IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received August 15, 2021, accepted September 3, 2021, date of publication September 7, 2021, date of current version September 14, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3110879

Dynamic and Updating Multigranulation Decision-Theoretic Rough Approximations When Adding or Deleting Objects

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ABSTRACT Along with the development of big data, knowledge updating occurs in various situations, some scholars had studied the dynamic method of updating knowledge in multigranulation decision-theoretic rough sets when adding or deleting granular structures. However, there is no study about the case of adding or deleting objects, which limits the development of multigranulation decision-theoretic rough sets. Based on the matrix method, the dynamic knowledge updating of optimistic multigranulation decision-theoretic rough set and pessimistic multigranulation decision-theoretic rough set were studied in this paper, then given the static algorithm and dynamic algorithms, and the time complexity of three algorithms was analyzed. The theory and experimental results show that two dynamic algorithms are both more effective than the static algorithm.

INDEX TERMS Incremental updating, knowledge discovery, multigranulation decision-theoretic rough sets.

I. INTRODUCTION

As a data analysing and processing theory, rough set theory was put forward by Z. Pawlak from Poland [1], it has made great progress in both theory and application, and has been applied to data mining [2], machine learning [3]–[5], knowledge discovery [6] and other fields. As the basic calculation of rough set, the calculation of lower and upper approximations are necessary steps to obtain other significant achievements, many scholars have popularized rough set in various aspects.

To overcome the influence of noise data, Yu *et al.* [7] proposed the decision-theoretic rough set, which simulates the process of human decision-making under uncertainty and risk, we can obtain the calculation of thresholds through decision risk minimization based on bayesian decision theory [8], conditional probability was estimated by naive bayesian model [9], and then given the concepts of positive domain, negative domain and edge domain, they are the basis of three-way decision [10], [11]. Therefore, decision-theoretic rough set is a model with solid theoretical foundation and practical application.

Pawlak rough set model and decision-theoretic rough set model are based on a single equivalence relation, due to practical needs, multigranulation rough set based on multiple equivalence relations was put forward by Qian *et al.* [12]. With the development of multigranulation rough set, several generalized multigranulation rough set models had been proposed to solve real-world problems with complex environments [13]–[17]. Multigranulation decision-theoretic rough set was proposed by Qian *et al.* [18], and several of its extended models were studied [19]–[23].

With the arrival of the big data, many scholars have studied incremental knowledge updating, using existing knowledge to update new knowledge, rather than recalculating, which will save a lot of time and space. There are three main aspects of the current dynamic knowledge updating: the dynamic change of objects, the dynamic change of attributes and the dynamic change of attribute values.

For the decision rules for the decision table with increase of objects, Liu *et al.* [24] proposed the corresponding incremental algorithm. Liang *et al.* [25] proposed an incremental feature selection algorithm based on rough set while increasing objects. Considering the addition and deletion of single object in neighborhood fuzzy decision system, Zeng *et al.* [26] proposed an incremental updating method of approximations in fuzzy rough set theory. Hu *et al.* [27] presented a method to update multigranulation approximations with the variation of granulars. Li *et al.* [28] proposed a matrix-based

The associate editor coordinating the review of this manuscript and approving it for publication was Sathish Kumar.

method of approximates when attribute values were updated dynamically in ordered information system. Li *et al.* [29] put forward the rule extraction algorithm based on the characteristic relation when adding and deleting multiple attributes. Qian *et al.* [30] proposed a solution for the attribute reduction problem to avoid redundant steps of incremental calculation with increasing attributes. Zeng *et al.* [31] proposed an incremental feature extraction algorithm for fuzzy rough set aiming at dynamic changes of attributes in mixed information system with many data types.

The rest of the paper is organized as follows. Section 2 briefly introduces the basic concepts of rough sets, multigranulation decision-theoretic rough sets and relation matrix. In section 3, the method of computing the lower and upper approximations are proposed by the matrix-based, then updating approximations when adding or deleting objects. Several dynamic algorithms are given in Section 4. In section 5, we verify the effectiveness of proposed dynamic algorithms experimentally. Finally, summarized the full text in Section 6 and put forward further directions of research.

II. PRELIMINARIES

A. ROUGH SETS

Give an information system $IS = \langle U, AT, V, f\rangle$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects called universe; AT is a non-empty finite set of attributes, $a \in AT$ is called an attribute; $V = \bigcup_{a\in AT} V_a$ is a set of attribute values, V_a is a non-empty set of values of attribute $a \in AT$, called the domain of a; $f : U \times AT \to V$ is an information function that maps an object in U to exactly one value in V_a such that $\forall a \in AT, x_i \in U, f(x_i, a) \in V_a$.

For a subset of attributes B, an indiscernibility relation R_B is defined as follows:

$$R_B = \{(x, y) \in U \times U : f(x, a) = f(y, a), \forall a \in B\}$$

where R_B is an equivalence relation on U. The equivalence relation R_B partitions the universe U into a family of disjoint subsets called equivalence classes, the equivalence class including x with respect to B is denoted as follows:

$$[x]_B = \{y \in U : (x, y) \in R_B\}$$

For a set $X \subseteq U$, the lower and upper approximations of X with repect to R are defined as follows:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$$
$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$$

where $[x]_R = \{y \in U|(x, y) \in R\}$ is a R-equivalence class containing x_i. If $\underline{R}(X) = \overline{R}(X)$, we say that X is a definable set; Otherwise, X is a rough set.

B. MULTIGRANULATION DECISION-THEORETIC ROUGH SETS (MG-DTRS)

Let $IS = \langle U, AT, V, f\rangle$ is an information system, $A_1, A_2, \ldots, A_m \subseteq AT$ and $\forall X \subseteq U$, the optimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{k=1}^m A_k}^{O,\alpha}(X)$ and $\overline{\sum_{k=1}^m A_k}^{O,\beta}(X)$, respectively, where

$$\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X) = \{x \in U : \bigvee_{i=1}^{m} P(X|[x]_{A_i}) \geq \alpha\}$$
$$\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X) = U - \{x \in U : \bigwedge_{i=1}^{m} P(X|[x]_{A_i}) \leq \beta\}$$

The pair $< \underline{\sum_{k=1}^m A_k}^{O,\alpha}(X), \overline{\sum_{k=1}^m A_k}^{O,\beta}(X) >$ is called the optimistic multi-granularity decision-theoretic rough sets of X.

For an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$ and $\forall X \subseteq U$, the pessimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{k=1}^m A_k}^{P,\alpha}(X)$ and $\overline{\sum_{k=1}^m A_k}^{P,\beta}(X)$, respectively, where

$$\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X) = \{x \in U : \bigwedge_{i=1}^{m} P(X|[x]_{A_i}) \geq \alpha\}$$
$$\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X) = U - \{x \in U : \bigvee_{i=1}^{m} P(X|[x]_{A_i}) \leq \beta\}$$

The pair $< \underline{\sum_{k=1}^m A_k}^{P,\alpha}(X), \overline{\sum_{k=1}^m A_k}^{P,\beta}(X) >$ is called the pessimistic multi-granularity decision-theoretic rough sets of X.

C. RELATION MATRIX

Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$ and $\forall X \subseteq U$, the characteristic function $F(X) = (f_1, f_2, \ldots, f_n)^T$ (T represents a transpose operation) of X is constructed as:

$$f_i = \begin{cases} 1, & x_i \in X \\ 0, & x_i \notin X \end{cases} \quad i = 1, 2, \ldots, n$$

For an information system $IS = \langle U, AT, V, f\rangle$, where $U = \{x_1, x_2, \ldots, x_n\}$, $A_1, A_2, \ldots, A_m \subseteq AT$, $\forall A_k \subseteq AT$, $k \in \{1, 2, \ldots, m\}$, the relation matrix $M_{A_k} = [m_{A_k}^{ij}]_{n\times n}$ of A_k is constructed as:

$$m_{A_k}^{ij} = \begin{cases} 1, & (x_i, x_j) \in R_{A_k} \\ 0, & (x_i, x_j) \notin R_{A_k} \end{cases} \quad i, j = 1, 2, \ldots, n$$

III. UPDATING MULTIGRANULATION DECISION-THEORETIC ROUGH APPROXIMATIONS WITH INCREASING OR DECREASING OF OBJECTS

A. MATRIX-BASED REPRESENTATION OF APPROXIMATIONS IN MG-DTRS

Due to the need of practical application, this section discusses multigranulation decision-theoretic rough approximations based on the matrix and give some properties.

Definition 1: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$. For a relation matrix $M_{A_k} = [m_{A_k}^{ij}]_{n\times n}$, a characteristic function $F(X) = (f_1, f_2, \ldots, f_n)^T$, a $n \times 1$ column vector $Q = (1, 1, \ldots, 1)^T$, two intermediate matrices $I^{\blacktriangle}_{A_k}(X) = [u^i_{A_k}]_{n\times 1}$ and $I^{\blacktriangledown}_{A_k} = [v^i_{A_k}]_{n\times 1}$ of A_k are defined by:

$$I^{\blacktriangle}_{A_k}(X) = M_{A_k} \times F(X)$$
$$I^{\blacktriangledown}_{A_k} = M_{A_k} \times Q$$

where $u^i_{A_k} = \sum_{j=1}^{n} m^{ij}_{A_k} f_j$, $v^i_{A_k} = \sum_{j=1}^{n} m^{ij}_{A_k}$, $(i, j = 1, 2, \ldots, n)$ and $\times$ denotes matrix multiplication.

Definition 2: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$. For two intermediate matrices $I^{\blacktriangle}_{A_k}(X) = [u^i_{A_k}]_{n\times 1}$ and $I^{\blacktriangledown}_{A_k} = [v^i_{A_k}]_{n\times 1}$, the basic matrix $I_{A_k}(X) = [h^i_{A_k}]_{n\times 1}$ of A_k is constructed as:

$$I_{A_k}(X) = I^{\blacktriangle}_{A_k}(X) ./ I^{\blacktriangledown}_{A_k}$$

where $h^i_{A_k} = u^i_{A_k}/v^i_{A_k}$ $(i = 1, 2, \ldots, n)$ and '$\times$' denotes matrix dot divide.

Definition 3: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$. For a basic matrix $I_{A_k}(X) = [h^i_{A_k}]_{n\times 1}$, two cut matrices of the lower and upper approximations of A_k are constructed as $I^{\downarrow}_{A_k}(X) = [h^{i\downarrow}_{A_k}]_{n\times 1}$ and $I^{\uparrow}_{A_k}(X) = [h^{i\uparrow}_{A_k}]_{n\times 1}$, where

$$h^{i\downarrow}_{A_k} = \begin{cases} 1, & h^i_{A_k} \geq \alpha \\ 0, & h^i_{A_k} < \alpha \end{cases} \quad i = 1, 2, \ldots, n$$

$$h^{i\uparrow}_{A_k} = \begin{cases} 1, & h^i_{A_k} > \beta \\ 0, & h^i_{A_k} \leq \beta \end{cases} \quad i = 1, 2, \ldots, n$$

Theorem 1: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$. For two cut matrices $I^{\downarrow}_{A_k}(X) = [h^{i\downarrow}_{A_k}]_{n\times 1}$ and $I^{\uparrow}_{A_k}(X) = [h^{i\uparrow}_{A_k}]_{n\times 1}$, the following results hold:

(1) $F(\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)) = max_{k=1}^{m}(I^{\downarrow}_{A_k}(X))$

(2) $F(\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)) = max_{k=1}^{m}(I^{\uparrow}_{A_k}(X))$

where 'max' indicate a maximum value among the corresponding position values of multiple matrices with same size.

Proof: (1) By the definitions of characteristic function and cut matrix, we can obtain $F(\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)) = (f_1, f_2, \ldots, f_n)^T$ and $I^{\downarrow}_{A_k}(X) = [h^{i\downarrow}_{A_k}]_{n\times 1}, \forall i \in \{1, 2, \ldots, n\}$,

$$\text{if } f_i = 1, \text{ then } x_i \in \underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)$$
$$\Leftrightarrow x_i \in \underline{A_k}^{\alpha}(X), \quad \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow P(X|[x_i]_{A_k}) = \frac{|X \cap [x_i]_{A_k}|}{|[x_i]_{A_k}|} \geq \alpha, \ \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^i_{A_k} = \frac{u^i_{A_k}}{v^i_{A_k}} \geq \alpha, \quad \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^{\downarrow i}_{A_k} = 1, \quad \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow max_{k=1}^{m} h^{\downarrow i}_{A_k} = 1, \ \text{then } f_i = max_{k=1}^{m} h^{\downarrow i}_{A_k};$$
$$\text{if } f_i = 0, \text{ then } x_i \notin \underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)$$
$$\Leftrightarrow x_i \notin \underline{A_k}^{\alpha}(X), \quad \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow P(X|[x_i]_{A_k}) = \frac{|X \cap [x_i]_{A_k}|}{|[x_i]_{A_k}|} < \alpha, \ \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^i_{A_k} = \frac{u^i_{A_k}}{v^i_{A_k}} < \alpha, \quad \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^{\downarrow i}_{A_k} = 0, \quad \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow max_{k=1}^{m} h^{\downarrow i}_{A_k} = 0, \ \text{then } f_i = max_{k=1}^{m} h^{\downarrow i}_{A_k}.$$

Thus, $F(\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)) = max_{k=1}^{m}(I^{\downarrow}_{A_k}(X))$.

(2) By the definitions of characteristic function and cut matrix, we can obtain $F(\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)) = (f_1, f_2, \ldots, f_n)^T$ and $I^{\uparrow}_{A_k}(X) = [h^{i\uparrow}_{A_k}]_{n\times 1}, \forall i \in \{1, 2, \ldots, n\}$,

$$\text{if } f_i = 1, \text{ then } x_i \in \overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)$$
$$\Leftrightarrow x_i \in \overline{A_k}^{\beta}(X), \quad \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow P(X|[x_i]_{A_k}) = \frac{|X \cap [x_i]_{A_k}|}{|[x_i]_{A_k}|} > \beta, \ \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^i_{A_k} = \frac{u^i_{A_k}}{v^i_{A_k}} > \beta, \ \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^{i\uparrow}_{A_k} = 1, \ \exists k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow max_{k=1}^{m} h^{i\uparrow}_{A_k} = 1, \ \text{then } f_i = max_{k=1}^{m} h^{i\uparrow}_{A_k};$$
$$\text{if } f_i = 0, \text{ then } x_i \notin \overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)$$
$$\Leftrightarrow x_i \notin \overline{A_k}^{\beta}(X), \quad \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow P(X|[x_i]_{A_k}) = \frac{|X \cap [x_i]_{A_k}|}{|[x_i]_{A_k}|} \leq \beta, \ \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^i_{A_k} = \frac{u^i_{A_k}}{v^i_{A_k}} \leq \beta, \quad \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow h^{i\uparrow}_{A_k} = 0, \quad \forall k \in \{1, 2, \ldots, m\}$$
$$\Leftrightarrow max_{k=1}^{m} h^{i\uparrow}_{A_k} = 0, \ \text{then } f_i = max_{k=1}^{m} h^{i\uparrow}_{A_k}.$$

Thus, $F(\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)) = max_{k=1}^{m}(I^{\uparrow}_{A_k}(X))$.

Theorem 2: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$. For two cut matrices $I^{\downarrow}_{A_k}(X) = [h^{i\downarrow}_{A_k}]_{n\times 1}$ and $I^{\uparrow}_{A_k}(X) = [h^{i\uparrow}_{A_k}]_{n\times 1}$, the following results hold:

(1) $F(\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X)) = min_{k=1}^{m}(I^{\downarrow}_{A_k}(X))$

(2) $F(\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X)) = min_{k=1}^{m}(I^{\uparrow}_{A_k}(X))$

where 'min' indicate a minimum value among the corresponding position values of multiple matrices with same size.

Proof: The process of proof is similar to Theorem 1.

The following example shows that there is no inclusion relationship between them.

Example 1: Give an information system.

$$U = \{x_1, x_2, x_3, x_4, x_5, x_6\}, \ AT = \{a_1, a_2, a_3, a_4\},$$
$$\text{let } A_1 = \{a_1\}, \ A_2 = \{a_2\}, \ A_3 = \{a_3\}, \ A_4 = \{a_4\},$$
$$X = \{x_3, x_4, x_5, x_7, x_8\}, \ \alpha = 0.8, \ \beta = 0.2.$$

TABLE 1. An information system.

U	a_1	a_2	a_3	a_4
x_1	1	1	1	3
x_2	1	1	1	3
x_3	2	1	1	3
x_4	3	2	1	3
x_5	2	3	2	2
x_6	1	2	1	1
x_7	1	3	2	3
x_8	3	3	2	2

According to the definition of characteristic function, we have:

$$F(X) = (0, 0, 1, 1, 1, 0, 1, 1)^T$$

According to the definition of relation matrix, we have:

$$M_{A_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_{A_2} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$M_{A_3} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$M_{A_4} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

According to Definition 1 and Definition 2, we can calculate intermediate matrices $I^{\blacktriangledown}_{A_1}(X)$, $I^{\blacktriangledown}_{A_1}$ and basic matrix $I_{A_1}(X)$ of A_1 as follows:

$$u^1_{A_1} = \sum_{j=1}^{8} m^{1j}_{A_1} f_j = 1, \quad v^1_{A_1} = \sum_{j=1}^{8} m^{1j}_{A_1} = 3,$$
$$u^2_{A_1} = \sum_{j=1}^{8} m^{2j}_{A_1} f_j = 2, \quad v^2_{A_1} = \sum_{j=1}^{8} m^{2j}_{A_1} = 3,$$
$$u^3_{A_1} = \sum_{j=1}^{8} m^{3j}_{A_1} f_j = 2, \quad v^3_{A_1} = \sum_{j=1}^{8} m^{3j}_{A_1} = 2,$$
$$u^4_{A_1} = \sum_{j=1}^{8} m^{4j}_{A_1} f_j = 2, \quad v^4_{A_1} = \sum_{j=1}^{8} m^{4j}_{A_1} = 3,$$
$$u^5_{A_1} = \sum_{j=1}^{8} m^{5j}_{A_1} f_j = 2, \quad v^5_{A_1} = \sum_{j=1}^{8} m^{5j}_{A_1} = 2,$$
$$u^6_{A_1} = \sum_{j=1}^{8} m^{6j}_{A_1} f_j = 1, \quad v^6_{A_1} = \sum_{j=1}^{8} m^{6j}_{A_1} = 3,$$
$$u^7_{A_1} = \sum_{j=1}^{8} m^{7j}_{A_1} f_j = 1, \quad v^7_{A_1} = \sum_{j=1}^{8} m^{7j}_{A_1} = 3,$$
$$u^8_{A_1} = \sum_{j=1}^{8} m^{8j}_{A_1} f_j = 2, \quad v^8_{A_1} = \sum_{j=1}^{8} m^{8j}_{A_1} = 3,$$

so $I^{\blacktriangle}_{A_1}(X) = (1, 2, 2, 2, 2, 1, 1, 2)^T$,

$$I^{\blacktriangledown}_{A_1} = (3, 3, 2, 3, 2, 3, 3, 3)^T,$$
$$I_{A_1}(X) = I^{\blacktriangle}_{A_1}(X)/.I^{\blacktriangledown}_{A_1} = (\frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, 1, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})^T.$$

According to Definition 3, we can calculate two cut matrices $I^{\downarrow}_{A_1}(X)$ and $I^{\uparrow}_{A_1}(X)$ as follows:

$$h^{1\downarrow}_{A_1} = 0,\ h^{2\downarrow}_{A_1} = 0,\ h^{3\downarrow}_{A_1} = 1,\ h^{4\downarrow}_{A_1} = 0,$$
$$h^{5\downarrow}_{A_1} = 1,\ h^{6\downarrow}_{A_1} = 0,\ h^{7\downarrow}_{A_1} = 0,\ h^{8\downarrow}_{A_1} = 0,$$
$$h^{1\uparrow}_{A_1} = 1,\ h^{2\uparrow}_{A_1} = 1,\ h^{3\uparrow}_{A_1} = 1,\ h^{4\uparrow}_{A_1} = 1,$$
$$h^{5\uparrow}_{A_1} = 1,\ h^{6\uparrow}_{A_1} = 1,\ h^{7\uparrow}_{A_1} = 1,\ h^{8\uparrow}_{A_1} = 1,$$

so $I^{\downarrow}_{A_1}(X) = (0, 0, 1, 0, 1, 0, 0, 0)^T$,

$$I^{\uparrow}_{A_1}(X) = (1, 1, 1, 1, 1, 1, 1, 1)^T;$$

Similarly, we can get cut matrices of X under $A2, A3, A4$ as follows:

$$I^{\downarrow}_{A_2}(X) = (0, 0, 0, 0, 1, 0, 1, 1)^T,$$
$$I^{\uparrow}_{A_2}(X) = (1, 1, 1, 1, 1, 1, 1, 1)^T,$$
$$I^{\downarrow}_{A_3}(X) = (0, 0, 0, 0, 1, 0, 1, 1)^T,$$
$$I^{\uparrow}_{A_3}(X) = (1, 1, 1, 1, 1, 1, 1, 1)^T,$$
$$I^{\downarrow}_{A_4}(X) = (0, 0, 0, 0, 1, 0, 0, 1)^T,$$
$$I^{\uparrow}_{A_4}(X) = (1, 0, 1, 1, 1, 0, 1, 1)^T.$$

Then $F(\underline{\sum_{k=1}^{m} A_k}^{O,0.8}(X)) = (0, 0, 1, 0, 1, 0, 1, 1)^T$,

$$F(\overline{\sum_{k=1}^{m} A_k}^{O,0.2}(X)) = (1, 1, 1, 1, 1, 1, 1, 1)^T,$$
$$F(\underline{\sum_{k=1}^{m} A_k}^{P,0.8}(X)) = (0, 0, 0, 0, 1, 0, 0, 0)^T,$$
$$F(\overline{\sum_{k=1}^{m} A_k}^{P,0.2}(X)) = (1, 0, 1, 1, 1, 0, 1, 1)^T.$$

So $\underline{\sum_{k=1}^{m} A_k}^{O,0.8}(X) = \{x_3, x_5, x_7, x_8\}$,

$$\overline{\sum_{k=1}^{m} A_k}^{O,0.2}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\},$$
$$\underline{\sum_{k=1}^{m} A_k}^{P,0.8}(X) = \{x_5\},$$
$$\overline{\sum_{k=1}^{m} A_k}^{P,0.2}(X) = \{x_1, x_3, x_4, x_5, x_7, x_8\}.$$

B. UPDATING MULTIGRANULATION DECISION-THEORETIC ROUGH APPROXIMATIONS WHILE INCREASING OBJECTS

In the previous section, we introduced multigranulation decision-theoretic rough approximations based on the matrix. In this section, we will introduce dynamic multigranulation decision-theoretic rough approximations with increasing objects.

Definition 4: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$. adding n^+ new objects to U, assume that the new universe is $U' = U \cup U^+$, where $U^+ = \{x_{n+1}, x_{n+2}, \ldots, x_{n+n^+}\}$. $\forall X^+ \subseteq U^+$, the characteristic function $F^+(X^+) = (f_{n+1}, f_{n+2}, \ldots, f_{n+n^+})^T$ of X^+ is constructed as:

$$f_i = \begin{cases} 1, & x_i \in X^+ \\ 0, & x_i \notin X^+ \end{cases} \quad i = n+1, n+2, \ldots, n+n^+$$

Theorem 3: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $U' = U \cup U^+$, $\forall X^+ \subseteq U^+$, if $X' = X \cup X^+$, then $F'(X') = [f_i']_{(n+n^+)\times 1}$ is the characteristic function of X' and the following result hold:

$$F'(X') = \begin{pmatrix} F(X) \\ F^+(X^+) \end{pmatrix}$$

Proof: It's obvious the theorem holds by Definition 4.

Definition 5: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$, R_{A_k} is the indiscernibility relation. Two incremental relation matrices $S_{A_k} = [s_{A_k}^{ij}]_{n\times n^+}$ and $T_{A_k} = [t_{A_k}^{ij}]_{n^+\times n^+}$ are constructed as:

$$s_{A_k}^{ij} = \begin{cases} 1, & (x_i, x_j) \in R_{A_k} \quad i = 1, 2, \ldots, n \\ 0, & (x_i, x_j) \notin R_{A_k} \quad j = n+1, n+2, \ldots, n+n^+ \end{cases}$$

$$t_{A_k}^{ij} = \begin{cases} 1, & (x_i, x_j) \in R_{A_k} \quad i = n+1, n+2, \ldots, n+n^+ \\ 0, & (x_i, x_j) \notin R_{A_k} \quad j = n+1, n+2, \ldots, n+n^+ \end{cases}$$

Theorem 4: Let $IS = \langle U, AT, V, f\rangle$ is an information system, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$, $S_{A_k} = [s_{A_k}^{ij}]_{n\times n^+}$ and $T_{A_k} = [t_{A_k}^{ij}]_{n^+\times n^+}$, $M'_{A_k} = [m_{A_k}^{ij}]_{(n+n^+)\times(n+n^+)}$ is the new relation matrix and the following result hold:

$$M'_{A_k} = \begin{pmatrix} M_{A_k} & S_{A_k} \\ S_{A_k}^T & T_{A_k} \end{pmatrix}$$

where $S_{A_k}^T$ is the transport matrix of S_{A_k}.

Proof: It's obvious the theorem holds by Definition 5.

Theorem 5: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$, $S_{A_k} = [s_{A_k}^{ij}]_{n\times n^+}$, $T_{A_k} = [t_{A_k}^{ij}]_{n^+\times n^+}$, $F^+(X^+) = (f_{n+1}, f_{n+2}, \ldots, f_{n+n^+})^T$, $Q^+ = (1, 1, \ldots, 1)^T$ is a $n^+ \times 1$ column vector, $I_{A_k}^{'\blacktriangle}(X') = [u_{A_k}^{'i}]_{(n+n^+)\times 1}$ and $I_{A_k}^{'\blacktriangledown} = [v_{A_k}^{'i}]_{(n+n^+)\times 1}$ are two new intermediate matrices of A_k, $\forall X \subseteq U$, if $X' = X \cup X^+$, then the following results hold:

$$I_{A_k}^{'\blacktriangle}(X') = \begin{pmatrix} I_{A_k}^{\blacktriangle}(X) + S_{A_k} \times F^+(X^+) \\ S_{A_k}^T \times F(X) + T_{A_k} \times F^+(X^+) \end{pmatrix}$$

$$I_{A_k}^{'\blacktriangledown} = \begin{pmatrix} I_{A_k}^{\blacktriangledown} + S_{A_k} \times Q^+ \\ S_{A_k}^T \times Q + T_{A_k} \times Q^+ \end{pmatrix}$$

where

$$u_{A_k}^{'i} = \begin{cases} u_{A_k}^i + \sum_{j=1}^{n^+} s_{A_k}^{ij} f_{n+j} \\ \qquad i = 1, 2, \ldots, n \\ \sum_{j=1}^{n} s_{A_k}^{j(i-n)} f_j + \sum_{j=1}^{n^+} t_{A_k}^{(i-n)j} f_{n+j} \\ \qquad i = n+1, n+2, \ldots, n+n^+ \end{cases}$$

$$v_{A_k}^{'i} = \begin{cases} v_{A_k}^i + \sum_{j=1}^{n^+} s_{A_k}^{ij} \\ \qquad i = 1, 2, \ldots, n \\ \sum_{j=1}^{n} s_{A_k}^{j(i-n)} + \sum_{j=1}^{n^+} t_{A_k}^{(i-n)j} \\ \qquad i = n+1, n+2, \ldots, n+n^+ \end{cases}$$

Proof: It's obvious the theorem holds by Definition 1.

Example 2 (Continued Example 1):
where $U^+ = \{x_9, x_{10}\}$, $X^+ = \{x_9\}$.

$$U' = U \cup U^+ = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$X' = X \cup X^+ = \{x_3, x_4, x_5, x_7, x_8, x_9\}.$$

TABLE 2. An information system with added objects.

U'	a_1	a_2	a_3	a_4
x_1	1	1	1	3
x_2	1	1	1	3
x_3	2	1	1	3
x_4	3	2	1	3
x_5	2	3	2	2
x_6	1	2	1	1
x_7	1	3	2	3
x_8	3	3	2	2
x_9	1	1	2	2
x_{10}	3	3	1	1

According to Theorem 3, we can get the characteristic function of X':

$$F'(X') = \begin{pmatrix} F(X) \\ F^+(X^+) \end{pmatrix} = (0, 0, 1, 1, 1, 0, 1, 1, 1, 0)^T$$

According to Theorem 4, we can calculate incremental relation matrices S_{A_1} and T_{A_1} as follows:

$$S_{A_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}^T$$

$$T_{A_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

According to Theorem 5, we can calculate new intermediate matrices $I_{A_1}^{'\blacktriangle}(X')$ and $I_{A_1}^{'\blacktriangledown}$ as follows:

$$u_{A_1}^{'1} = u_{A_1}^1 + \sum_{j=1}^{2} s_{A_1}^{1j} f_{8+j} = 2,$$

$$v_{A_1}^{'1} = v_{A_1}^1 + \sum_{j=1}^{2} s_{A_1}^{1j} = 4,$$

$$
\begin{aligned}
u'^{2}_{A_1} &= u^{2}_{A_1} + \sum\nolimits_{j=1}^{2} s^{2j}_{A_1} f_{8+j} = 2,\\
v'^{2}_{A_1} &= v^{2}_{A_1} + \sum\nolimits_{j=1}^{2} s^{2j}_{A_1} = 4,\\
u'^{3}_{A_1} &= u^{3}_{A_1} + \sum\nolimits_{j=1}^{2} s^{3j}_{A_1} f_{8+j} = 2,\\
v'^{3}_{A_1} &= v^{3}_{A_1} + \sum\nolimits_{j=1}^{2} s^{3j}_{A_1} = 2,\\
u'^{4}_{A_1} &= u^{4}_{A_1} + \sum\nolimits_{j=1}^{2} s^{4j}_{A_1} f_{8+j} = 2,\\
v'^{4}_{A_1} &= v^{4}_{A_1} + \sum\nolimits_{j=1}^{2} s^{4j}_{A_1} = 4,\\
u'^{5}_{A_1} &= u^{5}_{A_1} + \sum\nolimits_{j=1}^{2} s^{5j}_{A_1} f_{8+j} = 2,\\
v'^{5}_{A_1} &= v^{5}_{A_1} + \sum\nolimits_{j=1}^{2} s^{5j}_{A_1} = 2,\\
u'^{6}_{A_1} &= u^{6}_{A_1} + \sum\nolimits_{j=1}^{2} s^{6j}_{A_1} f_{8+j} = 2,\\
v'^{6}_{A_1} &= v^{6}_{A_1} + \sum\nolimits_{j=1}^{2} s^{6j}_{A_1} = 4,\\
u'^{7}_{A_1} &= u^{7}_{A_1} + \sum\nolimits_{j=1}^{2} s^{7j}_{A_1} f_{8+j} = 2,\\
v'^{7}_{A_1} &= v^{7}_{A_1} + \sum\nolimits_{j=1}^{2} s^{7j}_{A_1} = 4,\\
u'^{8}_{A_1} &= u^{8}_{A_1} + \sum\nolimits_{j=1}^{2} s^{8j}_{A_1} f_{8+j} = 2,\\
v'^{8}_{A_1} &= v^{8}_{A_1} + \sum\nolimits_{j=1}^{2} s^{8j}_{A_1} = 4;\\
u'^{9}_{A_1} &= \sum\nolimits_{j=1}^{8} s^{j1}_{A_1} f_j + \sum\nolimits_{j=1}^{2} t^{1j}_{A_1} f_{8+j} = 2,\\
v'^{9}_{A_1} &= \sum\nolimits_{j=1}^{8} s^{j1}_{A_1} + \sum\nolimits_{j=1}^{2} t^{1j}_{A_1} = 4,\\
u'^{10}_{A_1} &= \sum\nolimits_{j=1}^{8} s^{j2}_{A_1} f_j + \sum\nolimits_{j=1}^{2} t^{2j}_{A_1} f_{8+j} = 2,\\
v'^{10}_{A_1} &= \sum\nolimits_{j=1}^{8} s^{j2}_{A_1} + \sum\nolimits_{j=1}^{2} t^{2j}_{A_1} = 4;
\end{aligned}
$$

$$
\begin{aligned}
\text{so } I'^{\blacktriangle}_{A_1}(X') &= (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)^T\\
I'^{\blacktriangledown}_{A_1} &= (4, 4, 2, 4, 2, 4, 4, 4, 4, 4)^T\\
I'_{A_1}(X') &= (\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T
\end{aligned}
$$

According to Definition 3, we can calculate two new cut matrices $I^{\downarrow}_{A_1}(X')$ and $I^{\uparrow}_{A_1}(X')$ as follows:

$$
\begin{aligned}
&h^{1\downarrow}_{A_1} = 0, \quad h^{2\downarrow}_{A_1} = 0,\ h^{3\downarrow}_{A_1} = 1,\ h^{4\downarrow}_{A_1} = 0,\ h^{5\downarrow}_{A_1} = 1,\\
&h^{6\downarrow}_{A_1} = 0, \quad h^{7\downarrow}_{A_1} = 0, h^{8\downarrow}_{A_1} = 0,\ h^{9\downarrow}_{A_1} = 0, h^{10\downarrow}_{A_1} = 0,\\
&h^{1\uparrow}_{A_1} = 1, \quad h^{2\uparrow}_{A_1} = 1,\ h^{3\uparrow}_{A_1} = 1,\ h^{4\uparrow}_{A_1} = 1, h^{5\uparrow}_{A_1} = 1,\\
&h^{6\uparrow}_{A_1} = 1, \quad h^{7\uparrow}_{A_1} = 1,\ h^{8\uparrow}_{A_1} = 1,\ h^{9\uparrow}_{A_1} = 1, h^{10\uparrow}_{A_1} = 1,
\end{aligned}
$$

$$
\begin{aligned}
\text{so } I^{\downarrow}_{A_1}(X') &= (0, 0, 1, 0, 1, 0, 0, 0, 0, 0)^T,\\
I^{\uparrow}_{A_1}(X') &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T;
\end{aligned}
$$

Similarly, we can get new cut matrices of X under $A2, A3, A4$ as follows:

$$
\begin{aligned}
I^{\downarrow}_{A_2}(X') &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T,\\
I^{\uparrow}_{A_2}(X') &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T,\\
I^{\downarrow}_{A_3}(X') &= (0, 0, 0, 0, 1, 0, 1, 1, 1, 0)^T,\\
I^{\uparrow}_{A_3}(X') &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T,\\
I^{\downarrow}_{A_4}(X') &= (0, 0, 0, 0, 1, 0, 0, 1, 1, 0)^T,\\
I^{\uparrow}_{A_4}(X') &= (1, 0, 1, 1, 1, 0, 1, 1, 1, 0)^T.
\end{aligned}
$$

$$
\begin{aligned}
\text{Then } F(\underline{\sum\nolimits_{k=1}^{4} A_k}^{O,0.8}(X')) &= (0, 0, 1, 0, 1, 0, 1, 1, 1, 0)^T,\\
F(\overline{\sum\nolimits_{k=1}^{4} A_k}^{O,0.2}(X')) &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T,\\
F(\underline{\sum\nolimits_{k=1}^{4} A_k}^{P,0.8}(X')) &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T,\\
F(\overline{\sum\nolimits_{k=1}^{4} A_k}^{P,0.2}(X')) &= (1, 0, 1, 1, 1, 0, 1, 1, 1, 0)^T.
\end{aligned}
$$

$$
\begin{aligned}
\text{So } \underline{\sum\nolimits_{k=1}^{4} A_k}^{O,0.8}(X') &= \{x_3, x_5, x_7, x_8, x_9\},\\
\overline{\sum\nolimits_{k=1}^{4} A_k}^{O,0.2}(X') &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7,\\
&\qquad x_8, x_9, x_{10}\},\\
\underline{\sum\nolimits_{k=1}^{4} A_k}^{P,0.8}(X') &= \emptyset,\\
\overline{\sum\nolimits_{k=1}^{4} A_k}^{P,0.2}(X') &= \{x_1, x_3, x_4, x_5, x_7, x_8, x_9\}.
\end{aligned}
$$

C. UPDATING MULTIGRANULATION DECISION-THEORETIC ROUGH APPROXIMATIONS WHILE DECREASING OBJECTS

In the previous section, we introduced multigranulation decision-theoretic rough approximations based on the matrix. In this section, we will introduce dynamic multigranulation decision-theoretic rough approximations with decreasing objects.

Definition 6: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$. deleting n^- objects from U, assume that the new universe is $U' = U - U^-$, where $U^- = \{x_{n-n^-+1}, x_{n-n^-+2}, \ldots, x_n\}$. $\forall X^- \subseteq U^-$, the characteristic function $F^-(X^-) = (f_{n-n^-+1}, f_{n-n^-+2}, \ldots, f_n)^T$ of X^- is constructed as:

$$
f_i = \begin{cases} 1, & x_i \in X^- \\ 0, & x_i \notin X^- \end{cases} \quad i = n - n^- + 1, n - n^- + 2, \ldots, n
$$

Theorem 6: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $U' = U - U^-$, $\forall X \subseteq U$, if $X^- = X \cap U^-$ and $X' = X - X^-$, then $F'(X') = [f'_i]_{(n-n^-)\times 1}$ is the characteristic function of X' and the following result hold:

$$
F(X) = \begin{pmatrix} F'(X') \\ F^-(X^-) \end{pmatrix}
$$

Proof: It's obvious the theorem holds by Definition 6.

Definition 7: Let $IS = \langle U, AT, V, f\rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$, R_{A_k} is the indiscernibility relation. Two incremental relation matrices $S_{A_k} = [s^{ij}_{A_k}]_{(n-n^-)\times n^-}$ and

$T_{A_k} = [t_{A_k}^{ij}]_{n^- \times n^-}$ are constructed as:

$$s_{A_k}^{ij} = \begin{cases} 1, & (x_i, x_j) \in R_{A_k} \quad i = 1, 2, \ldots, n - n^- \\ 0, & (x_i, x_j) \notin R_{A_k} \quad j = n - n^- + 1, \ldots, n \end{cases}$$

$$t_{A_k}^{ij} = \begin{cases} 1, & (x_i, x_j) \in R_{A_k} \quad i = n - n^- + 1, \ldots, n \\ 0, & (x_i, x_j) \notin R_{A_k} \quad j = n - n^- + 1, \ldots, n \end{cases}$$

Theorem 7: Let $IS = \langle U, AT, V, f \rangle$ is an information system, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$, $S_{A_k} = [s_{A_k}^{ij}]_{(n-n^-)\times n^-}$ and $T_{A_k} = [t_{A_k}^{ij}]_{n^- \times n^-}$, $M'_{A_k} = [m_{A_k}^{ij}]_{(n-n^-)\times(n-n^-)}$ is the new relation matrix and the following result hold:

$$M_{A_k} = \begin{pmatrix} M'_{A_k} & S_{A_k} \\ S_{A_k}^T & T_{A_k} \end{pmatrix}$$

where $S_{A_k}^T$ is the transport matrix of S_{A_k}.

Proof: It's obvious the theorem holds by Definition 7.

Theorem 8: Let $IS = \langle U, AT, V, f \rangle$ is an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, $A_1, A_2, \ldots, A_m \subseteq AT$, $k \in \{1, 2, \ldots, m\}$, $S_{A_k} = [s_{A_k}^{ij}]_{(n-n^-)\times n^-}$, $T_{A_k} = [t_{A_k}^{ij}]_{n^- \times n^-}$, $F^-(X^-) = (f_{n-n^-+1}, f_{n-n^-+2}, \ldots, f_n)^T$, $Q^- = (1, 1, \ldots, 1)^T$ is a $n^- \times 1$ column vector, $Q' = (1, 1, \ldots, 1)^T$ is a $(n - n^-) \times 1$ column vector, $I'^{\blacktriangle}_{A_k}(X') = [u'^i_{A_k}]_{(n-n^-)\times 1}$ and $I'^{\blacktriangledown}_{A_k} = [v'^i_{A_k}]_{(n-n^-)\times 1}$ are two new intermediate matrices of A_k, $\forall X \subseteq U$, if $X^- = X \cap U^-$ and $X' = X - X^-$, then the following results hold:

$$I^{\blacktriangle}_{A_k}(X) = \begin{pmatrix} I'^{\blacktriangle}_{A_k}(X') + S_{A_k} \times F^-(X^-) \\ S_{A_k}^T \times F'(X') + T_{A_k} \times F^-(X^-) \end{pmatrix}$$

$$I^{\blacktriangledown}_{A_k} = \begin{pmatrix} I'^{\blacktriangledown}_{A_k} + S_{A_k} \times Q^- \\ S_{A_k}^T \times Q' + T_{A_k} \times Q^- \end{pmatrix}$$

where

$$u^i_{A_k} = \begin{cases} u'^i_{A_k} + \sum_{j=1}^{n^-} s_{A_k}^{ij} f_{n-n^-+j} \\ \qquad i = 1, 2, \ldots, n - n^- \\ \sum_{j=1}^{n-n^-} s_{A_k}^{j(i-n+n^-)} f_j + \sum_{j=1}^{n^-} t_{A_k}^{(i-n+n^-)j} f_{n-n^-+j} \\ \qquad i = n - n^- + 1, n - n^- + 2, \ldots, n \end{cases}$$

$$v^i_{A_k} = \begin{cases} v'^i_{A_k} + \sum_{j=1}^{n^-} s_{A_k}^{ij} \\ \qquad i = 1, 2, \ldots, n - n^- \\ \sum_{j=1}^{n-n^-} s_{A_k}^{j(i-n+n^-)} + \sum_{j=1}^{n^-} t_{A_k}^{(i-n+n^-)j} \\ \qquad i = n - n^- + 1, n - n^- + 2, \ldots, n \end{cases}$$

Proof: It's obvious the theorem holds by Definition 1.

Example 3 (Continued Example 1): where $U^- = \{x_7, x_8\}$.

$$\begin{aligned} U' &= U - U^- = \{x_1, x_2, x_3, x_4, x_5, x_6\}, \\ X^- &= U^- \cap X = \{x_7, x_8\}, \\ X' &= X - X^- = \{x_3, x_4, x_5\}. \end{aligned}$$

TABLE 3. An information system with added objects.

U'	a_1	a_2	a_3	a_4
x_1	1	1	1	3
x_2	1	1	1	3
x_3	2	1	1	3
x_4	3	2	1	3
x_5	2	3	2	2
x_6	1	2	1	1

According to Theorem 6, we can get the characteristic function of X^-:

$$F^-(X^-) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

According to Theorem 7, we can calculate incremental relation matrices S_{A_1} and T_{A_1} as follows:

$$S_{A_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}^T$$

$$T_{A_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

According to Theorem 8, we can calculate new intermediate matrices $I'^{\blacktriangle}_{A_1}(X')$ and $I'^{\blacktriangledown}_{A_1}$ as follows:

$$\begin{aligned} u'^1_{A_1} &= u^1_{A_1} - \sum_{j=1}^{2} s^{1j}_{A_1} f_{6+j} = 0, \\ v'^1_{A_1} &= v^1_{A_1} - \sum_{j=1}^{2} s^{1j}_{A_1} = 2, \\ u'^2_{A_1} &= u^2_{A_1} - \sum_{j=1}^{2} s^{2j}_{A_1} f_{6+j} = 2, \\ v'^2_{A_1} &= v^2_{A_1} - \sum_{j=1}^{2} s^{2j}_{A_1} = 4, \\ u'^3_{A_1} &= u^3_{A_1} - \sum_{j=1}^{2} s^{3j}_{A_1} f_{6+j} = 2, \\ v'^3_{A_1} &= v^3_{A_1} - \sum_{j=1}^{2} s^{3j}_{A_1} = 2, \\ u'^4_{A_1} &= u^4_{A_1} - \sum_{j=1}^{2} s^{4j}_{A_1} f_{6+j} = 2, \\ v'^4_{A_1} &= v^4_{A_1} - \sum_{j=1}^{2} s^{4j}_{A_1} = 4, \\ u'^5_{A_1} &= u^5_{A_1} - \sum_{j=1}^{2} s^{5j}_{A_1} f_{6+j} = 2, \\ v'^5_{A_1} &= v^5_{A_1} - \sum_{j=1}^{2} s^{5j}_{A_1} = 2, \\ u'^6_{A_1} &= u^6_{A_1} - \sum_{j=1}^{2} s^{6j}_{A_1} f_{6+j} = 2, \\ v'^6_{A_1} &= v^6_{A_1} - \sum_{j=1}^{2} s^{6j}_{A_1} = 4; \\ \text{so} I'^{\blacktriangle}_{A_1}(X') &= (0, 1, 2, 1, 2, 0)^T \\ I'^{\blacktriangledown}_{A_1} &= (2, 2, 2, 2, 2, 2)^T \\ I'_{A_1}(X') &= (0, \frac{1}{2}, 1, \frac{1}{2}, 1, 0)^T \end{aligned}$$

According to Definition 3, we can calculate two new cut matrices $I^{\downarrow}_{A_1}(X')$ and $I^{\uparrow}_{A_1}(X')$ as follows:

$$h^{1\downarrow}_{A_1} = 0, \; h^{2\downarrow}_{A_1} = 0, h^{3\downarrow}_{A_1} = 1, h^{4\downarrow}_{A_1} = 0,$$

$$
\begin{aligned}
&h_{A_1}^{5\downarrow} = 1,\ h_{A_1}^{6\downarrow} = 0,\\
&h_{A_1}^{1\uparrow} = 0,\ h_{A_1}^{2\uparrow} = 1, h_{A_1}^{3\uparrow} = 1, h_{A_1}^{4\uparrow} = 1,\\
&h_{A_1}^{5\uparrow} = 1,\ h_{A_1}^{6\uparrow} = 0,\\
\text{so } &I_{A_1}^{\downarrow}(X = (0, 0, 1, 0, 1, 0)^T,\\
&I_{A_1}^{\uparrow}(X') = (0, 1, 1, 1, 1, 0)^T;
\end{aligned}
$$

Similarly, we can get new cut matrices of X under $A2, A3, A4$ as follows:

$$
\begin{aligned}
I_{A_2}^{\downarrow}(X') &= (0, 0, 0, 0, 1, 0)^T,\\
I_{A_2}^{\uparrow}(X') &= (1, 1, 1, 1, 1, 1)^T,\\
I_{A_3}^{\downarrow}(X') &= (0, 0, 0, 0, 1, 0)^T,\\
I_{A_3}^{\uparrow}(X') &= (1, 1, 1, 1, 1, 1)^T,\\
I_{A_4}^{\downarrow}(X') &= (0, 0, 0, 0, 1, 0)^T,\\
I_{A_4}^{\uparrow}(X') &= (1, 0, 1, 1, 1, 0)^T.
\end{aligned}
$$

$$
\begin{aligned}
\text{Then } F(\underline{\sum_{k=1}^{4} A_k}^{O,0.8}(X')) &= (0, 0, 1, 0, 1, 0)^T,\\
F(\overline{\sum_{k=1}^{4} A_k}^{O,0.2}(X')) &= (1, 1, 1, 1, 1, 1)^T,\\
F(\underline{\sum_{k=1}^{4} A_k}^{P,0.8}(X')) &= (0, 0, 0, 0, 1, 0)^T,\\
F(\overline{\sum_{k=1}^{4} A_k}^{P,0.2}(X')) &= (0, 0, 1, 1, 1, 0)^T.\\
\text{So } \underline{\sum_{k=1}^{4} A_k}^{O,0.8}(X') &= \{x_3, x_5\},\\
\overline{\sum_{k=1}^{4} A_k}^{O,0.2}(X') &= \{x_1, x_2, x_3, x_4, x_5, x_6\},\\
\underline{\sum_{k=1}^{4} A_k}^{P,0.8}(X') &= \{x_5\},\\
\overline{\sum_{k=1}^{4} A_k}^{P,0.2}(X') &= \{x_3, x_4, x_5\}.
\end{aligned}
$$

IV. THE ALGORITHMS FOR UPDATING MULTIGRANULATION DECISION-THEORETIC APPROXIMATIONS WHILE ADDING OR DELETING OBJECTS

In this section, we give a static algorithm and two fast algorithms for updating multigranulation decision-theoretic rough approximations.

For Algorithm 1, Step 1 compute the characteristic function and it's time complexity is $O(|U|)$; Steps 2-12 calculate the relation matrix and it's time complexity is $O(m|U|^2)$; Steps 13-19 calculate two intermediate matrices and the basic matrix, it's time complexity is $O(m|U|^2)$; Steps 20-28 and Steps 29-37 calculate two cut matrices and their time complexity are $O(m|U|)$; Step 38 and Step 39 calculate the characteristic function of approximations and their time complexity are both $O(m|U|)$. Hence, the total time complexity of Algorithm 1 is $O(m|U|^2)$.

For Algorithm 2, Step 1 update the universe and the set X, it's time complexity is $O(1)$; Step 2 calculate the added characteristic function and it's time complexity is $O(|U^+|)$; Step 3 update the characteristic function and it's time complexity is $O(1)$; Steps 4-14 and Steps 15-25 calculate two intermediate matrices and their time complexity are $O(m|U||U^+|)$ and $O(m|U^+|^2)$ respectively; Steps 26-37 update two intermediate matrices and basic matrix, it's time complexity is $O(m|U'||U^+|)$; Steps 38-46 and Steps 47-55 calculate two new cut matrices and their time complexity are both $O(m|U'|)$; Step 56 and Step 57 update the characteristic function of approximations and their time complexity are both $O(m|U'|)$. Hence, the total time complexity of Algorithm 2 is $O(m|U'||U^+|)$.

Algorithm 1 Static Algorithm for Computing Multigranulation Decision-Theoretic Rough Approximations

Input: An information system $IS = \langle U, AT, V, f\rangle$, the set X, two thresholds α and β.

Output: $\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)$, $\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X)$.

1 **Compute** $F(X)$;
2 **for** k = 1 to m
3   **for** i = 1 to n
4   **for** j = 1 to n
5   **if** $(x_i, x_j) \in R_{A_k}$ **then**
6                 $m_{A_k}^{ij} = 1$;
7   **else** %Compute M_{A_k}
8                 $m_{A_k}^{ij} = 0$;
9   **end**
10   **end**
11   **end**
12 **end**
13 **for** k = 1 to m
14   **for** i = 1 to n
15         $u_{A_k}^{i} = \sum_{j=1}^{n} m_{A_k}^{ij} f_j$;
16         $v_{A_k}^{i} = \sum_{j=1}^{n} m_{A_k}^{ij}$; %Compute $I_{A_k}^{\blacktriangle}(X)$,$I_{A_k}^{\blacktriangledown}$,$I_{A_k}(X)$
17         $h_{A_k}^{i} = u_{A_k}^{i}/v_{A_k}^{i}$;
18   **end**
19 **end**
20 **for** k = 1 to m
21   **for** i = 1 to n
22   **if** $h_{A_k}^{i} \geq \alpha$ **then**
23             $h_{A_k}^{i\downarrow} = 1$;
24   **else** %Compute $I_{A_k}^{\downarrow}(X)$
25             $h_{A_k}^{i\downarrow} = 0$;
26   **end**
27   **end**
28 **end**
29 **for** k = 1 to m
30   **for** i = 1 to n
31   **if** $h_{A_k}^{i} > \beta$ **then**
32             $h_{A_k}^{i\uparrow} = 1$;
33   **else** %Compute $I_{A_k}^{\uparrow}(X)$
34             $h_{A_k}^{i\uparrow} = 0$;
35   **end**
36   **end**
37 **end**
38 **Compute** $F(\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)) = max_{k=1}^{m}(I_{A_k}^{\downarrow}(X))$, $F(\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)) = max_{k=1}^{m}(I_{A_k}^{\uparrow}(X))$;
39 **Compute** $F(\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X)) = min_{k=1}^{m}(I_{A_k}^{\downarrow}(X))$, $F(\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X)) = min_{k=1}^{m}(I_{A_k}^{\uparrow}(X))$;
40 **Return** $\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X)$, $\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X)$.

For Algorithm 3, Step 1 update the universe and the set X, it's time complexity is $O(1)$; Step 2 update the

Algorithm 2 The Incremental Algorithm for Updating Approximations of MG-DTRS While Adding Objects

Input: An information system $IS = \langle U, AT, V, f \rangle$, the target concept X, two thresholds α and β, U^+, X^+, $F(X)$, Q^+, $I^{\blacktriangle}_{A_k}(X)$, $I^{\blacktriangledown}_{A_k}$.
Output: $\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X')$, $\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X')$.

$U' = U \cup U^+, X' = X \cup X^+$;
Compute $F^+(X^+)$;
Update $F'(X') = (F(X), F^+(X^+))^T$;
for k = 1 to m
  for i = 1 to n
    for j = n+1 to n+n$^+$
      if $(x_i, x_j) \in R_{A_k}$ **then**
        $s^{ij}_{A_k} = 1$;
      else %Compute S_{A_k}
        $s^{ij}_{A_k} = 0$;
      end
    end
  end
end
for k = 1 to m
  for i = n+1 to n+n$^+$
    for j = n+1 to n+n$^+$
      if $(x_i, x_j) \in R_{A_k}$ **then**
        $t^{ij}_{A_k} = 1$;
      else %Compute T_{A_k}
        $t^{ij}_{A_k} = 0$;
      end
    end
  end
end
for k = 1 to m
  for i = 1 to n
    $u'^{i}_{A_k} = u^{i}_{A_k} + \sum_{j=1}^{n^+} s^{ij}_{A_k} f_{n+j}$;
    $v'^{i}_{A_k} = v^{i}_{A_k} + \sum_{j=1}^{n^+} s^{ij}_{A_k}$;
    $h'^{i}_{A_k} = u'^{i}_{A_k} / v'^{i}_{A_k}$;
  end %Compute $I'^{\blacktriangle}_{A_k}(X')$, $I'^{\blacktriangledown}_{A_k}$, $I'_{A_k}(X')$
  for i = n+1 to n+n$^+$
    $u'^{i}_{A_k} = \sum_{j=1}^{n} s^{j(i-n)}_{A_k} f_j + \sum_{j=1}^{n^+} t^{(i-n)j}_{A_k} f_{n+j}$;
    $v'^{i}_{A_k} = \sum_{j=1}^{n} s^{j(i-n)}_{A_k} + \sum_{j=1}^{n^+} t^{(i-n)j}_{A_k}$;
    $h'^{i}_{A_k} = u'^{i}_{A_k} / v'^{i}_{A_k}$;
  end
end
for k = 1 to m
  for i = 1 to n+n$^+$
    if $h^{i}_{A_k} \geq \alpha$ **then**
      $h^{i\downarrow}_{A_k} = 1$;
    else
      $h^{i\downarrow}_{A_k} = 0$; %Compute $I'^{\downarrow}_{A_k}(X')$
    end
  end
end
for k = 1 to m
  for i = 1 to n+n$^+$
    if $h^{i}_{A_k} > \beta$ **then**
      $h^{i\uparrow}_{A_k} = 1$;
    else
      $h^{i\uparrow}_{A_k} = 0$; %Compute $I'^{\uparrow}_{A_k}(X')$
    end
  end
end
Compute $F'(\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X')) = max_{k=1}^{m}(I'^{\downarrow}_{A_k}(X'))$,
$F'(\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X')) = max_{k=1}^{m}(I'^{\uparrow}_{A_k}(X'))$;
Compute $F'(\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X')) = min_{k=1}^{m}(I'^{\downarrow}_{A_k}(X'))$,
$F'(\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X')) = min_{k=1}^{m}(I'^{\uparrow}_{A_k}(X'))$;
Return $\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X')$, $\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X')$.

Algorithm 3 The Incremental Algorithm for Updating Approximations of MG-DTRS While Deleting Objects

Input: An information system $IS = \langle U, AT, V, f \rangle$, the target concept X, two thresholds α and β, U^-, X^-, $F(X)$, Q^-, $I^{\blacktriangle}_{A_k}(X)$, $I^{\blacktriangledown}_{A_k}$.
Output: $\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X')$, $\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X')$.

1 $U' = U - U^-, X' = X - X^-$;
2 **Update** $F^-(X^-)$ by $F(X)$;
3 **Compute** S_{A_k} by M_{A_k};
4 **for** k = 1 to m
5   **for** i = 1 to n-n$^-$
6     $u'^{i}_{A_k} = u^{i}_{A_k} - \sum_{j=1}^{n^-} s^{ij}_{A_k} f_{n-n^-+j}$;
7     $v'^{i}_{A_k} = v^{i}_{A_k} - \sum_{j=1}^{n^-} s^{ij}_{A_k}$;
8     $h'^{i}_{A_k} = u'^{i}_{A_k} / v'^{i}_{A_k}$;
9   **end** %Compute $I'^{\blacktriangle}_{A_k}(X')$, $I'^{\blacktriangledown}_{A_k}$, $I'_{A_k}(X')$
10 **end**
11 **for** k = 1 to m
12   **for** i = 1 to n-n$^-$
13   **if** $h^{i}_{A_k} \geq \alpha$ **then**
14       $h^{i\downarrow}_{A_k} = 1$;
15   **else**
16       $h^{i\downarrow}_{A_k} = 0$; %Compute $I'^{\downarrow}_{A_k}(X')$
17   **end**
18   **end**
19 **end**
20 **for** k = 1 to m
21   **for** i = 1 to n-n$^-$
22   **if** $h^{i}_{A_k} > \beta$ **then**
23       $h^{i\uparrow}_{A_k} = 1$;
24   **else**
25       $h^{i\uparrow}_{A_k} = 0$; %Compute $I'^{\uparrow}_{A_k}(X')$
26   **end**
27   **end**
28 **end**
29 **Compute** $F'(\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X')) = max_{k=1}^{m}(I'^{\downarrow}_{A_k}(X'))$,
$F'(\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X')) = max_{k=1}^{m}(I'^{\uparrow}_{A_k}(X'))$;
30 **Compute** $F'(\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X')) = min_{k=1}^{m}(I'^{\downarrow}_{A_k}(X'))$,
$F'(\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X')) = min_{k=1}^{m}(I'^{\uparrow}_{A_k}(X'))$;
31 **Return** $\underline{\sum_{k=1}^{m} A_k}^{O,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{O,\beta}(X')$, $\underline{\sum_{k=1}^{m} A_k}^{P,\alpha}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{P,\beta}(X')$.

TABLE 4. The description of data sets.

No.	Data sets	Objects	Attributes
1	Breast	286	10
2	Balance	625	5
3	Solar	1389	13
4	Chess	3196	37
5	Mushroom	8124	23
6	Nursery	12960	8

characteristic function and it's time complexity is $O(1)$; Step 3 compute the intermediate matrix and it's time complexity is $O(1)$; Steps 4-10 calculate two intermediate matrices and basic matrix, it's time complexity is $O(m|U'||U^-|)$; Steps 11-19 and Steps 20-28 calculate two new cut matrices and their time complexity are both $O(m|U'|)$; Step 29 and Step 30 update the characteristic function of approximations and their time complexity are both $O(m|U'|)$. Hence, the total time complexity of Algorithm 3 is $O(m|U'||U^-|)$.

TABLE 5. The comparison of static and incremental algorithms versus the size of added objects.

NO.	Breast		Balance		Solar		Chess		Mushroom		Nursery	
	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.
1	0.124	0.016	0.148	0.045	0.132	0.052	0.341	0.102	0.957	0.182	0.972	0.187
2	0.137	0.027	0.155	0.053	0.235	0.071	0.747	0.131	1.080	0.042	2.583	0.290
3	0.142	0.035	0.178	0.054	0.309	0.071	1.658	0.162	7.028	0.364	5.709	0.380
4	0.153	0.037	0.189	0.059	0.355	0.077	2.586	0.166	11.876	0.405	9.841	0.444
5	0.154	0.041	0.208	0.059	0.438	0.057	3.921	0.219	17.852	0.507	14.303	0.504
6	0.157	0.043	0.259	0.065	0.503	0.104	5.475	0.185	25.813	0.557	21.509	0.551
7	0.156	0.042	0.246	0.062	0.544	0.080	8.092	0.284	36.882	0.587	32.242	0.579
8	0.178	0.046	0.287	0.068	0.716	0.125	9.676	0.370	75.358	0.686	56.319	0.701
9	0.183	0.047	0.280	0.069	0.856	0.137	12.633	0.330	83.242	0.802	58.256	0.670
10	0.192	0.051	0.339	0.106	0.992	0.146	15.808	0.354	168.325	0.866	96.078	0.730

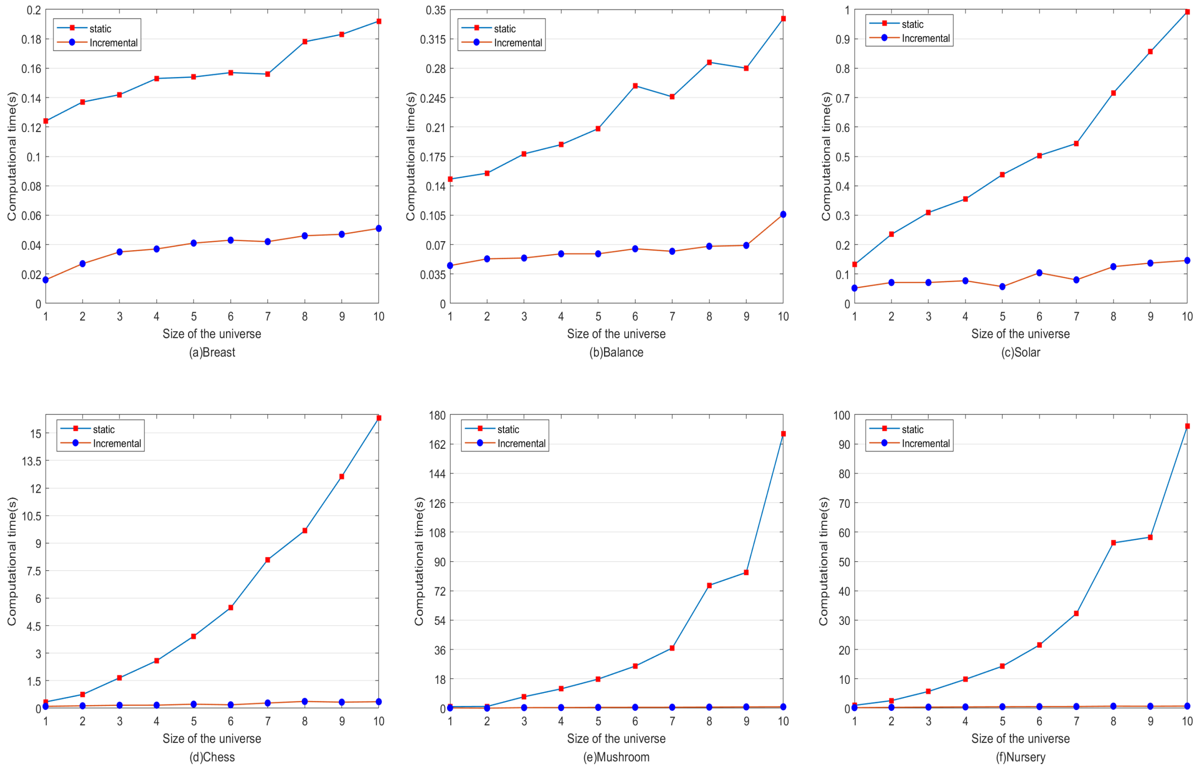

FIGURE 1. Computational time of static and incremental algorithms versus the size of added objects.

V. EXPERIMENTAL EVALUATION AND ANALYSIS

We conducted several experiments to evaluate the performance of the proposed incremental algorithms. From the UCI machine learning repository, the basic information of six data sets were wrote in Table 4, and experiments are implemented on a PC with Windows10, AMD Ryzen5 3550H CPU, 2.10 GHz and 16 GB memory, Algorithm 1 and Algorithm 2, Algorithm 3 were compared respectively. Each data set in Table 4 was divided into an average of 10 sub-data sets, and the first sub-data set was seen as the first basic data set, the combination of the first and second sub-data set was seen as the second basic data set, and so on.

A. EXPERIMENTS WITH DIFFERENT SIZED DATA SETS WHEN ADDING OBJECTS

In this subsection, for each basic data set, we randomly select 5% of the size of the basic data set from its complement set in the universe as the inserted new data set. By comparing the calculation time of Algorithm 1 and Algorithm 2, we show the efficiency of Algorithm 2 and the experimental results were listed in Table 5. With the increase of size for data set, the more detailed information of two algorithms were shown in Figure 1, it is easy to see from Figure 1 that the calculation time of two algorithms usually increase with the increase of the basic data set and Algorithm 2 is always faster than Algorithm 1, the larger the basic data set, the greater the difference in efficiency.

B. EXPERIMENTS WITH DIFFERENT SIZED DATA SETS WHEN DELETING OBJECTS

In this subsection, for each basic data set, we randomly select 5% of the size of the basic data set from its complement set in the universe as the inserted new data set. By comparing the

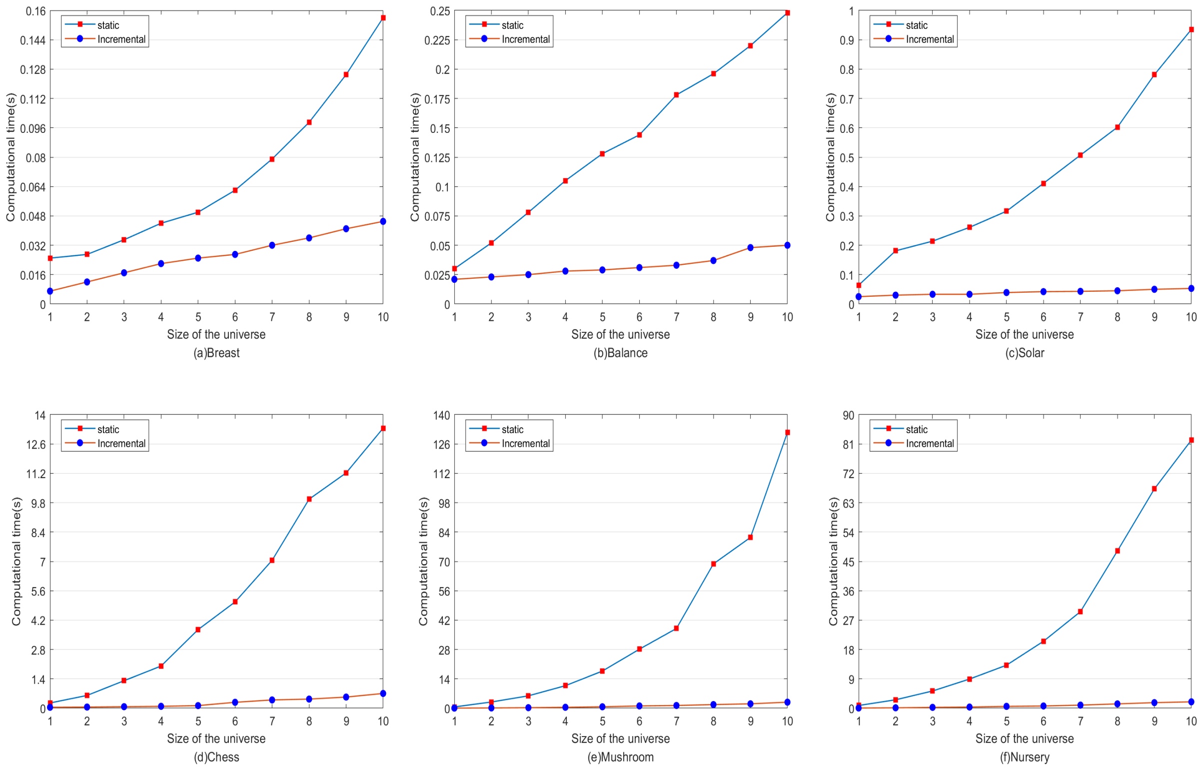

FIGURE 2. Computational time of static and incremental algorithms versus the size of deleted objects.

TABLE 6. The comparison of static and incremental algorithms versus the size of deleted objects.

NO.	Breast		Balance		Solar		Chess		Mushroom		Nursery	
	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.
1	0.025	0.007	0.030	0.021	0.064	0.025	0.250	0.044	0.615	0.049	0.829	0.057
2	0.027	0.012	0.052	0.023	0.181	0.030	0.609	0.055	2.961	0.127	2.584	0.127
3	0.035	0.017	0.078	0.025	0.214	0.033	1.311	0.072	5.904	0.255	5.278	0.259
4	0.044	0.022	0.105	0.028	0.261	0.033	2.011	0.091	10.738	0.440	8.941	0.360
5	0.050	0.025	0.128	0.029	0.316	0.039	3.748	0.124	17.711	0.690	13.189	0.568
6	0.062	0.027	0.144	0.031	0.411	0.042	5.068	0.282	28.161	1.091	20.518	0.687
7	0.079	0.032	0.178	0.033	0.507	0.043	7.053	0.394	38.015	1.302	29.607	0.948
8	0.099	0.036	0.196	0.037	0.602	0.045	9.968	0.435	68.844	1.727	48.201	1.326
9	0.125	0.041	0.220	0.048	0.781	0.050	11.22	0.531	81.432	2.098	67.229	1.710
10	0.156	0.045	0.248	0.050	0.935	0.053	13.354	0.704	131.54	2.876	82.189	1.957

calculation time of Algorithm 1 and Algorithm 3, we show the efficiency of Algorithm 3 and the experimental results were listed in Table 6. With the increase of size for data set, the more detailed information of two algorithms were shown in Figure 2, it is easy to see from Figure 2 that the calculation time of two algorithms usually increase with the increase of the basic data set and Algorithm 3 is always faster than Algorithm 1, the larger the basic data set, the greater the difference in efficiency.

VI. CONCLUSION

In this paper, we propose the method of computing approximations based on the matrix of multigranulation decision-theoretic rough sets. On this basis, the method of dynamic updating approximations with objects increased or deleted of multigranulation decision-theoretic rough sets are proposed, and some related properties are studied. For each case, an example is given to verify its validity. Finally, experimental studies show that two proposed incremental algorithms can significantly reduce unnecessary computing time.

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