2D in Fig(6-D).

VI. CONCLUSION

The described methods namely, the improved F-expansion method and generalized $\exp(-\phi(\zeta))$-expansion method have been effectively employed on the tmSKE and as consequences, abundant of different kinds of solitons and other waves solutions such as bright and dark solitons, multi-peak soliton, breather type waves, periodic solutions are obtained. The two-mode equation describes the spread of moving two-waves under the influence of dispersion, nonlinearity, and phase velocity factors. The obtained novel solitons and other wave results have significant applications in fluid dynamics, applied sciences and engineering. The Sawada-Kotera equations illustrating the non-linear wave phenomena in shallow water, ion-acoustic waves in plasmas, fluid dynamics, etc., and tmSKE also arising in fluid dynamics is addressed in this article. We may say that these two-waves solutions could be useful in many physical and engineering applications, for example, they can be used as barrier waves to strengthen the transmission of different signals data. Also, if a huge amount of data is complicated to pass on to a single router, it can be dispersed on two routers. The graphical moments of few solutions are depicted that helps the engineers and scientists for understanding the physical phenomena of this model. The restricted conditions are also added on solutions to avoid singularities. To explain the novelty between the present results and the previously attained results, a comparative study has been presented. The computational work and constructed results approve the effectiveness, simplicity, and impact of described techniques. Furthermore, the described techniques can be employed to any two-mode nonlinear PDEs and other models arising in fluid dynamics correlated with any physical and engineering problems to explore novel dual-wave and other wave solutions. The fractional derivative of this two-model will also consider to obtain such types of results by utilizing the described techniques. Our future work would be intense towards investigating the new dual-wave solutions by using different analytical, semi-analytical, and numerical methods to the tmSKE and fractional tmSKE.

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