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Stability Analysis of Two Kinds of Fractional-Order Neural Networks Based on Lyapunov Method

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ABSTRACT At present, the theory and application of fractional-order neural networks remain in the exploratory stage. We study the asymptotic stability of fractional-order neural networks with Riemann-Liouville (R-L) derivatives. For non-delayed and delayed systems, we propose an asymptotic stability criterion based on the combination of the Lyapunov method and linear matrix inequality (LMI) method. The highlights include the following: (1) for fractional-order neural networks with time delay, the existence and uniqueness of solutions are proven by using matrix analysis theory and contraction mapping theorem, and (2) based on the unique solution, a suitable Lyapunov functional is constructed. Based on the inequality theorem and LMI method, two sets of asymptotic stability criteria for fractional-order neural networks are proven, which avoids the difficulty of solving the fractional derivative by the Leibniz law. Finally, the results are verified using numerical simulations.

INDEX TERMS Fractional-order neural networks, time delay, asymptotic stability, Lyapunov, LMI.

I. INTRODUCTION

In recent years, neural networks have been used to solve many scientific problems in image, speech processing, combinatorial optimization, pattern recognition and other fields. As an extension of general neural networks, fractional-order neural networks have been widely used in intelligent control, nonlinear equation solving, data analysis, associative memory and optimization [1]–[3]. In contrast to general neural networks, fractional-order neural networks have two advantages. The first is that their infinite memory ability can describe the system model more accurately. The second is that the fractional-order system has more degrees of freedom, which can increase the flexibility of system parameter selection [4]. Therefore, people devote increasing attention to the combination of fractional-order calculus theory and neural networks [5]–[8].

Since the end of the 20th century, the combination of fractional-order calculus and neural networks has produced many achievements. Kaslik and Sivasundaram first studied the multiple dynamic properties of fractional-order Hopfield neural networks, and then the study of fractional-order chaotic neural networks was also carried out [9]–[12]. Research on fractional-order chaotic neural networks mainly obtains network parameters by numerical methods. At the same time, the limit cycle and bifurcation of fractional-order chaotic neural networks are also studied, and some valuable results are obtained [13], [14]. In the early stages of the research on fractional-order neural networks, only the global stability and system without time delay were studied, and the criteria and conclusions obtained exhibited strong limitations, such as limited application scope of the criteria and overly complex criteria expression.

With the development of fractional-order system theory, scholars have carried out an increasing number of studies on fractional-order neural networks, including global stability, exponential stability, asymptotic stability, synchronous stability, etc. The research system is also expanded from non-delayed and non-leakage systems to delayed systems and leakage systems and from linear systems to non-linear systems.

In the field of industry, scholars have studied the Mittag-Leffler stability, Lyapunov stability and asymptotic stability of fractional-order systems without delay. However,

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at present, the stability analysis of fractional-order neural networks still faces many challenges, such as excessive parameters of the stability criterion, strong limitations of the stability criterion, complex calculations and so on. Therefore, it is necessary to improve the relevant theories.

Many scholars have also studied the stability of fractional-order linear or nonlinear systems, including finite time stability, Mittag-Leffler stability, Lyapunov stability, asymptotic stability and uniform stability [15]–[18]. For example, in [19], the Mittag-Leffler stability criterion of the fractional Hopfield neural network is established. In [20], the global asymptotic stability of the improved Caputo fractional derivative recurrent neural network is studied. The robust stability of fractional-order memristor Hopfield neural networks with parameter perturbations is discussed in [21] and [22] established the LMI stability criterion for fractional-order nonlinear systems and applied it to the global Mittag-Leffler stability analysis of fractional-order neural networks. [23] uses the positive definite quadratic Lyapunov function, and the global Mittag-Leffler stability condition with impulse effect is obtained with LMI. Although these studies have produced some achievements, the research on robust stability has still not been fully discussed.

In the fractional-order neural network system, the delay will not only reduce the speed of network information transmission but also destroy the stability of the original network [24]–[28]. Therefore, the study of time-delay systems is of great significance in the theoretical research and practical application of fractional-order neural networks. To date, some achievements have been made in the study of the stability of fractional-order neural networks with time delay. [29] gives the asymptotic stability criteria of time-varying delay fractional-order neural networks. In [30], the Gronwall inequality is applied to discuss the finite time stability of neural networks with fractional delay. In [31], the global asymptotic stability of memory-based complex neural networks with fractional delay is studied. In [32], the global stability of Hopfield neural networks with fractional delay is studied. [33] studied the stability of nonlinear fractional-order cellular neural networks with multiple delays.

Based on the above analysis, although there are many research results on the stability of fractional-order neural networks [34]–[39], the existing research results still face many difficulties. For example, the practical application environment of fractional-order systems is not clear [34], [35], the suitable field is not wide enough [36], [37], and there are many constraints on the stability criteria [38], [39]. At the same time, because the fractional calculus does not meet the Leibniz law, as a result, the stability criterion of a fractional-order system cannot be directly applied to the analysis of a high-dimensional system [38]. To develop a more concise and effective stability analysis method for fractional-order neural networks, we need to further study new stability criteria of fractional-order neural networks.

In this paper, we focus on asymptotic stability analysis of nonlinear systems with time delay: several new stability criteria of fractional-order neural networks with delay are proposed, and the main idea is based on the combination of the LMI method and Lyaponov functional. Based on the R-L derivative, we study the asymptotic stability of two kinds of fractional-order neural networks, including delay systems and systems without delay. The basic idea is to extend the general neural network theory to the fractional-order neural network and further improve the stability analysis method of the fractional-order neural network.

Compared with the existing methods, first, our method can be used for robust stability analysis of complex nonlinear systems, including the stability analysis of non-delayed and delayed systems. Second, we use a more concise LMI expression for the stability criterion and use the Lyapunov direct method for theorem construction and proof. The proof is more concise and clearer, and the conclusion is more convincing. Finally, the obtained results have a wide range of applications and fewer restrictions on the criterion.

Our motivation can be described as follows. (1) The LMI is an effective method for the stability analysis of integer-order systems and has been successfully extended to fractional-order systems. In [34], [35], and [35], LMI stability criteria for fractional-order nonlinear systems are established and used for analyzing the global Mittag-Leffler stability of fractional-order neural networks [34]. By using a positive definite quadratic Lyapunov function, the global Mittag-Leffler stability criterion with impulsive effects is obtained in LMI form. Fractional-order neural network analysis based on LMI has the advantages of clear expression, simple calculation and strong practicability [35]. Therefore, the stability analysis of fractional-order systems based on LMI has received increasing attention from experts in this field [38]. (2) The Lyapunov direct method is a method that can determine the equilibrium point and judge the stability of the system without solving equations [29], [35]. It can be directly applied to the stability analysis of fractional-order systems. Therefore, this paper intends to combine the LMI method with the Lyapunov direct method, take the fractional-order neural network defined by the R-L derivative as the main object to study the stability of two kinds of fractional-order neural networks, and obtain some simple and practical stability criteria of fractional-order neural networks for non-delay and delay systems.

The main contributions of this paper are as follows: (1) for a fractional-order neural network with delay and non-delay, the existence and uniqueness of the system solution are proven by using matrix analysis theory and the contraction mapping theorem. (2) The Lyapunov correlation method is extended, the suitable Lyapunov function is constructed, and the selection range of the applicable function is expanded. The difficulty of solving the fractional-order partial derivative is avoided. By combining the Lyapunov functional with the LMI technique, two sets of stability criteria for fractional-order neural networks are proposed to avoid the difficulty of fractional-order partial derivation. (3) The validity of the criteria are proved by several numerical simulations.

The rest of this article is organized as follows. In Section II, the description of two types of fractional-order neural network models is given first, and the related definitions and lemmas that need to be used in this paper are given. Section III proves the existence of the solution of the fractional-order delay neural network and stability criteria based on the LMI and Lyapunov functions. In Section IV, some mathematical examples are given to verify the results, and the conclusion is given in Section V.

II. RELATED WORKS

A. MODEL

In this paper, two models are considered, described as follows.

Model 1: The following fractional-order neural network model is considered:

$$ {}_{0}^{RL}D_t^{\alpha}x(t) = -Bx(t) + Af(x(t)) + I, \quad t \geq 0, \tag{1}$$

where $x = [x_1, \cdots, x_n]^T \in R^n$ is the state vector, $B = diag\{b_i\}$, $b_i > 0$ is a positive definite diagonal matrix, $A = [A_{ij}]_{n\times n}$, $C = [C_{ij}]_{n\times n}$ represents the connection weight matrix, $f(x(t)) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^T$, which is the activation function, and ${}_{0}^{RL}D_t^{\alpha}$ is the type of fractional differential of the representation.

The activation function satisfies the following conditions:

$$ |f(x)| \leq k|x|, \tag{2}$$

where $k > 0$ is a known constant.

Model 2: The following fractional-order neural network with delay is also considered herein:

$$ {}_{0}^{RL}D_t^{\alpha}x(t) = -Bx(t) + Af(x(t)) + Cf(x(t-\tau)) + I, \quad t \geq 0, \tag{3}$$

where $x = [x_1, \cdots, x_n]^T \in R^n$ is the state vector, $B = diag\{b_i\}$, $b_i > 0$ is a positive definite diagonal matrix, $A = [A_{ij}]_{n\times n}$, $C = [C_{ij}]_{n\times n}$ represents the connection weight matrix, $f(x(t)) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^T$, $f_i(0) = 0$, which is the activation function, $\tau > 0$ represents the time lag, and ${}_{0}^{RL}D_t^{\alpha}$ is the type of fractional differential of the representation. The activation function satisfies eq.2.

There are two scalars $\underline{e_i}$ and $\overline{g_i}$ satisfying:

$$ \underline{e_i}(\eta - \nu)^2 \leq [f_i(\eta) - f_i(\gamma)] \cdot (\eta - \nu) \leq \overline{g_i}(\eta - \nu)^2. \tag{4}$$

B. RELATED DEFINITIONS AND LEMMAS

Definition 1 [40]: The type of R-L fractional α-order integrals of function $f(x)$ is defined as:

$$ {}_{t_0}D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)ds, \quad t > 0, \tag{5}$$

where $\alpha > 0$, $\Gamma(\cdot)$ is the gamma function, and $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt$.

Definition 2 [40]: Letting $0 \leq m-1 \leq q < m, m \in Z^+$, the R-L type of the function $f(t)$ is defined as

$$ {}_{t_0}^{RL}D_t^{-q}f(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s)ds, \quad t > 0. \tag{6}$$

Definition 3 [40]: The Caputo fractional α-order integrals of the function are defined as

$$ {}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}f^m(s)ds, \tag{7}$$

where $0 \leq m-1 \leq \alpha < m$.

Lemma 1 [41]: Fractional calculus satisfies the nature of linear operations:

$$ {}_aD_t^{\alpha}(uf(t) + vg(t)) = u_aD_t^{\alpha}f(t) + v_aD_t^{\alpha}g(t), \tag{8}$$

where $u, v, \alpha \in R$ and ${}_aD_t^{\alpha}$ for any fractional differential and integral under any definition.

Lemma 2 [41]: Letting $x(t): R^n \to R^n$ be a differentiable function vector and $t \geq t_0$, one has

$$ {}_{t_0}^{RL}D_t^{q}[x^T(t)Px(t)] \leq 2x(t)P{}_{t_0}^{RL}D_t^{q}x(t), \quad q \in (0,1), \tag{9}$$

where $P \in R^{n\times n}$ is a positive symmetrical square matrix.

Lemma 3: $\forall x, y \in R^n, \gamma > 0$, inequality $2x^Ty \leq \gamma x^Tx + \frac{1}{\gamma}y^Ty$ is established.

Lemma 4: $\forall x, y \in R^n$, where $Q_{n\times n}$ is a positive definite matrix, inequality $2x^Ty \leq x^TQx + y^TQ^{-1}y$ is established.

III. MAIN RESULTS

A. N-DIMENSIONAL FRACTIONAL-ORDER NEURAL NETWORK

To facilitate the establishment of the main results, we make the following assumptions.

H1: The neuron activation functions $f_i(.)(i = 1, 2, \cdots, n)$ are Lipschitz continuous. That is, there exist positive constants $M_i(j = 1, 2, \cdots, n)$ such that $\|f(u) - f(v)\| \leq M_1\|u - v\|$.

Theorem 1: If there are two positive definite matrices P and Q,

$$ \Omega = -2PB + Q + k^2PA^TQ^{-1}AP < 0, \tag{10}$$

and the system (eq.1) is then asymptotically stable at the equilibrium point.

Proof: First, the following Lyapunov function is constructed:

$$ V(x_t) = {}_0D_t^{-(1-\alpha)}(x^T(t)Px(t)), \tag{11}$$

where $0 < \alpha < 1$ and P is a positive definite matrix.

Owing to $P > 0$, $V(x_t) > 0$, according to *Definition 2* and *Lemma 2*, the derivative for $V(x_t)$ is

$$ \frac{dV(x_t)}{dt} = {}_0^{RL}D_t^{\alpha}(x^T(t)Px(t)) \leq 2x^T(t)P_0D_t^{\alpha}x(t), \tag{12}$$

and then

$$ \frac{dV(x_t)}{dt} \leq -2x^T(t)PBx(t) + 2x^T(t)PAf(x(t)). \tag{13}$$

According to *Lemma 4*,

$$ 2x^T(t)PAf(x(t)) \leq x^T(t)Qx(t) + f^T(x(t))A^TPQ^{-1}PAf(x(t)). \tag{14}$$

According to *Lemma 4* and eq.2,

$$f^T(x(t))f(x(t)) = |f(x(t))|^2 \le k^2|x(t)|^2. \tag{15}$$

Therefore, according to eq.14 and eq.15,

$$\frac{dV(x_t)}{dt} \le x^T(t)[-2PB + Q + k^2PA^TQ^{-1}AP]x(t). \tag{16}$$

By *Theorem 1*, it is known that

$$\frac{dV(x_t)}{dt} \le \lambda_{max}(\Omega)|x(t)|^2, \tag{17}$$

where λ represents the maximum eigenvalue of Ω, and

$$\Omega = -2PB + Q + k^2PA^TQ^{-1}AP < 0. \tag{18}$$

By using the Lyapunov direct method, the unique equilibrium point of the system (eq.1) is asymptotically stable.

Remark 1: In recent years, there have been many methods to analyze the stability of fractional-order neural networks [25]–[29]. Different from these methods, combined with LMI, we construct a Lyapunov functional in the sense of an R-L derivative and prove theorem 1. This criterion solves the problem that fractional-order calculus does not satisfy Leibniz's derivative rule, and greatly reduces the calculation.

Remark 2: The asymptotic stability studied in this paper is only related to the internal structure and parameters of the system itself. External input has no effect on the stability of the system.

B. N-DIMENSIONAL FRACTIONAL-ORDER NEURAL NETWORK WITH DELAY

Theorem 2: If there are two positive matrices P, Q and two constants β, γ, one has

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}{}^T & X_{22} \end{bmatrix} < 0, \tag{19}$$

where

$$X_{11} = -2PB + P(\frac{1}{\beta}AA^T + \frac{1}{\gamma}CC^T)P + \beta k^2 I_n + Q + A^TWA, \tag{20}$$

$$X_{12} = A^TWA_\tau, \quad X_{12}^T = A_\tau^TWA, \tag{21}$$

$$X_{22} = \gamma k^2 I_n - Q + A_\tau^TWA_\tau, \tag{22}$$

$$W = A_\tau^TVA_\tau, \quad V = \tau X. \tag{23}$$

Then, the system (eq.3) is asymptotically stable at the equilibrium point.

Proof: First, prove the existence and uniqueness of the system (eq.3) equilibrium point.

Let $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ be the equilibrium point of the system (eq.3),

$$-Bx^* + Af_1(x^*) + Cf_2(x^*) + I = 0, \tag{24}$$

let

$$D(x) = \begin{bmatrix} \frac{f_1(x_1)}{x_1} & & & 0 \\ & \frac{f_2(x_2)}{x_2} & & \\ & & \cdots & \\ 0 & & & \frac{f_n(x_n)}{x_n} \end{bmatrix}. \tag{25}$$

Define $E = diag\left(\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\right)$, $G = diag\left(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n\right)$.

Therefore, for the diagonal matrix $E \le D \le G$ satisfying $D(x^*)$, according to eq.25, we can obtain:

$$-Bx^* + (A+C)D(x^*)x^* + I = \left[-B + (A+C)D(x^*)\right]x^* + I = 0. \tag{26}$$

Using matrix analysis theory and Brouwer's fixed point theorem, the only necessary and sufficient condition for the existence of the equilibrium point can be obtained. The theorem is described as follows:

Theorem 3: For any I and any nonlinear function $f(.)$, the necessary and sufficient condition for the existence of a unique equilibrium point in the neural network (eq.3) is that the matrix is non-singular for all D satisfied.

Proof: First prove the necessity by contradiction.

Suppose that for a certain D_0 that satisfies $E \le D_0 \le F$, $-B + (A+B)D_0$ is singular, and the activation function can be constructed as $f(x)$: $f_i(x_i) = d_{i0}x_i, \quad i = 1, 2, \ldots, n$, the system (eq.3) becomes:

$${}_0^{RL}D_t^\alpha x(t) = -Bx(t) + AD_0x(t) + CD_0x(t-\tau) + I, \tag{27}$$

since the system has the equilibrium point, $[-B + A + CD_0]x^* + I = 0$ exists as a solution. $-B + A + CD_0$ is singular, so it can be determined that the system (eq.3) may have countless solutions or no solutions. Therefore, the system has the only balance point contradiction, so the hypothesis does not hold. $-B + (A+B)D$ is non-singular.

According to eq.26:

$$x = -[-B + (A+C)D(x)]^{-1}I, \tag{28}$$

mapping:

$$T(x) = -[-B + A + CD(x)]^{-1}I, \tag{29}$$

then

$$\|T(x)\| \le \|[-B + A + CD(x)]^{-1}\| \cdot \|I\|. \tag{30}$$

According to eq.4, with eq.30 bounded, let

$$\Omega = \{x \,|\, \|x\| \le M\}, \quad M = \max_{E \le D \le F} \left\|[-B + (A+C)D]^{-1}\right\| \, \|I\|, \tag{31}$$

then $\exists x^* \in \Omega$, so that

$$T(x^*) = x^* = -\left[-B + (A+C)D(x^*)\right]^{-1}I, \tag{32}$$

so the equilibrium point exists.

Supposing that there are two different equilibrium points $x^\star$ and $y^\star$, $x^* \neq y^*$ satisfies:

$$-Bx^* + (A+C)f\left(x^*\right) + I = 0, \tag{33}$$
$$-By^* + (A+C)f\left(y^*\right) + I = 0, \tag{34}$$

(eq.33)-(eq.34):

$$\left[-B + (A+C)D\left(x^* - y^*\right)\right]\left(x^* - y^*\right) = 0, \tag{35}$$

then

$$D\left(x^* - y^*\right) = \begin{bmatrix} \frac{f_1\left(x_1^*\right)-f_1\left(y_1^*\right)}{x_1^* - y_1^*} & & & 0 \\ & \frac{f_2\left(x_2^*\right)-f_2\left(y_2^*\right)}{x_2^* - y_2^*} & & \\ & & \cdots & \\ 0 & & & \frac{f_n\left(x_n^*\right)-f_n\left(y_n^*\right)}{x_n^* - y_n^*} \end{bmatrix}, \tag{36}$$

according to eq.4:

$$E \le D\left(x^* - y^*\right) \le F, \tag{37}$$

then $-B + (A+C)D(x^* - y^*)$ is non-singular. Therefore, $x^* - y^* = 0$ contradicts the assumption.

The above proves that the equilibrium point exists and is unique.

The following Lyapunov function is constructed:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t), \tag{38}$$
$$V_1(x_t) = {}_0D_t^{-(1-\alpha)}(x^T(t)Px(t)) + \int_0^t \int_{s-\tau}^s \dot{x}^T(\alpha)A_\tau^T XA_\tau\dot{x}(\alpha)d\alpha ds, \tag{39}$$
$$V_2(x_t) = \int_{-\tau}^0 \int_{t+\beta}^t \dot{x}(\alpha)A_\tau^T XA_\tau\dot{x}(\alpha)d\alpha d\beta, \tag{40}$$
$$V_3(x_t) = \int_{t-\tau}^t x^T(\alpha)Qx(\alpha)d\alpha. \tag{41}$$

Letting $\dot{x}(t) = Ax(t) + A_\tau x(t-\tau)$ and $0 < \alpha < 1$, P, Q are two positive definite matrices.

Owing to $P > 0, Q > 0$, $V(x_t) > 0$, according to the *Definition 2* derivative for $V(x_t)$,

$$\frac{dV_1(x_t)}{dt} = {}_0D_t^\alpha[x^T(t)Px(t)] + \int_{t-\tau}^t \dot{x}(\alpha)A_\tau^T XA_\tau\dot{x}(\alpha)d\alpha, \tag{42}$$
$$\frac{dV_2(x_t)}{dt} = \dot{x}(t)A_\tau^T VA_\tau\dot{x}(t) - \int_{t-\tau}^t \dot{x}(\alpha)A_\tau^T XA_\tau\dot{x}(\alpha)d\alpha, \tag{43}$$
$$\frac{dV_3(x_t)}{dt} = x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau), \tag{44}$$

one has

$$\begin{aligned}\frac{dV(x_t)}{dt} &= \frac{dV_1(x_t)}{dt} + \frac{dV_2(x_t)}{dt} + \frac{dV_3(x_t)}{dt} \\ &= {}_0D_t^\alpha x^T(t)Px(t) \\ &\quad + \dot{x}^T(t)A_\tau^T VA_\tau\dot{x}(t) \\ &\quad + x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau),\end{aligned} \tag{45}$$

according to *Lemma 3*;

$$\begin{aligned}{}_0D_t^\alpha x^T(t)Px(t) &\le 2x^T(t)P{}_0D_t^\alpha x(t) \\ &= -2x^T(t)PBx(t) \\ &\quad + 2x^T(t)PAf(x(t)) \\ &\quad + 2x^T(t)PCf(x(t-\tau)),\end{aligned} \tag{46}$$

according to *Lemma 4*; and

$$f^T(x(t))f(x(t)) = |f(x(t))|^2 \le \beta k^2|x(t)|^2, \tag{47}$$
$$\begin{aligned}\gamma f^T(x(t-\tau))f(x(t-\tau)) &= \gamma|f(x(t-\tau))|^2 \\ &\le \gamma k^2|x(t-\tau)|^2.\end{aligned} \tag{48}$$

From the above,

$$\begin{aligned}\frac{dV(x_t)}{dt} &\le x^T(t)[-2PB + P(\frac{1}{\beta}AA^T + \frac{1}{\gamma}CC^T)P \\ &\quad + (\beta+\gamma)k^2I_n + Q + W]x(t) \\ &\quad + x^T(t-\tau)[\gamma k^2I_n - Q + A_\tau^W A_\tau]x(t-\tau) \\ &\quad + A^Tx^T(t)WA_\tau x(t-\tau) \\ &\quad + A_\tau^Tx^T(t-\tau)WAx(t) \\ &\le \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22}\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix},\end{aligned} \tag{49}$$

where

$$\begin{aligned}X_{11} &= -2PB + P(\frac{1}{\beta}AA^T + \frac{1}{\gamma}CC^T)P \\ &\quad + \beta k^2I_n + Q + A^TWA,\end{aligned} \tag{50}$$
$$X_{12} = A^TWA_\tau, \quad X_{12}^T = A_\tau^TWA, \tag{51}$$
$$X_{22} = \gamma k^2I_n - Q + A_\tau^TWA_\tau, \tag{52}$$
$$W = A_\tau^TVA_\tau, \quad V = \tau X. \tag{53}$$

By *Theorem 1*, it is known that

$$\frac{dV(x_t)}{dt} \le \lambda_{max}(\Omega)|x(t)|^2, \tag{54}$$

where λ represents the maximum eigenvalue of Ω, and

$$\begin{bmatrix} X_{11} & X_{12} \\ {X_{12}}^T & X_{22}\end{bmatrix} < 0. \tag{55}$$

Then, the system (eq.3) is asymptotically stable at the equilibrium point.

Remark 3: In *Theorem 2*, the asymptotic stability criterion of fractional-order neural networks with time delay (eq.3) is established. This criterion is simple and clear and is described by LMI. On the one hand, it reduces the amount of calculation in the stability analysis; on the other hand, it fully reveals the logical relationship between network parameters.

Remark 4: In [29], Lyapunov function is constructed by fractional Razumikhin theorem. In [35], Lyapunov function is considered by applying a positive definite quadratic function $V(t) = x^T(t)Px(t)$. However, In the proof of *Theorem 2*, Lyapunov functional is composed of positive definite quadratic and functional with delay and nonlinear term. It is easy to see that the Lyapunov functional simplifies the amount of calculation and effectively solves the constant time delay problem of the fractional-order neural network. In addition, compared with the algebraic criterion in [34]–[37], the stability condition based on LMI in *Theorem 2* is less conservative and has great application value in solving simple practical engineering problems.

Remark 5: As we all know, LMI approach is a simple and intuitive tool to analyze system stability. In [34], the LMI stability condition of an uncertain fractional-order linear system under $1 < \alpha < 2$ is proposed. In [35], the global stability condition of the fractional-order neural network is given by using the LMI method, but the time delay is not discussed. In [38], the state estimation of FOBAM neural network is studied by using fractional Lyapunov direct method and LMI method. Inspired by [34]–[36], *Theorem 2* combines Lyapunov method with LMI method to propose a stability criterion for fractional-order neural network with time delay.

Compared with [34], [35], *Theorem 2* has simple conditions and is suitable for high-dimensional systems. Moreover, the criteria proposed in this paper can be extended to other types of fractional-order neural networks, such as the BAM neural network in [38].

IV. ILLUSTRATIVE EXAMPLES

To verify the correctness of the theoretical results in this paper, some numerical simulations are presented.

Example 1: Consider the R-L fractional neural networks described by

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6.5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & 1 & -1.5 \\ -1 & 2.5 & 2 \\ -2.5 & 2 & -1.5 \end{bmatrix}$$

$$\begin{cases} D^{\alpha}x_1(t) = -4x_1(t) + 5f(x_1(t)) + f(x_2(t)) \\ \qquad\qquad - 1.5f(x_3(t)) \\ D^{\alpha}x_2(t) = -6.5x_2(t) - f(x_1(t)) + 2.5f(x_2(t)) \\ \qquad\qquad + 2f(x_3(t)) \\ D^{\alpha}x_3(t) = -5x_3(t) - 2.5f(x_1(t)) + 2f(x_2(t)) \\ \qquad\qquad - 1.5f(x_3(t)), \end{cases} \tag{56}$$

where $x = (x_1, x_1, x_3)^T$ and activation function $f(.) = tanh(.)$ to meet system eq.1. The system has a unique equilibrium point $(x_1^{\star}, x_2^{\star}, x_3^{\star}) = (2.7243, -0.2312, -1.1346)$, and it is known that the conditions in eq.2 are satisfied.

Letting $k = 0.1 > 0, P = diag\{1, 1, 1\}, Q = diag\{1, 1, 1\}$, according to *Theorem 1*,

$$-2PB + Q + k^2PA^TQ^{-1}AP = \begin{bmatrix} -6.6650 & -0.0375 & -0.1175 \\ -0.0375 & -11.8875 & -0.0050 \\ -0.1175 & -0.0050 & -8.8750 \end{bmatrix} < 0.$$

Thus, the unique equilibrium point of the system is asymptotically stable. Then, the system initial values $(x_1, x_2, x_3) = (-8, 8, 8)$ under different fractional-order conditions $\alpha = 0.9, 0.4$ as shown in Figs. 1 and 2 are considered.

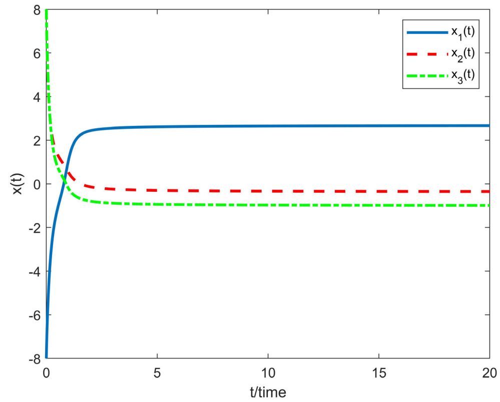

FIGURE 1. Trajectories of state variable $x(t)$ of neural networks (eq.56) with $\alpha = 0.9$, $(x_1, x_2, x_3) = (-8, 8, 8)$.

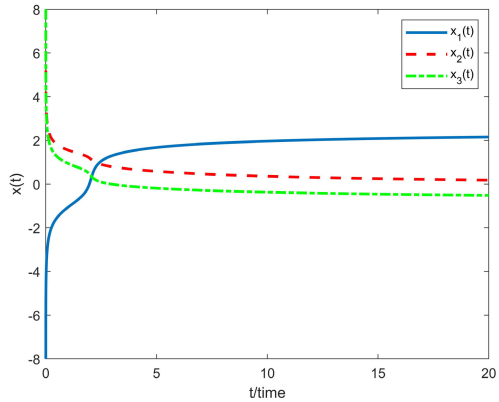

FIGURE 2. Trajectories of state variable $x(t)$ of neural networks (eq.56) with $\alpha = 0.4$, $(x_1, x_2, x_3) = (-8, 8, 8)$.

Remark 6: In [35], the numerical example results show that the convergence speed is affected by the order of the fractional-order system. As the fractional order α increases, the convergence speed becomes faster. For numerical *example 1*, different orders are selected to verify the validity of the stability criterion. It can be seen from Figs. 1 and 2 that the convergence speed of $\alpha = 0.9$ is faster than $\alpha = 0.4$, which is consistent with the conclusion of [35].

Example 2: Consider the R-L fractional neural networks described by

$$B = \begin{bmatrix} 1.2 & 0 \\ 0 & 2.3 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & -0.7 \\ 0.3 & 0.2 \end{bmatrix}$$

$$\begin{cases} D^{\alpha}x_1(t) = -1.2x_1(t) + 0.5f(x_1(t)) \\ \qquad\qquad - 0.7f(x_2(t)) \\ D^{\alpha}x_2(t) = -2.3x_2(t) + 0.3f(x_1(t)) \\ \qquad\qquad + 0.2f(x_2(t)), \end{cases} \tag{57}$$

where $\alpha = 0.8$ and activation function $f(.) = cos(.)$ to meet system eq.1. The system has a unique equilibrium point $(x_1^{\star}, x_2^{\star}) = (-0.3167, 0.3852)$. Letting $k = 1.5 > 0, P = diag\{1, 2\}, Q = diag\{1, 2\}$, according to *Theorem 1* one has

$$-2PB + Q + k^2PA^TQ^{-1}AP = \begin{bmatrix} -0.7363 & -1.4400 \\ -1.4400 & -2.6100 \end{bmatrix} < 0.$$

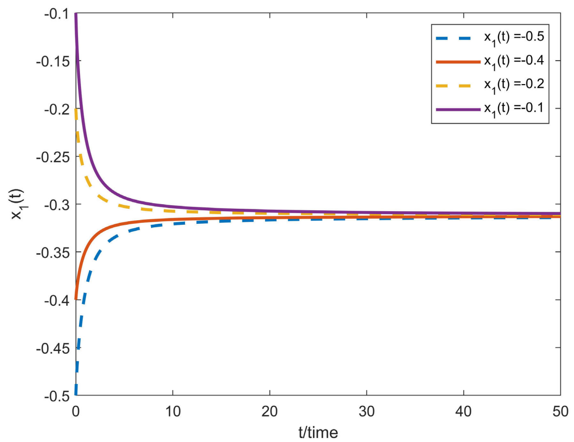

FIGURE 3. Trajectories of state variable $x_1(t)$ of neural networks (eq.57) with $\alpha = 0.8$ under different initial conditions.

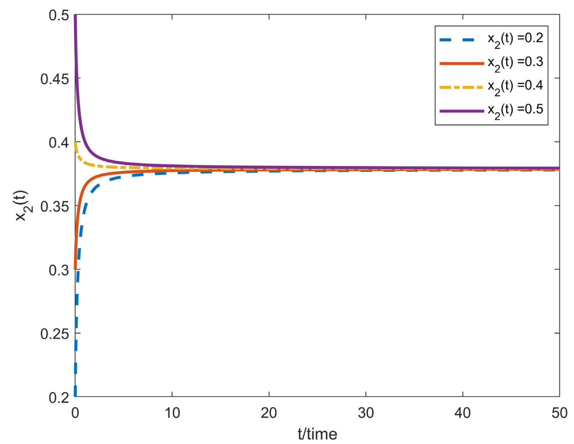

FIGURE 4. Trajectories of state variable $x_2(t)$ of neural networks (eq.57) with $\alpha = 0.8$ under different initial conditions.

Therefore, the unique equilibrium point of the system is asymptotically stable. Considering the different system initial values $(x_1, x_2) = (-0.5, 0.2), (x_1, x_2) = (-0.4, 0.3), (x_1, x_2) = (-0.2, 0.4), (x_1, x_2) = (-0.1, 0.5)$, as shown in Figs. 3 and 4, the unique equilibrium point $(x_1^\star, x_2^\star)$ in system (eq.57) can be obtained based on different initial values.

Remark 7: In [14] and [19], the results of numerical examples show that the fractional-order neural network is affected by the choice of activation function and has a longer convergence time. For numerical *example 2*, different initial conditions are selected to verify the validity of the stability criterion. It can be seen from Figs. 3 and 4 that the selection of the activation function is more suitable for fractional-order neural network systems, and the convergence time is shorter.

Example 3: Consider the R-L fractional delayed neural networks described by

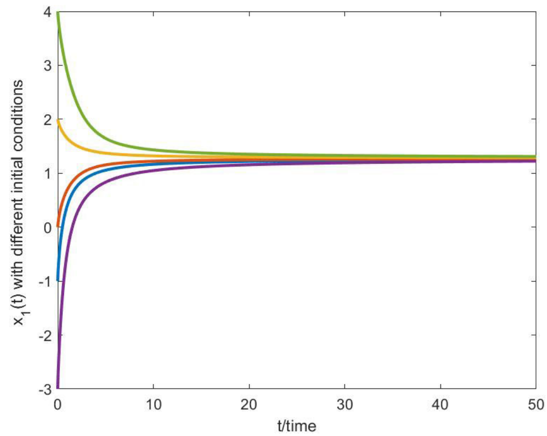

FIGURE 5. Trajectories of state variable $x_1(t)$ of neural networks (eq.58) with $\alpha = 0.8$, $\tau = 0.2$ under different initial conditions.

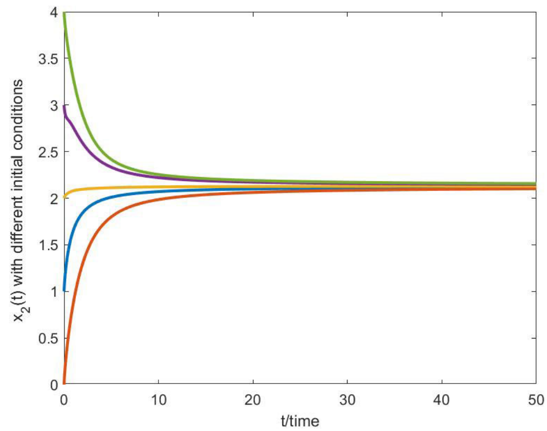

FIGURE 6. Trajectories of state variable $x_2(t)$ of neural networks (eq.58) with $\alpha = 0.8$, $\tau = 0.2$ under different initial conditions.

$$\begin{cases} D^\alpha x_1(t) = -0.7x_1(t) - 0.5sin(x_1(t)) + 0.2sin(x_2(t)) \\ \qquad + 0.1arctan(x_1(t-\tau)) \\ \qquad + 0.1arctan(x_2(t-\tau)) + 1 \\ D^\alpha x_2(t) = -0.6x_2(t) - 0.4sin(x_1(t)) + 0.3sin(x_2(t)) \\ \qquad + 0.2arctan(x_1(t-\tau)) \\ \qquad + 0.2arctan(x_2(t-\tau)) + 1, \end{cases} \tag{58}$$

where $\alpha = 0.8$ to meet system eq.2. Letting $k = 0.1 > 0$, $P = diag\{1, 1\}, Q = diag\{1, 1\}, \beta = 10, \gamma = 3, \tau = 0.2$ and

$$A_\tau = \begin{bmatrix} 0 & -0.1 \\ -0.1 & 0.2 \end{bmatrix},$$

according to *Theorem 2* one has

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}{}^T & X_{22} \end{bmatrix} = \begin{bmatrix} -0.2633 & 0.0389 & 2.0000 & -4.6000 \\ 0.0398 & -0.0478 & -2.2000 & 5.2000 \\ 2.0000 & -2.2000 & -0.9699 & -2.4000 \\ -4.6000 & 5.2000 & -2.4000 & -0.9694 \end{bmatrix} < 0.$$

$$B = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad A = \begin{bmatrix} -0.5 & 0.2 \\ -0.4 & 0.3 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, \quad I = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvalues are $\lambda = \{-0.2706, -0.0411, -0.9693, -0.9700\}$, so the unique equilibrium point of the system is

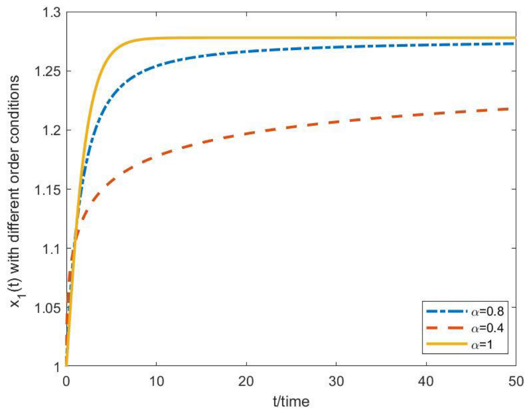

FIGURE 7. Trajectories of state variable $x_1(t)$ of neural networks (eq.58) with different order conditions.

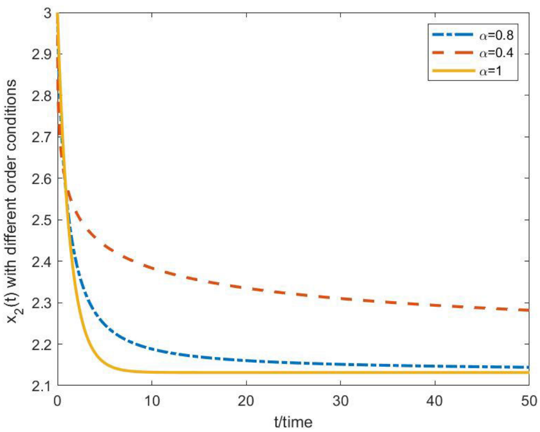

FIGURE 8. Trajectories of state variable $x_2(t)$ of neural networks (eq.58) with different order conditions.

asymptotically stable.Considering the different system initial values are depicted in Figs. 5 and 6. Moreover,the state trajectories of system (eq.58) under different fractional-order conditions $\alpha = 0.4, 0.8, 1.0$ are depicted in Figs. 7 and 8.

Example 4: Consider the R-L fractional delayed neural networks described by

$$B = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}, \quad A = \begin{bmatrix} 0.4 & -0.4 \\ 0.6 & 0.1 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.3 & -0.4 \\ 0.1 & 0.5 \end{bmatrix}, \quad I = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} D^{\alpha} x_1(t) = -0.9x_1(t) + 0.4tanh(x_1(t)) \\ \qquad - 0.4tanh(x_2(t)) - 0.3cos(x_1(t-\tau)) \\ \qquad - 0.4cos(x_2(t-\tau)) \\ D^{\alpha} x_2(t) = -1.1x_2(t) + 0.6tanh(x_1(t)) \\ \qquad + 0.1tanh(x_2(t)) + 0.1cos(x_1(t-\tau)) \\ \qquad + 0.5cos(x_1(t-\tau)), \end{cases} \tag{59}$$

where $\alpha = 0.4$ to meet system eq.2. Letting $k = 0.02 > 0$, $P = diag\{1, 1\}$, $Q = diag\{1, 1\}$, $\beta = 100$, $\gamma = 200$, $\tau = 0.5$, and

$$A_{\tau} = \begin{bmatrix} -0.2 & -0.3 \\ -1 & 0.2 \end{bmatrix},$$

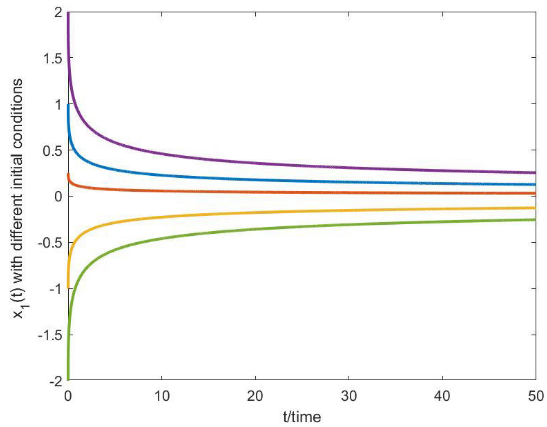

FIGURE 9. Trajectories of state variable $x_1(t)$ of neural networks (eq.59) with $\alpha = 0.4$, $\tau = 0.5$ under different initial conditions.

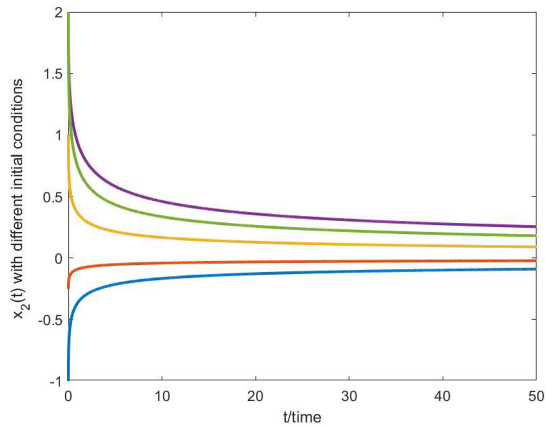

FIGURE 10. Trajectories of state variable $x_2(t)$ of neural networks (eq.59) with $\alpha = 0.4$, $\tau = 0.5$ under different initial conditions.

according to *Theorem 2* one has

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}{}^{T} & X_{22} \end{bmatrix} = \begin{bmatrix} -0.6826 & -0.0645 & -0.0442 & -0.0476 \\ -0.0645 & -1.0656 & 0.0085 & 0.0714 \\ -0.0442 & 0.0085 & -0.8622 & 0 \\ -0.0476 & 0.0714 & 0 & -0.8534 \end{bmatrix} < 0.$$

The eigenvalues are $\lambda = \{-0.6461, -1.0938, -0.8793, -0.8534\}$, so the unique equilibrium point of the system is asymptotically stable.Considering the different system initial values are depicted in Figs. 9 and 10. Moreover,the state trajectories of system (eq.59) under different fractional-order conditions $\alpha = 0.2, 0.5, 1.0$ are depicted in Figs. 11 and 12.

Remark 8: According to [32], for the case of $\alpha = 1$, system (eq.3) is simplified to an integer-order neural network with time delay. In order to verify the validity of the stability criterion, choose different orders in numerical *example 3* and *example 4*, and compare them with $\alpha = 1$. It can be seen from Figs. 7 and 11 that the conclusion is consistent with [32]. In addition, we can obtain the asymptotic stability criterion of integer-order neural networks with time delay from the results of this paper. Therefore, the stability criterion of this paper is the improvement and promotion of the classical integer-order neural network with time delay.

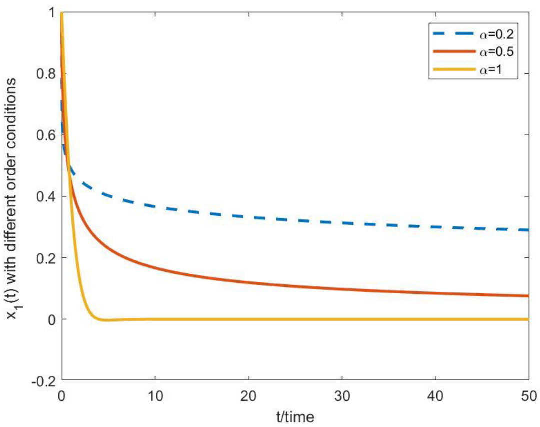

FIGURE 11. Trajectories of state variable $x_1(t)$ of neural networks (eq.59) with different order conditions.

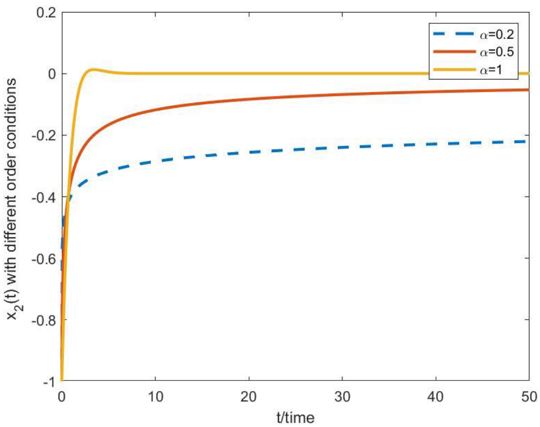

FIGURE 12. Trajectories of state variable $x_1(t)$ of neural networks (eq.59) with different order conditions.

TABLE 1. Comparison of convergence time of different literatures under different orders. (Through the comparative analysis of numerical examples in [14], [19], [32], and [35], it is concluded that the convergence time can be an important comparison index.Therefore, under the same initial conditions, different fractional orders are selected for comparison.)

Convergence time	$\alpha = 0.4$	$\alpha = 0.9$	$\alpha = 1$
Hai *et al.* [14]	110.35s	16.01s	6.3s
Zhang *et al.* [19]	53.49s	3.78s	1.97s
Wang *et al.* [32]	$431s$	$6.95s$	$2.8s$
Zhang *et al.* [35]	42.1s	3.42s	2.01s
Example 3 (eq.58)	$\mathbf{6}2.15$s	$\mathbf{9}.63$s	$\mathbf{4}.55$s

Based on the comparison results of the above numerical examples, it is found that the convergence time is an important index that can be used to compare the stability of fractional-order neural networks. Therefore, combined with *Remark6,Remark7* and *Remark8*, Tab. 1 is obtained through experimental verification for system (eq.58) under the same initial conditions. According to Tab. 1, the convergence of fractional order neural networks with time delay can be obtained intuitively.

V. CONCLUSION

In this paper, the asymptotic stability of two fractional-order neural networks with and without time delay in the sense of the R-L derivative is studied. Our contributions include: (1) the existence and uniqueness of equilibrium points for fractional-order neural networks are proven, and (2) by constructing the Lyapunov functional and combining the Lyapunov method and LMI method, two sets of asymptotic stability criteria are given. (3) Numerical examples verify the effectiveness of the results.

However, the stability criterion of the fractional-order system proposed by Lyapunov and LMI has some limitations: (1) this criterion can only be applied to the case in which the equilibrium point of the fractional-order neural network exists and is unique. If the system contains multiple equilibrium points, the criterion will not hold. (2) *Theorem 2* is not applicable to fractional-order neural networks with variable delays.

In this paper, we focus on the theoretical analysis of fractional-order neural networks. If the Lyapunov functions are chosen via the viewpoint of practical application, the choice of the Lyapunov function should be more flexible. For example, to facilitate the calculation, we can choose the approximate function of the theoretical function or discretize the function by numerical calculation, which will be our future work.

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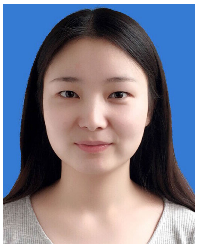

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