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A New Design to Finite-Time Stabilization of Nonlinear Systems With Applications to General Neural Networks

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ABSTRACT The finite-time stabilization problem of nonlinear systems is investigated in this paper. Firstly, to improve the precision of settling time of nonlinear system, a new finite-time stability theorem is established, and a higher precision settling time is derived from it. Moreover, by theoretical derivation, we prove that the corresponding settling time is more accurate than the existing results. Secondly, as an application, a new class of finite-time protocol framework, which unifies continuous protocol and discontinuous ones into a uniform formula, is designed to solve the finite-time stabilization problem of the general neural network system, and it can bring to a continuous control protocol and a discontinuous control protocol through choosing different design parameters. It is shown that the convergence rate is improved and also the corresponding settling time is upper bounded by some positive constant independent of initial conditions, which makes it convenient and flexible to adjust the settling time by adjusting design parameters. Finally, two numerical examples are provided to illustrate the effectiveness of our theoretical results.

INDEX TERMS Finite-time stability theorem, Filippov solution, general neural networks, Lyapunov function, settling time function, upper bound of settling time.

I. INTRODUCTION

In the past decades, the problems of stability analysis and stabilization problem of the nonlinear systems have been paid attention extensively in many disciplines, such as [1], [2] investigated finite-time control problem of switch systems and power systems, respectively. Stability analysis and stabilization design play vital roles in control theory and system identification. The stability of deterministic systems is usually classified into two cases, one is asymptotic stability and the other is finite-time stability. Asymptotic stability means that the state trajectory of the system tends to equilibrium as the time approaches infinity. Finite-time stability can ensure the state trajectory of the system reaches equilibrium in a finite time. Moreover, compared with asymptotic stability, finite-time stability has also many merits, such as higher accuracy, better robustness, and rejection of disturbance. Therefore, there is a rich of results [3]–[12] on finite-time stability of nonlinear systems that have been reported in the existing literature. Representative works can refer to [13]–[17]. In fact, finite-time stability is ubiquitous in practical applications, for example, control of the robot catching a "flying" ball or walking robot in between two impacts [7]. Finite-time attitude tracking problems for single and multiple spacecraft was dealt with in [33]. More recent results can refer to references [34]–[38].

To the best of our knowledge, techniques to investigate the finite-time stability problem of nonlinear systems mainly include two kinds. One is the homogeneity of function (or system), that is to say, the asymptotic stable system, owning negative homogeneity, is finite-time stable. Lots of efforts have been done to the finite-time stability problem of nonlinear systems by employing the technique. For example, [2], [9], [13], [27] studied the finite-time stability problem of nonlinear systems in virtue of homogeneity. However, one evident flaw of the technique is that corresponding settling time cannot be estimated explicitly in form of mathematics expression. The other method is that the corresponding

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Lyapunov function satisfies the differential inequalities: $\dot{V} + cV^p \leq 0$ or $\dot{V} + cV^p + dV \leq 0$ where $p \in (0, 1)$; $c > 0$, $d > 0$. As it is pointed out in [19], estimation of the settling time is one of the most important tasks for finite-time control of the system. Thus, to obtain settling time, the corresponding Lyapunov function has to satisfy inequalities: $\dot{V} + cV^p \leq 0$ or $\dot{V} + cV^p + dV \leq 0$, where $p \in (0, 1)$; $c, d > 0$, then the settling time $T(x_0)$ can be estimated as follow: $T(x_0) \leq \frac{V_0^{1-p}}{c(1-p)}$ or $T(x_0) \leq \frac{\ln\left(\frac{d}{c}V_0^{1-p}+1\right)}{d(1-p)}$. Obviously, from the above analysis, it can be seen that the corresponding settling time is usually an increasing function of the initial states of the system to some extent. Different initial values lead to different settling times and different estimations. At the same time, this feature becomes a shortcoming for some practical applications if the settling time needs to be obtained without initial conditions. Moreover, the initial states of many practical systems are often difficult to be obtained in advance. This leads to settling time unobtainable and deterioration of the system performance to some extent. To overcome this flaw, A special finite-time stability, which is called fixed-time stability, was proposed in [20], [21]. Subsequently, lots of interesting results are stimulated by fixed-time stability. For example, in [12], [20] and [38], authors derived some sufficient conditions to guarantee fixed-time stability of nonlinear systems, and some estimations of the settling time are provided, respectively. In addition, in [23], the implicit Lyapunov function method is applied to discuss the fixed-time stability of non-autonomous systems. Employing the inverse function of the Lyapunov function, Lu et. al. investigated the finite-time and fixed-time stability of the nonlinear systems in [22]. However, from the simulations of the above results, it can be found that corresponding settling time is estimated conservatively, that is to say, most of the estimations of settling time are rough compared with the real settling time. As we all know, the real settling time cannot be obtained accurately because of the zoom of inequality, and corresponding estimations of settling time are always conservative for the real settling time. Therefore, a natural question arises, how to obtain a more accurate estimation of settling time. Moreover, for the discontinuous nonlinear system, how to overcome the difficulty caused by the discontinuity is a challenging and interesting topic. The reason is that discontinuity leads to the nonexistence of the derivative of corresponding Lyapunov function at some special points, and analysis of stability cannot be performed as the continuous case. Thus, it is necessary to find a feasible method to analysize stability of discontinuous systems.

In addition, recently, control problem of network system became a hot topic, details can refer to references [34], [35]. Especially, as an application of finite/fixed-time stability, finite/fixed-time stabilization of neural networks has received tremendous attention from researchers. For stabilization problem of system, as one of the most important performance indexes, convergence time has been paid lots of attention in the process of studying the stabilization problem of network systems. Moreover, finite(fixed)-time synchronization or stabilization problems have been investigated well by Lyapunov methods. Nonlinear systems often include discontinuous system and continuous systems. And researchers often studied them respectively. Generally speaking, it is desired to study finite-time stabilization of nonlinear systems for the continuous and discontinuous system in a unified framework. What is more, to improve the precision of settling time, establishing new conditions for finite-time stability is also a challenging and interesting topic.

Motivated by the above observations, our objective is to establish new finite-time stability theorem, design control protocol and improve the precision of settling time for nonlinear system. Our contributions are highlighted as follows. Firstly, we will establish a new finite-time stability theorem for nonlinear continuous and discontinuous systems, which guarantees the associated settling time function is bounded by some constant regardless of initial conditions and in a concise form of framework. It is worthwhile noticing that settling time estimation is given in a concise form compared with [38]. Secondly, comparisons between the obtained estimation and the ones in [13] and [19] are carried out. Thirdly, based on the new finite/fixed-time stability theorem, a new class protocol framework is developed for the general neural network system, and the associated protocols not only solve the finite-time stabilization problem of the general networks system but also guarantees that the corresponding settling time has higher precision. Last but not least, some criteria are proposed to ensure general neural network systems realize finite-time stabilization under the associated protocols.

Notations are quite standard in this paper. R^n and $R^{n\times n}$ stand for n dimensional Euclidean space and the set of all $n\times n$ real matrices, respectively. A^T denotes transpose of matrix or vector A. For symmetric matrices A, B, $A \geq B$ $(A > B)$ means that $A - B$ is positive semi-definite(definite). I and 0 denote the identity matrix and a zero matrix, respectively. $diag\{\cdots\}$ represents a block-diagonal matrix; $sign(\cdot)$ is the sign function. $|v|$ denotes absolute value of v. $\|z\|$ denots 2 norm of vector z. co denotes convex hull and $S \subset \mathrm{R^n}$ denotes a set of measure zero. $\nabla V(\mu)$ denotes gradient of V at μ.

The rest of this paper is arranged as follows. In Section II, preliminaries and problem formulation are presented. Our main results are presented in Section III. Two simulation examples are performed in Section IV. Section V concludes this paper with some conclusions.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. SYSTEMS DESCRIPTION

In this paper, we consider general network system [31], whose dynamics is as follows

$$\dot{x}(t) = Ax(t) + Bf(x(t)) + J \tag{1}$$

where $x(t) = [x_1(t), \cdots, x_n(t)]^T \in R^n$ denotes the state vector, A is not only a diagonal matrix but also a negative definite one. $f(x(t))$ is a nonlinear function or vector field; and J denotes external disturbance vector. In this paper, our aim is to ensure system (1) stabilize to a desired state x^*. By the transformation $y(t) = x(t) - x^*$, one can shift the equilibrium point x^* to the origin of the following system.

$$\dot{y}(t) = Ay(t) + Bg(y(t)), \quad y(0) = y_0 \tag{2}$$

where function $g(y(t)) = f(x(t)) - f(x^*)$. Next, a control protocol $u(t)$ will be designed for the stabilization of system (2). The corresponding closed loop system can be written as follows

$$\dot{y}(t) = Ay(t) + Bg(y(t)) + u(t). \tag{3}$$

And the associated protocol $u(t)$ is designed as follows

$$u(t) = -k_1 y(t) - asign(y(t))|y(t)|^p - bsign(y(t))|y(t)|^{2-p}, \tag{4}$$

where parameters $k_1 > 0$, $a > 0$, $b > 0$ and $p \in [0, 1)$.

It is obvious that the system consisted of (3) and (4) with $0 < p < 1$ is a continuous system. And, if parameter $p = 0$ the control protocol (4) degenerates into

$$u(t) = -k_1 y(t) - asign(y(t)) - bsign(y(t))|y(t)|^2. \tag{5}$$

And the corresponding closed system becomes

$$\dot{y}(t) = Ay(t) + Bg(y(t)) - k_1 y(t) - asign(y(t)) - bsign(y(t))|y(t)|^2, \tag{6}$$

which is indeed a discontinuous system. The solution of (6) is understood under the framework of Filippov solutions [29]. According to proposition 1 in [30], there exists a solution defined on $[0, +\infty)$ for nonlinear system (6) in the sense of Filippov under the following assumption 1:

Assumption 1: Suppose that the nonlinear function $f(\cdot)$ satisfies the following inequality

$$\| f(u) - f(v) \| \leq \| M(u - v) \|, \quad \forall u, \ v \in R^n$$

where M is a known matrix.

Assumption 1 can be found in [31], thus, details on the Assumption 1 can be found in the reference [31]. To save space, we omit it here. Interested readers can refer to [31], where u represents a variable of function $f(u)$.

The implement of the proposed finite-time scheme (4) or (5) is performed as follows:

Step 1: Chose parameter ε and matrix P based on (31), and then derive the condition of parameter k_1.

Step2: Select parameters k_1, a and b.

Step 3: Construct controller by substituting the above parameters into (4) or (5).

It is worth noticing that parameters a and b impact the settling time of systems. Which can be seen from subsequent analysis.

Remark 1: The first step of implement of the scheme is very important, which is helpful to analyze the stability of the associated closed system. that is to say, parameter k_1 has to satisfy inequality (31), which is given in Theorem 5. Moreover, the choice of parameters p and $2 - p$ admits a merit, which can ensure corresponding system achieves both finite-time and fixed-time stability, in addition, the estimation of settling time has higher precision with respect to classic results.

Remark 2: To reduce the chattering phenomenon, we can choose some special functions to replace sign function, such as $sign(y(t))|y(t)|^p = (y(t))^{\frac{m}{n}}$, where m, n are both positive odd numbers and $m < n$, $p = \frac{m}{n}$. Then the chattering phenomenon will be reduced to some extent.

To state our main results, preliminaries, useful lemmas and necessary definitions are presented as follows.

Consider the following nonlinear system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{7}$$

where $x(t) \in R^n$ is state of the system (7), function $f(\cdot) : R^n \to R^n$ is a nonlinear function (or vector field). If $f(\cdot)$ is continuous, the existence of solution of system (equation) (7) can be ensured by the well Peano's theorem. If $f(\cdot)$ is discontinuous but locally measurable, the solution of the system (7) is understood in sense of the Filippov sense.

Definition 1: [19] System (3) is said to be stabilized in a finite time if, for any initial state $y(0)$, there exists a time T such that

$$\lim_{t \to T} \|y(t)\| = 0 \quad \text{and} \quad \|y(t)\| = 0, \quad \text{for } t > T,$$

where T is a positive constant. If T is independent of initial state $y(0)$, system (3) is said to be fixed-time stable.

B. SET VALUED LIE DERIVATIVE AND PRELIMINARIES

Set $E \subset R$, map $\mu \to \mathbb{F}(\mu)$ is said to be a set-valued map: $E \to \mathcal{B}(R^n)$, where $\mathcal{B}(R^n)$ denotes the set consisting of all the subsets of R^n. For every $\mu \in E$, there exists a nonempty set $\mathbb{F}(\mu)$ corresponding to μ.

For the following equation

$$\frac{d\mu}{dt} = l(\mu), \tag{8}$$

where $l(\cdot)$ is a discontinuous function. A solution in sense of Filippov of Cauchy problem of (7) with initial condition $\mu(0) = \mu_0$ is an absolutely continuous function $\mu(t)$, $t \in [0, T]$, such that

$$\frac{d\mu}{dt} \in \mathbb{F}(\mu), \tag{9}$$

where $\mathbb{F}(\mu) = \bigcap_{\delta>0} \cap_{\phi(N)=0} K[l(B(\mu, \delta) \backslash N)]$, $K[E]$ denotes the closure of the convex hull of set E, $B(\mu, \delta) = \{\omega : \|\omega - \mu\| \leq \delta\}$ and $\phi(N)$ denotes the Lebesgue measure of set N. Let $V : R^n \to R$ be a locally Lipchitz function, then the right direction derivative of V at μ and in the direction $v \in R^n$ is denoted by

$$V'_+(\mu, v) = \lim_{h \to 0^+} \frac{V(\mu + hv) - V(\mu)}{h}.$$

The upper generalized derivative [29] of V is defined as $\partial V(u) = \mathrm{co}\{\lim_{i\to\infty} \nabla V(\mu_i) : \mu_i \to \mu, \mu_i \notin S \cup \Omega_V\}$.

The set-valued Lie derivative of V versus l at μ is defined as $L_l V(\mu) = \{a \in R | \exists v \in \mathbb{F}(\mu) \text{ with } v^T\xi = a, \forall \xi \in \partial V(\mu)\}$.

Lemma 1: [32] Let real numbers $\xi_1, \cdots, \xi_n \geq 0, 0 < m \leq 1 < m_1$, then one has

$$\sum_{i=1}^{n} \xi_i^m \geq \left(\sum_{i=1}^{n} \xi_i\right)^m \text{ and } \sum_{i=1}^{n} \xi_i^{m_1} \geq n^{1-m_1} \left(\sum_{i=1}^{n} \xi_i\right)^{m_1}.$$

Lemma 2: [11], [13], [30] Suppose that function $V(x(t)) : R^n \rightarrow R$ is a C-regular, and that $x(t) : [0, +\infty) \rightarrow R^r$ is absolutely continuous on any compact interval of $[0, +\infty)$. If there exists a continuous function $\gamma : [0, +\infty) \rightarrow$ R, $\gamma(\sigma) > 0$ for $\sigma \in (0, +\infty)$ such that

$$\dot{V}(x(t)) < -\gamma(V(x(t))) \tag{10}$$

and

$$\int_0^{V(x(0))} \frac{1}{\gamma(\sigma)} d\sigma = t^* < +\infty. \tag{11}$$

Then, one has $V(t) = 0$ for $t \geq t^*$. In particular,

(1) If $\gamma(\sigma) = a\sigma^p + b\sigma$, for all $\sigma \in (0, +\infty)$, where $p \in (0, 1)$, $b > 0$, $a > 0$, then the associated settling time is estimated by

$$T_{f**} = \frac{1}{a(1-p)} \ln\left(\frac{bV_0^{1-p} + a}{a}\right). \tag{12}$$

(2) If $\gamma(\sigma) = a\sigma^p$, for all $\sigma \in (0, +\infty)$, where $1 > p > 0$, then the associated settling time is estimated by

$$T_{f1} = \frac{V_0^{1-p}}{a(1-p)}. \tag{13}$$

Lemma 3: [19] For system (7), if there exist a regular, positive definite and radially unbounded function $V(x) : R^n \rightarrow R$ and some real numbers $a > 0, b > 0, q > 1 > p \geq 0$ such that

$$\dot{V}(x(t)) \leq -\left(aV^p(x(t)) + bV^q(x(t))\right), \quad x(t) \in R^n \backslash \{0\},$$

then the origin of system (7) is fixed-time stable, and the associated settling time $T(x_0)$ is estimated by

$$T(x_0) \leq T_{\max}^4 = \frac{1}{a}\left(\frac{a}{b}\right)^{\frac{1-p}{q-p}} \left(\frac{1}{1-p} + \frac{1}{q-1}\right). \tag{14}$$

III. MAIN RESULTS

Before considering the finite-time stabilization problem of the system (3) with protocols (4) or (5), respectively, a new finite-time stability theorem is established as follows.

Theorem 1: Under the same conditions as that in Lemma 2, if $\gamma(\sigma) = a\sigma^p + b\sigma^{2-p}$, for $\sigma \in (0, +\infty)$, $a > 0$, $b > 0$, and $p \in [0, 1)$, then the origin of system (7) is finite-time stable, and the associated settling time can be estimated by

$$T(x_0) \leq T_{f*} = \frac{1}{\sqrt{ab}(1-p)} \arctan\left(\sqrt{\frac{b}{a}} V_0^{1-p}\right). \tag{15}$$

Proof: According to Lemma 2, it is easy to prove that $V(t)$ converges to zero in a finite time. Besides, the associated settling time can be estimated as follows.

$$\begin{aligned} T(x_0) &\leq \int_0^{V_0} \frac{1}{aV^p + bV^{2-p}} dV \\ &= \frac{1}{b(1-p)} \int_0^{V_0} \frac{1}{\frac{a}{b} + V^{2-2p}} d\left(V^{1-p}\right) \\ &= \frac{1}{\sqrt{ab}(1-p)} \arctan\left(\frac{\sqrt{b}V_0^{1-p}}{\sqrt{a}}\right) \\ &= T_{f*} \leq T_{max} = \frac{\frac{\pi}{2}}{\sqrt{ab}(1-p)}, \end{aligned} \tag{16}$$

where $V_0 = V(x(0))$. This completes the proof.

Remark 3: From the above proof of Theorem 1, it is easy to see that the proof process of Theorem 1 is simpler than that in [32], which gave a similar result. However, here parameter p can be chosen as zero, but it is unfeasible for that in [32]. Moreover, when $p = 0$, the corresponding estimation is estimated by $T_{f*} = \frac{1}{\sqrt{ab}} \arctan\left(\sqrt{\frac{b}{a}} V_0\right) \leq \frac{\frac{\pi}{2}}{\sqrt{ab}}$.

Remark 4: The parameter p is here obviously different from that in [32], where $p = \frac{p_0}{q_0}$, parameters p_0 and q_0 are positive odd numbers satisfying $p_0 < q_0$. Evidently, here the choice of parameter p is more flexible than that in [32]. If parameter $b = 0$, Theorem 1 will degenerate into Lemma 2. Thus, Theorem 1 can be seen an extension of Lemma 2.

Remark 5: It is obvious that the settling time T_{f*} is a bounded function on initial value V_0, and it is easy to prove that T_{f*} is less than T_{max}. It is worthwhile noticing that T_{max} is similar to that in [37], [38], but the result is concise. Thus, it brings convenience to adjust settling time even if initial states is unknown. Obviously, the power parameters of function $\gamma(\sigma) = a\sigma^p + b\sigma^{2-p}$ must be p and $2-p$, which limits the flexibility of choice of parameters. In other words, this is conservatism of Theorem 1 to some extent.

Based on the Theorem 1, the following result can be derived directly.

Corollary 1: Consider the following scalar system:

$$\dot{y}(t) = -a\,\text{sign}(y(t))|y(t)|^p - b\,\text{sign}(y(t))|y(t)|^{2-p}, \tag{17}$$

where $y(t) \in R$ and parameters $a > 0$, $b > 0$, $0 \leq p < 1$, then the origin of system (17) is finite-time stable and the associated settling time satisfies $T(y_0) \leq T_{f*}$, where

$$T_{f*} = \frac{1}{\sqrt{ab}(1-p)} \arctan\left(\frac{\sqrt{b}|y(0)|^{1-p}}{\sqrt{a}}\right), \tag{18}$$

Proof: Choose the candidate Lyapunov function $V(y(t)) = |y(t)|$. It is easy to verify that $V(y(t))$ is a C-regular, positive definite and radially unbounded function. If parameter $0 < p < 1$, the right hand of system (17) is continuous and the following equality holds:

$$\frac{dV(y(t))}{dt} = \text{sign}(y(t))\dot{y}(t). \tag{19}$$

If parameter $p = 0$, system (17) is a discontinuous one, then one has

$$\partial V(y(t)) = \begin{cases} 1, & \text{if } y(t) < 0 \\ [-1, 1], & \text{if } y(t) = 0 \\ -1, & \text{if } y(t) > 0 \end{cases}, \tag{20}$$

in other words, for any $v(t) \in \partial V(y(t))$, $v(t) = \text{sign}(y(t))$ for $y(t) \neq 0$; and $v(t)$ can be any number in the interval [-1, 1] for $y(t) = 0$. Particularly, $v(t)$ is chosen as $v(t) = \text{sign}(y(t))$, then

$$\frac{dV(y(t))}{dt} = \partial V(y(t))\,\text{sign}(y(t))\dot{y}(t). \tag{21}$$

For any $0 \leq p < 1$, according to proof in [20], it is easy to obtain $T(y_0) \leq T_{f*}$, where

$$T_{f*} = \frac{1}{\sqrt{ab}(1-p)} \arctan\left(\frac{\sqrt{b}|y(0)|^{1-p}}{\sqrt{a}}\right) < T_{max}. \tag{22}$$

This is complete proof.

Remark 6: Evidently, if $p = \frac{m}{n}$, where m, n are positive integers satisfying $m < n$, from Lemma 3, the origin of system (17) is still finite-time stable. and the settling time $T(y_0)$ satisfies

$$\begin{aligned} T(y_0) \leq T^4_{\max} &= \frac{1}{b}\left(\frac{b}{a}\right)^{\frac{1}{2}}\left(\frac{1}{1-\frac{m}{n}} + \frac{1}{1-\frac{m}{n}}\right) \\ &= \frac{1}{\sqrt{ab}}\frac{2}{1-p}. \end{aligned} \tag{23}$$

In fact, in [20] the stability problem of system (17) was discussed, in which m, n are required to be positive odd integers and the settling time satisfies $T(y_0) \leq T^1_{\max} = \frac{1}{(1-p)}\left(\frac{1}{a} + \frac{1}{b}\right)$. Combing results in [19], it is easy to prove that $T_{f*} \leq T^4_{\max} \leq T^1_{\max}$ for any $a, b > 0$, that is, the estimation of the settling time in this paper is more accurate than existing ones.

To highlight the merit of Theorem 3.1, some comparison between our results and existing results in literature are to provided. Firstly, to compare with the results in Lemma 2, the associated conclusion is given in the form of theorems as follows.

Theorem 2: $T_{f*} < T_{f1}$.

Proof: Due to T_{f*} and T_{f1} are positive numbers, according to the basic inequality $\arctan\theta < \theta$ for $\theta > 0$, we have

$$\frac{T_{f*}}{T_{f1}} < \frac{\frac{1}{\sqrt{ab}}\frac{V_0^{1-p}}{\sqrt{\frac{a}{b}}}}{\frac{1}{a}V_0^{1-p}} = 1. \tag{24}$$

Thus, we have $\frac{T_{f*}}{T_{f1}} < 1$ that is to say $T_{f*} < T_{f1}$. This completes the proof of the Theorem.

Moreover, another similar results is presented as follows.

Theorem 3: $T_{f*} > T_{f**}$ if $V_0 \in [0, 1)$, and $T_{f*} < T_{f**}$ if $V_0 \in [1, +\infty)$.

Proof: Evidently, one can obtain

$$\begin{aligned} T_{f**} - T_{f*} = &\frac{1}{\sqrt{b}(1-p)}\left[\frac{1}{\sqrt{b}}\ln\left(1 + \frac{b}{a}V_0^{1-p}\right) - \right] \\ &- \frac{1}{\sqrt{b}(1-p)}\left[\frac{1}{\sqrt{a}}\arctan\left(\sqrt{\frac{b}{a}}V_0^{1-p}\right)\right]. \end{aligned}$$

For the sake of simplicity, let $V_0^{1-p} = u$ and one has

$$f(u) = \frac{1}{\sqrt{b}}\ln\left(1 + \frac{b}{a}u\right) - \frac{1}{\sqrt{a}}\arctan\left(\sqrt{\frac{b}{a}}u\right). \tag{25}$$

It is easy to verify that $T_{f**} - T_{f*} > 0 \Leftrightarrow f(u) > 0$. The derivative of $f(u)$ versus u can be calculated as follows.

$$\dot{f}(u) = \frac{1}{\sqrt{b}}\frac{r^2}{(1 + r^2u)} - \frac{1}{\sqrt{a}}\frac{r}{(1 + r^2u^2)}, \tag{26}$$

where $r = \sqrt{\frac{b}{a}}$. To simplify the expression of function $\dot{f}(u)$, we let

$$\dot{f}(u) = \frac{r}{\sqrt{ab}(1 + r^2u)(1 + r^2u^2)}h(u),$$

where

$$\begin{aligned} h(u) &= r\sqrt{a}\left(1 + r^2u^2\right) - \sqrt{b}\left(1 + r^2u\right) \\ &= \sqrt{a}r^3u^2 - \sqrt{b}r^2u + (r\sqrt{a} - \sqrt{b}). \end{aligned}$$

One can obtain $\dot{f}(u) > 0 \Leftrightarrow h(u) > 0$. In addiiton, it is easy to calculate that

$$h(0) = r\sqrt{a}\left(1 + r^2 \times 0\right) - \sqrt{b}\left(1 + r^2 \times 0\right) = 0, \tag{27}$$

and

$$\begin{aligned} h(1) &= r\sqrt{a}\left(1 + r^2\right) - \sqrt{b}\left(1 + r^2\right) \\ &= \sqrt{a}\left(\sqrt{\frac{b}{a}}\right)^3 - \sqrt{b}\left(\sqrt{\frac{b}{a}}\right)^2 \\ &\quad + \left(\sqrt{\frac{b}{a}}\sqrt{a} - \sqrt{b}\right) = 0. \end{aligned} \tag{28}$$

According to the property of quadratic function, we have $h(u) < 0$ in the interval (0, 1) and $h(u) > 0$ in the interval $(1, +\infty)$. So function $f(u)$ is a decreasing function in the interval (0, 1) and it is an increasing function in the interval $(1, +\infty)$. Due to $h(0) = 0$ and $h(1) = 0$, one has $f(u) < 0$ in the interval (0, 1) and $f(u) > 0$ in the interval $(1, +\infty)$. Therefore, we have $T_{f*} > T_{f**}$ in the interval (0, 1), and $T_{f*} < T_{f**}$ in the interval $(1, +\infty)$. Because $u \in (0, 1)$ if and only if $V_0 \in (0, 1)$ and $u \in (1, +\infty)$ if and only if $V_0 \in (1, +\infty)$, thus $T_{f*} > T_{f**}$ if $V_0 \in [0, 1)$, and $T_{f*} < T_{f**}$ if $V_0 \in [1, +\infty)$. This completes the proof.

Remark 7: The comparison between T_{f*} and T_{f**} demonstrates that initial value V_0 plays a vital role to obtain the high precise settling time. To get high precise settling time estimation, the first step is to calculate the initial value V_0. In addition, it worthwhile noticing that the comparisons between T_{f*} and T_{f**} is independent of the parameter p, $p \in [0, 1)$.

Remark 8: Recalling T_{f*} in the above two theorems, we can find that the bound of the settling time is not only dependent on the control strengths a, b and the initial value $y(0)$, but also on the parameter p. It is well known that, for the given $y(0)$ and p, the larger control strength will result in the smaller settling time. In reverse, assume the control strengths a, b are invariant, we need find the relationship between settling time T_{f*} and parameter p. It is not easy to analyze the relationship between T_{f*} and p directly, but we can analyze the relationship between T_{max} and p. The derivative of T_{max} versus p can be calculated as following.

$$\frac{dT_{\max}}{dp} = \frac{1}{\sqrt{ab}}\frac{\pi}{2} \times \frac{1}{(1-p)^2} > 0. \tag{29}$$

Thus, $T_{\max}$ is an increasing function of parameter p, and T_{max} reaches its minimum $\frac{\pi}{2\sqrt{ab}}$ when $p = 0$. Thus, to obtain the smaller T_{max}, parameter p should be chosen as small as possible.

Moreover, another conclusion is given as follows.

Theorem 4: $T_{f*} < T^4_{\max}$.

Proof: Due to

$$\begin{aligned} T_{f*} &= \int_0^{V_0} \frac{1}{aV^p + bV^{2-p}} dV \\ &< \int_0^{\infty} \frac{1}{aV^p + bV^{2-p}} dV \le T^4_{\max}. \end{aligned} \tag{30}$$

Thus, we have $T^4_{\max} > T_{f*}$. This is complete proof.

Remark 9: Theorem 4 demonstrates that the estimation $T^4_{\max}$ is conservative with respect to T_{f*}. The reason is that inequality $T_{f*} < T^4_{\max}$ always holds whether initial conditions are known or not.

Remark 10: In fact, in [37], $T^4_{\max}$ has been compared with the classic results in [20], where function $\gamma(\sigma) = a\sigma^p + b\sigma^q$, $0 < p < 1 < q$, $a > 0$, $b > 0$. Corresponding settling time is estimated by $T(x_0) \le \frac{1}{a(1-p)} + \frac{1}{b(q-1)}$. [19] has proved that $T^4_{\max}$ has higher precision than the one in [20]. Therefore T_{f*} has higher precision than estimation in [20].

In next subsection, we are to consider the finite-time stabilization issue of system (3) under (4) with $0 < p < 1$.

Theorem 5: Suppose Assumption 1 holds and $0 < p < 1$. Then the system (3) under protocol (4) is finite-time stable if there exist one positive constant ε and one positive definite matrix P such that

$$PA + A^TP - 2k_1P + \varepsilon^{-1}PBBP + \varepsilon M^TM \le 0. \tag{31}$$

In addition, the associated settling time can be estimated by

$$\begin{aligned} T(x_0) &\le T_{f_c^*} \\ &= \frac{2}{\sqrt{a_1b_1}(1-p)} \tan^{-1}\left(\frac{(\lambda_{\max}(P)\,|y_0|)^{1-p}}{\sqrt{\frac{a_1}{b_1}}}\right), \end{aligned} \tag{32}$$

where $b_1 = 2b\lambda_{min}(P)n^{\frac{p-1}{2}}[\lambda_{max}(P)]^{\frac{p-3}{2}}$, $\lambda_{max}(P)$ is the maximum eigenvalue of matrix P, $\lambda_{min}(P)$ is the minimum eigenvalue of matrix P, and $a_1 = 2a\lambda_{min}(P)[\lambda_{max}(P)]^{-\frac{p+1}{2}}$.

Proof: Consider the candidate Lyapunov function

$$V(y(t)) = y^T(t)Py(t),$$

where matrix P satisfies inequality (31). The derivative of $V(y(t))$ along system (3) can be calculated as follows

$$\begin{aligned} \dot{V}(y(t)) &= 2y^T(t)PS - 2y^T(t)Pk_3\,\mathrm{sign}(y(t))|y(t)|^{2-p} \\ &= 2y^T(t)P(A - k_1I)\,y(t) + 2y^T(t)PBg(y(t)) \\ &\quad - 2ay^T(t)P\,\mathrm{sign}(y(t))|y(t)|^p \\ &\quad - 2by^T(t)P\,\mathrm{sign}(y(t))|y(t)|^{2-p} \end{aligned} \tag{33}$$

where $S = (A - k_1I)\,y(t) + Bg(y(t)) - k_2\,\mathrm{sign}(y(t))|y(t)|^p$, I is compatible identity matrix. From the Assumption 1 and the inequality $x^Ty + y^Tx \le \varepsilon x^Tx + \varepsilon^{-1}y^Ty$, where parameter $\varepsilon > 0$ is an arbitrary constant, one has

$$\begin{aligned} 2y^T(t)PBg(y(t)) &\le \varepsilon^{-1}y^T(t)PBB^TPy(t) \\ &\quad + \varepsilon g^T(y(t))g(y(t)) \\ &\le \varepsilon^{-1}y^T(t)PBB^TPy(t) \\ &\quad + \varepsilon y^T(t)M^TMy(t). \end{aligned} \tag{34}$$

Combining (18) and (19), one has

$$\begin{aligned} \dot{V}(y(t)) &\le 2y^T(t)P(A - k_1I)\,y(t) + \varepsilon^{-1}y^T(t)PBB^TPy(t) \\ &\quad + \varepsilon y^T(t)M^TMy(t) \\ &\quad - 2ay^T(t)P\,\mathrm{sign}(y(t))|y(t)|^\alpha \\ &\quad - 2by^T(t)P\,\mathrm{sign}(y(t))|y(t)|^{2-p} \\ &\quad - 2a\lambda_{\min}(P)\sum_{i=1}^n |y_i(t)|^{p+1} \\ &\quad - 2b\lambda_{\min}(P)\sum_{i=1}^n |y_i(t)|^{3-p} \\ &\le -2a\lambda_{\min}(P)\sum_{i=1}^n |y_i(t)|^{p+1} \\ &\quad - 2b\lambda_{\min}(P)\sum_{i=1}^n |y_i(t)|^{3-p}. \end{aligned} \tag{35}$$

From $0 < p < 1$ and Lemma 1, we have the following inequalities

$$\begin{aligned} \sum_{i=1}^n |y_i(t)|^{p+1} &= \sum_{i=1}^n \left((y_i(t))^2\right)^{\frac{p+1}{2}} \\ &\ge \left(\sum_{i=1}^n (y_i(t))^2\right)^{\frac{p+1}{2}} \\ &\ge \left[y^T(t)y(t)\right]^{\frac{p+1}{2}}. \end{aligned} \tag{36}$$

$$\sum_{i=1}^{n}|y_i(t)|^{3-p} = \sum_{i=1}^{n}\left((y_i(t))^2\right)^{\frac{3-p}{2}} \geq n^{\frac{p-1}{2}}\left(\sum_{i=1}^{n}(y_i(t))^2\right)^{\frac{3-p}{2}} = n^{\frac{p-1}{2}}\left[y^T(t)y(t)\right]^{\frac{3-p}{2}}. \tag{37}$$

In addition, due to matrix P is a positive definite, one can obtain the following inequality

$$\lambda_{\max}(P)y^T(t)y(t) \geq V(y(t)) = y^T(t)Py(t) \geq \lambda_{\min}(P)y^T(t)y(t). \tag{38}$$

Thus one has

$$\frac{V(y(t))}{\lambda_{\max}(P)} \leq y^T(t)y(t) \leq \frac{V(y(t))}{\lambda_{\min}(P)}. \tag{39}$$

Combining Lemma 1, (38) and (39), we have

$$\sum_{i=1}^{n}|y_i(t)|^{p+1} \geq \left[y^T(t)y(t)\right]^{\frac{p+1}{2}} \geq \left[\frac{V(y(t))}{\lambda_{\max}(P)}\right]^{\frac{p+1}{2}} \tag{40}$$

and

$$\sum_{i=1}^{n}|y_i(t)|^{3-p} \geq n^{\frac{p-1}{2}}\left[y^T(t)y(t)\right]^{\frac{3-p}{2}} \geq n^{\frac{p-1}{2}}\left[\frac{V(y(t))}{\lambda_{\max}(P)}\right]^{\frac{3-p}{2}}. \tag{41}$$

Combining (35), (36), (39), (40) and (41), we can obtain

$$\begin{aligned}\dot{V}(y(t)) &\leq -2a\lambda_{\min}(P)\sum_{i=1}^{n}|y_i(t)|^{p+1} - 2b\lambda_{\min}(P)\sum_{i=1}^{n}|y_i(t)|^{3-p}\\ &\leq -2a\lambda_{\min}(P)\left[\frac{V(y(t))}{\lambda_{\max}(P)}\right]^{\frac{p+1}{2}} - 2b\lambda_{\min}(P)n^{1-\frac{3-\alpha}{2}}\left[\frac{V(y(t))}{\lambda_{\max}(P)}\right]^{\frac{3-p}{2}}\\ &= -2a\lambda_{\min}(P)\left[\frac{V(y(t))}{\lambda_{\max}(P)}\right]^{\frac{p+1}{2}} - 2b\lambda_{\min}(P)n^{1-\frac{3-\alpha}{2}}\left[\frac{V(y(t))}{\lambda_{\max}(P)}\right]^{\frac{3-p}{2}}\\ &= -a_1(V(y(t)))^{p^0} - b_1(V(y(t)))^{q^0}.\end{aligned} \tag{42}$$

where $p^0 = \frac{1+p}{2} \in [0.5, 1)$, $q^0 = 2 - p^0 > 1$. From Theorem 1 and the above inequality, $V(y(t))$ converges to zero in a finite time, that is, the closed loop system (3) is finite-time stable, and the associated settling time is estimated by

$$\begin{aligned}T(x_0) &\leq \frac{1}{\sqrt{a_1b_1}\left(1-p^0\right)}\arctan\left(\frac{V_0^{1-p^0}}{\sqrt{\frac{a_1}{b_1}}}\right)\\ &= \frac{2}{\sqrt{a_1b_1}(1-p)}\arctan\left(\frac{\left(\lambda_{\max}(P)\,\|y_0\|^2\right)^{\frac{1-p}{2}}}{\sqrt{\frac{a_1}{b_1}}}\right)\\ &= T_{fc*} \leq T_{max} = \frac{\pi}{\sqrt{a_1b_1}(1-p)}.\end{aligned} \tag{43}$$

This is complete proof.

Remark 11: From Theorem 5, it is easy to see that the settling time is upper bounded by a constant $T_{\max}$ regardless of initial states. Thus this makes it flexible to adjust settling time without initial conditions.

For the case $p = 0$, the corresponding system (3) with (5) is a discontinuous one. Subsequently, we will prove that it is still finite-time stable by the Filippov solutions. And the corresponding result is presented as follows.

Theorem 6: Under the same condition in Theorem 5, system (3) under protocol (5) is finite-time stable. Moreover, the settling time can be estimated by

$$T_{fd*} = \frac{2}{\sqrt{a_0b_0}}\arctan\left(\frac{\theta^{\frac{1}{2}}\,\|y_0\|}{\sqrt{\frac{a_0}{b_0}}}\right), \tag{44}$$

where parameters $a_0 = 2a\theta^{\frac{1}{2}}$, $b_0 = 2b\theta^{-\frac{1}{2}}n^{-\frac{1}{2}}$.

Proof: To deal with the discontinuous system (3) with (5), we denote

$$h(y) = (A - k_1I)\,y(t) + By(t) - a\text{sign}(y(t)) - b\,\text{sign}(y(t))|y(t)|^2, \tag{45}$$

and corresponding system can be written as $\dot{y}(t) = h(y)$. Consider candidate Lyapunov function

$$V(y(t)) = y^T(t)Py(t),$$

where $P = \theta I$, where $0 < \theta \in R$. Based on the property of set-valued map $\mathbb{F}$, the set-value Lie derivative of $V(y(t))$ with respect to $\mathbb{F}(h)$, which is denoted by $L_hV(y(t))$, where

$$\mathbb{F}(h) = (A - k_1I)\,y(t) + By(t) - a\,\text{Sign}(y(t)) - b\,\text{Sign}(y(t))|y(t)|^2 \tag{46}$$

Sign(x) = *sign*(x), if $x \neq 0$, and *Sign*$(x) \in [0, 1]$ if $x = 0$. Thus, the set-value Lie derivative can be calculated as following

$$\begin{aligned}L_hV &= \left(\frac{\partial V}{\partial y}\right)^T\mathbb{F}(h)(y) \subseteq 2y^T(t)\theta\,(A - k_1I)\,y(t) + 2y^T(t)\theta Bg(y(t)) - 2\theta a\sum_{i=1}^{n}y_i(t)\,\text{sign}\,(y_i(t)) - 2\theta b\sum_{i=1}^{n}|y_i(t)|^3\\ &= 2y^T(t)\theta\,(A - k_1I)\,y(t) + 2y^T(t)\theta Bg(y(t)) - 2\theta a\sum_{i=1}^{n}|y_i(t)| - 2\theta b\sum_{i=1}^{n}|y_i(t)|^3\end{aligned} \tag{47}$$

which is indeed a scalar function. Moreover, according to Lemma 1, one can obtain the following inequalities

$$\sum_{i=1}^{n}|y_i(t)| \geq \left(\sum_{i=1}^{n} y_i^2(t)\right)^{\frac{1}{2}} \tag{48}$$

and

$$\sum_{i=1}^{n}\left|y_i^3(t)\right| \geq n^{-\frac{1}{2}}\left(\sum_{i=1}^{n} y_i^2(t)\right)^{\frac{3}{2}}. \tag{49}$$

Combing (29), (30) and (31), one has

$$\begin{aligned}\dot{V}(y) = L_h V(y) &= 2y^T(t)\theta\,(A - k_1 I)\,y(t) + 2y^T(t)\theta B g(y(t)) \\ &\quad - 2\theta a \sum_{i=1}^{n}|y_i(t)| - 2\theta b \sum_{i=1}^{n}|y_i(t)|^3 \\ &\leq y^T(t)\left[\theta A + \theta A^T - 2k_1\theta I + \epsilon^{-1}\theta^2 BB^T + \epsilon MM^T\right] \\ &\quad \times y(t) - 2\theta a\left(\sum_{i=1}^{n} y_i^2(t)\right)^{\frac{1}{2}} \\ &\quad - 2b\theta n^{-\frac{1}{2}}\left(\sum_{i=1}^{n} y_i^2(t)\right)^{\frac{3}{2}}. \end{aligned} \tag{50}$$

Then, in the same line as the proof of Theorem 5, one can obtain the following inequality

$$\begin{aligned}\dot{V}(y) &\leq -2\theta a\left(\sum_{i=1}^{n} y_i^2(t)\right)^{\frac{1}{2}} - 2\theta b n^{-\frac{1}{2}}\left(\sum_{i=1}^{n} y_i^2(t)\right)^{\frac{3}{2}} \\ &\leq -2\theta a\left(\frac{V(y)}{\theta}\right)^{\frac{1}{2}} - 2\theta b n^{-\frac{1}{2}}\left(\frac{V(y)}{\theta}\right)^{\frac{3}{2}} \\ &= -2\theta^{\frac{1}{2}} a (V(y))^{\frac{1}{2}} - 2\theta^{-\frac{1}{2}} b n^{-\frac{1}{2}} (V(y))^{\frac{3}{2}} \\ &= -a_0 (V(y))^{\frac{1}{2}} - b_0 (V(y))^{\frac{3}{2}}, \end{aligned} \tag{51}$$

where parameters $a_0 = 2\theta^{\frac{1}{2}}a$, $b_0 = 2\theta^{-\frac{1}{2}}bn^{-\frac{1}{2}}$. According to Theorem 1 and inequality (33), we can conclude that system (3) is finite-time table, and the associated settling time can be estimated by

$$\begin{aligned} T\,(y_0) \leq T_{fd*} &= \frac{2}{\sqrt{a_0 b_0}} \arctan\left(\frac{V_0^{\frac{1}{2}}}{\sqrt{\frac{a_0}{b_0}}}\right) \\ &= \frac{2}{\sqrt{a_0 b_0}} \arctan\left(\frac{\theta^{\frac{1}{2}}\,\|y_0\|}{\sqrt{\frac{a_0}{b_0}}}\right) \\ &\leq T_{max} = \frac{\pi}{2\sqrt{a_0 b_0}}. \end{aligned} \tag{52}$$

This is complete proof.

Remark 12: From the process of the above proof, one can find that the control strength parameters k_1, a, b in the protocol $u(t)$ play different roles, and the inequality (31) plays a critical role in analysis of finite-time stability. k_1 can be obtained from (31), but parameters a, b are independent of (31). Thus, we can firstly fix parameters a, b and focus on the design of k_1. In other words, the desired parameter k_1 can be obtained by linear matrix inequality (31), which was proposed in [33].

To verify our theoretical results and compare with the latest works in references [19], [20], [37], [38], in the section of Numberical Simulation we give the corresponding examples to demonstrate the merits of our theoretical results.

IV. NUMERICAL SIMULATIONS

In this section, two examples are provided to demonstrate the effectiveness of the obtained theoretical results in Section III.

A. EXAMPLE 4.1

To verify the Theorem 1 and Theorem 3.7, we consider the system (17), which was ever discussed in [19], where parameters are chosen as $a = 5$, $b = 1$, initial states are set as $y_1 = -20$, $y_2 = 10$, $y_3 = 30$, $y_4 = -100$, $y_5 = 50$ and $y_6 = 150$.

From Corollary 1, the origin of system (17) is finite-time stable. To illustrate the effectiveness of protocol (4) and (5), respectively, two sets of power parameter p are chosen as (1) $p = 0$; (2) $p = 0.6$.

For the case (1) $p = 0$, the associated initial states are chosen as $y_1 = -20$, $y_2 = 10$, $y_3 = 30$. By computation, one gets $T_{f*}(y_1) = 0.6527$, $T_{f*}(y_2) = 0.6041$, $T_{f*}(y_3) = 0.6692$, $T_{max} = 0.705$, $T_{max}^4 = 0.8944$, $T_{max}^1 = 1.2$. These computation results accords with our theoretical results.

For the case (2), here initial states are chosen as $y_1 = -20$, $y_2 = 10$, and $y_3 = 30$. The magnitude of the associated settling time can be obtained as following: $T_{f*}(y_1) = 1.0926$, $T_{f*}(y_2) = 0.943$, $T_{f*}(y_3) = 1.1739$, $T_{max} = 1.7562$, $T_{max}^4 = 2.2361$, $T_{max}^1 = 3$. Obviously, the settling time given in Theorem 1 is the smallest. Moreover, larger initial values are chosen as $y_4 = -100$, $y_5 = 50$, $y_6 = 150$. we obtain the following estimations of settling time $T_{f*}(y_4) = 1.3754$, $T_{f*}(y_5) = 1.2672$, $T_{f*}(y_6) = 1.429$, $T_{max} = 1.75621$, $T_{max}^4 = 2.236$, $T_{max}^1 = 3$. The corresponding settling time supports our theoretical results as well. State trajectories of the system (17) are shown in Figures.1-2. These computation and simulation results illustrate that whether $p = 0$ or $p = 0.6$, inequalities $T_{f*}(y_i) < T_{max} < T_{max}^4 < T_{max}^1$ always holds for $i = 1, \cdots, 6$, which also shows that the settling time function T_{f*} in this paper has higher precision than those in [18], [20] and [19]. As is shown in Figure 2, the merit is obvious for the larger initial values. These simulation results support our theoretical results perfectly.

B. EXAMPLE 4.2

To verify effectiveness of the protocol (4) and (5), respectively, we simulate the general neural network system (3) with the two protocols. Corresponding parameters are chosen as following

Matrices

$$A = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix},$$

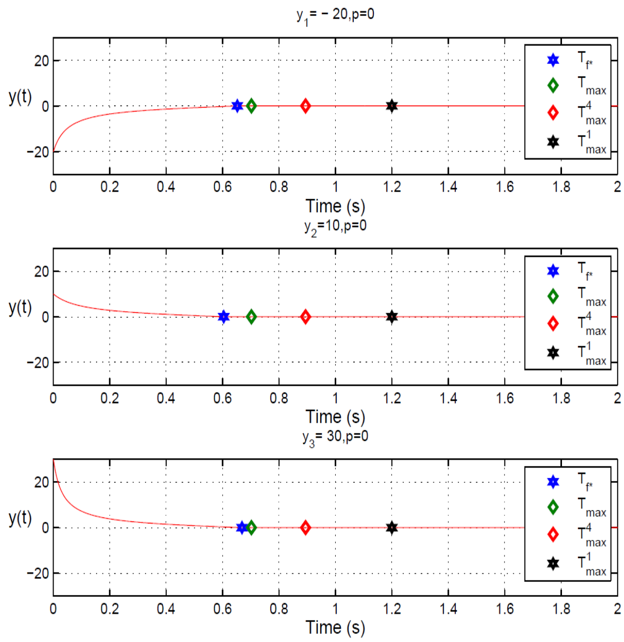

FIGURE 1. State trajectory of system (17) with $p = 0$.

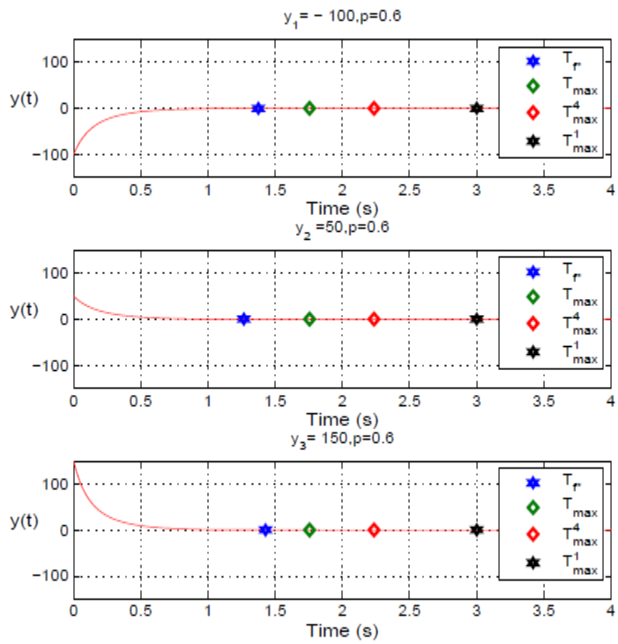

FIGURE 2. State trajectory of system (17) with $p = 0.6$.

$$J = (0, 0, 0)^T,$$

$$B = \begin{bmatrix} 1 & -0.1 & 0.1 \\ -0.1 & 1 & 0 \\ 0.1 & 0 & -0.1 \end{bmatrix},$$

and function $f(\theta) = \tanh(\theta)$. Then, it is obvious that matrix M can be chosen as identity matrix I_3 for the inequality (31). By solving linear matrix inequality (31), we get parameters $\theta = 2.42$, $K = 4.9.74$, $\varepsilon = 10.04$, then one can get $k_1 = \theta^{-1}K = 4.02$. From Theorem 3.13 and Theorem 3.15, the corresponding closed loop system (3) is finite-time stable. Here, to show state trajectories of the general neural networks system (3), three sets of initial states are taken as $x_1(0) = (-3, -1, 3)^T$, $x_2(0) = (-15, -5, 15)^T$, and $x_3(0) = (-30, -10, 30)^T$, parameters $a = 3.2$ and $b = 1$. Firstly, to verify effectiveness of continuous protocol, we set power parameter $p = 0.6$, by simple computation, one gets the associated settling time as follows.

$T_{f_c*}(x_1) = 0.5296$, $T_{f_c*}(x_2) = 0.7193$, $T_{f_c*}(x_3) = 0.7835$, $T_{max} = 1.0114$, $T^4_{max} = 1.2877$, $T^1_{max} = 1.608$. Figure.3 is the corresponding state trajectory of the system (3). The simulation results illustrate that protocol (4) is feasible. In addition, it is worth noticing that the precision of settling time is higher when initial state is smaller. Even if initial states are large, the precision is still higher than exiting results. To verify the effectiveness of the discontinuous protocol (5), pick $p = 0$, other conditions are same to the case $p = 0.6$. By computation, one gets the following estimations of settling time: $T_{f_d*}(x_1) = 0.3756$, $T_{f_d*}(x_2) = 0.4560$, $T_{f_d*}(x_3) = 0.4665$, $T_{max} = 0.4770$, $T^4_{max} = 0.6074$, $T^1_{max} = 0.8439$. Figure.4 is the associated state trajectory of the system (3). From figure 3 and figure 4, one can find that

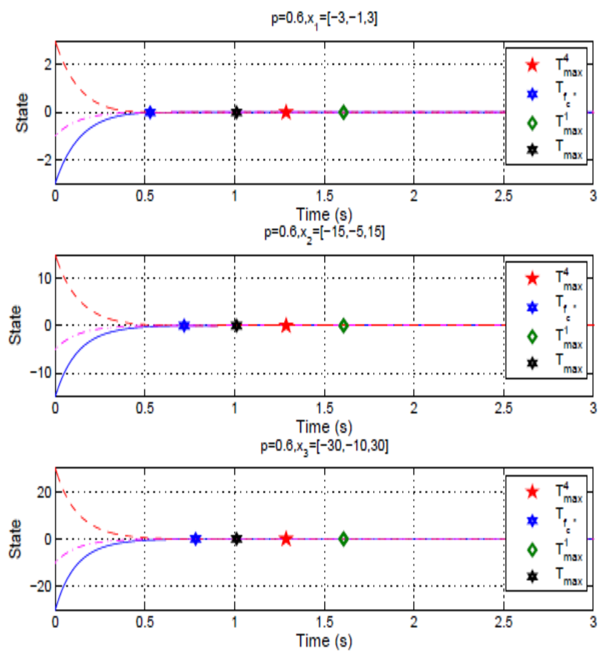

FIGURE 3. State trajectories of system (3) under control protocol (4).

$T_{f*}(x_i) < T_{max} < T^4_{max} < T^1_{max}$ holds for $i = 1, \cdots, 6$ whether $p \in (0, 1)$ or $p = 0$. These simulation results support our theoretical analysis perfectly. From figures 1-4, it can be seen that the precise is higher than the existing results. Especially, for the small initial states, the estimation of settling time has higher precision than the ones in references [13], [19], and concise estimate form than the ones in [37], [38], which estimated by special functions.

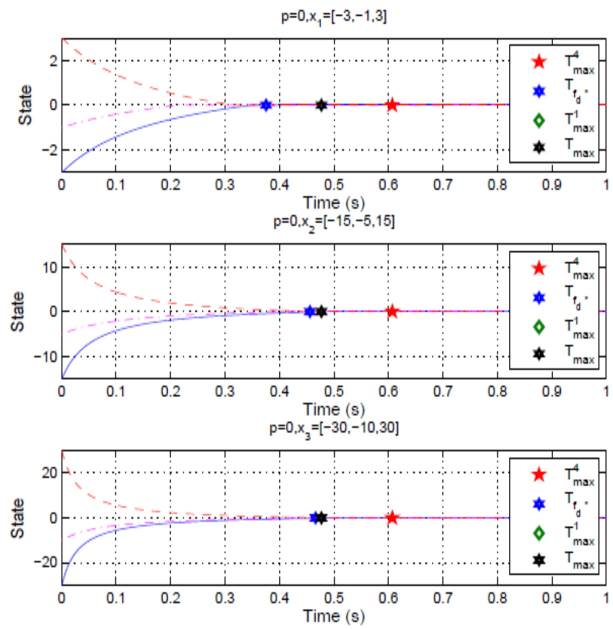

FIGURE 4. State trajectories of system (3) under control protocol (5).

V. CONCLUSION

The stability problem of nonlinear system and stabilization problem of general neural network are investigated in this paper, respectively. By the method of Lyapunov function, a new finite-time stability theorem is established. In addition, some comparisons between Theorem 3.1 and existing classic results on fixed-time stability theorem are carried out. It is shown that the precision of settling time is improved a lot and the estimation of settling time is closer to real settling time and it is upper bounded by a constant independent of the initial states of the controlled system. In particular, we proved that the settling time has higher accuracy than those in [18]–[20], [37]. As a practical application, a new class of protocol framework is designed to solve finite-time stabilization problem of general neural networks. However, there are still some remained work to study, for example, the cases that dynamic of the system including disturbance, delay, and so on, is not involved still here. In addition, similar to reference [36], which dealt with practical finite-time stable of system, how to, by means of similar method and the obtained theoretical results, deal with practical finite-time stable problem of switching delay system is an open problem. We are to explore these problems in future as well.

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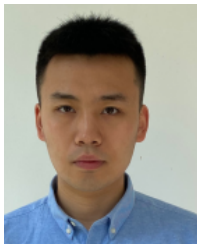

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