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Robustness Analysis of Exponential Stability of Neutral-Type Nonlinear Systems With Multi-Interference

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ABSTRACT With a view to the unfavorable impact of the inevitable exogenous interferences for the practical engineering and signal transmission, here we focus on the robustness of global exponential stability for nonlinear dynamical systems subject to piecewise constant arguments, neutral terms and stochastic disturbances (SNPNDS). A new troublesome problem is that the neutral terms appeared in the derivative part affected on the other two interference factors is not a simple accumulation, so the Lipchitz condition is adopted to establish the ternary transcendental equations. However, different from previous transcendental equations with single or double variables, solving the transcendental equations with three variables becomes a bottleneck again. Hence, the special independent parameters & interdependent variables method targeted for SNPNDS is adopted here: firstly, all relative independent parameters are fixed. Next, the upper bounds of these three interdependent variables are orderly derived by their coupling relationship. Therein, the optimal constraint conditions for piecewise constant arguments and neutral terms are deduced. Through the strategies mentioned above, a class of algebraic problems of estimating three upper bounds by solving transcendental equations with three variables is settled. Besides, the main method ensures that the relationship built among these interference factors is mutually restrictive and dynamic. Meanwhile, the optimized constraints make the linkage effect more comprehensive and valid. Furthermore, the established mechanism is practical enough to be generalized to more multivariable systems. Finally, the numerical simulation comparisons are given to illustrate the validity of the derived results.

INDEX TERMS Robustness, nonlinear system, neutral term, piecewise constant argument, stochastic disturbance.

I. INTRODUCTION

Nonlinear dynamic systems have received more and more attention due to the changeable dynamic properties, variety of model forms and arbitrary switching patterns. Up to now, there have been quite a few characteristic investigations of control methods and dynamical behaviors for nonlinear systems [1]–[20], such as fixed-time control [1], event-triggered adaptive control [2], distributed control [3], piecewise control [4], fuzzy control [5], horizon control [6], U-control [7], passivity cascade technique-based control [8], stabilization control [9], iterative learning control [10], sliding set design [11], robustness control [12] and so forth. In addition, various dynamical behaviors of nonlinear systems have been explored [13]–[20], such as asymptotic stability [13], Mittag-Leffler stability [14], globally exponential stabilization [15], [16], synchronization [13], [17], dissipativity [18], robustness analysis [19], [20], etc. Nowadays, the characteristic application scenarios of nonlinear systems widely appear in reality, such as the computer-node information transmission, the circuit conduction, robot joint control, drive-by-wire control systems and so forth [21]–[23]. Therefore, in terms of the mutability and practicability of nonlinear dynamical systems, a further research on more

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feasible control methods and dynamical behaviors should be carried out, such as the following robustness and global exponential stability (GES).

Robustness and GES for nonlinear systems are two hot research fields no matter in the past or present [15], [16], [19], [20], [24]–[29]. On the one hand, robustness is usually endowed with different meanings in diverse application scenarios. The robustness principle selected in this paper is inspired by the classical one discussed in [24], that is, all upper bounds of the interference factors that the disturbed system can tolerate to maintain stability again should be technically derived on the basis of the original stable system. If the interference intensities are all lower than the deduced threshold values, the system will be stable. Or else, as long as one of the interferences is too large, the system will be unstable and even tremulous. On the other hand, GES can ensure that a system decreases at an exponential decay rate after a rapid response and finally stabilizes at the same equilibrium point. Compared with the general stability, the decay rate of GES can be precisely captured by the final derivation [30]. Therefore, the investigations on the robustness of global exponential stability (RoGES) of nonlinear systems will manifest a great superiority.

Nonlinear dynamic systems are almost inevitably accompanied by sundry disturbances. Thus three categories of crucial interferences widely applied for practical engineering applications are considered here: piecewise constant arguments (PCAs), neutral terms (NTs), and stochastic disturbances (SDs). For the first interference type-piecewise constant argument (PCA), due to the different switching speeds of components at different locations and the discrepant signal conduction times, time-delay is such a common interference factor that it can cause the reception hysteresis, oscillation or behavioral bifurcation of the dynamic system, which is a kind of ubiquitous and indispensable interference factor in communication engineering, image processing, secure communication, etc. PCA, as for an upgraded form of time-delay, originates from [31] and [32], and then Akhmet *et al.* [30] used equivalent integral equations to study the stability of differential equations by constructing the Lyapunov function, which complements the knowledge territory of the solution channels towards the stability issues with PCA. At present, PCA has been gradually applied to various nonlinear dynamical systems [33]–[40], such as Cohen-Grossberg neural networks [33], BAM systems [34], memristor-based dynamic systems [35], cellular neural networks [36], [37], nonlinear differential equations [38], [39], fuzzy neurodynamic systems [40], etc. For the second interference type-neutral term (NT), NT is the interference factor located in the derivative part of the system. Neutral-type nonlinear systems have some specific physical application scenarios, such as the electrical interconnect and the electromagnetic interference design in digital computers [41], that is, an equivalent circuit neutral-type nonlinear system is favorable to handle a class of electromagnetic problems. In addition, chemical reaction processes, fluid flow processes, and turbojet rotation processes can all be modeled by using neutral-type nonlinear systems. The dynamical behaviors for neutral-type nonlinear systems include: stability [42], [43], controllability [44], stabilisation [45], passiveness [46], [47], asymptotic behavior of solutions [48], [49], roubustness [50], [51] and so forth. Therefore, it is practical to consider the GES of nonlinear systems with NTs. For the third interference type-stochastic disturbances (SD), SD actually refers to the noise derived from random Brownian motion in real world. The noise acting on multifarious dynamical systems is usually pervasive and everlasting, but there were no appropriate mathematical tools to characterize this variable originally until the appearance of It$\hat{o}$ integral defined by K.It$\hat{o}$ in 1949 [52]. Later, Shen and Wang [26] and Zhang [28] further used the analytical method of equivalent integral equations and stochastic differential equations to establish a new measurement by estimating the upper bound of SDs to analyze the robustness of the systems.

So far, with a view to these three kinds of interference factors: PCA, NT and SD, some investigators researched the robustness or GES of nonlinear systems influenced by single interference factor, such as [6], [20], [24]–[26], [30], [33], [50], etc. Some investigators explored the robustness or GES of systems affected by dual interference factors, such as [16], [27]–[29], [51], [53] and so forth. In the above literatures, Ref. [6] established an approximation method based on Monte Carlo simulation to solve the receding nonlinear stochastic control stability problems. Ref. [33] studied the interval fuzzy robust exponential stability of Cohen-Grossberg networks by means of the comparison principle. Ref. [50] solved the controller gains problems for the robustness of the fractional-order NDS according to the LMI method and the cone complementarity linearization method. Ref. [16] proposed a split-step θ-method and presented comparison results of the numerical and real solutions for the exponential stability of a class of stochastic differential equations. Ref. [51] raised the existence & uniqueness theorem for the stochastic neutral functional differential equations with infinite delays and tested the almost sure stability performance of the states with the general rate of decay. Ref. [53] considered the pth moment exponential stability equivalence among the stochastic differential equations and its derivative systems by Euler-Maruyama method. In addition, Ref. [20], [24]–[29] investigated the robustness of dynamical systems by deriving the upper bounds of the single interference factors [20], [24]–[26] or dual interference factors [27]–[29]. However, very few studies have been done to explore the RoGES of nonlinear systems subject to all these three interference factors. Hence, it is a requisite and realistic issue to explore that the perturbed nonlinear systems can tolerate how much the interval length of these interferences to maintain the stability again based on the original stable system.

Synthesize the above, in accordance with [29], we will further explore the RoGES of nonlinear dynamical system with piecewise constant arguments, neutral terms and stochastic disturbances (SNPNDS). (NPNDS is the system which removes the stochastic disturbances of SNPNDS). Estimating the upper bounds of all these three interference factors is the final aim. Thus, the main contributions of this paper are listed: (1) An "independent parameters & interdependent variables" method targeted for SNPNDS model is created for this paper. Previous method of solving the transcendental equations with single variable and double variables are all quite unsuitable to solve the ternary transcendental equations in this paper, so the above method is proposed, which means that some parameters independent of three variables can be fixed firstly, and then the upper bounds of interference factors are given by the corresponding transcendental equations orderly rather than simultaneously (more details can be seen in Remark 11 and Fig. 2). Therefore, on the one hand, an algebraic problem of estimating the upper bounds by solving transcendental equations with three variables is solved. On the other hand, the relationship built among the three interference factors of SNPNDS by this method is dynamic and efficient. (2) The neutral terms that appear in the derivative part is a troublesome problem for deriving the upper bounds because the effect of neutral terms on the other two factors is not just a simple accumulation. More importantly, the interference interaction relationship between neutral terms and the other two interference factors is unpredictable. So we adopted lipchitz conditions to construct the relationship between the neutral functions and the current states, which brings great convenience to establish transcendental equations with these three interference factors. (3) We considered the more comprehensive constraint conditions for PCAs and NTs compared with the work [24]–[28], such as the θ_1, θ_2 and k set in Theorem 1 and Theorem 2, which guarantee the preciseness of the upper bounds of PCAs and NTs and the higher validity of the linkage relationship of PCAs, NTs and SDs. (4) The final dynamic data groups and the numerical simulations intuitively demonstrated that the new NPNDS and SNPNDS have strong robustness under the optimized constraint conditions of theorems, that is, these conditions can automatically adjust the relative size of these three interference factors. Then if the actual jamming intensities of the PCAs, NTs and SDs are all lower than the derived upper bounds, the disturbed system will be GES. Or else, the system will be unstable and even appear oscillation phenomenon. Apparently, the method established in this paper is practical enough to be generalized from three-variable systems to multivariable systems. One of the practical applications of this paper is the stability of signal transmission in large-scale power system which is susceptible to external interferences. For example, it is worth thinking about the robustness of a repaired system after some remote signal interruptions. Furthermore, the handling method for multi-interference systems appeared in this paper can be used to measure the robustness level of more other neutral-type nonlinear systems subject to various exogenous factors by estimating their threshold values.

The composition of the remaining parts is organized as follows. Some necessary mathematical notations are shown in part II. Lemma 1 and the main Theorem 1 to illustrate the RoGES of NPNDS are given in part III. Lemma 2 and the main Theorem 2 to illustrate the RoGES of SNPNDS are given in part IV. The intuitive numerical simulation comparisons are given to demonstrate the benign robustness of NPNDS and SNPNDS in part V. Final part VI is a summary of this paper and a further vision of the future work.

II. PRELIMINARIES

Throughout the paper, let $\boldsymbol{N}$ be the natural number set. For given countable constant n, denote $\underline{n} = \{1, 2, \cdots, n\}$. Denote $\boldsymbol{R}$ as the real number set. Let $\boldsymbol{Z}^+$ be the positive integer set. $\mathbb{R}^n$ stands for n-dimensional Euclidean space and $\mathbb{R}^+$ is a non-negative real number space. For a n-dimensional vector $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$, where T is the transpose of a vector, the vector norm of ψ is recorded as $\|\psi\| = \sum_{i=1}^{n} |\psi_i|$. $\{\theta_p\}$, $\{\eta_p\}$ are two real-valued sequences, $p \in \boldsymbol{N}$, satisfying $\theta_p \leq \eta_p \leq \theta_{p+1}$, such that $\lim_{p\to\infty} \eta_p = +\infty$. Besides, denote that $\{\mathfrak{F}_t\}_{t\geq t_0\geq 0}$ is a filtration which is right-continuous and increasing with $\{\mathfrak{F}_0\}$ which includes every P-null set on Φ, where $\Phi = (\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq t_0\geq 0}, P)$ is the representation of a complete probability space, whereupon $B(t) \in \Phi$ is a scalar Brownian motion. $E\{\cdot\}$ means the mathematical expectation defined on sapce Φ.

III. THE ROBUSTNESS OF NPNDS

The main work of nonlinear dynamical systems with neutral terms and generalized piecewise constant arguments (NPNDS) will be explained in the following section:

$$\begin{cases} d[r(t) - G(r(t))] = f(r(t), r(\rho(t)), t)dt, \ t \geq t_0, \\ r(t_0) = r_0, \end{cases} \tag{1}$$

where $G(r(t)) : \mathbb{R}^+ \to \mathbb{R}^n \to \mathbb{R}^n$ is the neutral-type function. $f(r(t), r(\rho(t)), t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is activation function about state $r(t)$ and $r(\rho(t))$, where $r(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is the current state and $r(\rho(t)) : \mathbb{R}^+ \to \mathbb{R}^n \to \mathbb{R}^n$ is the PCA state. Besides, $\rho(t) = \eta_p$ when $\theta_p \leq t < \theta_{p+1}, p \in \boldsymbol{N}$.

Remark 1: Understandably, if we consider the dynamic behavior characteristic of NPNDS in the interval $[\theta_p, \theta_{p+1})$, $p \in \boldsymbol{N}$. NPNDS (1) is an advanced system when $\theta_p \leq t < \eta_p$, and NPNDS (1) is a hysteretic system when $\eta_p \leq t < \theta_{p+1}$. Therefore, system (1) is a mixed system that unifies advanced and hysteretic time deviation.

Without piecewise constant arguments and neutral terms, NPNDS (1) can be regarded as the following nonlinear dynamic system (NDS):

$$\begin{cases} \dot{u}(t) = f(u(t), u(t), t), \ t \geq t_0, \\ u(t_0) = u_0 = r_0. \end{cases} \tag{2}$$

In terms of [54], the solution to NDS (2) exists and be unique, and NDS (2) contains the origin solution.

Subsequently, the exponential stable expression of the state of the system (2) is given by the following Definition 1.

Definition 1: If there are $\alpha > 0$ and $\beta > 0$ such that $\|u(t;t_0,u_0)\| \le \alpha\|u(t;t_0,u_0)\|\exp\{-\beta(t-t_0)\}$ is credible for any $t \ge t_0$, then the state $u(t;t_0,u_0)$ of NDS (2) can achieve GES.

Throughout the section III, some rational assumptions are given below:

(A1) Suppose that $f(\cdot)$ is local lispchitz, $f(0,0,t)=0$, there are lipschitz constants $k_1 > 0$ and $k_2 > 0$ such that

$$\|f(\varrho_1,\varsigma_1,t)-f(\varrho_2,\varsigma_2,t)\| \le k_1\|\varrho_1-\varrho_2\|+k_2\|\varsigma_1-\varsigma_2\|,$$

for any $\varrho_1,\varrho_2,\varsigma_1,\varsigma_2 \in \mathbb{R}^n$, and $t \in \mathbb{R}^+$.

(A2) There is a Lipschitz constant k such that

$$\|G(\varrho)-G(\varsigma)\| \le k\|\varrho-\varsigma\|,\ k\in(0,1),\ k\in\boldsymbol{R}$$

holds for all $\varrho,\varsigma \in \mathbb{R}^n$.

(A3) Assume that $\theta_{p+1}-\theta_p \le \theta$ for any $p \in \boldsymbol{N}$, $\theta > 0$.

$$\text{(A4)}\ \frac{k_1\theta(1+k+k_2\theta)}{(1-k)^2}\exp\left\{\frac{k_1\theta}{1-k}\right\}+\frac{k_2\theta}{1-k}<1.$$

$$\text{(A5)}\ \alpha exp\{-\beta(T-\theta)\}+\frac{1}{1-k}\left[k\alpha\exp(-\beta\theta)+k+\frac{2k_2\alpha}{\beta(1-k)}\right]exp\left\{\frac{2T[k_1+3k_2-k(k_1+k_2)]}{(1-k)^2}\right\}<1.$$

Remark 2: (1) "k" appeared in (A2) is named as the neutral term (NT) compressibility coefficient, and the upper bound of NTs expressed throughout this paper exactly refers to the threshold value of k. (2) "θ" appeared in (A3) represents piecewise constant argument (PCA) interval length, and the upper bound of PCAs expressed throughout this paper exactly refers to the threshold value of θ.

Remark 3: In section III, for notational brevity, let

$$\begin{aligned}
\hat{M} &= \alpha\exp\{-\beta(T-\theta)\}+\omega_1\exp\{2T\omega_2\},\\
\omega_1 &= \frac{1}{1-k}\left\{k\alpha\exp(-\beta\theta)+k+\frac{k_2\alpha(1+\lambda)}{\beta}\right\},\\
\omega_2 &= \frac{k_1+(2+\lambda)k_2}{1-k},\\
\lambda &= \frac{1+k}{(1-\xi)(1-k)},\\
\xi &= \frac{k_1\theta(1+k+k_2\theta)}{(1-k)^2}\exp\left\{\frac{k_1\theta}{1-k}\right\}+\frac{k_2\theta}{1-k},\\
T &> (ln\alpha)/\beta>0,\quad T\in\boldsymbol{R}.
\end{aligned}$$

Then the following Lemma 1 aims to clarify the relationship between the generalized PCA state $r(\rho(t))$ and the current state $r(t)$.

Lemma 1: If $r(t)=(r_1(t),r_2(t),\ldots,r_n(t))^T$ is a solution of NPNDS (1), $n\in\underline{n}$. Then inequality

$$r(\rho(t)) \le \lambda\|r(t)\| \tag{3}$$

holds based on (A1)-(A4) for any $t \ge 0$, where

$$\lambda = \frac{1+k}{(1-\xi)(1-k)}, \tag{4}$$

$$\xi = \frac{k_1\theta(1+k+k_2\theta)}{(1-k)^2}\exp\left\{\frac{k_1\theta}{1-k}\right\}+\frac{k_2\theta}{1-k}. \tag{5}$$

Proof: From (1), for any $t\in[\theta_p,\theta_{p+1})$, when $\rho(t)=\eta_p$, $p\in\boldsymbol{N}$, we have

$$\begin{aligned}
r(t)-r(\eta_p)&+G(r(\eta_p))-G(r(t))\\
&=\int_{\eta_p}^t f(r(s),r(\eta_p),s)ds.
\end{aligned} \tag{6}$$

On the one hand, taking the absolute value and using the norm inequality on both sides of (6), we get

$$\begin{aligned}
\|r(t)-r(\eta_p)\|&-\|G(r(\eta_p))-G(r(t))\|\\
&\le\int_{\eta_p}^t \|f(r(s),r(\eta_p),s)\|ds.
\end{aligned} \tag{7}$$

Then according to (A1)-(A3), for $\theta_p \le \eta_p < t < \theta_{p+1}$, we have

$$\begin{aligned}
\|r(t)\| &\le \|r(\eta_p)\|+k\|r(\eta_p)-r(t)\|+k_1\int_{\eta_p}^t\|r_1(s)\|ds\\
&\quad+k_2\int_{\eta_p}^t\|r(\eta_p)\|ds\\
&\le \|r(\eta_p)\|+k\|r(\eta_p)\|+k\|r(t)\|+k_1\int_{\eta_p}^t\|r(s)\|ds\\
&\quad+k_2\theta\|r(\eta_p)\|.
\end{aligned} \tag{8}$$

Directly, from (8), we get

$$\|r(t)\| \le \frac{1+k+k_2\theta}{1-k}\|r(\eta_p)\|+\frac{k_1}{1-k}\int_{\eta_p}^t\|r(s)\|ds$$

with $k\in(0,1)$. Then the Gronwall-Bellman Lemma yields that

$$\|r(t)\| \le \frac{1+k+k_2\theta}{1-k}\exp\left\{\frac{k_1\theta}{1-k}\right\}\|r(\eta_p)\|. \tag{9}$$

Symmetrically, from (6), we further have

$$G(r(\eta_p))-G(r(t))=r(\eta_p)-r(t)+\int_{\eta_p}^t f(r(s),r(\eta_p),s)ds. \tag{10}$$

Taking the absolute value and using the norm inequality on both sides of (10), according to (A2), we obtain

$$\begin{aligned}
\|r(\eta_p)-r(t)&+\int_{\eta_p}^t f(r(s),r(\eta_p),s)ds\|\\
&\le k\|r(\eta_p)\|+k\|r(t)\|.
\end{aligned} \tag{11}$$

Using the norm inequality on the left of (11) again, so

$$\begin{aligned}
\|r(\eta_p)\|-\|r(t)\|&-\int_{\eta_p}^t\|f(r(s),r(\eta_p),s)\|ds\\
&\le k\|r(\eta_p)\|+k\|r(t)\|.
\end{aligned} \tag{12}$$

Combining the similar terms in (12), we have

$$(1-k)\|r(\eta_p)\| \le (1+k)\|r(t)\|+\int_{\eta_p}^t(f(r(s),r(\eta_p),s)ds. \tag{13}$$

From (13), together with (A1), (A3) and (9), we get

$$\begin{aligned}\|r(\eta_p)\| \leq\ & \frac{1+k}{1-k}\|r(t)\| + \frac{1}{1-k}\Big(k_1\int_{\eta_p}^{t}\|r(s)\|ds \\ & + k_2\int_{\eta_p}^{t}\|r(\eta_p)\|ds\Big) \\ \leq\ & \frac{1+k}{1-k}\|r(t)\| + \frac{k_1\theta}{1-k}\Big[\frac{1+k+k_2\theta}{1-k} \\ & \times \exp\Big\{\frac{k_1\theta}{1-k}\Big\}\|r(\eta_p)\|\Big] + \frac{k_2\theta}{1-k}\|r(\eta_p)\| \\ \leq\ & \frac{1+k}{1-k}\|r(t)\| + \Big\{\frac{k_1\theta(1+k+k_2\theta)}{(1-k)^2}\exp\Big\{\frac{k_1\theta}{1-k}\Big\} \\ & + \frac{k_2\theta}{1-k}\Big\}\|r(\eta_p)\| \\ =:\ & \frac{1+k}{1-k}\|r(t)\| + \xi\|r(\eta_p)\|, \end{aligned} \tag{14}$$

where

$$\xi = \frac{k_1\theta(1+k+k_2\theta)}{(1-k)^2}\exp\Big\{\frac{k_1\theta}{1-k}\Big\} + \frac{k_2\theta}{1-k}, \quad k \in (0,1).$$

Therefore, according to (14), when $\rho(t) = \eta_p$, $t \in [\theta_p, \theta_{p+1})$, $p \in \boldsymbol{N}$, it follows that

$$\begin{aligned}\|r(\eta_p)\| \leq\ & \frac{1}{1-\xi}\cdot\frac{1+k}{1-k}\|r(t)\| \\ =:\ & \lambda\|r(t)\|,\end{aligned}$$

where

$$\lambda = \frac{1}{1-\xi}\cdot\frac{1+k}{1-k}.$$

Hence, for $t \in [\theta_p, \theta_{p+1})$, because of the arbitrariness of t and p, (3) is valid for any $t \geq 0$. And so far we accomplish the proof.

Theorem 1: Supposed (A1)-(A5) is attainable and NDS (2) can achieve GES. Then NPNDS (1) can achieve GES if the bound of neutral term compressibility coefficient $k < \bar{k}$, where $\bar{k}$ is the supremum of the following inequality

$$\begin{aligned}\alpha\exp(-\beta T) + \frac{1}{1-\bar{k}}\Big\{\bar{k}\alpha + \bar{k} + \frac{2k_2\alpha}{\beta(1-\bar{k})}\Big\} \\ \times\exp\Big\{\frac{2T[k_1+3k_2-\bar{k}(k_1+k_2)]}{(1-\bar{k})^2}\Big\} < 1,\end{aligned} \tag{15}$$

and the interval length of the piecewise constant arguments satisfies

$$0 < \theta < \theta_4 = \min\Big\{\frac{T}{2},\ \theta_1,\ \theta_2,\ \theta_3\Big\}, \tag{16}$$

where θ_1 and θ_2 are the upper bounds that satisfy assumptions (A4) and (A5) respectively, θ_3 is the unique positive solution of the following transcendental equation

$$\alpha\exp\{-\beta(T-\theta)\} + \omega_1\exp\{2T\omega_2\} = 1, \tag{17}$$

where

$$\begin{aligned}\lambda &= \frac{1+k}{(1-\xi)(1-k)}, \\ \xi &= \frac{k_1\theta(1+k+k_2\theta)}{(1-k)^2}\exp\Big\{\frac{k_1\theta}{1-k}\Big\} + \frac{k_2\theta}{1-k}, \\ \omega_1 &= \frac{1}{1-k}\Big\{k\alpha\exp(-\beta\theta) + k + \frac{k_2\alpha(1+\lambda)}{\beta}\Big\}, \\ \omega_2 &= \frac{k_1+(2+\lambda)k_2}{1-k},\end{aligned}$$

and α, β, k_1, k_2 are all known constants, $T > (ln\alpha)/\beta > 0$, $k \in (0,1)$.

Proof: For briefly record, set $r(t, t_0, r_0)$ as $r(t)$ and $u(t, t_0, u_0)$ as $u(t)$, from (1) and (2), according to (A1)-(A3) and (3), for any $t \geq t_0 \geq 0$, we can get

$$\begin{aligned}&\|r(t)-u(t)\| \\ &= \|G(r(t)) - G(r_0) + \int_{t_0}^{t}[f(r(s), r(\rho(s)), s) \\ &\quad -f(u(s), u(s), s)]ds\| \\ &\leq \|G(r(t)) - G(r_0)\| + k_1\int_{t_0}^{t}\|r(s)-u(s)\|ds \\ &\quad + k_2\int_{t_0}^{t}\|r(\rho(s)) - r(s) + r(s) - u(s)\|ds \\ &\leq k\|r(t)-r_0\| + (k_1+k_2)\int_{t_0}^{t}\|r(s)-u(s)\|ds \\ &\quad + k_2\int_{t_0}^{t}\|r(\rho(s)-r(s))\|ds \\ &\leq k\|r(t)\| + k\|r_0\| + (k_1+k_2)\int_{t_0}^{t}\|r(s)-u(s)\|ds \\ &\quad + k_2\int_{t_0}^{t}[\|r(\rho(s))\| + \|r(s)\|]ds \\ &\overset{Lem1}{\leq} k\|r(t)-u(t)\| + k\|u(t)\| + k\|r_0\| + (k_1+k_2) \\ &\quad \times\int_{t_0}^{t}\|r(s)-u(s)\|ds + k_2(1+\lambda)\int_{t_0}^{t}\|r(s)-u(s) \\ &\quad + u(s)\|ds \\ &\leq k\|r(t)-u(t)\| + k\|u(t)\| + k\|r_0\| + [k_1+(2+\lambda)k_2] \\ &\quad \int_{t_0}^{t}\|r(s)-u(s)\|ds + k_2(1+\lambda)\int_{t_0}^{t}\|u(s)\|ds.\end{aligned} \tag{18}$$

From Definition 1, on the interval $[t_0-\theta, t_0+\theta]$, we have

$$\begin{aligned}&\|r(t)-u(t)\| \\ &\leq k\|r(t)-u(t)\| + k\alpha\exp(-\beta(t-t_0))\|u_0\| + k\|r_0\| \\ &\quad + [k_1+(2+\lambda)k_2]\times\int_{t_0}^{t}\|r(s)-u(s)\|ds \\ &\quad + \frac{k_2\alpha(1+\lambda)}{\beta}\|u_0\| \\ &\leq k\|r(t)-u(t)\| + \Big\{k\alpha\exp(-\beta\theta) + k \\ &\quad + \frac{k_2\alpha(1+\lambda)}{\beta}\Big\}\|y_0\| + [k_1+(2+\lambda)k_2] \\ &\quad \times\int_{t_0}^{t}\|r(s)-u(s)\|ds.\end{aligned} \tag{19}$$

From (19), directly, we have

$$\begin{aligned}
&\|r(t)-u(t)\| \\
&\leq \frac{1}{1-k}\left\{k\alpha\exp(-\beta\theta)+k+\frac{k_2\alpha(1+\lambda)}{\beta}\right\}\|r_0\| \\
&\quad+\frac{k_1+(2+\lambda)k_2}{1-k}\int_{t_0}^{t}\|r(s)-u(s)\|ds \\
&\leq \omega_1\|y_0\|+\omega_2\int_{t_0}^{t}\|r(s)-u(s)\|ds,
\end{aligned}$$

where

$$\begin{aligned}
\omega_1 &= \frac{1}{1-k}\left\{k\alpha\exp(-\beta\theta)+k+\frac{k_2\alpha(1+\lambda)}{\beta}\right\}, \\
\omega_2 &= \frac{k_1+(2+\lambda)k_2}{1-k}.
\end{aligned}$$

According to Gronwall inequation, for $t_0+\theta \leq t \leq t_0+2T$,

$$\begin{aligned}
\|r(t)-x(t)\| &\leq \omega_1\|r_0\|\exp\{\omega_2(t-t_0)\} \\
&\leq \omega_1\exp\{2T\omega_2\}\sup_{t_0-\theta\leq t\leq t_0+\theta}\|r(t)\|, \quad (20)
\end{aligned}$$

Subsequently, from (20), we get

$$\begin{aligned}
\|r(t)\| &\leq \|r(t)-u(t)\|+\|u(t)\| \\
&\leq \omega_1\exp\{2T\omega_2\}\sup_{t_0-\theta\leq t\leq t_0+\theta}\|r(t)\|+\|u(t)\|.
\end{aligned}$$

Note that $\theta < \frac{T}{2}$, according to the global exponential stability of (2), when $t_0-\theta+T \leq t \leq t_0-\theta+2T$, we further obtain

$$\begin{aligned}
\|r(t)\| &\leq \left\{\alpha\exp\{-\beta(T-\theta)\}+\omega_1\exp\{2T\omega_2\}\right\} \\
&\quad\times\sup_{t_0-\theta\leq t\leq t_0-\theta+T}\|r(t)\| \\
&=: \hat{M}\sup_{t_0-\theta\leq t\leq t_0-\theta+T}\|r(t)\|, \quad (21)
\end{aligned}$$

where

$$\hat{M}=\alpha\exp\{-\beta(T-\theta)\}+\omega_1\exp\{2T\omega_2\}.$$

Denote a function

$$\begin{aligned}
F(\lambda(\theta),k) &= \alpha\exp\{-\beta(T-\theta)\}+\frac{1}{1-k} \\
&\quad\times\left\{k\alpha\exp(-\beta\theta)+k+\frac{k_2\alpha(1+\lambda(\theta))}{\beta}\right\} \\
&\quad\times\exp\left\{\frac{2T[k_1+(2+\lambda(\theta))k_2]}{1-k}\right\}, \quad (22)
\end{aligned}$$

where

$$\lambda(\theta)=\frac{1+k}{(1-k)(1-\xi(\theta))}.$$

On the one hand, the bound of θ can be derived here. From (A4), surely we know

$$0<\xi(\theta)=\frac{k_1\theta(1+k+k_2\theta)}{(1-k)^2}\exp\left\{\frac{k_1\theta}{1-k}\right\}+\frac{k_2\theta}{1-k}<1, \quad (23)$$

assume θ_1 is the unique solution of $\xi(\theta)=1$, obviously

$$\lambda(\theta)\in(\frac{1+k}{1-k},+\infty)$$

holds for any $\theta \in (0,\theta_1)$. Thus there exists the unique upper bound θ_1 that satisfy (23), i.e. (A4).

Invoking (22) and (A5), we have

$$\begin{aligned}
&F(\frac{1+k}{1-k},k) \\
&= \alpha\exp\{-\beta(T-\theta)\}+\frac{1}{1-k}\left[k\alpha\exp(-\beta\theta)+k\right. \\
&\quad\left.+\frac{2k_2\alpha}{\beta(1-k)}\right]\exp\left\{\frac{2T[k_1+3k_2-k(k_1+k_2)]}{(1-k)^2}\right\} \\
&< 1, \quad (24)
\end{aligned}$$

and

$$F(\infty,k)>1. \quad (25)$$

Combining (24), (25) and the monotone increasing property of $F(\lambda(\theta),k)$, there exists

$$\hat{\lambda}(\theta)\in(\frac{1+k}{1-k},+\infty)$$

such that

$$F(\hat{\lambda}(\theta),k)=1. \quad (26)$$

holds. Based on the above analysis, there exists a unique $\hat{\theta} \in (0,\theta_1)$, which makes $\lambda(\theta)=\hat{\lambda}(\theta)$ true. Accordingly, there is a unique θ_2 which makes $F(\hat{\lambda}(\theta),k)=1$ true, θ_2 is the upper bound of Assumption (A5). Suppose θ_3 is the unique positive solution of (22). Choose

$$\theta_4=\min\left\{\frac{T}{2},\theta_1,\theta_2,\theta_3\right\}, \quad (27)$$

Hence, (A4), (A5) and $0<\hat{M}=F(\lambda(\theta),k)<1$ have been all satisfied when $0<\theta<\theta_4$, that is, the bound of θ is obtained.

On the other hand, the bound of θ can be derived here. Since $\frac{\partial\lambda(\theta)}{\partial\theta}>0$, $\lambda(\theta)$ increases monotonically with respect to θ ($\theta>0$). So there exists

$$\lambda(\theta)\big|_{\theta=0}=\lambda(0)=\frac{1+k}{1-k}.$$

Besides, from (22), since $\frac{\partial F(\lambda(\theta),k)}{\partial\theta}>0$, surely there also exists

$$F(\lambda(\theta),k)\big|_{\theta=0}=F(\lambda(0),k)=F(\frac{1+k}{1-k},k)\big|_{\theta=0}.$$

The upper bound of k is recorded as $\bar{k}$, $k<\bar{k}$, where $\bar{k}$ can be given by $F(\lambda(\theta),\bar{k})\big|_{\theta=0}<1$, that is

$$\begin{aligned}
&\alpha\exp(-\beta T)+\frac{1}{1-\bar{k}}\left\{\bar{k}\alpha+\bar{k}+\frac{2k_2\alpha}{\beta(1-\bar{k})}\right\} \\
&\quad\times\exp\left\{\frac{2T[k_1+3k_2-\bar{k}(k_1+k_2)]}{(1-\bar{k})^2}\right\}<1 \quad (28)
\end{aligned}$$

and $\bar{k} \in (0, 1)$. Thus from (28), we obtain the upper bound of neutral terms.

Synthesize (27) and (28), we obtain the upper bounds of NTs and PCAs, that is, $\bar{k}$ and θ_4.

Next, if we set $\kappa = -ln\hat{M}/T$, from (21), we can obtain

$$\sup_{t_0-\theta+T\le t\le t_0-\theta+2T} \|r(t, t_0, r_0)\| \le \exp(-\kappa T) \sup_{t_0-\theta\le t\le t_0-\theta+T} \|r(t, t_0, r_0)\|. \quad (29)$$

Consequently, for any positive integer $l \in \underline{n}$, when $t \geq t_0 - \theta + (l-1)T$, in accordance with the fluidity of the state trajectory [26], we have

$$r(t; t_0, r_0) = r(t; t_0 - \theta + (l-1)T, r(t_0 - \theta + (l-1)T; t_0, r_0)). \quad (30)$$

Combining (29) and (30), we can get

$$\begin{aligned}
&\sup_{t_0-\theta+lT\le t\le t_0-\theta+(l+1)T} \|r(t; t_0, r_0)\| \\
&= \sup_{t_0-\theta+(l-1)T+T\le t\le t_0-\theta+lT+T} \|r(t; t_0 - \theta + (l-1)T, \\
&\quad r(t_0 - \theta + (l-1)T; t_0, r_0))\| \\
&\le \exp(-\kappa T) \sup_{t_0-\theta+(l-1)T\le t\le t_0-\theta+lT} \|r(t; t_0, r_0)\| \\
&\le \exp(-\kappa lT) \sup_{t_0-\theta\le t\le t_0-\theta+T} \|r(t; t_0, r_0)\| \\
&= M\exp(-\kappa lT), \quad (31)
\end{aligned}$$

where $M = \sup_{t_0-\theta\le t\le t_0-\theta+T} \|r(t, t_0, r_0)\|$. Hence there exists $l \in \boldsymbol{N}$ such that for any $t_0 - \theta + lT \le t \le t_0 - \theta + (l+1)T$, we have $-lT \le -(t - t_0) + (T - \theta)$, then

$$\begin{aligned}
\|r(t)\| &\le M\exp(-\kappa lT) \\
&\le M \exp\{\kappa[T - \theta - (t - t_0)]\} \\
&\le M\exp(\kappa T)\exp(-\kappa(t - t_0)). \quad (32)
\end{aligned}$$

In this way, the state of NPNDS (1) achieves global exponential stability.

Remark 4: Here is an explanation, one can see Fig. 1, which is about the comparison about the relative length relation among the time intervals in the proof of Theorem 1. Due to $\theta < T/2$, the relative size of the intervals among $[t_0-\theta, t_0+\theta]$ (the piecewise argument interval used in (19)), $[t_0 + \theta, t_0 + 2T]$ (used in (20)), $[t_0 - \theta, t_0 - \theta + T]$ and $[t_0 - \theta + T, t_0 - \theta + 2T]$ (used in (21)) is shown.

Remark 5: (28) is an unary transcendental inequality about $\bar{k}$ since other parameters α, β, T, k_1, k_2 can be listed in advance, thus we can obtain the upper bound of k by MATLAB precisely.

Remark 6: This is an illustration about assumptions (A1)-(A5) in part III and (B1)-(B2) in part IV. First, (A1) and (A2) are the qualitative requirements of activation function $f(\cdot)$ and neutral function $G(\cdot)$ throughout the whole text. Next, (A3) is a symbolic representation of the interval length $[\theta_p, \theta_{p+1})$. Then, (A4) is an assumption which makes

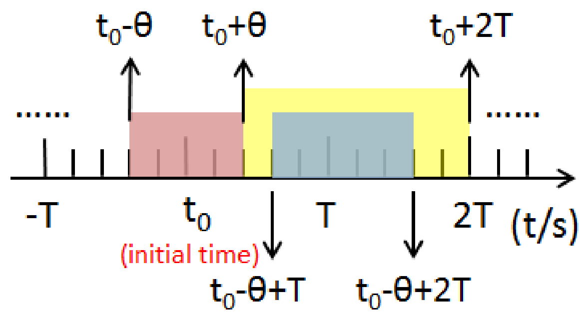

FIGURE 1. Relative length comparison of intervals containing the argument time.

"$1 - \xi > 0$" true appeared in the denominator of (4) in Lemma 1, and (A4) is also the origin of θ_1 in (16). Besides, (A5) is an assumption which makes "$F(1 + k/1 - k, k) < 1$" true in (24) so as to ensure the existence of (26), and (A5) is also the origin of θ_2 in (16). In addition, (B1)-(B2) for Theorem 2 in part IV have the same function as (A4)-(A5) for Theorem 1 in part III. Thus each of these assumptions plays an important role and they are indispensable for this paper. However, as for the conservatism of these assumptions, future work only needs to focus on the optimization of (A1)-(A2) in these assumptions. More recommendations on the feasibility of future work can be seen in part VI.

IV. THE ROBUSTNESS OF SNPNDS

In this section, we will consider the following nonlinear dynamical system with generalized piecewise constant arguments, neutral terms and stochastic disturbances (SNPNDS) as

$$\begin{cases} d[r(t)-G(r(t))] = [f(r(t), r(\rho(t)), t)]dt + \sigma r(t)dB(t), \\ \qquad\qquad t \ge t_0, \\ r(t_0) = r_0, \end{cases} \quad (33)$$

where $G(r(t)) : \mathbb{R}^+ \to \mathbb{R}^n \to \mathbb{R}^n$ is the neutral-type function. $f(r(t), r(\rho(t)), t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is activation function about state $r(t)$ and $r(\rho(t))$, where $r(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is the current state and $r(\rho(t)) : \mathbb{R}^+ \to \mathbb{R}^n \to \mathbb{R}^n$ is the PCA state. Besides, $\rho(t) = \eta_p$ when $\theta_p \le t < \theta_{p+1}$, $p \in \boldsymbol{N}$. $B(t) \in \Phi$ presented in SNPNDS (33) is a scalar Brownian motion defined on complete probability space Φ. And σ shows the intensity of exogenous stochastic disturbances (SDs).

Similar to Remark 2, the upper bound of SDs expressed throughout this paper exactly refers to the threshold value of σ.

Remark 7: Obviously, if we fix $p \in \boldsymbol{N}$, and consider the differential system on the interval $[\theta_p, \theta_{p+1})$. SNPNDS (33) is an advanced system when $\theta_p \le t < \eta_p$, and SNPNDS (33) is a hysteretic system when $\eta_p \le t < \theta_{p+1}$. Therefore, system (33) is a hybrid system which unifies advanced and hysteretic time deviation under external disturbance in the stochastic environment.

Without three interference factors: PCAs, NTs and SDs, SNPNDS (33) evolves into the generalized NDS as

$$\begin{cases} \dot{u}(t) = f(u(t), u(t), t), \ t \geq t_0, \\ u(t_0) = u_0 = r_0, \end{cases} \tag{34}$$

The exponential stability Definition of (34) is depicted in Definition 1. But in the stochastic probability environment, the corresponding exponential stability definitions of SNPNDS (33) are shown in the following Definition 2 and Definition 3.

Definition 2: If there are $\tilde{\alpha} > 0$ and $\tilde{\beta} > 0$ such that $\|r(t; t_0, r_0)\| \leq \tilde{\alpha}\|r_0\| \exp\{-\tilde{\beta}(t - t_0)\}$ is valid for all $t \geq t_0$, $r_0 \in \mathbb{R}^n$, then the state $r(t; t_0, r_0)$ of (33) is called almost surely exponential stability (ASES).

Definition 3: If there are $\tilde{\alpha} > 0$ and $\tilde{\beta} > 0$ such that $E\|r(t; t_0, x_0)\|^2 \leq \tilde{\alpha}\|r_0\|^2 \exp\{-\tilde{\beta}(t - t_0)\}$ is valid for all $t \geq t_0$, $r_0 \in \mathbb{R}^n$, then the state $r(t; t_0, r_0)$ of (33) is called mean square exponential stability (MSES).

In addition to assumptions (A1)-(A3) in section III, here are the additional assumptions (B1)-(B2) shown below:

$$\begin{aligned} &(B1) 8k_2^2\theta + (8k_1^2\theta^2 + 4\sigma^2\theta)\left[\frac{4 + 8k^2 + 8\theta^2 k_2^2}{1 - 8k^2} \right. \\ &\qquad \left. \times \exp\left\{\frac{8\theta^2\, k_1^2 + 4\sigma^2\theta}{1 - 8k^2}\right\}\right] < 1 - 8k^2, \\ &(B2) \frac{2}{1 - 12k^2}\left[12k^2\alpha^2 \exp(-2\beta\theta) + \left(\frac{120k_2^2\theta}{1 - 8k^2} + 3\sigma^2\right)\right. \\ &\qquad \left. \times \alpha^2/\beta + 6k^2\right] \exp\left\{\frac{2T}{1 - 12k^2} \times \left[\frac{240k_2^2\theta}{1 - 8k^2} + 6\sigma^2\right.\right. \\ &\qquad \left.\left. + 6\theta(k_1^2 + 2k_2^2)\right]\right\} + 2\alpha^2 \exp(-2\beta(T - \theta)) < 1. \end{aligned}$$

Remark 8: In section IV, for notational brevity, let

$$\begin{aligned} \omega_3 &= \frac{1}{1 - 12k^2}\left\{12k^2\alpha^2 \exp(-2\beta\theta) \right. \\ &\qquad \left. + \left[48k_2^2\theta(1 + \Gamma) + 6\sigma^2\right] \times \alpha^2/2\beta + 6k^2\right\}, \\ \omega_4 &= \frac{1}{1 - 12k^2}\left\{48k_2^2\theta(1 + \Gamma) + 6\sigma^2 + 6\theta(k_1^2 + 2k_2^2)\right\}, \\ \Gamma &= \frac{4 + 8k^2}{1 - 8k^2 - \breve{o}}, \\ \breve{o} &= 8k_2^2\theta^2 + (8k_1^2\theta^2 + 4\sigma^2\theta)\left[\frac{4 + 8k^2 + 8\theta^2 k_2^2}{1 - 8k^2} \right. \\ &\qquad \left. \times \exp\left\{\frac{8\theta^2\, k_1^2 + 4\sigma^2\theta}{1 - 8k^2}\right\}\right], \\ T &> (ln2\alpha^2)/2\beta > 0, \quad T \in \boldsymbol{R}. \end{aligned}$$

Then the following Lemma 2 aims to clarify the relationship between the generalized PCA state $r(\rho(t))$ and the current state $r(t)$.

Lemma 2: Let (A1)-(A3) and (B1) hold, and $r(t) = (r_1(t), r_2(t), \ldots, r_n(t))^T$ is a solution of (33), $n \in \underline{n}$. Then the following inequality

$$E\|r(\rho(t))\|^2 \leq \Gamma E\|r(t)\|^2 \tag{35}$$

holds for all $t \geq 0$, where

$$\begin{aligned} \Gamma &= \frac{4 + 8k^2}{1 - 8k^2 - \breve{o}}, \\ \breve{o} &= 8k_2^2\theta^2 + (8k_1^2\theta^2 + 4\sigma^2\theta)\left[\frac{4 + 8k^2 + 8\theta^2 k_2^2}{1 - 8k^2} \right. \\ &\qquad \left. \times \exp\left\{\frac{8\theta^2\, k_1^2 + 4\sigma^2\theta}{1 - 8k^2}\right\}\right]. \end{aligned}$$

and $8k^2 < 1$.

Proof: Fix $t \in \mathbb{R}^+$, $p \in \boldsymbol{N}$, for any $t \in [\theta_p, \theta_{p+1})$, when $\rho(t) = \eta_p$, $p \in \boldsymbol{N}$, we have

$$\begin{aligned} r(t) &= r(\eta_p) + G(r(t)) - G(r(\eta_p)) + \int_{\eta_p}^{t} f(r(s), r(\rho(s)), s)ds \\ &\quad + \int_{\eta_p}^{t} \sigma r(s)dB(s), \end{aligned} \tag{36}$$

Applying the mathematical expectation modulus inequality and (A2), we have

$$\begin{aligned} E\|r(t)\|^2 &= E\|r(\eta_p) + G(r(t)) - G(r(\eta_p)) + \int_{\eta_p}^{t} f(r(s), \\ &\qquad r(\rho(s)), s)ds + \int_{\eta_p}^{t} \sigma r(s)dB(s)\|^2 \\ &\leq 4\Big[E\|r(\eta_p)\|^2 + E\|G(r(t)) - G(r(\eta_p))\|^2 \\ &\qquad + E\|\int_{\eta_p}^{t} f(r(s), r(\rho(s)), s)ds\|^2 \\ &\qquad + E\|\int_{\eta_p}^{t} \sigma r(s)dB(s)\|^2\Big] \\ &\leq 4\Big[E\|r(\eta_p)\|^2 + k^2E\|r(t) - r(\eta_p)\|^2 \\ &\qquad + E\|\int_{\eta_p}^{t} f(r(s), r(\rho(s)), s)ds\|^2 \\ &\qquad + E\|\int_{\eta_p}^{t} \sigma r(s)dB(s)\|^2\Big] \\ &\leq 4\Big[E\|r(\eta_p)\|^2 + k^2\Big(2E\|r(t)\|^2 + 2E\|r(\eta_p)\|^2\Big) \\ &\qquad + E\|k_1\int_{\eta_p}^{t} r(s)ds + k_2\int_{\eta_p}^{t} r(\rho(s))ds\|^2 \\ &\qquad + E\|\int_{\eta_p}^{t} \sigma r(s)dB(s)\|^2\Big] \\ &\leq 4\Big[E\|r(\eta_p)\|^2 + 2k^2(E\|r(t)\|^2 + E\|r(\eta_p)\|^2) \\ &\qquad + 2k_1^2E\|\int_{\eta_p}^{t} r(s)ds\|^2 + 2k_2^2\theta^2E\|r(\eta_p)\|^2 \\ &\qquad + E\|\int_{\eta_p}^{t} \sigma r(s)dB(s)\|^2\Big]. \end{aligned} \tag{37}$$

So by virtue of the Cauchy-Schwarz inequality and the isometric property of Itô integral, (37) can evolve into

$$\begin{aligned} E\|r(t)\|^2 \leq & 4\Big[E\|r(\eta_p)\|^2 + 2k^2(E\|r(t)\|^2 + E\|r(\eta_p)\|^2) \\ & + 2k_1^2 E\int_{\eta_p}^t 1^2 ds \int_{\eta_p}^t \|r(s)\|^2 ds + 2k_2^2\theta^2 E\|r(\eta_p)\|^2 \\ & + \sigma^2\int_{\eta_p}^t E\|r(s)\|^2 ds\Big] \\ \leq & 4\Big[E\|r(\eta_p)\|^2 + 2k^2 E\|r(t)\|^2 + 2k^2 E\|r(\eta_p)\|^2 \\ & + 2k_1^2\theta\int_{\eta_p}^t E\|r(s)\|^2 ds + 2k_2^2\theta^2 E\|r(\eta_p)\|^2 \\ & + \sigma^2\int_{\eta_p}^t E\|r(s)\|^2 ds\Big] \\ \leq & (4 + 8k^2 + 8\theta^2 k_2^2)E\|r(\eta_p)\|^2 + 8k^2 E\|r(t)\|^2 \\ & + (8\theta k_1^2 + 4\sigma^2)\int_{\eta_p}^t E\|r(s)\|^2 ds. \end{aligned} \tag{38}$$

Directly, by uniting the similar terms on both sides of (38), we obtain

$$\begin{aligned} E\|r(t)\|^2 \leq & \frac{4 + 8k^2 + 8\theta^2 k_2^2}{1 - 8k^2} E\|r(\eta_p)\|^2 + \frac{8\theta k_1^2 + 4\sigma^2}{1 - 8k^2} \\ & \times \int_{\eta_p}^t E\|r(s)\|^2 ds, \end{aligned} \tag{39}$$

where $8k^2 < 1$, and by virtue of Gronwall-Bellman Lemma, (39) become

$$\begin{aligned} E\|r(t)\|^2 \leq & \frac{4 + 8k^2 + 8\theta^2 k_2^2}{1 - 8k^2} \\ & \times \exp\left\{\frac{8\theta^2 k_1^2 + 4\sigma^2\theta}{1 - 8k^2}\right\} E\|r(\eta_p)\|^2. \end{aligned} \tag{40}$$

Besides, from (36), we obtain

$$\begin{aligned} r(\eta_p) = & r(t) - [G(r(t)) - G(r(\eta_p))] - \int_{\eta_p}^t f(r(s), r(\rho(s)), s)ds \\ & - \int_{\eta_p}^t \sigma r(s)dB(s). \end{aligned}$$

From (37), similarly, we get

$$\begin{aligned} E\|r(\eta_p)\|^2 \leq & E\|r(t) - [G(r(t)) - G(r(\eta_p))] - \int_{\eta_p}^t f(r(s), \\ & r(\rho(s)), s)ds - \int_{\eta_p}^t \sigma r(s)dB(s)\|^2 \\ \leq & 4\Big[E\|r(t)\|^2 + E\|G(r(t)) - G(r(\eta_p))\|^2 \\ & + E\|\int_{\eta_p}^t f(r(s), r(\rho(s)), s)ds\|^2 \\ & + E\|\int_{\eta_p}^t \sigma r(s)dB(s)\|^2\Big]. \end{aligned} \tag{41}$$

In terms of (A2) and the mathematical expectation norm inequality, we have

$$\begin{aligned} E\|r(\eta_p)\|^2 \leq & 4\Big[E\|r(t)\|^2 + 2k^2(E\|r(t)\|^2 + E\|r(\eta_p)\|^2) \\ & + E\|k_1\int_{\eta_p}^t r(s)ds + k_2\int_{\eta_p}^t r(\rho(s))ds\|^2 \\ & + E\|\int_{\eta_p}^t \sigma r(s)dB(s)\|^2\Big] \\ \leq & 4\Big[E\|r(t)\|^2 + 2k^2 E\|r(t)\|^2 + 2k^2 E\|r(\eta_p)\|^2 \\ & + 2k_1^2\theta\int_{\eta_p}^t E\|r(s)\|^2 ds + 2k_2^2\theta^2 E\|r(\eta_p)\|^2 \\ & + \sigma^2\int_{\eta_p}^t E\|r(s)\|^2 ds\Big] \\ \leq & 4\Big[(1 + 2k^2)E\|r(t)\|^2 + 2k^2 E\|r(\eta_p)\|^2 \\ & + 2k_1^2\theta\int_{\eta_p}^t E\|r(s)\|^2 ds + 2k_2^2\theta^2 E\|r(\eta_p)\|^2 \\ & + \sigma^2\int_{\eta_p}^t E\|r(s)\|^2 ds\Big] \\ \leq & (4 + 8k^2)E\|r(t)\|^2 + 8k^2 E\|r(\eta_p)\|^2 \\ & + 8k_1^2\theta\int_{\eta_p}^t E\|r(s)\|^2 ds + 8k_2^2\theta^2 E\|r(\eta_p)\|^2 \\ & + 4\sigma^2\int_{\eta_p}^t E\|r(s)\|^2 ds. \end{aligned} \tag{42}$$

For the convenience later, merging partial terms on both sides of (42), we directly get

$$\begin{aligned} & (1 - 8k^2)E\|r(\eta_p)\|^2 \\ & \leq (4 + 8k^2)E\|r(t)\|^2 + (8k_1^2\theta + 4\sigma^2)\int_{\eta_p}^t E\|r(s)\|^2 ds \\ & \quad + 8k_2^2\theta^2 E\|r(\eta_p)\|^2. \end{aligned} \tag{43}$$

Therefore, applying the Gronwall-Bellman lemma, it follows that

$$\begin{aligned} & (1 - 8k^2)E\|r(\eta_p)\|^2 \\ & \leq (4 + 8k^2)E\|r(t)\|^2 + (8k_1^2\theta^2 + 4\sigma^2\theta) \\ & \quad \times \left[\frac{4 + 8k^2 + 8\theta^2 k_2^2}{1 - 8k^2}\exp\left\{\frac{8\theta^2 k_1^2 + 4\sigma^2\theta}{1 - 8k^2}\right\}\right] E\|r(\eta_p)\|^2 \\ & \quad + 8k_2^2\theta^2 E\|r(\eta_p)\|^2 \\ & \leq (4 + 8k^2)E\|r(t)\|^2 + \Big\{8k_2^2\theta^2 + (8k_1^2\theta^2 + 4\sigma^2\theta) \\ & \quad \times \left[\frac{4 + 8k^2 + 8\theta^2 k_2^2}{1 - 8k^2}\exp\left\{\frac{8\theta^2 k_1^2 + 4\sigma^2\theta}{1 - 8k^2}\right\}\right]\Big\} E\|r(\eta_p)\|^2 \\ & \leq (4 + 8k^2)E\|r(t)\|^2 + \check{\delta} E\|r(\eta_p)\|^2, \end{aligned} \tag{44}$$

where

$$\breve{o} = 8k_2^2\theta^2 + (8k_1^2\theta^2 + 4\sigma^2\theta) \times\left[\frac{4+8k^2+8\theta^2k_2^2}{1-8k^2}\exp\left\{\frac{8\theta^2k_1^2+4\sigma^2\theta}{1-8k^2}\right\}\right]. \quad (45)$$

Disposing the inequality (44), we obtain

$$\left\{1-8k^2-\breve{o}\right\}E\|r(\eta_p)\|^2 \leq (4+8k^2)E\|r(t)\|^2. \quad (46)$$

Finally, in combination with (B1), it follows that

$$\begin{aligned} E\|r(\eta_p)\|^2 &\leq \frac{4+8k^2}{1-8k^2-\breve{o}}E\|r(t)\|^2 \\ &\leq \Gamma E\|r(t)|^2, \end{aligned} \quad (47)$$

where

$$\Gamma = \frac{4+8k^2}{1-8k^2-\breve{o}}.$$

Remark 9: In the scaling process from (42) to (43), we do not merge all of the terms which include $E\|r(\eta_p)\|^2$ to the left side of inequation (43), accordingly we obtain inequality (46), this point is to pave the way for (60), which is also the origin of Assumption (B1).

Theorem 2: If (A1)-(A3) and (B1)-(B2) hold and system (34) can achieve GES. Then system (33) can achieve ASES and even MSES if the following conditions hold: firstly, if neutral term compressibility coefficient k satisfy the following inequality:

$$\begin{aligned} 0 < k &< \bar{k} \\ &= \min\left\{\begin{array}{l} \sqrt{3}/6, \\ \sup\left\{\bar{k}\,\middle|\,\{1-12\bar{k}\}^{-1}\times\{24\bar{k}^2\alpha^2+12\bar{k}^2\} \right. \\ \left. \qquad +2\alpha^2\exp(-2\beta T)<1\right\}. \end{array}\right. \end{aligned} \quad (48)$$

Besides, stochastic disturbance intensity $|\sigma| < \bar{\sigma}, \bar{\sigma} > 0$ stands for the upper bound of inequality (49)

$$\frac{2}{1-12k^2}\left(12k^2\alpha^2+3\bar{\sigma}\alpha^2/\beta+6k^2\right)\exp\left\{\frac{12T\bar{\sigma}^2}{1-12k^2}\right\} +2\alpha^2\exp(-2\beta T) = 1. \quad (49)$$

Additionally, if the interval length of piecewise constant arguments

$$\theta < \theta_4 = \min\left\{\frac{T}{2}, \theta_1, \theta_2, \theta_3\right\}, \quad (50)$$

where θ_1 and θ_2 are the upper bounds which satisfy assumptions (B1) and (B2) respectively, $\theta_3 > 0$ stands for the unique solution for below transcendental equation:

$$2\omega_3\exp\{2T\omega_4\} + 2\alpha^2\exp\{-2\beta(T-\theta)\} = 1, \quad (51)$$

where

$$\begin{aligned} \omega_3 &= \frac{1}{1-12k^2}\Big\{12k^2\alpha^2\exp(-2\beta\theta) \\ &\quad + \left[48k_2^2\theta(1+\Gamma)+6\sigma^2\right]\times\alpha^2/2\beta+6k^2\Big\} \\ \omega_4 &= \frac{1}{1-12k^2}\Big\{48k_2^2\theta(1+\Gamma)+6\sigma^2+6\theta(k_1^2+2k_2^2)\Big\}, \\ \Gamma &= \frac{4+8k^2}{1-8k^2-\breve{o}}, \\ \breve{o} &= 8k_2^2\theta^2+(8k_1^2\theta^2+4\sigma^2\theta)\left[\frac{4+8k^2+8\theta^2k_2^2}{1-8k^2}\right. \\ &\quad \left.\times\exp\left\{\frac{8\theta^2\,k_1^2+4\sigma^2\theta}{1-8k^2}\right\}\right], \end{aligned}$$

with $T > (ln2\alpha^2)/2\beta > 0$ and α, β are known constants.

Proof: For briefly record, we denote $r(t; t_0, r_0)$ as $r(t)$ and $u(t; t_0, u_0)$ as $u(t)$, from (33) and (34), when $t \geq t_0 > 0$, we have

$$\begin{aligned} r(t)-u(t) &= G(r(t))-G(r_0)+\int_{t_0}^{t} f(r(s), r(\rho(s)), s) \\ &\quad -f(u(s), u(s), s)ds + \sigma\int_{t_0}^{t} r(s)dB(s). \end{aligned} \quad (52)$$

So according to (A2), the mathematical expectation inequality and the isometric property of Itô integral, we get

$$\begin{aligned} &E\|r(t)-u(t)\|^2 \\ &= E\|G(r(t))-G(r_0)+\int_{t_0}^{t} f(r(s), y(\rho(s)), s) \\ &\quad -f(u(s), u(s), s)ds+\sigma\int_{t_0}^{t} r(s)dB(s)\|^2 \\ &\leq 3E\|G(r(t))-G(r_0)\|^2+3E\|\int_{t_0}^{t} f(r(s), r(\rho(s)), s) \\ &\quad -f(u(s), u(s), s)ds\|^2+3E\|\sigma\int_{t_0}^{t} r(s)dB(s)\|^2 \\ &\leq 3k^2E\|r(t)-r_0\|^2+3E\|\int_{t_0}^{t} f(r(s), r(\rho(s)), s) \\ &\quad -f(u(s), u(s), s)ds\|^2+3\sigma^2E\int_{t_0}^{t}\|r(s)\|^2ds. \end{aligned} \quad (53)$$

In combination with Cauchy-Schwarz inequation and (A1) in section III, for any $t \geq t_0 \geq 0$, we get

$$\begin{aligned} &3E\|\int_{t_0}^{t} f(r(s), r(\rho(s)), s)-f(u(s), u(s), s)ds\|^2 \\ &\leq 3E\int_{t_0}^{t} 1^2ds\int_{t_0}^{t}\|f(r(s), r(\rho(s)), s) \\ &\quad -f(u(s), u(s), s)ds\|^2ds \\ &\leq 3(t-t_0)\int_{t_0}^{t} E\|k_1[r(s)-u(s)]+k_2[r(\rho(s))-u(s)]\|^2ds \end{aligned}$$

$$\begin{aligned}
&\leq 3(t-t_0)\Big[2k_1^2\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds\\
&\quad +2k_2^2\int_{t_0}^{t}E\|r(\rho(s))-r(s)+r(s)-u(s)\|^2ds\Big]\\
&\leq 6(t-t_0)\Big\{k_1^2\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds\\
&\quad +2k_2^2\int_{t_0}^{t}E\|r(\rho(s))-r(s)\|^2+E\|r(s)-u(s)\|^2ds\Big\}\\
&\leq 6(t-t_0)\Big\{(k_1+2k_2^2)\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds\\
&\quad +4k_2^2\Big[\int_{t_0}^{t}E\|r(\rho(s))\|^2+E\|r(s)\|^2ds\Big]\Big\}\\
&\leq 6(t-t_0)(k_1+2k_2^2)\int_{t_0}^{t}E\|r(s)-x(s)\|^2ds\\
&\quad +24k_2^2(t-t_0)\Big[\int_{t_0}^{t}E\|r(\rho(s))\|^2+E\|r(s)\|^2ds\Big]\\
&\overset{Lem2}{\leq} 6(t-t_0)(k_1+2k_2^2)\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds\\
&\quad +24k_2^2(t-t_0)(1+\Gamma)\int_{t_0}^{t}E\|r(s)\|^2ds. \qquad (54)
\end{aligned}$$

Substitute (54) into (53), for $t\in[t_0-\theta,t_0+\theta]$, we derive

$$\begin{aligned}
&E\|r(t)-u(t)\|^2\\
&\leq 3k^2E\|r(t)-r_0\|^2+6(t-t_0)(k_1+2k_2^2)\\
&\quad\times\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds+24k_2^2(t-t_0)(1+\Gamma)\\
&\quad\times\int_{t_0}^{t}E\|r(s)\|^2ds+3\sigma^2E\int_{t_0}^{t}\|r(s)\|^2ds\\
&\leq 6k^2E\|r(t)\|^2+6k^2E\|r_0\|^2+\Big[24k_2^2(t-t_0)(1+\Gamma)\\
&\quad+3\sigma^2\Big]\int_{t_0}^{t}E\|r(s)-u(s)+u(s)\|^2ds\\
&\quad+6(t-t_0)(k_1+2k_2^2)\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds\\
&\leq 6k^2E\|r(t)-u(t)+u(t)\|^2+6k^2\|r_0\|^2\\
&\quad+\Big[24k_2^2\theta(1+\Gamma)+3\sigma^2\Big]\\
&\quad\times\int_{t_0}^{t}\Big[2E\|r(s)-u(s)\|^2+2E\|u(s)\|^2\Big]ds\\
&\quad+6\theta(k_1+2k_2^2)\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds\\
&\leq 12k^2E\|r(t)-u(t)\|^2+12k^2E\|u(t)\|^2+6k^2\|r_0\|^2\\
&\quad+\Big\{48k_2^2\theta(1+\Gamma)+6\sigma^2+6\theta(k_1+2k_2^2)\Big\}\\
&\quad\times\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds+\Big[48k_2^2\theta(1+\Gamma)\\
&\quad+6\sigma^2\Big]\int_{t_0}^{t}E\|u(s)\|^2ds. \qquad (55)
\end{aligned}$$

By virtue of the stability property of system (34), when $t_0+\theta\leq t\leq t_0+2T$, it yields that

$$\begin{aligned}
&E\|r(t)-u(t)\|^2\\
&\leq 12k^2E\|r(t)-u(t)\|^2+12k^2\alpha^2\exp(-2\beta(t-t_0))\|u_0\|^2\\
&\quad+\Big\{48k_2^2\theta(1+\Gamma)+6\sigma^2+6\theta(k_1^2+2k_2^2)\Big\}\\
&\quad\times\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds+\Big[48k_2^2\theta(1+\Gamma)+6\sigma^2\Big]\\
&\quad\times\int_{t_0}^{t}\alpha^2\|u_0\|^2\exp(-2\beta(s-t_0))ds+6k^2\|r_0\|^2\\
&\leq 12k^2E\|r(t)-x(t)\|^2+12k^2\alpha^2\exp(-2\beta\theta)\|u_0\|^2\\
&\quad+\Big\{48k_2^2\theta(1+\Gamma)+6\sigma^2+6\theta(k_1^2+2k_2^2)\Big\}\\
&\quad\times\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds+\Big[48k_2^2\theta(1+\Gamma)+6\sigma^2\Big]\\
&\quad\times\alpha^2\|u_0\|^2/2\beta+6k^2\|r_0\|^2\\
&\leq 12k^2E\|r(t)-u(t)\|^2+\Big\{12k^2\alpha^2\exp(-2\beta\theta)\\
&\quad+\Big[48k_2^2\theta(1+\Gamma)+6\sigma^2\Big]\alpha^2/2\beta+6k^2\Big\}\|r_0\|^2\\
&\quad+\Big\{48k_2^2\theta(1+\Gamma)+6\sigma^2+6\theta(k_1^2+2k_2^2)\Big\}\\
&\quad\times\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds. \qquad (56)
\end{aligned}$$

Directly, merging the similar terms on both sides of (56), we further have

$$\begin{aligned}
&E\|r(t)-u(t)\|^2\\
&\leq \frac{1}{1-12k^2}\Big\{12k^2\alpha^2\exp(-2\beta\theta)+\Big[48k_2^2\theta(1+\Gamma)+6\sigma^2\Big]\\
&\quad\times\alpha^2/2\beta+6k^2\Big\}\|r_0\|^2+\frac{1}{1-12k^2}\Big\{48k_2^2\theta(1+\Gamma)+6\sigma^2\\
&\quad+6\theta(k_1^2+2k_2^2)\Big\}\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds\\
&=:\omega_3\|r_0\|^2+\omega_4\int_{t_0}^{t}E\|r(s)-u(s)\|^2ds \qquad (57)
\end{aligned}$$

where $12k^2<1$ and

$$\begin{aligned}
\omega_3&=\frac{1}{1-12k^2}\Big\{12k^2\alpha^2\exp(-2\beta\theta)\\
&\quad+\Big[48k_2^2\theta(1+\Gamma)+6\sigma^2\Big]\times\alpha^2/2\beta+6k^2\Big\}\\
\omega_4&=\frac{1}{1-12k^2}\Big\{48k_2^2\theta(1+\Gamma)+6\sigma^2+6\theta(k_1^2+2k_2^2)\Big\}. \qquad (58)
\end{aligned}$$

Applying the Gronwall inequality to (57), when $t_0+\theta\le t\le t_0+2T$ (i.e., $\theta<T/2$), it follows that

$$\begin{aligned}&E\|r(t)-u(t)\|^2\\&\quad\le\omega_3\exp\{\omega_4(t-t_0)\}\sup_{t_0-\theta\le t\le t_0+\theta}E\|r(t)\|^2\\&\quad\le\omega_3\exp\{2T\omega_4\}\sup_{t_0-\theta\le t\le t_0+\theta}E\|r(t)\|^2.\end{aligned}$$

Subsequently, for any $t_0-\theta+T\le t\le t_0-\theta+2T$, it follows that

$$\begin{aligned}E\|r(t)\|^2&=E\|r(t)-u(t)+u(t)\|^2\\&\le 2E\|r(t)-u(t)\|^2+2E\|u(t)\|^2\\&\le 2\omega_3\exp\{2T\omega_4\}\sup_{t_0-\theta\le t\le t_0+\theta}E\|r(t)\|^2\\&\quad+2\alpha^2\|x_0\|^2\exp(-2\beta(T-\theta))\\&\le\Big[2\omega_3\exp\{2T\omega_4\}+2\alpha^2\exp(-2\beta(T-\theta))\Big]\\&\quad\times\sup_{t_0-\theta\le t\le t_0-\theta+T}E\|r(t)\|^2\\&=:\hat{C}\sup_{t_0-\theta\le t\le t_0-\theta+T}E\|r(t)\|^2,\end{aligned}\tag{59}$$

where

$$\hat{C}=2\omega_3\exp\{2T\omega_4\}+2\alpha^2\exp(-2\beta(T-\theta)).$$

Similarly, the relative length relation among the time intervals appeared in the proof of Theorem 2 is shown in Fig. 1 in section III.

According to (B1), the following inequality (60) holds:

$$\begin{aligned}\breve{o}(\theta)=8k_2^2\theta+(8k_1^2\theta^2+4\sigma^2\theta)\Bigg[\frac{4+8k^2+8\theta^2k_2^2}{1-8k^2}\\\times\exp\left\{\frac{8\theta^2k_1^2+4\sigma^2\theta}{1-8k^2}\right\}\Bigg]<1-8k^2.\end{aligned}\tag{60}$$

Assume there exists $\theta_1>0$, which is the unique solution of $\breve{o}(\theta)=1-8k^2$. And apparently

$$\Gamma(\theta)=\frac{4+8k^2}{1-8k^2-\breve{o}(\theta)}\in\left(\frac{4+8k^2}{1-8k^2},\infty\right),\tag{61}$$

where $\breve{o}$ is the same as the one defined in Lemma 2. And due to the monotonicity of $\breve{o}(\theta)$, θ_1 satisfy (60), i.e. the origin of (B1), which is mentioned in Remark 9.

Similarly, denote a function

$$\begin{aligned}&F(\Gamma(\theta),k,\sigma)\\&\quad=\frac{2}{1-12k^2}\Bigg\{12k^2\alpha^2\exp(-2\beta\theta)\\&\qquad+\Big[48k_2^2\theta(1+\Gamma)+6\sigma^2\Big]\alpha^2/2\beta+6k^2\Bigg\}\\&\qquad\times\exp\left\{\frac{2T}{1-12k^2}\Big[48k_2^2\theta(1+\Gamma)+6\sigma^2\right.\\&\qquad\left.+6\theta(k_1^2+2k_2^2)\Big]\right\}+2\alpha^2\exp(-2\beta(T-\theta)),\end{aligned}\tag{62}$$

where Γ is the same as the one defined in Lemma 2.

At first, from (B2), we get

$$\begin{aligned}&F(\frac{4+8k^2}{1-8k^2},k,\sigma)\\&\quad=\frac{2}{1-12k^2}\Bigg[12k^2\alpha^2\exp(-2\beta\theta)\\&\qquad+\left(\frac{120k_2^2\theta}{1-8k^2}+3\sigma^2\right)\alpha^2/\beta+6k^2\Bigg]\exp\Bigg\{\frac{2T}{1-12k^2}\\&\qquad\times\left[\frac{240k_2^2\theta}{1-8k^2}+6\sigma^2+6\theta(k_1^2+2k_2^2)\right]\Bigg\}\\&\qquad+2\alpha^2\exp(-2\beta(T-\theta))<1,\end{aligned}\tag{63}$$

and

$$F(\infty,k,\sigma)>1,$$

thus there exists the unique

$$\tilde{\Gamma}(\theta)\in(\frac{4+8k^2}{1-8k^2},+\infty)$$

such that $F(\tilde{\Gamma}(\theta),k,\sigma)=1$.

Hence, there exists a $\tilde{\theta}\in(0,\theta_1)$, which makes $\Gamma(\theta)=\tilde{\Gamma}(\theta)$ true. Besides, let θ_2 be the supremum which satisfy Assumption (B2) and θ_3 is the unique positive solution of equation (62). Select

$$\theta_4=\min\left\{\frac{T}{2},\theta_1,\theta_2,\theta_3\right\},$$

in terms of the continuous monotonicity of $\Gamma(\cdot)$ and $F(\cdot)$, we have $0<\Gamma(\theta)<\tilde{\Gamma}(\theta)$ and $0<\tilde{C}=F(\Gamma(\theta),k,\sigma)<1$ when $0<\theta<\theta_4$. Hence, we obtain the upper bound of the interval length of piecewise arguments θ.

Furthermore, from the characteristic of the function $\Gamma(\cdot)$ and $F(\cdot)$, it follows that

$$\begin{aligned}F(\Gamma(\theta),k,\sigma)\Big|_{\theta,\sigma=0}&=\frac{2}{1-12k^2}\Big(12k^2\alpha^2+6k^2\Big)\\&\quad+2\alpha^2\exp\{-2\beta T\}<1.\end{aligned}\tag{64}$$

Moreover, by virtue of (39) and (58), the bound of k must satisfy $k^2<\min\{1/8,1/12\}$, so

$$0<k<\frac{\sqrt{3}}{6}.\tag{65}$$

Therefore, we get the bound of neutral term k, i.e., the bound of k is symbolized as $\bar{k}$, the upper bound of k can be given by

$$\begin{aligned}&0<k<\bar{k}\\&\quad=\min\left\{\begin{aligned}&\sqrt{3}/6,\\&\sup\Big\{\bar{k}\Big|\{1-12\bar{k}\}^{-1}\times\{24\bar{k}^2\alpha^2+12\bar{k}^2\}\\&\qquad\qquad+2\alpha^2\exp(-2\beta T)<1\Big\},\end{aligned}\right.\end{aligned}\tag{66}$$

where α and β are known constants.

Finally, stochastic disturbance intensity $|\sigma| < \bar{\sigma}, \bar{\sigma} > 0$ repersents the only solution of transcendental equation (67):

$$\begin{aligned} &F(\Gamma(\theta), k, \bar{\sigma})\Big|_{k\neq 0,\theta=0} \\ &= \frac{2}{1-12k^2}\left(12k^2\alpha^2 + 3\bar{\sigma}\alpha^2/\beta + 6k^2\right)\exp\left\{\frac{12T\bar{\sigma}^2}{1-12k^2}\right\} \\ &\quad +2\alpha^2\exp(-2\beta T) = 1. \end{aligned} \tag{67}$$

So far we obtain the upper bound of NTs, SDs and PCAs, i.e. $\bar{k}$, $\bar{\sigma}$ and θ_4.

Let $\bar{\kappa} = -ln\hat{C}/T$, from (59), we further get

$$\begin{aligned} &\sup_{t_0-\theta+T\le t\le t_0-\theta+2T} E\|r(t)\|^2 \\ &\le \exp(-\bar{\kappa}T) \sup_{t_0-\theta\le t\le t_0-\theta+T} E\|r(t)\|^2 \\ &\le C\exp(-\bar{\kappa}T), \end{aligned} \tag{68}$$

where

$$C = \sup_{t_0-\theta\le t\le t_0-\theta+T} E\|r(t)\|^2.$$

Since the rest is the same as Theorem 1, it is omitted here. Thus for any $t_0 - \theta \le t \le t_0 - \theta + T$ and $q \in \boldsymbol{N}$, when $t_0 - \theta + qT \le t \le t_0 - \theta + (q+1)T$, we further get

$$\|r(t)\| \le C\exp(\bar{\kappa}(T-\theta))\exp(-\bar{\kappa}(t-t_0)).$$

So the state of SNPNDS (33) achieves globally exponential stability.

Remark 10: The Table 1 is a short comparison of the most relevant research work in the past few years. The elements involved in this comparison include: global exponential stability, robustness, PCA, NT and SD. In the latest [29], the RoGES of recurrent neural networks with PCA and NT has been newly analyzed, which is extremely important bridging foundation work for this paper. Compared with [29], on the one hand, due to the more generalized modeling form and wider applications for practical engineering and computing, nonlinear system investigated here further reveals a greater superiority. On the other hand, the stochastic perturbation caused by random Brownian motion is such a pervasive element that it increases the rationality and adaptability for SNPNDS in reality.

TABLE 1. The progress of related work on SNPNDS.

	GES	Robustness	Nonlinear system	PCA	NT	SD
Akhmet2010 [30]	-	-	-	✓	-	-
Shen2012 [24]	✓	✓	-	-	-	✓
Shen2013 [25]	✓	✓	✓	-	✓	-
Shen2016 [26]	✓	✓	✓	-	-	✓
Wu2015 [27]	✓	✓	-	-	✓	✓
Zhang2017 [28]	✓	✓	✓	✓	-	✓
Our recent work [29]	✓	✓	-	✓	✓	-
This paper	✓	✓	✓	✓	✓	✓

(Note: The work in Part III includes the first five elements; and the work in Part IV includes all these six elements.)

Remark 11: The sufficient conditions for the RoGES of system (1) and (33) are given in Theorem 1 and Theorem 2, respectively. Taking Theorem 2 as an example, the detailed calculative steps of the "independent parameters & interdependent variables method" are as follows: firstly, fix the parameters α, β, k_1, k_2 and T in advance, named independent parameters. Secondly, these parameters are substituted into (48) to obtain the upper bound of the neutral term k: $\bar{k}$. The third step is to select the appropriate $k < \bar{k}$, and put k and other parameters into (49) to obtain the upper bound of the stochastic disturbance σ: $\bar{\sigma}$. Finally, the selected $k < \bar{k}$ and $\sigma < \bar{\sigma}$ are substituted into (50) to obtain the bound of the piecewise constant argument θ: θ_4. So far, the upper bounds of all three interference factors ($\bar{k}$, $\bar{\sigma}$ and θ_4) are intuitively obtained.

A more fluent and intuitive flow chart to explain the "independent parameters & interdependent variables method" for Theorem 1 and Theorem 2 can be seen in Figure 2, the blue shadow represents the independent parameters part, the yellow shadow represents the dynamic interdependent variables part.

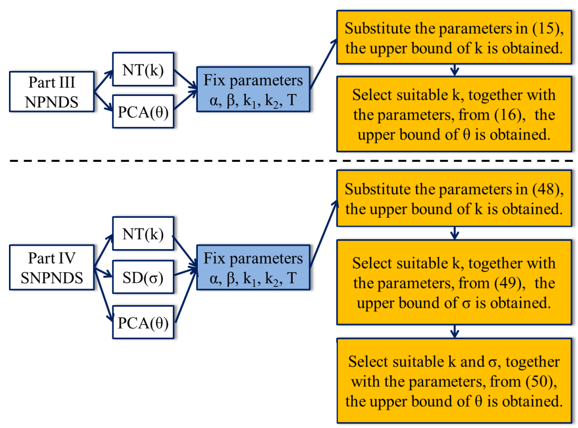

FIGURE 2. Dynamic calculation process about Theorem 1 and Theorem 2.

V. NUMERICAL EXAMPLES

The results contained above will be confirmed in this following section. Several comparable numerical simulations are given to illustrate the efficiency of deduced results.

Example 1: Consider a two-state NPNDS:

$$\begin{cases} \frac{d}{dt}\big(r_1(t) + kcos(r_1(t))\big) \\ \quad = -r_1(t) - 0.05sin^2(r_1(\rho(t)/2)) \\ \qquad -0.01sin^2(r_2(\rho(t)/3)), \\ \frac{d}{dt}\big(r_2(t) + kcos(r_2(t))\big) \\ \quad = -r_2(t) - 0.01sin^2(r_1(\rho(t)/2)) \\ \qquad -0.05sin^2(r_2(\rho(t)/3)), \end{cases} \tag{69}$$

where $\{\theta_p\} = \{\frac{p}{5}\}$, $\{\eta_p\} = \{\frac{2p+1}{10}\}$, and the piecewise constant argument function $\rho(t) = \eta_p$ when $t \in [\theta_p, \theta_{p+1})$, $p \in \boldsymbol{N}$.

Select $\rho(t) = t$, the undisturbed system of (69) is shown below

$$\begin{cases} \dot{u}_1(t) = -u_1(t) - 0.05sin^2(u_1(t)/2) \\ \qquad\quad -0.01sin^2(u_2(t)/3), \\ \dot{u}_2(t) = -u_2(t) - 0.01sin^2(u_1(t)/2) \\ \qquad\quad -0.05sin^2(u_2(t)/3), \end{cases} \tag{70}$$

By virtue of the comparison principle in [55], system (70) is globally exponential stability with $\alpha = 1.2$, $\beta = 0.9$, the convergent behaviors of $u_1(t)$ and $u_2(t)$ of (70) are portrayed in Fig. 3.

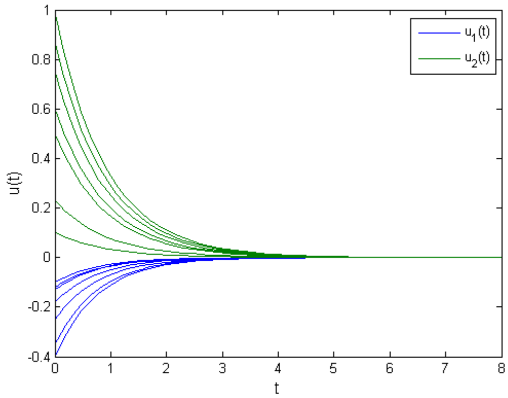

FIGURE 3. The stable trajectory of $x_1(t)$ and $x_2(t)$ of (70).

According to Theorem 1, let $T = 1 > (ln1.2)/0.9 = 0.2026$. Set $k_1 = 0.1$, $k_2 = 0.03$ by (A1). Substitute the parameters into (15), the upper bound of k can be calculated by MATLAB as

$$k < \bar{k} = 0.096.$$

If we set $k = 0.08$, put it together with other needed parameters into (A4), (A5) and (17), we obtain respectively $\theta_1 = 3.9825$, $\theta_2 = 0.2128$, $\theta_3 = 0.2417$, hence we get

$$\theta_4 = \min\left\{\frac{T}{2}, \theta_1, \theta_2, \theta_3\right\} = 0.2128.$$

So if we select $k = 0.08 < 0.096$ and $\theta = 0.2 < \theta_4 = 0.2128$, NPNNN system (1) will converge to the same equilibrium point, the states $r_1(t)$ and $r_2(t)$ of (69) are depicted in Fig. 4 and Fig. 5.

Remark 12: In accordance with the theoretically analysis and multiple confirmation by MATLAB, $\theta_1 = 3.9825$ is not a wrong data point. That is because condition (A4) is weaker than conditions (A5) and (17). Thus the bound derived by (A4) is relatively broader than the bounds derived by (A5) and (17). This point can also be testified in the following Example 2.

Remark 13: The Table 2 shows the dynamic process that θ_4 varies with k in NPNDS (69). By (15), we get $k < 0.096$, hence we choose $k = 0.08, 0.06, 0.04, 0.02$ and 0 to see the variation of θ_4. It is clear that the final θ_4 will correspondingly increase when the selected k decreases. Furthermore, due to the restriction of condition (16), θ_4 will not increase infinitely,

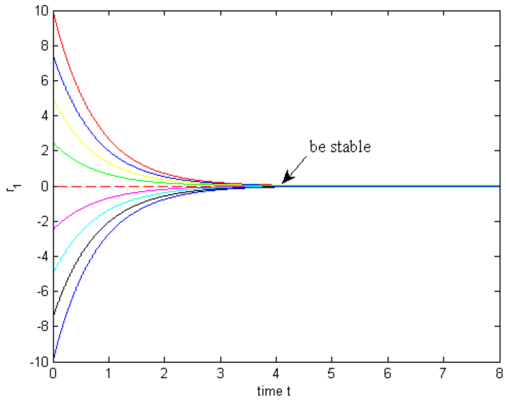

FIGURE 4. The convergent behavior of the state $r_1(t)$ in (69)with $k = 0.08$ and $\theta = 0.2$.

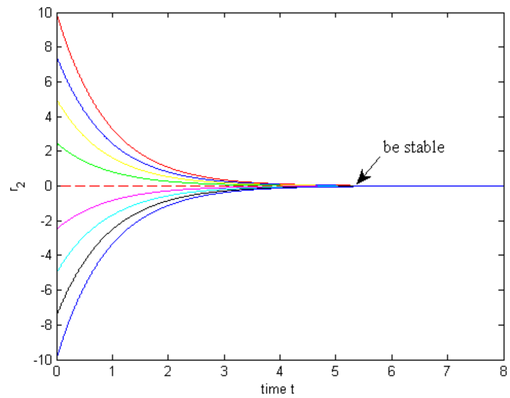

FIGURE 5. The convergent behavior of the state $r_2(t)$ in (69)with $k = 0.08$ and $\theta = 0.2$.

TABLE 2. The effect of different selected values $k \in (0, 0.096)$ on θ_4.

	k	T/2	θ_1	θ_2	θ_3	θ_4
1✓	0.08	0.5	3.9825	0.2128	0.2417	0.2128
2	0.06	0.5	4.1569	0.3760	0.3856	0.3760
3	0.04	0.5	4.3351	0.4928	0.4931	0.4928
4	0.02	0.5	4.5171	0.5842	0.5791	0.5
5	0	0.5	4.7030	0.6592	0.6509	0.5

which also demonstrates the effectiveness of the sufficient conditions in Theorem 1 ("✓" means the data group used in Table 2-4 in Part V).

Example 2: Consider a one-state original nonlinear system

$$\dot{u}(t) = -4.3u(t) + 0.3f(u(t)). \tag{71}$$

Nonlinear function $f(u) = tanh(u(t))$ here, it is easy to know that system (71) can achieve exponential stability when $\alpha = 1$, $\beta = 3$. The stable state trajectory is shown in Fig. 6.

Then if we put three important interference factors: NT, PCA and SD into (71), SNPNDS (72) is shown as below:

$$\frac{d}{dt}(r(t) - ksin(r(t))) = -4.3r(t) + 0.298tanh(r(t)) \\ +0.002tanh(r(\gamma(t))) + \sigma r(t)dB(t), \tag{72}$$

TABLE 3. The effect of selected values $\sigma \in (0, 0.2616)$ on θ_4 with fixed $k = 0.1$.

	k	$\bar{\sigma}$	σ	T/2	θ_1	θ_2	θ_3	θ_4
1✓	0.1	0.2616	0.04	0.25	0.0299	0.001229	0.001219	0.001219
2	0.1	0.2616	0.08	0.25	0.0298	0.001147	0.001139	0.001139
3	0.1	0.2616	0.12	0.25	0.0297	0.001011	0.001004	0.001004
4	0.1	0.2616	0.16	0.25	0.0296	0.000820	0.000816	0.000816
5	0.1	0.2616	0.20	0.25	0.0294	0.000576	0.000574	0.000574

TABLE 4. The effect of selected values $k \in (0, 0.1387)$ on θ_4 with fixed $\sigma = 0.04$.

	k	$\bar{\sigma}$	σ	T/2	θ_1	θ_2	θ_3	θ_4
1✓	0.10	0.2616	0.04	0.25	0.0299	0.001229	0.001219	0.001219
2	0.07	0.3394	0.04	0.25	0.0312	0.002179	0.002149	0.002149
3	0.05	0.3733	0.04	0.25	0.0319	0.002703	0.002657	0.002657
4	0.03	0.3954	0.04	0.25	0.0323	0.003083	0.003022	0.003022
5	0	0.4076	0.04	0.25	0.0326	0.003308	0.003237	0.003237

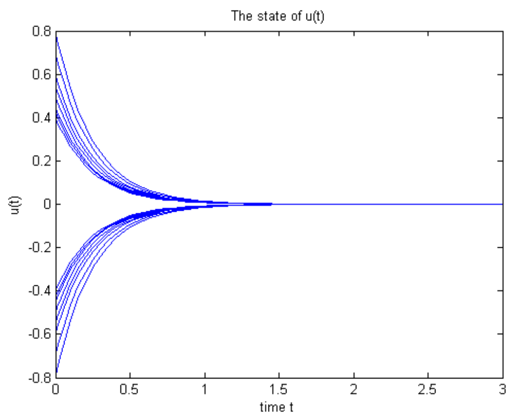

FIGURE 6. The stable trajectory of $u(t)$ in (71).

if we fix $\{\theta_p\} = \{\frac{p}{1000}\}$, $\{\eta_p\} = \{\frac{2p+1}{2000}\}$, $p \in \boldsymbol{N}$. The piecewise constant argument function $\rho(t) = \eta_p$ when $t \in [\theta_p, \theta_{p+1})$, $p \in \boldsymbol{N}$. From Theorem 2, $T = 0.5 > (ln2\alpha^2)/2\beta = 0.1155$. Set $k_1 = 4.3$, $k_2 = 1$. We substitute the parameters in (48), then the bound of k is shown as

$$k < \bar{k} = 0.1387.$$

If we set $k = 0.1$ and put it together with other fixed parameters into (49), accordingly we get the stochastic disturbance intensity:

$$\sigma < \bar{\sigma} = 0.2616.$$

Next, substitute $k = 0.1$, $\sigma = 0.04$ with other parameters into (B1), (B2) and (51), then we will get the upper bound of θ.

(i) From Assumption (B1), we have $\theta_1 = 0.02992$;

(ii) From (B2), we get $\theta_2 = 0.001229$;

(iii) From Theorem 2, we get $\theta_3 = 0.001219$, which is the unique positive solution that satisfies equation (51).

Hence, we can obtain the supremum of piecewise argument θ, that is

$$\theta_4 = \min\left\{\frac{T}{2}, \theta_1, \theta_2, \theta_3\right\}$$

$$= \min\left\{0.25, 0.0299, 0.001229, 0.001219\right\}$$

$$= 0.001219.$$

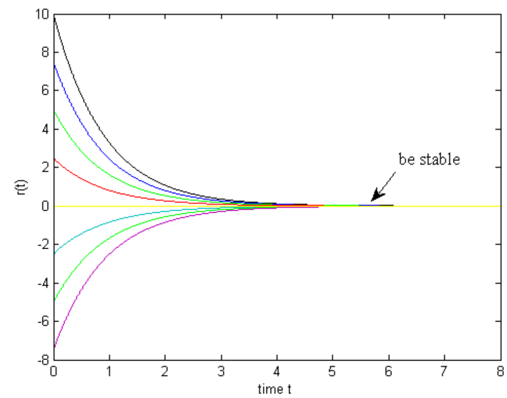

FIGURE 7. The convergent behavior trajectory of $r(t)$ in (72).

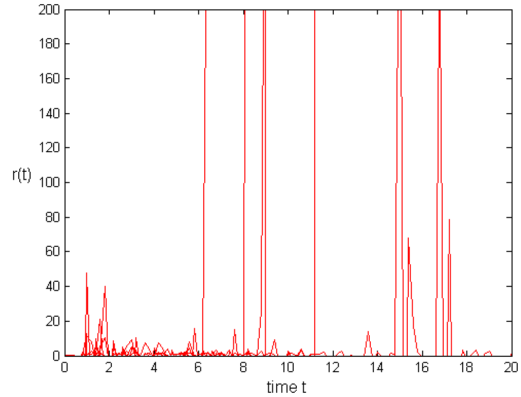

FIGURE 8. Instability of $r(t)$ with $k = 0.5$, $\sigma = 0.04$, $\theta = 0.001$ in (72).

Accordingly, if we select three indicators: $k = 0.1 < 0.1387$, $\sigma = 0.04 < 0.2616$ and $\theta = 0.001 < 0.001219$, the stable trajectory of $r(t)$ is provided in Fig. 7.

Remark 14: Table 3 and Table 4 show the dynamic process that θ_4 varies with σ, and σ varies with k in SNPNDS (72), the effect of selected values $\sigma \in (0, 0.2616)$ on θ_4 with fixed $k = 0.1$ is shown in Table 3, and the effect of selected values $k \in (0, 0.1387)$ on θ_4 with fixed $\sigma = 0.04$ is contained in Table 4. It is known that the relationship built in these three interference factors in SNPNDS is dynamic and the derived bounds $\bar{k}$, $\bar{\sigma}$ and θ_4 are visualized by Theorem 2.

Additionally, some unstable cases are given accordingly. The first category: one of the conditions of (48), (49) and (50) in Theorem 2 is not satisfied.

(1) For $k = 0.5 > 0.1387$, the unstable trajectory of state $r(t)$ of (72) can be seen in Fig. 8.

(2) For $\sigma = 0.3 > 0.2616$, the unstable trajectory of state $r(t)$ of (72) can be seen in Fig. 9.

(3) For $\theta = 0.2 > 0.0012$, the unstable trajectory of state $r(t)$ of (72) can be seen in Fig. 10.

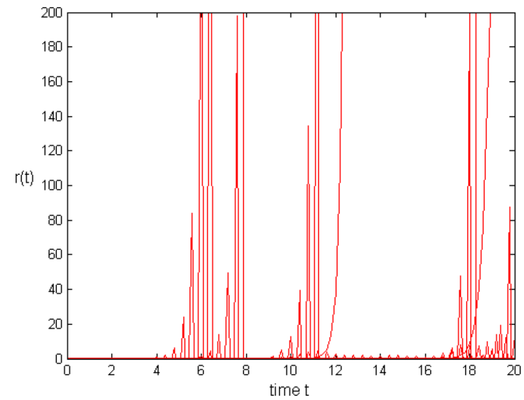

FIGURE 9. Instability of $r(t)$ with $k = 0.1$, $\sigma = 0.3$, $\theta = 0.001$ in (72).

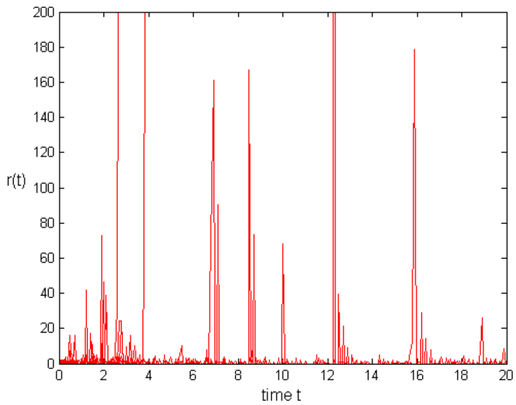

FIGURE 10. Instability of $r(t)$ with $k = 0.1$, $\sigma = 0.04$, $\theta = 0.2$ in (72).

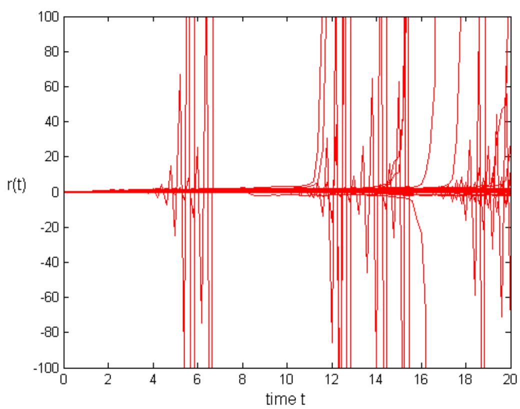

FIGURE 11. Instability of $r(t)$ with $k = 0.5$, $\sigma = 0.3$, $\theta = 0.001$ in (72).

The second category: any two conditions of (48), (49) and (50) in Theorem 2 are not satisfied.

(4) In Fig. 11, since $k = 0.5 > 0.1387$, $\sigma = 0.3 > 0.2616$, the behavior of state $r(t)$ in (72) is unstable.

(5)In Fig. 12, since $k = 0.5 > 0.1387$, $\theta = 0.2 > 0.0012$, the behavior of state $r(t)$ in (72) is unstable.

(6)In Fig. 13, since $\sigma = 0.3 > 0.2616$, $\theta = 0.2 > 0.0012$, the behavior of state $r(t)$ in (72) is unstable.

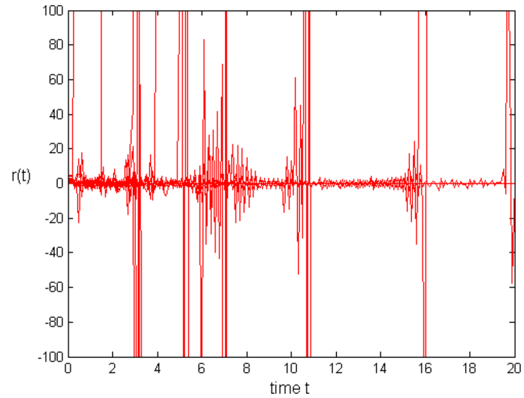

FIGURE 12. Instability of $r(t)$ with $k = 0.5$, $\sigma = 0.1$, $\theta = 0.2$ in (72).

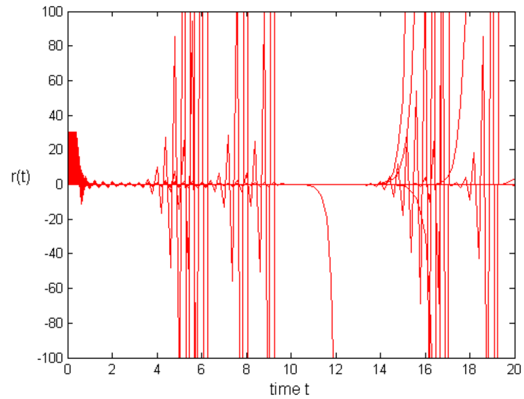

FIGURE 13. Instability of $r(t)$ with $k = 0.1$, $\sigma = 0.3$, $\theta = 0.2$ in (72).

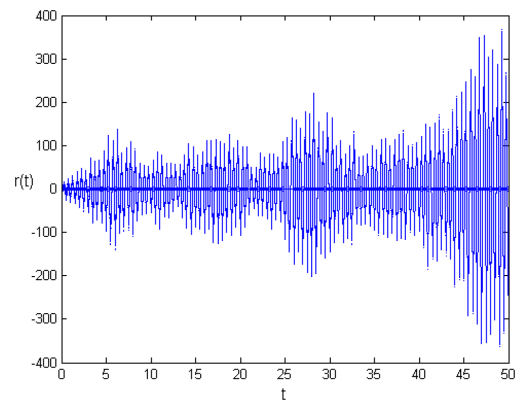

FIGURE 14. Instability of $r(t)$ with $k = 0.5$, $\sigma = 0.3$, $\theta = 0.2$ in (72).

The third category: (48), (49) and (50) in Theorem 2 are all not satisfied.

(7)Finally, if we set $k = 0.5 > 0.1387$, $\sigma = 0.3 > 0.2616$, $\theta = 0.2 > 0.0012$, the instability imgicon is shown as Fig. 14.

Remark 15: Compared with the simulation results of [26] and [28], it can be seen that the neutral terms and the stochastic disturbances will interact with each other, producing the unknown higher levels of randomness such as the tremor phenomenon in Fig. 8-Fig. 14, which further indicates that the influence of the neutral terms on the system cannot be ignored. At the same time, we have experimentally proved that if the real disturbance values are all lower than the derived results, SNPNDS will be stable; otherwise, as long as one of real interference values exceeds the threshold values $\bar{k}$, θ_4 and $\bar{\sigma}$, SNPNDS will be unstable. In addition, unstable images Fig. 8-Fig. 14 are also a favourable reference frame to testify the efficiency of derived results. And the striking contrast between 7 and Fig. 8-Fig. 14 exactly proves the strong robustness of SNPNDS under the deduced constraint conditions in Theorem 2.

VI. CONCLUSION

In summary, we have explored the RoGES of NPNDS and SNPNDS. Firstly, we provided the infrastructural lemmas to illustrate the relationship between PCA state $r(\rho(t))$ and current state $r(t)$ for NPNDS and SNPNDS. Secondly, the special ''independent parameters & interdependent variables'' method and optimal constraints targeted for NPNDS and SNPNDS are adopted to get the upper bounds of all perturbation factors. Besides, we theoretically proved that NPNDS and SNPNDS can be exponential stability if the selected disturbed values of the interference factors are all lower than the deduced results given in this paper. Eventually, two comprehensive examples demonstrated the efficiency of the robustness of the NPNDS and SNPNDS, respectively.

The following statement is a further vision of future work. On the one hand, some improvements for this work can be implemented. Firstly, future work may optimize the assumptions (A1)-(A2) used in the Theorem 1-Theorem 2. Secondly, the classical LMI method or Lyapunov construction method can be considered to be used here to optimize the calculation process. On the other hand, some available application directions are conceived. Firstly, future work can consider more diverse interference factors on SNPNDS, such as impulses, multiple time delays, Markov jumps and so on. Besides, other control measures can be adopted, such as fixed-time control, finite-time control, fuzzy control, etc. In addition, the independent parameters & interdependent variables method provided in this paper may provide a new idea for solving a class of algebraic problems with multivariable transcendental equations. Furthermore, SNPNDS explored here can be extended to high-dimensional, low-dimensional spaces or complex networked systems, such as the fractional-order SNPNDS or complex-value SNPNDS.

REFERENCES

[1] X. Jin, ''Adaptive fixed-time control for MIMO nonlinear systems with asymmetric output constraints using universal barrier functions,'' *IEEE Trans. Autom. Control*, vol. 64, no. 7, pp. 3046–3053, Jul. 2019.

[2] L. Xing, C. Wen, Z. Liu, H. Su, and J. Cai, ''Event-triggered adaptive control for a class of uncertain nonlinear systems,'' *IEEE Trans. Autom. Control*, vol. 62, no. 4, pp. 2071–2076, Apr. 2017.

[3] K. Sun, H. Yu, and X. Xia, ''Distributed control of nonlinear stochastic multi-agent systems with external disturbance and time-delay via event-triggered strategy,'' *Neurocomputing*, vol. 452, pp. 275–283, Sep. 2021.

[4] M. Hui, N. Luo, Q. Wu, R. Yao, and L. Bai, ''New results of finite-time synchronization via piecewise control for memristive Cohen–Grossberg neural networks with time-varying delays,'' *IEEE Access*, vol. 7, pp. 79173–79185, Jun. 2019.

[5] J. Zhang, W.-H. Chen, X. Lu, and J. Wen, ''Fuzzy controller synthesis for nonlinear neutral state-delayed systems with impulsive effects,'' *Inf. Sci.*, vol. 555, pp. 293–313, May 2021.

[6] F. Bertoli and A. N. Bishop, ''Nonlinear stochastic receding horizon control: Stability, robustness and Monte Carlo methods for control approximation,'' *Int. J. Control*, vol. 91, no. 10, pp. 2387–2402, Oct. 2018.

[7] W. Zhang, Q. Zhu, S. Mobayen, H. Yan, J. Qiu, and P. Narayan, ''U-model and U-control methodology for nonlinear dynamic systems,'' *Complexity*, vol. 2020, Jun. 2020, Art. no. 1050254.

[8] R. Rahmani, S. Mobayen, A. Fekih, and J.-S. Ro, ''Robust passivity cascade technique-based control using RBFN approximators for the stabilization of a cart inverted pendulum,'' *Mathematics*, vol. 9, no. 11, p. 1229, May 2021, doi: 10.3390/math9111229.

[9] F. Tao, M. Li, and Z. Fu, ''Robust stabilization control for uncertain nonlinear systems based on two-step coprime factorization,'' *J. Franklin Inst.*, vol. 357, no. 18, pp. 13666–13686, Dec. 2020.

[10] X. Yang, X. Ruan, and D. Li, ''Iterative learning control for nonlinear switched systems with constant time delay and noise,'' *IEEE Access*, vol. 8, pp. 3827–3836, Dec. 2020.

[11] M. Jafari and S. Mobayen, ''Second-order sliding set design for a class of uncertain nonlinear systems with disturbances: An LMI approach,'' *Math. Comput. Simul.*, vol. 156, pp. 110–125, Feb. 2019.

[12] G. Pujol-Vazquez, S. Mobayen, and L. Acho, ''Robust control design to the Furuta system under time delay measurement feedback and exogenous-based perturbation,'' *Mathematics*, vol. 8, no. 12, Nov. 2020, Art. no. 2131, doi: 10.3390/math8122131.

[13] X. Liu, Q. Song, X. Yang, Z. Zhao, Y. Liu, and F. E. Alsaadi, ''Asymptotic stability and synchronization for nonlinear distributed-order system with uncertain parameters,'' *Neurocomputing*, vol. 404, pp. 276–282, Sep. 2020.

[14] A. Lahrouz, R. Hajjami, M. El Jarroudi, and A. Settati, ''Mittag-Leffler stability and bifurcation of a nonlinear fractional model with relapse,'' *J. Comput. Appl. Math.*, vol. 386, Apr. 2021, Art. no. 113247, doi: 10.1016/j.cam.2020.113247.

[15] Y. Sun, J. Zhang, and X. Zhao, ''Nonlinearly exponential stability for the compressible Navier-Stokes equations with temperature-dependent transport coefficients,'' *J. Differ. Equ.*, vol. 286, pp. 676–709, Jun. 2021.

[16] H. Yang, M. Song, and M. Liu, ''Strong convergence and exponential stability of stochastic differential equations with piecewise continuous arguments for non-globally Lipschitz continuous coefficients,'' *Appl. Math. Comput.*, vol. 341, pp. 111–127, Jan. 2019.

[17] H. Hamiche, H. Takhi, M. Messadi, K. Kemih, O. Megherbi, and M. Bettayeb, ''New synchronization results for a class of nonlinear discrete-time chaotic systems based on synergetic observer and their implementation,'' *Math. Comput. Simul.*, vol. 185, pp. 194–217, Jul. 2021.

[18] H. Pang, S. Tan, and C. Li, ''Exponential quasi-dissipativity and boundedness property for switched nonlinear systems,'' *Nonlinear Anal.: Hybrid Syst.*, vol. 40, May 2021, Art. no. 101012, doi: 10.1016/j.nahs.2021.101012.

[19] L. Cuevas, D. Nešić, and C. Manzie, ''Robustness analysis of nonlinear observers for the slow variables of singularly perturbed systems,'' *Int. J. Robust Nonlinear Control*, vol. 30, no. 14, pp. 5628–5656, Sep. 2020.

[20] Y. Hu, F. Wu, and C. Huang, ''Robustness of exponential stability of a class of stochastic functional differential equations with infinite delay,'' *Automatica*, vol. 45, no. 11, pp. 2577–2584, Nov. 2009.

[21] F. Kasolis and M. Clemens, ''Information-based model reduction for nonlinear electro-quasistatic problems,'' *J. Comput. Phys.*, vol. 404, Mar. 2020, Art. no. 109118, doi: 10.1016/j.jcp.2019.109118.

[22] E. Madsen, O. S. Rosenlund, D. Brandt, and X. Zhang, ''Comprehensive modeling and identification of nonlinear joint dynamics for collaborative industrial robot manipulators,'' *Control Eng. Pract.*, vol. 101, Aug. 2020, Art. no. 104462, doi: 10.1016/j.conengprac.2020.104462.

[23] O. Mokhiamar and S. Amine, ''Lateral motion control of skid steering vehicles using full drive-by-wire system,'' *Alexandria Eng. J.*, vol. 56, no. 4, pp. 383–394, Dec. 2017.

[24] Y. Shen and J. Wang, ''Robustness analysis of global exponential stability of recurrent neural networks in the presence of time delays and random disturbances,'' *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 1, pp. 87–96, Jan. 2012.

[25] Y. Shen and J. Wang, ''Robustness analysis of global exponential stability of non-linear systems with time delays and neutral terms,'' *IET Control Theory Appl.*, vol. 7, no. 9, pp. 1227–1232, Jun. 2013.

[26] Y. Shen and J. Wang, "Robustness of global exponential stability of nonlinear systems with random disturbances and time delays," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 46, no. 9, pp. 1157–1166, Sep. 2016.
[27] C. Wu, J. Hu, and Y. Li, "Robustness analysis of hybrid stochastic neural networks with neutral terms and time-varying delays," *Discrete Dyn. Nature Soc.*, vol. 2015, Oct. 2015, Art. no. 278571.
[28] J.-E. Zhang, "Robustness analysis of global exponential stability of nonlinear systems with deviating argument and stochastic disturbance," *IEEE Access*, vol. 5, pp. 13446–13454, 2017.
[29] W. Si, T. Xie, and B. Li, "Exploration on robustness of exponentially global stability of recurrent neural networks with neutral terms and generalized piecewise constant arguments," *Discrete Dyn. Nature Soc.*, vol. 2021, Jul. 2021, Art. no. 9941881, doi: 10.1155/2021/9941881.
[30] M. U. Akhmet, D. Aruğaslan, and E. Yilmaz, "Stability analysis of recurrent neural networks with piecewise constant argument of generalized type," *Neural Netw.*, vol. 23, no. 7, pp. 805–811, Sep. 2010.
[31] S. M. Shah and J. Wiener, "Advanced differential equations with piecewise constant argument deviations," *Int. J. Math. Math. Sci.*, vol. 6, no. 4, pp. 671–703, Nov. 1983.
[32] K. L. Cooke and J. Wiener, "Retarded differential equations with piecewise constant delays," *J. Math. Anal. Appl.*, vol. 99, no. 1, pp. 265–297, Mar. 1984.
[33] G. Bao, S. P. Wen, and Z. G. Zeng, "Robust stability analysis of interval fuzzy Cohen–Grossberg neural networks with piecewise constant argument of generalized type," *Neural Netw.*, vol. 33, pp. 32–41, Sep. 2012.
[34] H. Zhang, H. Ji, Z. Ye, and T. Senping, "Passivity analysis of Markov jump BAM neural networks with mode-dependent mixed time-delays via piecewise-constant transition rates," *J. Franklin Inst.*, vol. 353, no. 6, pp. 1436–1459, Apr. 2016.
[35] M. C. Tan and D. S. Xu, "Multiple μ-stability analysis for memristor-based complex-valued neural networks with nonmonotonic piecewise nonlinear activation functions and unbounded time-varying delays," *Neurocomputing*, vol. 275, pp. 2681–2701, Jan. 2018.
[36] W. Zhou, B. Li, and J.-E. Zhang, "Matrix measure approach for stability and synchronization of complex-valued neural networks with deviating argument," *Math. Problems Eng.*, vol. 2020, Nov. 2020, Art. no. 8877129.
[37] L. Wan and A. Wu, "Multistability in Mittag-Leffler sense of fractional-order neural networks with piecewise constant arguments," *Neurocomputing*, vol. 286, pp. 1–10, Apr. 2018.
[38] M. Milošević, "The Euler–Maruyama approximation of solutions to stochastic differential equations with piecewise constant arguments," *J. Comput. Appl. Math.*, vol. 298, pp. 1–12, May 2016.
[39] L. Barreira and C. Valls, "Stable invariant manifolds for delay equations with piecewise constant argument," *J. Difference Equ. Appl.*, vol. 24, no. 1, pp. 148–163, Nov. 2017.
[40] A. Wu and Z. Zeng, "Output convergence of fuzzy neurodynamic system with piecewise constant argument of generalized type and time-varying input," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 46, no. 12, pp. 1689–1702, Dec. 2016.
[41] A. Bellen, N. Guglielmi, and M. Zennaro, "On the contractivity and asymptotic stability of systems of delay differential equations of neutral type," *BIT Numer. Math.*, vol. 39, pp. 1–24, Mar. 1999.
[42] J. Zhao, Y. Yi, and Y. Xu, "Strong convergence and stability of the split-step theta method for highly nonlinear neutral stochastic delay integro differential equation," *Appl. Numer. Math.*, vol. 157, pp. 385–404, Nov. 2020.
[43] H. Y. Mo, L. N. Liu, M. L. Xing, F. Q. Deng, and B. Zhang, "Exponential stability of implicit numerical solution for nonlinear neutral stochastic differential equations with time-varying delay and Poisson jumps," *Math. Methods Appl. Sci.*, vol. 44, no. 7, pp. 5574–5592, Dec. 2020.
[44] B. S. Vadivoo, R. Ramachandran, J. Cao, H. Zhang, and X. Li, "Controllability analysis of nonlinear neutral-type fractional-order differential systems with state delay and impulsive effects," *Int. J. Control, Autom. Syst.*, vol. 16, no. 2, pp. 659–669, Apr. 2018.
[45] M. Shen, C. Fei, W. Fei, and X. Mao, "Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations," *Syst. Control Lett.*, vol. 137, Mar. 2020, Art. no. 104645, doi: 10.1016/j.sysconle.2020.104645.
[46] Q. Song, S. Chen, Z. Zhao, Y. Liu, and F. E. Alsaadi, "Passive filter design for fractional-order quaternion-valued neural networks with neutral delays and external disturbance," *Neural Netw.*, vol. 137, pp. 18–30, May 2021.
[47] W. S. Wang and S.-F. Li, "Dissipativity of Runge–Kutta methods for neutral delay differential equations with piecewise constant delay," *Appl. Math. Lett.*, vol. 21, no. 9, pp. 983–991, Sep. 2008.
[48] W. Wang, Q. Fan, Y. Zhang, and S. Li, "Asymptotic stability of solution to nonlinear neutral and volterra functional differential equations in Banach spaces," *Appl. Math. Comput.*, vol. 237, pp. 217–226, Jun. 2014.
[49] T. Li and Y. V. Rogovchenko, "On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations," *Appl. Math. Lett.*, vol. 105, Jul. 2020, Art. no. 106293, doi: 10.1016/j.aml.2020.106293.
[50] Z. S. Aghayan, A. Alfi, and J. A. Tenreiro Machado, "Robust stability analysis of uncertain fractional order neutral-type delay nonlinear systems with actuator saturation," *Appl. Math. Model.*, vol. 90, pp. 1035–1048, Feb. 2021.
[51] F. Wu, S. Hu, and C. Huang, "Robustness of general decay stability of nonlinear neutral stochastic functional differential equations with infinite delay," *Syst. Control Lett.*, vol. 59, nos. 3–4, pp. 195–202, Mar. 2010.
[52] X. Mao, "Stability and stabilisation of stochastic differential delay equations," *IET Control Theory Appl.*, vol. 1, no. 6, pp. 1551–1566, Nov. 2007.
[53] M. Song, Y. Geng, and M. Liu, "Stability equivalence among stochastic differential equations and stochastic differential equations with piecewise continuous arguments and corresponding Euler–Maruyama methods," *Appl. Math. Comput.*, vol. 400, Jul. 2021, Art. no. 125813, doi: 10.1016/j.amc.2020.125813.
[54] S. Reich, "On an existence and uniqueness theory for nonlinear differential-algebraic equations," *Circuits, Syst., Signal Process.*, vol. 10, no. 3, pp. 343–359, Sep. 1991.
[55] X. X. Liao, *Theory and Application of Stability for Dynamical Systems*. Beijing, China: Nat. Defence Ind., 2000.

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