# Low-Hit-Zone Wide Gap Frequency Hopping Sequence Sets With Optimal Average Hamming Correlation

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**ABSTRACT** In frequency hopping (FH) sequence design, the wide gap is an important performance indicator. In this paper, the average Hamming correlation of wide gap FH sequence (WG-FH sequence) sets with low hit zone, which has not yet been reported previously, is studied. A lower bound on the average Hamming auto-correlation and the average Hamming cross-correlation of low-hit-zone FH sequence sets is firstly derived. The new lower bound includes the bound for conventional FH sequence sets derived by Peng *et al.* as a special case. Then a construction of WG-FH sequence sets with multiple low hit zones is presented, which are optimal by the new bound for these low hit zones. Finally, another class of optimal WG-FH sequence sets with multiple low hit zones is presented, which have larger wide gap.

**INDEX TERMS** Frequency hopping sequences, low hit zone, average Hamming correlation, wide gap, quasi-synchronous frequency hopping systems.

## I. INTRODUCTION

In frequency hopping (FH) multiple-access (MA) spread-spectrum systems, the frequencies used are chosen pseudo-randomly by a code called FH sequence. As is often the case, in MA environments, mutual interference occurs when two or more transmitters transmit on the same frequency at the same time. It is desirable to keep the mutual interference between transmitters at a level as low as possible. The degree of the mutual interference is clearly related to the Hamming cross-correlation properties of the FH sequences [8], [11], [13]. In addition, it is also required that the FH sequences have good Hamming auto-correlation properties so as to minimize the ambiguity of the source identity. Thus, the design of an FH sequence set with good Hamming correlation properties is an important problem. There are two kinds of Hamming correlations, i.e., maximum Hamming correlation and average Hamming correlation. There are many FH sequences with

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optimal maximum Hamming correlation in the literature [1], [2], [5]–[7], [9], [10], [17], [20], [25], [28]–[30]. In recent years, some conventional FH sequences with optimal average Hamming correlation were reported in the literature [4], [12], [16], [27].

Different from conventional FH sequence design, the FH sequence design with no hit zone or low hit zone aims at making Hamming correlation values equal to zero or a very low value within a correlation zone. The significance of no (low) hit zone is that, even there are relative delays between the transmitted FH sequences, there will be no hit or the number of hits will be kept at a very low level between different sequences as long as the relative delay does not exceed certain limit (zone), thus reducing or eliminating the mutual interference. The corresponding FH sequences are called low-hit-zone FH sequences. There are some low-hit-zone FH sequences with optimal maximum Hamming correlation in the literature [3], [18], [19], [21], [23], [24]. However, the average Hamming correlation of low-hit-zone FH sequences has not yet been studied.

Wide gap FH sequences (WG-FH sequences) are effective to reduce narrowband interference, track jamming, and broadband blocking jamming. Due to these advantages, they have been used in practical systems widely. Recently, the researchers obtained some bounds on WG-FH sequences [14], [15]. However, it is difficult to design WG-FH sequences with good maximum Hamming correlation properties. For average Hamming correlation of conventional WG-FH sequences, it is meaningless to be studied since they have optimal average Hamming correlation as long as each frequency slot appears same number in the sequence set [27]. But for WG-FH sequences with low hit zone, it needs to meet the condition in Theorem 3.3 if they are optimal with respect to average Hamming correlation. Thus, the study of WG-FH sequences with low hit zone is meaningful.

In this paper, we study the average Hamming correlation of WG-FH sequence sets with low hit zone. First, we derive a lower bound on the average Hamming auto-correlation and the average Hamming cross-correlation of low-hit-zone FH sequence sets. Then we give a construction of WG-FH sequence sets with multiple low hit zones, which are optimal with respect to the new bound. Further, another construction of optimal WG-FH sequence sets with multiple low hit zones is presented which have larger wide gap.

The rest of this paper is organized as follows. In Section II, the related definitions and notations are introduced. In Section III, a lower bound on the average Hamming auto-correlation and the average Hamming cross-correlation of low-hit-zone FH sequence sets is derived. In Section IV, a construction of WG-FH sequence sets with multiple low hit zones is presented. In Section V, a class of WG-FH sequence sets with multiple low hit zones which have larger wide gap is constructed. Finally, the correspondence concluded with some remarks.

## II. PRELIMINARIES

Let $V = \{l_0, l_1, \cdots, l_{v-1}\}$ be a frequency slot set with size $v$ and $F = \{F^{(0)}, F^{(1)}, \cdots, F^{(K-1)}\}$ a set of FH sequences over $V$, where $F^{(i)} = (f_0^i, f_1^i, \cdots, f_{L-1}^i)$ for $i = 0, 1, \cdots, K - 1$. Let $c(x, y) = 1$ for $x = y$ and $c(x, y) = 0$ for $x \neq y$. The periodic Hamming correlation of $F^{(i)}$ and $F^{(j)}$ at time delay $\tau$ is given as follows:

$$C_{F^{(i)}F^{(j)}}(\tau) = \sum_{k=0}^{L-1} c(f_k^i, f_{k+\tau}^j), \qquad (1)$$

where the subscript addition $k + \tau$ is performed modulo $L$.

The $C_{F^{(i)}F^{(j)}}(\tau)$ is called the periodic Hamming auto-correlation function when $i = j$ and the periodic Hamming cross-correlation function when $i \neq j$.

For a conventional FH sequence set $F$, the overall number of Hamming auto-correlation and Hamming cross-correlation are defined as follows, respectively:

$$R_a(F) = \sum_{F^{(i)} \in F, 1 \leq \tau \leq L-1} C_{F^{(i)}F^{(i)}}(\tau), \qquad (2)$$

$$R_c(F) = \sum_{F^{(i)}, F^{(j)} \in F, i \neq j, 0 \leq \tau \leq L-1} C_{F^{(i)}F^{(j)}}(\tau). \qquad (3)$$

The average Hamming auto-correlation and the average Hamming cross-correlation of $F$ are defined as follows, respectively:

$$A_a(F) = \frac{R_a(F)}{K(L-1)}, \qquad (4)$$

$$A_c(F) = \frac{R_c(F)}{K(K-1)L}. \qquad (5)$$

In 2010, Peng *et al.* [27] established the following lower bound on the average Hamming auto-correlation and the average Hamming cross-correlation of a conventional FH sequence set.

*Lemma 2.1:* Let $F$ be a conventional FH sequence set with family size $K$ and sequence length $L$ over a given frequency slot set $V$ with size $v$. $A_a(F)$ and $A_c(F)$ are average Hamming auto-correlation and average Hamming cross-correlation of $F$ respectively. We have

$$v(L-1)A_a(F) + v(K-1)LA_c(F) \geq KL^2 - vL. \qquad (6)$$

For a low-hit-zone FH sequence set $F$ with low hit zone $Z$, $0 \leq Z \leq L - 1$, the overall number of Hamming auto-correlation and Hamming cross-correlation are defined as follows, respectively:

$$R_a(F) = \sum_{F^{(i)} \in F, 1 \leq \tau \leq Z} C_{F^{(i)}F^{(i)}}(\tau), \qquad (7)$$

$$R_c(F) = \sum_{F^{(i)}, F^{(j)} \in F, i \neq j, 0 \leq \tau \leq Z} C_{F^{(i)}F^{(j)}}(\tau). \qquad (8)$$

The average Hamming auto-correlation and the average Hamming cross-correlation of $F$ are defined as follows, respectively:

$$A_a(F) = \frac{R_a(F)}{KZ}, \qquad (9)$$

$$A_c(F) = \frac{R_c(F)}{K(K-1)(Z+1)}. \qquad (10)$$

When $Z = L - 1$, the low-hit-zone FH sequence set $F$ degenerates into a conventional FH sequence set.

For simplicity, we denote $R_a = R_a(F)$, $R_c = R_c(F)$, $A_a = A_a(F)$, and $A_c = A_c(F)$.

Then we give the definition of the wide gap of FH sequence sets whether they have low hit zone or not.

*Definition 2.2:* An FH sequence set $F$ over $V = \{0, 1, \cdots, v - 1\}$ is said to have wide gap $w$, where $w$ is an integer, if for every $(f_0, f_1, \cdots, f_{L-1}) \in F$ the following inequality holds:

$$|f_{j+1} - f_j| > w, \quad j = 0, 1, \cdots, L-1.$$

We call $F$ a WG-FH sequence set.

## III. LOWER BOUND ON THE AVERAGE HAMMING CORRELATIONS OF LOW-HIT-ZONE FH SEQUENCE SETS

In this section, we derive a bound on the average Hamming auto-correlation and the average Hamming cross-correlation of low-hit-zone FH sequence sets which includes the bound for conventional FH sequence sets derived by Peng *et al.* as a special case.

*Theorem 3.1:* Let $F$ be a low-hit-zone FH sequence set of family size $K$ and length $L$ over a given frequency slot set $V$ with size $v$, and $Z$ the low hit zone of $F$. $A_a$ and $A_c$ are average Hamming auto-correlation and average Hamming cross-correlation of $F$ respectively. We have

$$vZA_a + v(K-1)(Z+1)A_c \geq KL(Z+1) - vL. \quad (11)$$

*Proof:* We have

$$\sum_{F^{(i)},F^{(j)}\in F} \sum_{0\leq\tau\leq Z} C_{F^{(i)}F^{(j)}}(\tau)$$

$$= \sum_{F^{(i)}\in F} C_{F^{(i)}F^{(i)}}(0) + \sum_{F^{(i)}\in F} \sum_{1\leq\tau\leq Z} C_{F^{(i)}F^{(i)}}(\tau)$$

$$+ \sum_{F^{(i)},F^{(j)}\in F, i\neq j} \sum_{0\leq\tau\leq Z} C_{F^{(i)}F^{(j)}}(\tau)$$

$$= KL + R_a + R_c. \quad (12)$$

By Lemma 6 in [26], we have

$$\sum_{F^{(i)},F^{(j)}\in F} \sum_{0\leq\tau\leq Z} C_{F^{(i)}F^{(j)}}(\tau) \geq \frac{K^2 L(Z+1)}{v}. \quad (13)$$

By (9), (10), (12), and (13), we get that

$$vZA_a + v(K-1)(Z+1)A_c \geq KL(Z+1) - vL.$$

$\square$

Putting $Z = L - 1$ in Theorem 3.1, one can obtain the bound on average Hamming correlation of conventional FH sequence sets which was derived by Peng *et al.* [27] in 2010.

*Corollary 1:* Let $F$ be a conventional FH sequence set of family size $K$ and length $L$ over a given frequency slot set $V$ with size $v$. $A_a$ and $A_c$ are average Hamming auto-correlation and average Hamming cross-correlation of $F$ respectively. We have

$$v(L-1)A_a + v(K-1)LA_c \geq KL^2 - vL.$$

This result is same as that in Lemma 2.1. That is, the bound (11) includes the bound (6) for conventional FH sequence sets derived by Peng *et al.* as a special case.

*Definition 3.2:* A low-hit-zone FH sequence set is said to be optimal with respect to average Hamming correlation if the bound (11) is met with equality. A conventional FH sequence set is said to be optimal with respect to average Hamming correlation if the bound (6) is met with equality.

For a low-hit-zone FH sequence set $F = \{F^{(0)}, F^{(1)}, \cdots, F^{(K-1)}\}$ with low hit zone $Z$ over a given frequency slot set

$V = \{l_0, l_1, \cdots, l_{v-1}\}$, where $F^{(i)} = (f_0^i, f_1^i, \cdots, f_{L-1}^i)$ for $0 \leq i \leq K - 1$, let

$$\rho_k^j(F) = \sum_{i=0}^{K-1} \sum_{\tau=0}^{Z} c(l_j, f_{k+\tau}^i) \quad (14)$$

where $0 \leq j \leq v - 1$, $0 \leq k \leq L - 1$, and the subscript addition $k + \tau$ is performed modulo $L$.

The following theorem gives a sufficient and necessary condition for a low-hit-zone FH sequence set which is optimal with respect to average Hamming correlation [22].

*Theorem 3.3:* Let $F = \{F^{(0)}, F^{(1)}, \cdots, F^{(K-1)}\}$ be an FH sequence set with low hit zone $Z$ over a given frequency slot set $V = \{l_0, l_1, \cdots, l_{v-1}\}$, where $F^{(i)} = (f_0^i, f_1^i, \cdots, f_{L-1}^i)$ for $0 \leq i \leq K - 1$. The low-hit-zone FH sequence set $F$ is optimal with respect to average Hamming correlation if and only if $\rho_k^j(F) + \rho_{k+Z}^j(F) = \frac{2K(Z+1)}{v}$ for any $0 \leq j \leq v - 1$ and $0 \leq k \leq L - 1$, where $v|2K(Z+1)$.

*Remark 1:* For FH sequence sets with low hit zone, it needs to meet the condition in Theorem 3.3 if they are optimal with respect to average Hamming correlation.

## IV. CONSTRUCTION FOR WG-FH SEQUENCE SETS WITH MULTIPLE LOW HIT ZONES

Now we give a construction of WG-FH sequence sets in this section, which have multiple low hit zones. Therein, they are optimal with respect to average Hamming correlation for these low hit zones.

*Construction 1:*

**Step 1:** Let $V = \{0, 1, \cdots, q-1\}$ be a frequency slot set with size $q$ and $M$, $n$, $w$ be three integers such that $\gcd(M + w, q)|M$ and $M \leq \frac{q}{2} - w$. Construct an $M \times \frac{nq}{\gcd(M+w,q)}$ matrix over $V$ as follows:

$$A = \begin{pmatrix} b_0^0 & b_1^0 & \cdots & b_{\frac{nq}{\gcd(M+w,q)}-1}^0 \\ b_0^1 & b_1^1 & \cdots & b_{\frac{nq}{\gcd(M+w,q)}-1}^1 \\ \vdots & \vdots & & \vdots \\ b_0^{M-1} & b_1^{M-1} & \cdots & b_{\frac{nq}{\gcd(M+w,q)}-1}^{M-1} \end{pmatrix},$$

where

$$\{b_j^0, b_j^1, \cdots, b_j^{M-1}\}$$
$$= \{j(M+w), j(M+w)+1, \cdots, j(M+w)+(M-1)\} \quad (15)$$

for $j = 0, 1, \cdots, \frac{nq}{\gcd(M+w,q)} - 1$ and the operations in (15) are performed modulo $q$. Note that (15) represents that $b_j^0, b_j^1, \cdots, b_j^{M-1}$ are in any permutation of $j(M+w), j(M+w)+1, \cdots, j(M+w)+(M-1)$. Thereby, $b_{j_1}^i$ may be different from $b_{j_2}^i$ for $\frac{q}{\gcd(M+w,q)}|(j_1 - j_2), j_1 \neq j_2$, and $i = 0, 1, \cdots, M-1$.

**Step 2:** Construct an FH sequence set $F = \{F^{(0)}, F^{(1)}, \cdots, F^{(M-1)}\}$ with low hit zones $\frac{kq}{\gcd(M+w,q)} - 1$, $k = 1, 2, \cdots, n$, where

$$F^{(i)} = (b_0^i, b_1^i, \cdots, b_{\frac{nq}{\gcd(M+w,q)}-1}^i)$$

for $0 \leq i \leq M - 1$.

*Remark 2:* The FH sequence set $F$ generated by Construction 1 has multiple low hit zones $\frac{kq}{\gcd(M+w,q)} - 1, k = 1, 2, \cdots, n$. For these low hit zones, it has optimal average Hamming correlation which will be described by the following theorem.

*Theorem 4.1:* The FH sequence set $F$ generated by Construction 1 is a WG-FH sequence set with wide gap $w$ and optimal with respect to average Hamming correlation by the bound (11) for low hit zones $\frac{kq}{\gcd(M+w,q)} - 1, k = 1, 2, \cdots, n$.

*Proof:* First, we prove that the FH sequence set $F$ has wide gap $w$. For $F^{(i)} = (b_0^i, b_1^i, \cdots, b_{\frac{nq}{\gcd(M+w,q)}-1}^i), 0 \leq i \leq M-1$, we have

$$|b_{j+1}^i - b_j^i|$$
$$= |\langle (j+1)(M+w) + \sigma_1 \rangle_q - \langle j(M+w) + \sigma_2 \rangle_q| \quad (17)$$

where $j = 0, 1, \cdots, \frac{nq}{\gcd(M+w,q)} - 1$ and $0 \leq \sigma_1, \sigma_2 \leq M-1$. Let $j(M+w) + \sigma_2 = \alpha_1 q + \beta_1$ and $(j+1)(M+w) + \sigma_1 = \alpha_2 q + \beta_2, \alpha_1, \alpha_2 \geq 0, 0 \leq \beta_1, \beta_2 \leq q-1$. Then (17) can be rewritten as

$$|b_{j+1}^i - b_j^i| = |\beta_2 - \beta_1|. \quad (18)$$

Since $M \leq \frac{q}{2} - w$, we have

$$(j+1)(M+w) + \sigma_1 - [j(M+w) + \sigma_2]$$
$$= M + w + \sigma_1 - \sigma_2$$
$$\leq q - w - 1$$
$$< q. \quad (19)$$

Thus, we discuss it into the following two cases.

*Case 1.* $\alpha_1 = \alpha_2$. By (18), we have

$$|b_{j+1}^i - b_j^i| = |(j+1)(M+w) + \sigma_1 - [j(M+w) + \sigma_2]|$$
$$= |M + w + \sigma_1 - \sigma_2|$$
$$> w.$$

*Case 2.* $\alpha_1 = \alpha_2 - 1$. In this case, (18) becomes

$$|b_{j+1}^i - b_j^i| = |(j+1)(M+w) + \sigma_1 - [j(M+w) + \sigma_2] - q|$$
$$= |M + w + \sigma_1 - \sigma_2 - q|.$$

Together with (19), we have $M + w + \sigma_1 - \sigma_2 - q \leq -w - 1$. This implies that

$$|b_{j+1}^i - b_j^i| \geq w + 1 > w.$$

Hence, it has wide gap $w$. Now we consider its average Hamming correlation. For $Z = \frac{kq}{\gcd(M+w,q)} - 1, k = 1, 2, \cdots, n$, by (7) and (8) we have

$$R_a + R_c + M \cdot \frac{nq}{\gcd(M+w,q)}$$
$$= \sum_{F^{(i)}, F^{(j)} \in F} \sum_{\tau=0}^{Z} C_{F^{(i)}F^{(j)}}(\tau)$$

$$= \sum_{F^{(i)}, F^{(j)} \in F} \sum_{\tau=0}^{Z} \sum_{l=0}^{q-1} \sum_{\kappa=0}^{\frac{nq}{\gcd(M+w,q)}-1} c(l, b_\kappa^i) \cdot c(l, b_{\kappa+\tau}^j)$$

$$= \sum_{l=0}^{q-1} \sum_{\kappa=0}^{\frac{nq}{\gcd(M+w,q)}-1} \sum_{i=0}^{M-1} c(l, b_\kappa^i) \cdot \left( \sum_{\tau=0}^{Z} \sum_{j=0}^{M-1} c(l, b_{\kappa+\tau}^j) \right). \quad (20)$$

Consider the variable $(\kappa + \tau)(M + w) + \sigma$ where $\tau = 0, 1, \cdots, \frac{kq}{\gcd(M+w,q)} - 1$ and $\sigma = 0, 1, \cdots, M - 1$. For $k' = 0, 1, \cdots, k-1$ and $m = 0, 1, \cdots, \frac{M}{\gcd(M+w,q)} - 1$, we prove that $(\kappa + \tau_1)(M + w) + \sigma_1 \neq (\kappa + \tau_2)(M + w) + \sigma_2 \mod q$ where $(\tau_1 - \tau_2)^2 + (\sigma_1 - \sigma_2)^2 \neq 0, \frac{k'q}{\gcd(M+w,q)} \leq \tau_1, \tau_2 \leq \frac{(k'+1)q}{\gcd(M+w,q)} - 1, m \cdot \gcd(M + w, q) \leq \sigma_1, \sigma_2 \leq (m + 1) \cdot \gcd(M + w, q) - 1$. Suppose that $(\kappa + \tau_1)(M + w) + \sigma_1 = (\kappa + \tau_2)(M + w) + \sigma_2 \mod q$ which is equivalent to $(\tau_1 - \tau_2)\frac{M+w}{\gcd(M+w,q)} = \frac{\sigma_2-\sigma_1}{\gcd(M+w,q)} \mod \frac{q}{\gcd(M+w,q)}$. If $\sigma_1 \neq \sigma_2$, then $\gcd(M + w, q) \nmid (\sigma_2 - \sigma_1)$ which contradicts the hypothesis. If $\sigma_1 = \sigma_2$, then $\frac{q}{\gcd(M+w,q)} \nmid (\tau_1 - \tau_2)$ for $\tau_1 \neq \tau_2$. Note that $\frac{q}{\gcd(M+w,q)}$ and $\frac{M+w}{\gcd(M+w,q)}$ are coprime. Then the hypothesis still does not hold. So $(\kappa + \tau)(M + w) + \sigma$ takes the value of each element once in $\{0, 1, \cdots, q-1\}$ with $\frac{k'q}{\gcd(M+w,q)} \leq \tau \leq \frac{(k'+1)q}{\gcd(M+w,q)} - 1$ and $m \cdot \gcd(M + w, q) \leq \sigma \leq (m + 1) \cdot \gcd(M + w, q) - 1$. It follows that

$$\sum_{\tau=0}^{Z} \sum_{j=0}^{M-1} c(l, b_{\kappa+\tau}^j) = \sum_{k'=0}^{k-1} \sum_{m=0}^{\frac{M}{\gcd(M+w,q)}-1} 1$$
$$= \frac{kM}{\gcd(M + w, q)}.$$

Then (20) becomes

$$R_a + R_c + M \cdot \frac{nq}{\gcd(M + w, q)}$$

$$= \frac{kM}{\gcd(M + w, q)} \cdot \sum_{l=0}^{q-1} \sum_{\kappa=0}^{\frac{nq}{\gcd(M+w,q)}-1} \sum_{i=0}^{M-1} c(l, b_\kappa^i)$$

$$= \frac{kM}{\gcd(M + w, q)} \cdot \sum_{l=0}^{q-1} \frac{nM}{\gcd(M + w, q)}$$

$$= \frac{qnkM^2}{[\gcd(M + w, q)]^2}.$$

By (9), (10), and (20), we have

$$\left[ \frac{kq^2}{\gcd(M + w, q)} - q \right] A_a + \frac{kq^2(M - 1)}{\gcd(M + w, q)} A_c$$
$$= \frac{Mnkq^2}{[\gcd(M + w, q)]^2} - \frac{nq^2}{\gcd(M + w, q)}.$$

This implies that the bound (11) is met. $\square$

$$A = \begin{pmatrix} 2 & 7 & 1 & 4 & 12 & 5 & 11 & 0 & 6 & 12 & 4 & 10 & 3 & 11 & 0 & 8 & 13 & 5 & 10 & 3 & 9 \\ 3 & 6 & 13 & 7 & 10 & 3 & 9 & 1 & 9 & 13 & 7 & 11 & 5 & 8 & 2 & 6 & 0 & 4 & 12 & 5 & 8 \\ 1 & 8 & 0 & 5 & 13 & 2 & 10 & 3 & 8 & 0 & 6 & 12 & 4 & 9 & 1 & 9 & 12 & 7 & 11 & 4 & 10 \\ 0 & 9 & 12 & 6 & 11 & 4 & 8 & 2 & 7 & 1 & 5 & 13 & 2 & 10 & 3 & 7 & 1 & 6 & 13 & 2 & 11 \end{pmatrix}. \quad (16)$$

**TABLE 1.** The optimality of the WG-FH sequence set *F* in each low hit zone.

| Low hit zone $Z$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| According to bound (11) | N | N | N | N | O | N | N | N | N | N | N | O | N | N | N | N | N | N | O |

*Example 1:* Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ be its frequency slot set. For $M = 4$, $w = 2$, and $n = 3$, construct a $4 \times 21$ matrix $A$ as (16), shown at the bottom of the previous page. Then the desired FH sequence set $F$ can be got as follows:

$F = \{(2, 7, 1, 4, 12, 5, 11, 0, 6, 12, 4, 10, 3, 11, 0, 8, 13, 5,$
$\quad 10, 3, 9), (3, 6, 13, 7, 10, 3, 9, 1, 9, 13, 7, 11, 5, 8, 2, 6,$
$\quad 0, 4, 12, 5, 8), (1, 8, 0, 5, 13, 2, 10, 3, 8, 0, 6, 12, 4,$
$\quad 9, 1, 9, 12, 7, 11, 4, 10), (0, 9, 12, 6, 11, 4, 8, 2, 7, 1,$
$\quad 5, 13, 2, 10, 3, 7, 1, 6, 13, 2, 11)\}.$

It is easy to check that $F$ has wide gap $w = 2$. For low hit zone $Z = 6$, we can calculate that $A_a = \frac{7}{24}$ and $A_c = \frac{11}{12}$. By bound (11)

$$84A_a + 294A_c \geq 294.$$

Then the inequality is met. Similarly, we can calculate that $A_a = \frac{31}{52}$ and $A_c = \frac{221}{168}$ for low hit zone $Z = 13$. By bound (11)

$$182A_a + 588A_c \geq 882.$$

The inequality is also met. For low hit zone $Z = 20$, we can get that $A_a = \frac{29}{40}$ and $A_c = \frac{181}{126}$. Note that $F$ degenerates into a conventional FH sequence set for $Z = 20$. By bound (6) we have

$$280A_a + 882A_c \geq 1470.$$

The inequality is met. Therefore, $F$ is optimal with respect to average Hamming correlation for low hit zones $Z = 6, 13, 20$. Besides, one can check that for any low hit zone $Z \neq 6, 13, 20$, $F$ is not optimal with respect to average Hamming correlation. This is described in Table 1, where "N" and "O" represent "not optimal" and "optimal" respectively.

## V. SECOND CLASS OF WG-FH SEQUENCE SETS WITH MULTIPLE LOW HIT ZONES

In this section, we give another construction of WG-FH sequence sets which have larger wide gap.

*Construction 2:*

**Step 1:** Let $V = \{0, 1, \cdots, q - 1\}$ be a frequency slot set with size $q$ and $l, M, n, w'$ be four integers such that $l|M$,

$\gcd(M + w', q)|M$ and $M \leq \frac{q}{2} - w'$. Construct $l \times \frac{nq}{\gcd(M+w',q)}$ matrices over $V$ as follows:

$$B^h = \begin{pmatrix} b_0^{hl} & b_1^{hl} & \cdots & b_{\frac{nq}{\gcd(M+w',q)}-1}^{hl} \\ b_0^{hl+1} & b_1^{hl+1} & \cdots & b_{\frac{nq}{\gcd(M+w',q)}-1}^{hl+1} \\ \vdots & \vdots & & \vdots \\ b_0^{hl+l-1} & b_1^{hl+l-1} & \cdots & b_{\frac{nq}{\gcd(M+w',q)}-1}^{hl+l-1} \end{pmatrix},$$

where $h = 0, 1, \cdots, \frac{M}{l} - 1$ and

$$\{b_j^{hl}, b_j^{hl+1}, \cdots, b_j^{hl+l-1}\}$$
$$= \{j(M + w') + hl,$$
$$\quad j(M + w') + hl + 1, \cdots, j(M + w') + hl + l - 1\}$$

(21)

for $j = 0, 1, \cdots, \frac{nq}{\gcd(M+w',q)} - 1$, and the operations in (21) are performed modulo $q$. Note that (21) represents that $b_j^{hl}, b_j^{hl+1}, \cdots, b_j^{hl+l-1}$ are in any permutation of $j(M+w') + hl, j(M + w') + hl + 1, \cdots, j(M + w') + hl + l - 1$. Then $b_{j_1}^i$ may be different from $b_{j_2}^i$ for $\frac{q}{\gcd(M+w',q)}|(j_1 - j_2), j_1 \neq j_2$, and $i = 0, 1, \cdots, M - 1$.

**Step 2:** Construct an FH sequence set $F = \{F^{(0)}, F^{(1)}, \cdots, F^{(M-1)}\}$ with low hit zones $\frac{kq}{\gcd(M+w',q)} - 1, k = 1, 2, \cdots, n$, where

$$F^{(i)} = (b_0^i, b_1^i, \cdots, b_{\frac{nq}{\gcd(M+w',q)}-1}^i)$$

for $0 \leq i \leq M - 1$.

*Remark 3:* The FH sequence set $F$ generated by Construction 2 has wide gap $w' + M - l$ which is larger than that generated by Construction 1.

*Theorem 5.1:* The FH sequence set $F$ generated by Construction 2 is a WG-FH sequence set with wide gap $w' + M - l$ and optimal with respect to average Hamming correlation by the bound (11) for low hit zones $\frac{kq}{\gcd(M+w',q)} - 1, k = 1, 2, \cdots, n$.

*Proof:* We first prove that the FH sequence set $F$ has wide gap $w' + M - l$. For $F^{(i)} = (b_0^i, b_1^i, \cdots, b_{\frac{nq}{\gcd(M+w',q)}-1}^i)$,

$$B^0 = \begin{pmatrix} 1 & 6 & 13 & 5 & 10 & 3 & 9 & 1 & 7 & 12 & 4 & 11 & 2 & 8 & 0 & 6 & 12 & 5 & 11 & 3 & 8 \\ 0 & 7 & 12 & 4 & 11 & 2 & 8 & 0 & 6 & 13 & 5 & 10 & 3 & 9 & 1 & 7 & 13 & 4 & 10 & 2 & 9 \end{pmatrix}. \quad (22)$$

$$B^1 = \begin{pmatrix} 2 & 9 & 0 & 7 & 13 & 4 & 10 & 2 & 8 & 1 & 7 & 12 & 4 & 11 & 3 & 8 & 0 & 6 & 12 & 5 & 10 \\ 3 & 8 & 1 & 6 & 12 & 5 & 11 & 3 & 9 & 0 & 6 & 13 & 5 & 10 & 2 & 9 & 1 & 7 & 13 & 4 & 11 \end{pmatrix}. \quad (23)$$

$0 \leq i \leq M - 1$, we have

$$|b_{j+1}^i - b_j^i| = |\langle (j+1)(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_1 \rangle_q$$
$$- \langle j(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_2 \rangle_q | \quad (24)$$

where $j = 0, 1, \cdots, \frac{nq}{\gcd(M+w',q)} - 1$ and $0 \leq \sigma_1, \sigma_2 \leq l - 1$. Let $j(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_2 = \alpha_1 q + \beta_1$ and $(j+1)(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_1 = \alpha_2 q + \beta_2$, $\alpha_1, \alpha_2 \geq 0$, $0 \leq \beta_1, \beta_2 \leq q - 1$. Then (24) can be rewritten as

$$|b_{j+1}^i - b_j^i| = |\beta_2 - \beta_1|. \quad (25)$$

Since $M \leq \frac{q}{2} - w'$, we have

$$(j+1)(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_1 - [j(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_2]$$
$$= M + w' + \sigma_1 - \sigma_2$$
$$\leq q - w' - 1$$
$$< q. \quad (26)$$

Thus, we discuss it into the following two cases.
*Case 1.* $\alpha_1 = \alpha_2$. By (25), we have

$$|b_{j+1}^i - b_j^i|$$
$$= |(j+1)(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_1 - [j(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_2]|$$
$$= |M + w' + \sigma_1 - \sigma_2|$$
$$> M + w' - l.$$

*Case 2.* $\alpha_1 = \alpha_2 - 1$. In this case, (25) becomes

$$|b_{j+1}^i - b_j^i|$$
$$= |(j+1)(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_1 - [j(M+w') + l \left\lfloor \frac{i}{l} \right\rfloor + \sigma_2] - q|$$
$$= |M + w' + \sigma_1 - \sigma_2 - q|.$$

Note that

$$M + w' + \sigma_1 - \sigma_2 - q \leq M + w' + l - 1 - q$$
$$\leq M + w' + l - 1 - 2(M + w')$$
$$= l - 1 - (M + w').$$

This implies that

$$|b_{j+1}^i - b_j^i| \geq M + w' - l + 1 > M + w' - l.$$

Hence, it has wide gap $M + w' - l$. By (21), we have that

$$\{b_j^0, b_j^1, \cdots, b_j^{M-1}\}$$
$$= \bigcup_{h=0,1,\cdots,\frac{M}{l}-1} \{b_j^{hl}, b_j^{hl+1}, \cdots, b_j^{hl+l-1}\}$$
$$= \{j(M+w'), j(M+w')+1, \cdots, j(M+w')+M-1\}.$$

Thus, by the similar discussion in the proof of Theorem 4.1 we can get that $F$ is optimal with respect to average Hamming correlation by the bound (11) for low hit zones $\frac{kq}{\gcd(M+w',q)} - 1$ for $k = 1, 2, \cdots, n$. $\quad\square$

*Example 2:* Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ be its frequency slot set. For $l = 2$, $M = 4$, $w' = 2$, and $n = 3$, construct two $2 \times 21$ matrices $B^0$, $B^1$ as (22) and (23), shown at the bottom of the previous page. Then the desired FH sequence set $F$ can be got as follows:

$F = \{(1, 6, 13, 5, 10, 3, 9, 1, 7, 12, 4, 11, 2, 8, 0, 6, 12, 5$
$, 11, 3, 8), (0, 7, 12, 4, 11, 2, 8, 0, 6, 13, 5, 10, 3, 9, 1,$
$7, 13, 4, 10, 2, 9), (2, 9, 0, 7, 13, 4, 10, 2, 8, 1, 7, 12, 4,$
$11, 3, 8, 0, 6, 12, 5, 10), (3, 8, 1, 6, 12, 5, 11, 3, 9, 0, 6,$
$13, 5, 10, 2, 9, 1, 7, 13, 4, 11)\}.$

It is easy to check that $F$ has wide gap $w = 4$. For low hit zones $Z = 6, 13, 20$, one can check that $F$ is optimal with respect to average Hamming correlation, and not optimal for any low hit zone $Z \neq 6, 13, 20$.

## VI. CONCLUSION
In this paper, we first derived a lower bound on the average Hamming auto-correlation and the average Hamming cross-correlation of low-hit-zone FH sequence sets. The lower bound includes the bound for conventional FH sequence sets derived by Peng *et al.* as a special case. Then we presented a construction of WG-FH sequence sets with multiple low hit zones, which are optimal with respect to the new lower bound. Finally, we gave a construction of low-hit-zone WG-FH sequence sets with larger wide gap, which are optimal for multiple low hit zones. It is expected that the proposed lower bound will be useful in evaluating new low-hit-zone FH sequence designs and the proposed low-hit-zone WG-FH sequence sets will be useful in quasi-synchronous FH-MA systems to eliminate the narrowband interference and track jamming.

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