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Filtered Observer-Based IDA-PBC Control for Trajectory Tracking of a Quadrotor

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ABSTRACT In this paper, a new filtered observer-based IDA-PBC (Interconnection and Damping Assignment-Passivity Based Control) strategy is developed for trajectory tracking of a quadrotor in the presence of disturbances and model uncertainties. The proposed algorithm allows the control of the quadrotor in all its states. It can deal with the noisy output measurements and uncertainties in the translational and rotational dynamics, as unmodeled dynamics inherent to real systems or unknown external signals (exogenous signals). The designed filtered observer estimates the state from noisy output measurements, and it depends only on two design parameters. Numerical simulation tests are carried out to highlight the overall controller approach in a realistic scenario.

INDEX TERMS Observer-based control IDA-PBC, quadrotor, filtered observer, model uncertainties.

I. INTRODUCTION

Quadrotors are widely used in everyday life, they have been used in many applications, e.g., surveillance, photography, transport of packages, agriculture, rescue, etc, [1], [2]. However, the performance of these vehicles can be degraded by the presence of disturbances, such as measurement noise and uncertainties. For the control of quadrotors, it is necessary to have specific characteristics of smoothness in both control signals and sensors, for this has been used innumerable techniques using filters [3], [4].

The use and research of observers for state estimation, perturbations, and the calculation of output controllers in drones is still very active, e.g., [5]–[8]. Given the complexity and its enormous applications, this research is still open [9]–[12]. To the best of our knowledge, there are few works oriented to the use of the novel filtered observers to applications in drones or quadrotors as [13].

Thus, the design problem of controllers and observers for quadrotors has been widely researched in the literature and is a novel subject in the current research. Within the framework of this theme, a continuous high-gain observer-based PD (Proportional Derivative) control for a quadrotor vehicle is synthesized in [14], where the high-gain observer estimates the state vector from the position coordinates and only one angle (yaw), the interesting of this paper is that the continuous-time estimation (the roll and pitch angles and all velocities) is achievable by using sampled measurements and a PD control. Other recent work [15] includes the problem of fault estimation for a quadrotor using a robust $\mathcal{H}_\infty$ observer, where the supposed faults are external disturbances, parameter uncertainties, and nonlinear terms. In [16] a cascade active disturbance rejection control and a backstepping sliding-mode control are applied to achieve the desired trajectory tracking performance of a quadrotor in the presence of external disturbances and model uncertainties. [17] presents an integral sliding mode and backstepping sliding mode controllers in a double loop control structure, which ensure the trajectory tracking capability for the desired position of a quadrotor. Also, [18] developed an observer to estimate the external wrench forces applied on a UAV (Unmanned Aerial Vehicle). The lagrangian representation of the UAV is used, which permits to shape the interactive behavior of the quadrotor using an IDA-PBC (Interconnection and Damping Assignment Passivity Based Control).

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The IDA-PBC strategy has been successfully applied to mechatronic systems, for example: [19] presents a dynamic feedback control by the IDA-PBC algorithm, to stabilize a class of weakly-coupled electromechanical systems as the electrostatic MEMS devices and the magnetic levitation. [20] designs two adaptive control techniques to handle uncertainties caused by parametric and modeling errors in the Acrobot and non-prehensile planar rolling robotic (disk-on-disk) systems. The techniques use the PCH (Port-controlled Hamiltonian) modelling framework and the IDA-PBC algorithm. [21] introduces an alternative construction that yields a continuous IDA-PBC law.

Other important works are: [22], [23]. [22] presents a linear matrix inequality-based second-order sliding set control for the uncertain systems with time-varying uncertainties, nonlinearities and external disturbances. [23] develops a passivity cascade technique-based control for a nonlinear inverted pendulum system with uncertainties and exogenous disturbances.

Other approaches have addressed the observer-based control design. For example, in [24] the design problem of the observer-based adaptive sliding mode control is studied and applied to a fixed-wing UAV. In [25] an observer-based backstepping control for a quadrotor in the presence of disturbances and parametric uncertainties is proposed. The disturbance observer is used to compensate the unknown disturbance, however, it only compensates small measurement noises. [26] develops an observed-based control strategy for position-yaw tracking of quadrotors. A disturbance observer-based nonlinear controller for the trajectory tracking of a quadrotor in the presence of noisy output measurements is introduced in [27], where the translational positions have a good performance, however, the rotational positions have highly significant variations.

It is noteworthy that several relatively recent works have been focused on the improvement of the high-gain observer performances, especially in the presence of measurement noises, by proposing versions with a time-varying design parameter or by slightly changing the structure of the gain in the observer [13], [28], [29]. This new work falls within the framework of this last approach.

A very common practical situation in the real-time implementation of any observer-based control is that output measurements can be strongly disturbed by noisy measurements, resulting in high estimation errors, or worse, unstable estimation behavior. In this situation, the main problem is to prove the exponential error convergence to a neighborhood of the origin for any initial conditions of a proposed observer that provides a free noise estimation of the state vector for a quadrotor. In this sense, the main focus of this work consists in the synthesis of a novel filtered observer-based IDA-PBC control for trajectory tracking of a quadrotor, which by the observer will be capable of dealing with the noisy output measurements and uncertainties in the translational and rotational dynamics, as unmodeled dynamics inherent to real systems or unknown external signals (exogenous signals).

There is currently little work on control algorithms applied to quadrotors that can handle and thus perform well in the presence of relatively large noise signals in the output measurements and uncertainties, in fact, most of the few current designs perform well only for small values of noise signals that are present in the output measurements, e.g. [14], [25]. Furthermore, current approaches do not consider all the system dynamics to propose estimates, e.g. [24]. Moreover, most of the works consider only the presence of noise in the rotation dynamics, e.g. [27].

Unlike many works published in the literature, the proposed observer system only needs two tuning parameters, one to adjust the state estimate and the other to filter the measurement noise. Traditional observer systems use several tuning parameters and most of them perform a linearization, e.g. [9], [10]. Also, some works consider more than 4 (out of 12) elements of the state vector as measured output, e.g. [26].

Thus, motivated by the aforementioned works, in this paper, we offer an alternative approach for trajectory tracking of a quadrotor in the presence of measurement noise and uncertainty in the states by using an observer-based IDA-PBC strategy. In order to design a filtered observer that estimates the state from noisy output measurements, one represents the quadrotor dynamic model as a non-uniformly observable nonlinear system with coupled blocks, based on the proposed observer from [14], for nonlinear systems with block-state-affine cascade structure, where a sufficient persistent excitation condition is provided. In the case when the noise signal is of high intensity or is strongly variable, the existing observer-based control design results fail to provide a good behavior. However, this paper introduces an algorithm that works for relatively large noise signals, which are present in the output measurements, and it also deals with uncertainties.

The primary aim of unmanned aerial vehicles is to follow a trajectory within the desired operating and safety limits. However, in many cases this is not possible due to the presence of measurement noise and uncertainty. It is evident that the quadrotor needs a robust controller to compensate for disturbances. Faced with this problem, the novelty of this work is to design a robust observer-based IDA-PBC control with respect to uncertainty in states and external signals that disturb the system, such as measurement noise, for trajectory tracking of a quadrotor. The contribution is then, the observer design that considers the nonlinearities of the model and that is able to estimate with a good performance, despite the presence of relatively large measurement noise and uncertainty in the state, this causes the controller performance to be adequate, i.e. even if the controller is not robust to measurement noise and uncertainty in the states, the observer will feedback to the controller signals free of measurement noise and uncertainty. In the case of many physical systems, the above is very important, for example, in the quadrotor the noise in the tracker positions has a significant impact on the accuracy of the position measurement.

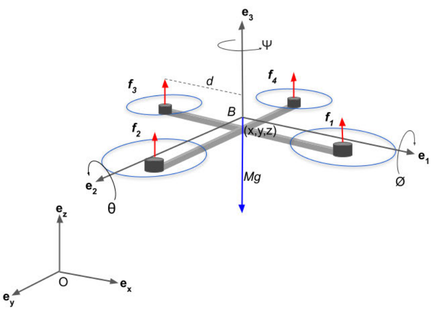


FIGURE 1. Quadrotor UAV.

The paper is organized as follows: Section II presents the quadrotor dynamic model. In Section 3, the IDA-PBC strategy is developed. The filtered observer design is designed in Section IV. Before addressing the observer design for a quadrotor, some fundamental definitions are given. The performances of the proposed observer-based control are shown in Section V. Finally, conclusion is given in Section VI.

II. QUADROTOR DYNAMIC MODEL

In this section, the quadcopter dynamic model is introduced. Figure 1 shows the free-body diagram of a quadrotor, which has six degrees of freedom and only four control inputs.

As reported in Figure 1, $O = \{e_x, e_y, e_z\}$ is the inertial coordinate frame fixed to the ground and $B = \{e_1, e_2, e_3\}$ is the body fixed coordinate frame. The vector $q = \begin{bmatrix} \nu & \varrho \end{bmatrix}^T \in \mathbb{R}^6$, where $\nu = \begin{bmatrix} x & y & z \end{bmatrix}^T$ denotes the translational position of the vehicle. The rotational position of the quadcopter is represented by the Euler angles: roll, pitch and yaw, i,e,, $\varrho = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^T$, the distance between the motors and the gravity center is represented by d. Finally, f_1, f_2, f_3 and f_4 are the thrust forces provided by each rotor.

The control input is defined as $u = \begin{bmatrix} f & \tau \end{bmatrix}^T \in \mathbb{R}^4$, where f is the total thrust magnitude, $\tau = \begin{bmatrix} \tau_\psi & \tau_\theta & \tau_\phi \end{bmatrix}^T$ is the input torques vector. Particularly for a quadcopter, the relationships between the input torques and forces are defined as

$$\begin{bmatrix} f \\ \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} k_i \, \omega_i^2 \\ k_d \left(\omega_1{}^2 - \omega_2{}^2 + \omega_3{}^2 - \omega_4{}^2\right) \\ (k_2 \, \omega_2^2 - k_4 \, \omega_4^4) d \\ (k_3 \, \omega_3^2 - k_1 \, \omega_1^1) d \end{bmatrix}$$

where $k_i \, \omega_i^2$ defines the thrust of the propeller of motor i w.r.t. angular velocity ω_i, on each rotor. R is the rotational matrix from the body frame to the inertial one, where the shorthand notations for trigonometric functions $s_\theta = \sin(\theta)$ and $c_\theta = \cos(\theta)$ are used, this also applies for all angles, it yields:

$$R = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\phi s_\theta & s_\psi s_\phi + c_\psi c_\phi s_\theta \\ c_\theta s_\psi & s_\psi s_\theta s_\phi + c_\psi c_\phi & -c_\psi s_\phi + s_\psi s_\theta c_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

The mathematical model of the quadcopter for the translational and rotational motions is represented as follows [14]:

$$\begin{cases} \ddot{\nu} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \begin{bmatrix} s_\psi s_\phi + c_\psi c_\phi s_\theta \\ -c_\psi s_\phi + s_\psi s_\theta c_\phi \\ c_\theta c_\phi \end{bmatrix} \dfrac{f}{m}, \\ \ddot{\varrho} = \begin{bmatrix} a_1 \dot{\theta} \dot{\psi} + b_1 \tau_\phi \\ a_2 \dot{\phi} \dot{\psi} + b_2 \tau_\theta \\ a_3 \dot{\theta} \dot{\phi} + b_3 \tau_\psi \end{bmatrix} \end{cases} \tag{1}$$

where the mass of the quadcopter is introduced by m, the gravity acceleration is represented by g, the constants a_1, a_2, a_3, b_1, b_2 and b_3 are obtained as:

$$a_1 = \frac{I_y - I_z}{I_x}, \quad a_2 = \frac{I_z - I_x}{I_y}, \quad a_3 = \frac{I_x - I_y}{I_z},$$
$$b_1 = \frac{1}{I_x}, \quad b_2 = \frac{1}{I_y}, \quad b_3 = \frac{1}{I_z}$$

where I_x, I_y and I_z represent the moments of inertia of the vehicle with respect to frame B.

According to $q = [\nu \quad \varrho]^T$, the mathematical model (1) can be rewritten as follows:

$$\begin{cases} \ddot{x} = \dfrac{f}{m}\left(s_\psi s_\phi + c_\psi c_\phi s_\theta\right) & \ddot{\phi} = a_1 \dot{\theta} \dot{\psi} + b_1 \tau_\phi \\ \ddot{y} = -\dfrac{f}{m}\left(-c_\psi s_\phi + s_\psi s_\theta c_\phi\right) & \ddot{\theta} = a_2 \dot{\phi} \dot{\psi} + b_2 \tau_\theta \\ \ddot{z} = -g + \dfrac{f}{m} c_\theta c_\phi & \ddot{\psi} = a_3 \dot{\theta} \dot{\phi} + b_3 \tau_\psi \end{cases} \tag{2}$$

III. IDA-PBC STRATEGY FOR A QUADROTOR

We can represent (2) as:

$$M\ddot{q} + C(\dot{q})\dot{q} + G = B(q)u$$

where the inertia matrix M is constant and independent of q, $C(\dot{q})$ represents the Coriolis matrix, G is the gravitational vector and $B(q)$ takes the form:

$$B(q) = \begin{bmatrix} s_\psi s_\phi + c_\psi c_\phi s_\theta & 0 & 0 & 0 \\ -c_\psi s_\phi + s_\psi s_\theta c_\phi & 0 & 0 & 0 \\ c_\theta c_\phi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The total energy is

$$H(q, p) = \frac{1}{2} p^T M^{-1} p + V(q) \tag{3}$$

where the total energy $H(q, p)$ is a Hamiltonian function, $q \in \mathbb{R}^6$ represents the generalized position and $p \in \mathbb{R}^6$ introduces the generalized momentum. Then, the dynamic system is defined as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u \tag{4}$$

where $I_n \in \mathbb{R}^{6 \times 6}$ is the identity matrix, $\nabla_q H = \partial H / \partial q$ and $\nabla_p H = \partial H / \partial p$.

The desired Hamiltonian function can be written as:

$$H_d(q,p) = \frac{1}{2}p^T {M_d}^{-1} p + V_d(q) \tag{5}$$

where $M_d = M_d^T > 0$ introduces the closed loop inertia matrix and V_d expresses the desired potential energy. It is necessary that the energy V_d have an isolated minimum at q_*, i.e., $q_* = \arg\min V_d(q)$.

The PBC control signal is the sum of energy shaping and injects the damping:

$$u = u_{es}(q,p) + u_{di}(q,p) \tag{6}$$

The desired port controlled Hamiltonian dynamics is taken as [30]:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = [J_d(q,p) - R_d(q,p)] \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{7}$$

where

$$J_d = -J_d^T = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2(q,p) \end{bmatrix}$$
$$R_d = R_d^T = \begin{bmatrix} 0 & 0 \\ 0 & B(q)K_v B(q)^T \end{bmatrix} \geq 0$$

J_2 is a skew-symmetric matrix and $K_v = K_v^T > 0$, both containing design parameters.

The damping injection term is given by

$$u_{di}(q,p) = -K_v B(q)^T \nabla_p H_d \tag{8}$$

To compute u_{es} we substitute (6) and (8) in (4) and make it equal to (7), then

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u_{es} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M^{-1}M_d & J_2(q,p) \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{9}$$

One can observe from (9) that the first row is clearly satisfied. In the second set of equations the PDEs (Partial Differential Equations) yield to the energy shaping expression given by

$$u_{es} = \left(B(q)^T B(q)\right)^{-1} B(q)^T \times \left(\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p\right) \tag{10}$$

The PDEs (9) can be separated into the equations

$$B(q)^{\perp}\left\{\nabla_q(p^T M^{-1} p) - M_d M^{-1} \nabla_q (p^T M_d^{-1} p) + 2J_2 M_d^{-1} p\right\} = 0$$
$$B(q)^{\perp}\left\{\nabla_q V - M_d M^{-1} \nabla_q V_d\right\} = 0 \tag{11}$$

where $B(q)^{\perp}$ is the full rank left annihilator of $B(q)$.

To obtain the energy-shaping, we consider $J_2 = 0$ and M_d a constant matrix of the form:

$$\mathcal{M}_d = \begin{bmatrix} a_1 & 0 & 0 & a_7 & 0 & 0 \\ 0 & a_2 & 0 & 0 & a_8 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ a_7 & 0 & 0 & a_4 & 0 & 0 \\ 0 & a_8 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix},$$
$$a_1 a_2 a_3 a_4 a_5 a_6 > 0 \tag{12}$$

With $B(q)^{\perp} = \begin{bmatrix} c_\phi c_\psi & c_\phi s_\psi & -s_\phi & 0 & 0 & 0 \end{bmatrix}$, the potential energy PDE (11) takes the form:

$$c_\phi s_\psi \left(\frac{a_2}{m}\frac{\partial V_d}{\partial y} + \frac{a_8}{I_y}\frac{\partial V_d}{\partial \phi}\right) + c_\phi c_\psi \left(\frac{a_1}{m}\frac{\partial V_d}{\partial z} + \frac{a_7}{I_x}\frac{\partial V_d}{\partial \theta}\right) + Mg s_\phi - \frac{a_3}{m} s_\phi \frac{\partial V_d}{\partial z} = 0 \tag{13}$$

which is solved and thus the desired potential energy is computed by

$$V_d = \frac{mgI_y}{a_8 c_\psi} \ln c_\phi + \Phi(\bullet) \tag{14}$$

where $\Phi(q)$ is an arbitrary differentiable function. For this, the necessary condition $\nabla_q V_d(q_*) = 0$ and the sufficient condition $\nabla_q^2 V_d(q_*) > 0$ will hold if the Hessian of $\Phi(q)$ at q_* is positive. We choose $\Phi(q)$ to be a quadratic function which leads to

$$V_d = \frac{mgI_y}{a_8 c_\psi} \ln c_\phi + \frac{k_{px}}{2}\left(\theta - \frac{a_7 m}{a_1 I_x} x\right)^2 + \frac{k_{py}}{2}\left(\phi - \frac{a_8 m}{a_2 I_y} y\right)^2 + \frac{k_{pz}}{2}\left(z - \frac{mI_y}{a_3 a_8 c_\psi} \ln c_\phi\right)^2 \tag{15}$$

where (x_d, y_d, z_d) denotes the equilibrium configuration and the k_{px}, k_{py}, k_{pz} terms are proportional gains and are used as tuning parameters. In order to determine the final control law, we first compute the energy-shaping expression u_{es} from (10) and second, we obtain the damping injection expression u_{di} from (8).

IV. DESIGN OF THE FILTERED OBSERVER

The excessive cost of the motion capture systems have motivated researchers to propose estimation and control approaches, which are able to handle this condition. Thus, a simple and cost-effective way to obtain the position ν of the quadrotor is to use a Global Positioning System (GPS), in addition, the yaw angle ψ can be obtained from a digital compass. Another obstacle of the robust control design for the quadrotor is the presence of the adverse effects as disturbance, uncertainty and noisy measurements. The output measurements are usually transmitted with measurement noise, there is therefore a peaking phenomenon of the state estimates affecting the control performance. Then, in order to propose an observer with filtering capabilities for the system (2), firstly, one defines the continuous-time measured output of the quadrotor system as: $Y = [x\ y\ z\ \psi]$.

With the aim of providing a filtered observer based IDA-PBC control for the system (2) and (3), one should consider a change of coordinates $\Phi : K \subset \mathbb{R}^{12} \rightarrow M \subset \mathbb{R}^{12}$, these set are compact and where $Z = \Phi(q)$, in such a way that the quadrotor system (2) in their coordinates can be rendered a canonical form. Thus, the new vector of coordinates is given: $z = \left[z^1\ z^2\ z^3\ z^4\right]^T$, where the first two variables is $z^j = \left[z_1^j\ z_2^j\ z_3^j\ z_4^j\right]^T$, $j = 1, 2$ and the other two is $z^i = [z_1^i, z_2^i]^T$, $i = 3, 4$.

Now, the new system of coordinates is defined as follows:

$$
\begin{array}{lll}
z_1^1 = x & z_1^2 = y & z_1^3 = z \\
z_2^1 = \dot{x} & z_2^2 = \dot{y} & z_2^3 = \dot{z} \\
z_3^1 = s_\theta & z_3^2 = -c_\theta s_\phi & z_1^4 = \psi \\
z_4^1 = \dot{\theta} s_\theta & z_4^2 = \dot{\theta} s_\theta s_\phi - \dot{\phi} c_\theta c_\phi & z_2^4 = \dot{\psi}
\end{array}
$$

Consider the quadrotor positions $y_1 = x$ and $y_2 = y$ be the first two measured outputs, therefore, z^j for $j = 1, 2$ and it yields:

$$
\left\{ \dot{z}_1^j = z_2^j, \quad \dot{z}_2^j = \frac{f}{m} z_3^j, \quad \dot{z}_3^j = z_4^j, \quad \dot{z}_4^j = \varphi_4^j(z, u) \right.
$$

where $\varphi_4^1(z, u) = (a_2 \dot{\phi} \dot{\psi} + b_2 \tau_\psi) c_\theta - \dot{\theta}^2\ s_\theta$ and $\varphi_4^2(z, u) = (a_2 \dot{\phi} \dot{\psi} + b_2 \tau_\psi) s_\theta s_\phi + \dot{\theta}^2\ c_\theta s_\phi + 2 \dot{\phi} \dot{\theta} s_\theta c_\phi - (a_1 \dot{\theta} \dot{\psi} + b_1 \tau_\theta) c_\theta c_\phi + \dot{\phi}^2\ c_\theta s_\phi$.

Thus, the first two subsystems of the coordinates can be written as:

$$
\begin{cases} \dot{z}^j = A_j(u) z^j + \varphi^j(z, u), \\ y_j = C_j z^j, \quad \text{for } j = 1, 2 \end{cases} \tag{16}
$$

where $C_j = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ and:

$$
A_j(u) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{f}{m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \varphi_j(z, u) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varphi_4^1(z, u) \end{bmatrix}
$$

The same procedure is also possible for the other two states, therefore the quadrotor position $y_3 = z$ and the angular position $y_4 = \psi$, thus:

$$
\begin{cases} \dot{z}^i = A_i(u) z^i + \varphi^i(z, u), \\ y_i = C_i z^i, \quad \text{for } i = 3, 4 \end{cases} \tag{17}
$$

where

$$
A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix},
$$
$$
\varphi^i(z, u) = \begin{bmatrix} 0 \\ \varphi_2^i(z, u) \end{bmatrix}
$$

where the nonlinear terms are $\varphi_2^3(z, u) = -g + c_\theta c_\phi \frac{u_1}{m}$ and $\varphi_2^4(z, u) = a_3 \dot{\phi} \dot{\theta} + b_3\ u_4 4$.

Based on the obtained nonlinear subsystems (16)-(17), the purpose of this work is to propose an observer for a quadrotor, which is subject to unknown uncertainties and noise signals, i.e., these signals are present in each subsystem. This leads to the following class of block state affine nonlinear uncertain systems:

$$
\begin{cases} \dot{x} = A(u) x + \varphi(x, u) + B\varepsilon \\ y = Cx + w \end{cases} \tag{18}
$$

where $x = (x^1, \ldots, x^q)^T \in \mathbb{R}^n$ represents the state vector, with $x^k = (x_1^k, x_2^k, \ldots, x_{\lambda_k}^k)^T \in \mathbb{R}^{n_k}$, where $x_i^k = (x_{i,1}^k, \ldots, x_{i,p_k}^k)^T \in \mathbb{R}^{p_k}$ with $x_{i,j}^k \in \mathbb{R}$ for $k = 1, \ldots, q$, $i = 1, \ldots, \lambda_k$, $j = 1, \ldots, p_k$ with $\sum_{k=1}^{q} n_k = \sum_{k=1}^{q} p_k \lambda_k = n$; $p_k \geq 1$ and $\lambda_k \geq 2$. $u(t) \in \mathbb{R}^m$ defines the input vector. Finally, $y = (y_1, \ldots, y_q)^T \in \mathbb{R}^p$ denotes the output vector, with $y_k \in \mathbb{R}^{p_k}$ for $k = 1, \ldots, q$ and $\sum_{k=1}^{q} p_k = p$. The unknown function is $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^q)^T \in \mathbb{R}^n$ and the noise signal is $w = (w^1, \ldots, w^q)^T \in \mathbb{R}^p$.

The matrix of the functions A and the matrix C are expressed as follows:

$$
A(u) = \operatorname{diag}\left[A_1(u) \cdots A_q(u)\right], \quad C = \operatorname{diag}\left[C_1 \cdots C_q\right]
$$
$$
A_k(u) = \begin{bmatrix} 0 & A_1^k(u) & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & & & A_{\lambda_k - 1}^k(u) \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$
$$
C_k = \begin{bmatrix} I_{p_k} & 0 & \cdots & 0 \end{bmatrix}
$$

The matrix B is defined as follows:

$$
B = \begin{bmatrix} B_1 \cdots B_q \end{bmatrix}^T \quad B_k = \begin{bmatrix} 0 & \cdots & 0 & I_{p_k \times 1} \end{bmatrix}^T
$$

The nonlinear function $\varphi(u, x)$ is given by the expression:

$$
\varphi(u, x) = \begin{pmatrix} \varphi^1(u, x) \\ \varphi^2(u, x) \\ \vdots \\ \varphi^q(u, x) \end{pmatrix} \in \mathbb{R}^n,
$$
$$
\varphi^k(u, x) = \begin{pmatrix} \varphi_1^k(u, x) \\ \varphi_2^k(u, x) \\ \vdots \\ \varphi_{\lambda_k}^k(u, x) \end{pmatrix} \in \mathbb{R}^{n_k}
$$

where the function $\varphi_i^k(u, x) \in \mathbb{R}^{p_k}$ is differentiable w.r.t. x and assumes a structural dependence on the state variables.

Additionally, in order to provide a filtered observer, the following classical assumptions are considered (see [13])

A1 The state vector and the control input are bounded, i.e., $x \in X$ and $u \in U$.

A2 The function $\varphi(x, u)$ is Lipschitz w.r.t. x uniformly w.r.t. u, with a Lipschitz constant L_φ.

A3 The upper bound of the function $A_k(u)$ is denoted by $A_M = \sup_{t \geq 0} \|A_k(u)\|$.

A4 The unknown function ε and the noise signal w are essentially bounded, i.e., $\exists \bar{\varepsilon} > 0$,

$\text{Sup.Ess.}_{t\geq 0} \|\varepsilon\| \leq \bar{\varepsilon}$; and $\exists \bar{w} > 0$, $\text{Sup.Ess.}_{t\geq 0} \|w\| \leq \bar{w}$.

A. SOME DEFINITIONS AND NOTATIONS

Let us introduce the following diagonal matrix for filtered observer synthesis:

$$\Delta_k(\Theta) = diag\left[I_{p_k} \quad I_{p_k}/\Theta^{\delta_k} \quad \cdots \quad I_{p_k}/\Theta^{\delta_k(\lambda_k-1)} \right] \tag{19}$$

for $k = 1, \ldots, q$ and where the tuning parameter is $\Theta \geq 1$ and one defines the powers δ_k such as:

$$\begin{cases} \delta_k = 2^{q-k}\left(\prod_{i=k+1}^{q}\left(\lambda_i - \frac{3}{2}\right)\right) & for\ k = 1, \ldots, q-1, \\ \delta_q = 1 \end{cases}$$

For any $k = 1, \ldots, q-1$, one has:

$$\frac{\delta_k}{2} = \left(\lambda_{k+1} - \frac{3}{2}\right)\delta_{k+1} \tag{20}$$

Since $\lambda_k \geq 2$, one has $(\lambda_{k+1} - 3/2) \geq 1/2$ and therefore the δ_k's is a positive decreasing sequence of real numbers, it means that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_q = 1$.

Thus, the following sequence of scalar numbers for $k = 1, \ldots, q$ and $i = 1, \ldots, \lambda_k$ is denoted as:

$$\sigma_i^k = \sigma_1^k + (i-1)\delta_k \tag{21}$$

where $\sigma_1^k = -(\lambda_k - 1)\delta_k + (\lambda_1 - 1)\delta_1 + \eta(1 - \frac{1}{2^{k-1}})$, where the term η is chosen arbitrarily within the interval $\eta \in (0, 1]$.

Now, similarly to the Δ_k's, one defines for $k = 1, \ldots, q$ the diagonal matrices Λ_k's as: $\Lambda_k(\Theta) = \Theta^{-\sigma_1^k}\Delta_k(\Theta)$. Thus, considering the forms of $\Lambda_k(\Theta)$, $\Delta_k(\Theta)$, the matrices A_k and C_k, one obtains the next equalities:

$$\begin{aligned} \Lambda_k(\Theta)A_k(u)\Lambda_k(\Theta)^{-1} &= \Delta_k(\Theta)A_k(u)\Delta_k(\Theta)^{-1} \\ &= \Theta^{\delta_k}A_k(u) \\ \Theta^{-\sigma_1^k}C_k\Lambda_k^{-1}(\Theta) &= C_k\Delta_k^{-1} = C_k \\ \delta_k K_k(S_k) &= K_k(S_k)D_k(\Theta) \end{aligned} \tag{22}$$

The main drawback for synthesising an observer for system (18) is that its observability depends on the input. Therefore, this should be ensured by a condition of the persistent excitation. In this sense, one needs to introduce a state transition matrix for system (18), as follows:

$$\dot{\xi}_k(t) = A_k(u)\xi_k(t) \tag{23}$$

where $\xi_k \in \mathbb{R}^n$ and u denotes the input. The matrix $\Phi_u^k(t, s)$ is defined as follows

$$\frac{d\Phi_u^k(t,s)}{dt} = A_k(u)\Phi_u^k(t,s), \quad \forall t \geq s \geq 0,\ \Phi_u^k(t,t) = I_n, \quad \forall t \geq 0$$

Then, the following condition should be satisfied

A5 The input u satisfies the following persistent excitation condition:

$$\begin{aligned} &\int_{t-1/\Theta^{\delta_k}}^{t} \Phi_u^k(s,t)^T C_k^T C_k \Phi_u^k(s,t)ds \\ &\geq \frac{\delta_0}{\Theta^{\delta_k}\alpha(\Theta)}\Delta_k(\Theta)^2 \\ &\exists \Theta^* > 0, \quad \exists \delta_0 > 0,\ \forall \Theta \geq \Theta^*, \\ &\forall k = 1, \ldots, q, \quad \forall t \geq 1/\Theta^{\delta_k}, \end{aligned} \tag{24}$$

where $\alpha(\Theta) \geq 1$ is such that $\lim_{\Theta\to\infty} \frac{\alpha(\Theta)}{\Theta^{\frac{\eta}{2^{q-1}}}} = 0$.

Our main result of the work is addressed to the design of an observer providing an estimation of the state in presence of the perturbed output, as noisy output measurements, also, the nonlinear system considering some uncertainties. The candidate observer for system (18) is defined as follows:

$$\dot{\hat{x}}^k = A_k(u)\hat{x}^k + \varphi^k(u,\hat{x}) - \Theta^{\delta^k}\Delta_k^{-1}(\Theta)K_k(S_k)\eta_k \tag{25}$$
$$\dot{S}_k = \Theta^{\delta_k}\left(-S_k - A_k(u)^T S_k - S_k A_k(u) + C_k^T C_k\right) \tag{26}$$
$$\dot{\eta}_k = \mu^{\delta_k}\Theta^{\delta_k}\left((\Theta^{\delta_k}A_k^T - I_k)\eta_k + C_{\eta_k}^T(C_k\hat{x}^k - y_k)\right) \tag{27}$$

for $k = 1, \ldots, k$, where $\hat{x}^k = (\hat{x}_1^k, \hat{x}_2^k, \ldots, \hat{x}_{\lambda_k}^k)^T \in \mathbb{R}^{n_k}$ denotes the state estimate and the tuning parameters $\Theta > 0$ and $\mu > 0$. The matrices A_k^T, I_k (identity matrix) and C_{η_k} with dimensions $n_k \times n_k$, $n_k \times n_k$ and $1 \times n_k$ and, respectively, are given:

$$A_k = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & & & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{n_k} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

The observer gain is

$$K_k(S_k) = diag\left(S_k^{-1}C_k^T\right) \tag{28}$$

Following the above, the next main theorem can be established:

Theorem 1: Consider the uncertain nonlinear system with noisy measurement outputs (18), satisfying assumptions **A1-A5**. If there exists two tuning parameters Θ^* and μ^* for sufficiently values of Θ and μ, such that $\Theta > \Theta^*$ and $\mu > \mu^*$, system (25)-(27) provides a free noise estimation of the state vector for system (18) with an exponential error convergence to a neighborhood of the origin for any initial conditions $(\hat{x}^k(0)) \in X$ and $\eta(0) = 0$, thus, the observation error $\hat{x}^k - x^k$ closes to zero, depending on $\bar{\varepsilon}$ that is the ultimate limit of uncertainties and the noise signal $\bar{w}$.

Proof of Theorem 1: Now, we prove the convergence to a neighborhood of the origin of the i-th component of the observation error $\tilde{e}^k \in \mathbb{R}^{n_k}$ where $\tilde{e}^k = \hat{x}^k - x^k$. Let $\tilde{e}^k$ be the k-th subcomponent of $\tilde{e}$, which is the estimation error $\tilde{e} = \hat{x} - x$. Now we obtain:

$$\dot{\tilde{e}}^k = A_k(u)\tilde{e}^k + \tilde{\varphi}^k(u,\hat{x},x) - \Theta^{\delta_k}\Delta_k^{-1}(\Theta)K_k(S_k)\eta_k - B_k\varepsilon_k \tag{29}$$

where $\tilde{\varphi}^k(u,\hat{x},x) = \varphi^k(u,\hat{x}) - \varphi^k(u,x)$.

For $k = 1, \ldots, q$ set:

$$\bar{e}^k = \Lambda_k(\Theta)\tilde{e}^k \quad \text{and } \bar{\eta}^k = D_k(\Theta)\tilde{\eta}^k \tag{30}$$

where $\Lambda_k(\Theta)$ is given by (19) and $D_k(\Theta)$ is given:

$$D_k(\Theta) = diag\left[I_{p_1} \quad I_{p_1}/\Theta^{\delta_k} \quad \cdots \quad I_{p_1}/\Theta^{\delta_k(\lambda_k-1)} \right] \tag{31}$$

It is therefore easy to deduce that $D_k(\Theta)C_{\eta_k}^T C_k = C_k C_{\eta_k}^T \Lambda_k(\Theta)$.

Using the identities (22) and (30), into equation (29), it yields:

$$\begin{aligned}
\dot{\bar{e}}^k &= \Theta^{\delta_k} A_k(u)\bar{e}^k - \Theta^{\delta_k}\Lambda_k(\Theta)K_k(S)\eta_k \\
&\quad + \Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\varepsilon_k \\
&= \Theta^{\delta_k} A_k(u)\bar{e}^k - \Theta^{\delta_k} K_k(S)D_k(\Theta)\eta_k \\
&\quad + \Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\varepsilon_k \\
&= \Theta^{\delta_k} A_k(u)\bar{e}^k - \Theta^{\delta_k} K_k(S)\bar{\eta}_k \\
&\quad + \Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\varepsilon_k
\end{aligned} \tag{32}$$

Now, from the equations (18) and (27), and also using the identifies (29), one gets:

$$\begin{aligned}
\dot{\bar{\eta}}_k &= -\Theta^{\delta_k}\mu^{\delta_k}\left((I_k - A_k^T)\bar{\eta}_k - D_k(\Theta)C_{\eta_k}^T C_k(\tilde{e}^k - w_k)\right) \\
&= -\Theta^{\delta_k}\mu^{\delta_k}\left((I_k - A_k^T)\bar{\eta}_k + C_{\eta_k}^T C_k\Lambda_k(\Theta)(\tilde{e}^k - w_k)\right) \\
&= -\Theta^{\delta_k}\mu^{\delta_k}\left((I_k - A_k^T)\bar{\eta}_k - C_{\eta_k}^T C_k\bar{e}^k + C_{\eta_k}^T C_k w_k\right)
\end{aligned} \tag{33}$$

Adding and subtracting the term $\Theta^{\delta_k} S_k^{-1} C_k^T C_k \bar{e}^k$, one obtains the following error dynamics:

$$\begin{aligned}
\dot{\bar{e}}^k &= \Theta^{\delta_k}\left(A_k(u) - S_k^{-1}C_k^T C_k\right)\bar{e}^k - \Theta^{\delta_k} K_k(S_k)\bar{\eta}_k \\
&\quad + \Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\varepsilon_k \\
&\quad + \Theta^{\delta_k} S_k^{-1} C_k^T C_k \bar{e}^k
\end{aligned} \tag{34}$$

From the observation gain $K_k(S_k)$ in (28), one has:

$$K_k(S_k)U_k = S_k^{-1}C_k^T C_k \tag{35}$$

which permits to rewrite the error dynamics:

$$\begin{aligned}
\dot{\bar{e}}^k &= \Theta^{\delta_k}\left(A_k(u) - S_k^{-1}C_k^T C_k\right)\bar{e}^k - \Theta^{\delta_k} K_k(S_k)\bar{z}_k \\
&\quad + \Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\varepsilon_k
\end{aligned} \tag{36}$$

where $\bar{z}^k = \bar{\eta}_k - U_k\bar{x}^k$.

From the equations (33) and (36), one obtains the time derivative of $\bar{z}^k$, as follows:

$$\begin{aligned}
\dot{\bar{z}}^k &= \dot{\bar{\eta}}_k - U_k\dot{\bar{e}}^k \\
&= -\Theta^{\delta_k}\mu^{\delta_k}\left((I_k - A_k^T)\bar{\eta}_k - C_{\eta_k}^T C_k\bar{e}^k + C_{\eta_k}^T C_k w_k\right) \\
&\quad - \Theta^{\delta_k} U_k A_k(u)\bar{e}^k - U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) \\
&\quad + \Theta^{\delta_k} U K_k(S_k)\bar{\eta}_k + U_k B_k\varepsilon_k \\
&= \Theta^{\delta_k}\left(-\mu^{\delta_k}(I_k - A_k^T) + U_k K_k(S_k)\right)\bar{\eta}_k \\
&\quad + \Theta^{\delta_k}\left(\mu^{\delta_k} C_{\eta_k}^T C_k - U_k A_k(u)\right)\bar{e}^k \\
&\quad - U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Theta^{\delta_k}\mu^{\delta_k} C_{\eta_k}^T C_k w_k \\
&= \Theta^{\delta_k}\left(-\mu^{\delta_k}(I_k - A_k^T) + U_k K_k(S_k)\right)(\bar{z}^k + U_k\bar{e}^k) \\
&\quad + \Theta^{\delta_k}\left(\mu^{\delta_k} C_{\eta_k}^T C_k - U_k A_k(u)\right)\bar{e}^k \\
&\quad - U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Theta^{\delta_k}\mu^{\delta_k} C_{\eta_k}^T C_k w_k \\
&= \Theta^{\delta_k}\left(-\mu^{\delta_k}(I_k - A_k^T) + U_k K_k(S_k)\right)\bar{z}^k \\
&\quad + \Theta^{\delta_k}\left(\mu^{\delta_k} C_{\eta_k}^T C_k - U_k A_k(u) - \mu^{\delta_k}(I_k - A_k^T)U_k\right. \\
&\quad \left. + U_k K_k(S_k)U_k\right)\bar{e}^k - U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) \\
&\quad - \Theta^{\delta_k}\mu^{\delta_k} C_{\eta_k}^T C_k w_k
\end{aligned} \tag{37}$$

From (35), one verifies that $(I_k - A_k^T)U_k = C_{\eta_k}^T C_k$, therefore one gets the following dynamic equation:

$$\begin{aligned}
\dot{\bar{z}}^k &= \Theta^{\delta_k}\mu^{\delta_k}\left(-(I_k - A_k^T) + \frac{1}{\mu^{\delta_k}}U_k K_k(S_k)\right)\bar{z}^k \\
&\quad + \Theta^{\delta_k}\left(-U_k A_k(u) + U_k K_k(S_k)U_k\right)\bar{e}^k \\
&\quad - U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}^k, x^k) - \Theta^{\delta_k}\mu^{\delta_k} C_{\eta_k}^T C_k w_k
\end{aligned} \tag{38}$$

Now, consider the following Lyapunov candidate function:

$$\begin{aligned}
V(\bar{e}, \bar{z}, t) &= V_1(\bar{x}) + V_2(\bar{z}) \\
&= \bar{e}^T S\bar{e} + \bar{z}^T\bar{z} \\
&= \sum_{k=1}^{q} V_{1,k}(\bar{e}^k) + \sum_{k=1}^{q} V_{2,k}(\bar{z}^k)
\end{aligned} \tag{39}$$

where $V_{1,k}(\bar{e}^k) = \bar{e}^{k^T} S_k\bar{e}^k$ and $V_{2,k}(\bar{e}^k) = \bar{z}^{k^T}\bar{z}^k$.

Firstly, one focuses on the term $V_{1,k}(\bar{e}^k)$, differentiating $V_{1,k}(\bar{e}^k)$ along the trajectories of the system:

$$\dot{V}_{1,k}(\bar{e}^k) = 2\bar{e}^{k^T} S_k(t)\dot{\bar{e}}^k + \bar{e}^{k^T}\dot{S}_k(t)\bar{e}^k \tag{40}$$

where S_k is given by (26).

Introducing the ODE Lyapunov equation (36) into (40), it yields:

$$\begin{aligned}
\dot{V}_{1,k}(\bar{e}^k) &= -\Theta^{\delta_k}\bar{e}^{k^T} S_k\bar{e}^k - \Theta^{\delta_k}\bar{e}^k C_k^T C_k\bar{e}^k \\
&\quad + 2\bar{e}^{k^T} S_k\Lambda_k(\Theta)\tilde{\varphi}^k(u, \hat{x}, x) \\
&\quad - 2\bar{e}^{k^T} S_k\Lambda_k(\Theta)B_k\varepsilon_k - \Theta^{\delta_k}\bar{e}^{k^T} S_k\bar{e}^k K_k(S_k)\bar{z}^k \\
&\leq -\Theta^{\delta_k}\bar{e}^{k^T} S_k\bar{e}^k \\
&\quad + 2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\sum_{i=1}^{\lambda_k}\frac{1}{\Theta^{\sigma_i^k}}\left\|\tilde{\varphi}_i^k(u, \hat{x}^k, x^k)\right\| \\
&\quad + 2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\frac{\bar{\varepsilon}}{\Theta^{\sigma_1^k + (\lambda_k-1)\delta_k}} \\
&\quad + 2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\|K_k(S_k)\|\|\bar{z}^k\|
\end{aligned} \tag{41}$$

where $\sigma_i^k = \sigma_1^k + (i-1)\delta_k$.

In addition, considering Assumption **A1-A2**, i.e. the boundedness of $\frac{\partial\varphi}{\partial x}$, one has:

$$\begin{aligned}
\dot{V}_{1,k}(\bar{e}^k) &\leq -\Theta^{\delta_k} V_{1,k}(\bar{e}^k) \\
&\quad + 2L_{\tilde{\varphi}^k}\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)} \\
&\quad \times \sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\chi_{l,j}^{k,i}\Theta^{\sigma_j^l - \sigma_i^k}\left\|\bar{e}_j^l\right\|
\end{aligned}$$

$$+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\frac{\bar{\varepsilon}}{\Theta^{\sigma_1^k+(\lambda_k-1)\delta_k}}$$
$$+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\|K_k(S_k)\|\|\tilde{z}^k\| \quad (42)$$

where $L_{\tilde{\varphi}^k}$ comes from Assumption **A2** and the $\chi_{l,j}^{k,i}$'s are defined as in Lemma 1 in [31].

According to assumption **A5**, one clearly has

$$\lambda_m(S_k) \geq \frac{e^{-1}\delta_0}{\alpha(\Theta)} \quad (43)$$

Also, σ_j^l is defined as in (21):

$$\sigma_j^l = \sigma_1^l + (j+1)\delta_l \quad (44)$$

Finally, one obtains:

$$\begin{aligned}&\dot{V}_{1,k}(\bar{e}^k)\\ &\leq -\Theta^{\delta_k}V_{1,k}(\bar{e}^k) + 2L_{\tilde{\varphi}^k}\mu_s\\ &\times\sqrt{V_{1,k}(\bar{e}^k)}\sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\chi_{l,j}^{k,i}\theta^{\sigma_j^l-\sigma_i^k-\frac{\delta_l}{2}-\frac{\delta_k}{2}}\sqrt{\Theta^{\delta_l}V_l^1(\bar{e}_{1,l})}\\ &+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\frac{\bar{\varepsilon}}{\Theta^{\sigma_1^k+(\lambda_k-1)\delta_k}}\\ &+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\|K_k(S_k)\|\|\tilde{z}^k\|\end{aligned} \quad (45)$$

where:

$$\mu_s = \sqrt{\frac{\lambda_M(S_k)}{\lambda_m(S_k)}} \quad (46)$$

Moreover, according to Lemma 1 in [31], one has:

$$\begin{aligned}\sigma_j^l - \sigma_i^k - \frac{\delta_l}{2} - \frac{\delta_k}{2} &\leq -\frac{\eta}{2^q}\\ \sigma_1^k + (\lambda_k-1)\delta_k &\geq (\lambda_1-1)\delta_1\end{aligned} \quad (47)$$

Considering (45) and (47), one gets

$$\begin{aligned}&\dot{V}_{1,k}(\bar{e}^k)\\ &\leq -\Theta^{\delta_k}V_{1,k}(\bar{e}^k)\\ &+2L_{\tilde{\varphi}^k}\mu_s\Theta^{-\frac{\eta}{2^q}}\sqrt{\Theta^{\delta_k}V_{1,k}(\bar{e}^k)}\sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{\Theta^{\delta_l}V_l^1(\bar{e}_l)}\\ &+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\|K_k(S_k)\|\|\tilde{z}^k\|\end{aligned} \quad (48)$$

Moreover, let

$$V_{1,k}^*(\bar{e}^k) = \Theta^{\delta_k}V_{1,k}(\bar{e}^k) \quad V_1^*(\bar{e}) = \sum_{k=1}^{q}V_{1,k}^*(\bar{e}^k) \quad (49)$$

Since $\delta_q = 1$, one has

$$V_1^*(\bar{e}) \geq \Theta^{\delta q}V_1(\bar{e}) = \Theta V_1(\bar{e}) \quad (50)$$

And inequality (48) becomes:

$$\begin{aligned}\dot{V}_{1,k} &\leq -V_{1,k}^*\\ &\quad+2L_{\tilde{\varphi}^k}\mu_s\Theta^{-\frac{\eta}{2^q}}\sqrt{V_{1,k}^*(\bar{e}^k)}\sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{V_{1,l}^*(\bar{e}_l)}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\|K_k(S_k)\|\|\tilde{z}^k\|\\ &= -V_{1,k}^*\\ &\quad+2L_{\tilde{\varphi}^k}\mu_s\lambda_k\Theta^{-\frac{\eta}{2^q}}\sqrt{V_{1,k}^*(\bar{e}^k)}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{V_{1,l}^*(\bar{e}_l)}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_{1,k}(\bar{e}^k)}\|K_k(S_k)\|\|\tilde{z}^k\|\end{aligned} \quad (51)$$

Then,

$$\begin{aligned}\dot{V}_{1,k} &\leq -V_{1,k}^* + 2L_{\tilde{\varphi}^k}\mu_s\lambda_k\Theta^{-\frac{\eta}{2^q}}\sqrt{V_{1,k}^*(\bar{e}^k)}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{V_1^*(\bar{e}_l)}\\ &\quad+2\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_1(\bar{e})}\|K_k(S_k)\|\|\tilde{z}^k\|\\ &\leq -V_{1,k}^* + 2nL_{\tilde{\varphi}^k}\mu_s\lambda_k\Theta^{-\frac{\eta}{2^q}}\sqrt{V_{1,k}^*(\bar{e}^k)}\sqrt{V_1^*(\bar{e}_l)}\\ &\quad+2\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_1(\bar{e})}\|K_k(S_k)\|\|\tilde{z}^k\|\\ &\leq -V_{1,k}^* + 2nL_{\tilde{\varphi}^k}\mu_s\lambda_k\Theta^{-\frac{\eta}{2^q}}V_1^*(\bar{e})\\ &\quad+2\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_1(\bar{e})}\|K_k(S_k)\|\|\tilde{z}^k\|\end{aligned} \quad (52)$$

Hence,

$$\begin{aligned}\dot{V}_1(\bar{e}) &\leq -V_1^*(\bar{e}) + 2n^2L_{\tilde{\varphi}}\mu_s\Theta^{-\frac{\eta}{2^q}}V_1^*(\bar{e})\\ &\quad+2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_1(\bar{e})}\|K_k(S_k)\|\|\tilde{z}^k\|\\ &\leq -\left(1-2n^2\mu_s\Theta^{-\frac{\eta}{2^q}}L_{\tilde{\varphi}}\right)V_1^*(\bar{e})\\ &\quad+2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\\ &\quad+2\sqrt{\lambda_M(S_k)}\sqrt{V_1(\bar{e})}\|K_k(S_k)\|\|\tilde{z}^k\|\end{aligned} \quad (53)$$

where $L_{\tilde{\varphi}} = \max\left\{L_{\tilde{\varphi}^k};\, 1 \leq k \leq q\right\}$.

Furthermore, using (50) for θ high enough such that

$$\left(1 - n^2\mu_s\Theta^{-\frac{\eta}{2^q}}L_{\tilde{\varphi}}\right) > 0 \quad (54)$$

According to assumption **A3** and (54), one gets:

$$\dot{V}_1(\bar{e}) \leq -\Theta\left(1-\sqrt{\frac{\alpha(\Theta)}{\Theta^{\frac{\eta}{2q-1}}}}\sqrt{\frac{n^4 e\lambda_M(S)}{\delta_0}}L_{\tilde{\varphi}}\right)V_1(\bar{e}) + 2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}} + 2\sqrt{\lambda_M(S_k)}\sqrt{V_1(\bar{e})}\|K_k(S_k)\|\|\bar{z}^k\| \tag{55}$$

Now, we focus on the term $V_{2,k}(\bar{z}^k) = \bar{z}^{k^T}\bar{z}^k = \|\bar{z}^k\|^2$, one gets the following differential equation:

$$\begin{aligned}\dot{V}_{2,k}(\bar{z}^k) &= \Theta^{\delta_k}\mu^{\delta_k}\bar{z}^{k^T}\left(-(2I_k-(A_k^T+A_k))\right)\bar{z}^k \\ &\quad + 2\Theta^{\delta_k}\bar{z}^{k^T}U_kK_k(S_k)\bar{z}^k \\ &\quad + 2\Theta^{\delta_k}\bar{z}^{k^T}\left(-U_kA_k(u)+U_kK_k(S_k)U_k\right)\bar{e}^k \\ &\quad - 2\bar{z}^{k^T}U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u,\hat{x}^k,x^k) \\ &\quad - 2\Theta^{\delta_k}\bar{z}^{k^T}\mu^{\delta_k}C_{\eta_k}^T C_k w_k \\ &\leq -\Theta^{\delta_k}\mu^{\delta_k}\rho_k\|\bar{z}^k\|^2 + 2\Theta^{\delta_k}\|U_kK_k(S_k)\|\|\bar{z}^k\|^2 \\ &\quad + 2\Theta^{\delta_k}\|U_kA_k(u)+U_kK_k(S_k)U_k\|\|\bar{e}^k\|\|\bar{z}^k\| \\ &\quad - 2\bar{z}^{k^T}U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u,\hat{x}^k,x^k) \\ &\quad - 2\Theta^{\delta_k}\bar{z}^{k^T}\mu^{\delta_k}C_{\eta_k}^T C_k w_k \end{aligned}\tag{56}$$

where ρ_k denotes the smallest eigenvalue of the SPD matrix $2I_k-(A_k^T+A_k)$.

From Assumption **A2**, one obtains the following expressions:

$$\begin{aligned}&2\bar{z}^{k^T}U_k\Lambda_k(\Theta)\tilde{\varphi}^k(u,\hat{x}^k,x^k) \\ &\quad \leq 2n\Theta^{-\frac{\eta}{2q}}L_{\tilde{\varphi}}\sqrt{V_{2,k}(\bar{z}^k)}\frac{\sqrt{V_{1,k}(e)}}{\sqrt{\lambda_m(S_k)}}\end{aligned}\tag{57}$$

$$\begin{aligned}&2\Theta^{\delta_k}\|U_kA_k(u)+U_kK_k(S_k)U_k\|\|\bar{e}^k\|\|\bar{z}^k\| \\ &\quad \leq 2\Theta^{\delta_k}\sqrt{n}(A_M+K_M)\sqrt{V_{2,k}(\bar{z}^k)}\frac{\sqrt{V_{1,k}(\bar{e})}}{\sqrt{\lambda_m(S_k)}}\end{aligned}\tag{58}$$

where $A_M = sup_{u\in U}A(u)$ and $K_M = sup_{t\geq 0}\|K_k(S_k)\|$.

Then, substituting the above expression into (56), it yields:

$$\begin{aligned}\dot{V}_{2,k}(\bar{z}^k) &\leq -\Theta^{\delta_k}\mu^{\delta_k}\rho_k V_{2,k}(\bar{z}^k) + 2\Theta^{\delta_k}\sqrt{n}K_M V_{2,k}(\bar{z}^k) \\ &\quad + 2\Theta^{\delta_k}\sqrt{n}(A_M+K_M)\sqrt{V_{2,k}(\bar{z}^k)}\frac{\sqrt{V_{1,k}(e)}}{\sqrt{\lambda_m(S_k)}} \\ &\quad + 2n\Theta^{-\frac{\eta}{2q}}L_{\tilde{\varphi}}\sqrt{V_{2,k}(\bar{z}^k)}\frac{\sqrt{V_{1,k}(\bar{e})}}{\sqrt{\lambda_m(S_k)}} \\ &\quad + 2\Theta^{\delta_k}\mu^{\delta_k}\sqrt{V_{2,k}(\bar{z}^k)}\bar{w} \\ &\leq -\Theta^{\delta_k}\mu^{\delta_k}\rho_k\left(1-2\frac{K_M\sqrt{n}}{\mu^{\delta_k}\rho_k}\right)V_{2,k}(\bar{z}^k) \\ &\quad + 2\sqrt{n}\left(\sqrt{n}\Theta^{-\frac{\eta}{2q}}L_{\tilde{\varphi}}+\Theta^{\delta_k}(A_M+K_M)\right) \\ &\quad \times\sqrt{V_{2,k}(\bar{z}^k)}\frac{\sqrt{V_{1,k}(\bar{e})}}{\sqrt{\lambda_m(S_k)}} + 2\Theta^{\delta_k}\mu^{\delta_k}\sqrt{V_{2,k}(\bar{z}^k)}\bar{w}\end{aligned}\tag{59}$$

The value of μ can be chosen as $\left(1-2\frac{K_M\sqrt{n}}{\mu^{\delta_k}\rho_k}\right)\geq\frac{1}{2}$, thus, $\mu\geq 2\sqrt{n}\frac{K_M}{\rho_k}$ and $\Theta^{\delta_k}\leq 1$. Therefore, the above inequality can be rewritten as:

$$\begin{aligned}\dot{V}_{2,k}(\bar{z}^k) &\leq -\frac{\Theta^{\delta_k}\mu^{\delta_k}\rho_k}{2}V_{2,k}(\bar{z}^k) + 2\Theta\mu\sqrt{V_{2,k}(\bar{z}^k)}\bar{w} \\ &\quad + 2\sqrt{\frac{ne}{\delta_0}}\left(\sqrt{n}\Theta^{-\frac{\eta}{2q}}L_{\tilde{\varphi}}+\Theta^{\delta_k}(A_M+K_M)\right) \\ &\quad \times\sqrt{\alpha(\Theta)}\sqrt{V_{2,k}(\bar{z}^k)}\sqrt{V_{1,k}(\bar{e})} \\ &\leq -\frac{\Theta^{\delta_k}\mu^{\delta_k}\rho_k}{2}V_{2,k}(\bar{z}^k) + 2\Theta^{\delta_k}\mu^{\delta_k}\sqrt{V_{2,k}(\bar{z}^k)}\bar{w} \\ &\quad + 2\sqrt{\frac{ne}{\delta_0}}\left(\sqrt{n}\Theta^{-\left(\frac{\eta}{2q}+\delta_k\right)}L_{\tilde{\varphi}}+(A_M+K_M)\right) \\ &\quad \times\Theta^{\delta_k}\sqrt{\alpha(\Theta)}\sqrt{V_{2,k}(\bar{z}^k)}\sqrt{V_{1,k}(\bar{e})} \\ &\leq -\frac{\Theta^{\delta_k}\mu^{\delta_k}\rho_k}{2}V_{2,k}(\bar{z}^k) + 2\Theta^{\delta_k}\mu^{\delta_k}\sqrt{V_{2,k}(\bar{z}^k)}\bar{w} \\ &\quad + 2\sqrt{\frac{ne}{\delta_0}}\left(\sqrt{n}L_{\tilde{\varphi}}+(A_M+K_M)\right) \\ &\quad \times\Theta^{\delta_k}\sqrt{\alpha(\Theta)}\sqrt{V_{2,k}(\bar{z}^k)}\sqrt{V_{1,k}(\bar{e})}\end{aligned}\tag{60}$$

Similarly as in the previous term $V_{1,k}^*$, one considers that $V_{2,k}^*(\bar{z}^k) = \Theta^{\delta_k}V_k^2(\bar{z}^k)$, therefore, $V_2^*(\bar{z}) = \sum_{k=1}^{q}V_{2,k}^*(\bar{z}^k)$, and knowing that $\delta_q = 1$, one deduces that

$$V_2^*(\bar{z}) \geq \Theta^{\delta_q}V_2(\bar{z}) = \Theta V_2(\bar{z}) \tag{61}$$

Hence,

$$\begin{aligned}\dot{V}_2(\bar{z}) &\leq -\frac{\Theta\mu\rho}{2}V_2(\bar{z}) + 2\Theta\mu\sqrt{V_2(\bar{z})}\bar{w} \\ &\quad + 2\sqrt{\frac{ne}{\delta_0}}\left(\sqrt{n}L_{\tilde{\varphi}}+(A_M+K_M)\right) \\ &\quad \times\Theta\sqrt{\alpha(\Theta)}\sqrt{V_2(\bar{z})}\sqrt{V_1(\bar{e})} \\ &\leq -\frac{\Theta\mu\rho}{2}V_2(\bar{z}) + 2\Theta\mu\sqrt{V_2(\bar{z})}\bar{w} \\ &\quad + c_0\Theta\sqrt{\alpha(\Theta)}\sqrt{V_2(\bar{z})}\sqrt{V_1(\bar{e})}\end{aligned}\tag{62}$$

where $c_0 = 2\sqrt{\frac{ne}{\delta_0}}\left(\sqrt{n}L_{\tilde{\varphi}}+(A_M+K_M)\right)$.

Now, using inequality (55), it leads to:

$$\begin{aligned}\dot{V}_1(\bar{e}) &\leq -\Theta\left(1-\sqrt{\frac{\alpha(\Theta)}{\Theta^{\frac{\eta}{2q-1}}}}\sqrt{\frac{n^4 e\lambda_M(S)}{\delta_0}}L_{\tilde{\varphi}}\right)V_1(\bar{e}) \\ &\quad + 2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}} \\ &\quad + 2K_M\sqrt{\lambda_M(S_k)}\sqrt{V_1(\bar{e})}\sqrt{V_2(\bar{z})} \\ &\leq -\Theta\omega(\Theta)V_1(\bar{e}) + c_1\sqrt{V_1(\bar{e})}\sqrt{V_2(\bar{z})} \\ &\quad + 2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\end{aligned}\tag{63}$$

where $c_1 = 2\,K_M\sqrt{\lambda_M(S_k)}$ and

$$\omega(\Theta) = \left(1-\sqrt{\frac{\alpha(\Theta)}{\Theta^{\frac{\eta}{2q-1}}}}\sqrt{\frac{n^4 e\lambda_M(S)}{\delta_0}}L_{\tilde{\varphi}}\right). \tag{64}$$

Accordingly, it is possible to set:

$$\bar{V}(\bar{e},\bar{z},t) = \bar{V}_1(\bar{e}) + \bar{V}_2(\bar{z}) \tag{65}$$

where $\bar{V}_1 = \Theta\omega(\Theta)V_1(\bar{e})$ and $\bar{V}_2 = \frac{\Theta\mu\rho}{2}V_2(\bar{z})$. Therefore, one states $V_1(\bar{e}) \leq \bar{V}(\bar{e},\bar{z},t)$ and $V_2(\bar{z}) \leq \bar{V}(\bar{e},\bar{z},t)$.

Thus,

$$\begin{aligned}\dot{V}_1(\bar{e}) &\leq -\bar{V}_1(\bar{e}) + c_1\Theta\sqrt{\frac{\bar{V}_1(\bar{e})}{\Theta\omega(\Theta)}}\sqrt{\frac{\bar{V}_2(\bar{z})}{\frac{\Theta\mu\rho}{2}}} \\ &\quad + 2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}} \\ &= -\bar{V}_1(\bar{e}) + \frac{c_1}{\sqrt{\mu}}\sqrt{\frac{2}{\omega(\Theta)\rho}}\sqrt{\bar{V}_1(\bar{e})}\sqrt{\bar{V}_2(\bar{z})} \\ &\quad + 2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\end{aligned} \tag{66}$$

Since $V_1(\bar{e}) \leq \bar{V}(\bar{e},\bar{z},t)$, it yields:

$$\begin{aligned}\dot{V}_1(\bar{e}) &= -\bar{V}_1(\bar{e}) + \frac{c_1}{\sqrt{\mu}}\sqrt{\frac{2}{\omega(\Theta)\rho}}\bar{V}(\bar{e},\bar{z},t) \\ &\quad + 2q\sqrt{\lambda_M(S)}\sqrt{V_1(\bar{e})}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}}\end{aligned} \tag{67}$$

Similarly, one obtains

$$\begin{aligned}\dot{V}_2(\bar{z}) &\leq -\bar{V}_2(\bar{z}) + 2\Theta\mu\sqrt{V_2(\bar{z})}\bar{w} \\ &\quad + c_0\Theta\sqrt{\alpha(\Theta)}\sqrt{\frac{\bar{V}_1(\bar{e})}{\Theta\omega(\Theta)}}\sqrt{\frac{\bar{V}_2(\bar{z})}{\frac{\Theta\mu\rho}{2}}} \\ &\leq -\bar{V}_2(\bar{z}) + 2\Theta\mu\sqrt{V_2(\bar{z})}\bar{w} \\ &\quad + c_0\sqrt{\frac{\alpha(\Theta)}{\mu}}\sqrt{\frac{2}{\omega(\Theta)\rho}}\bar{V}(\bar{e},\bar{z},t)\end{aligned} \tag{68}$$

Set $V(\bar{e},\bar{z},t) = V_1(\bar{x}) + V_2(\bar{z})$ as the candidate Lyapunov equation. From equations (67) and (68), and considering $\sqrt{V_1} \leq V$ and $\sqrt{V_2} \leq V$, one gets:

$$\begin{aligned}\dot{V}(\bar{e},\bar{z},t) &= -\bar{V} + \sqrt{\frac{2}{\omega(\Theta)\rho}}\sqrt{\frac{1}{\mu}}\left(c_1 + c_0\sqrt{\alpha(\Theta)}\right)\bar{V} \\ &\quad + 2\left(q\sqrt{\lambda_M(S)}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}} + \Theta\mu\bar{w}\right)\sqrt{\bar{V}}\end{aligned} \tag{69}$$

For Θ and μ sufficiently high, one has:

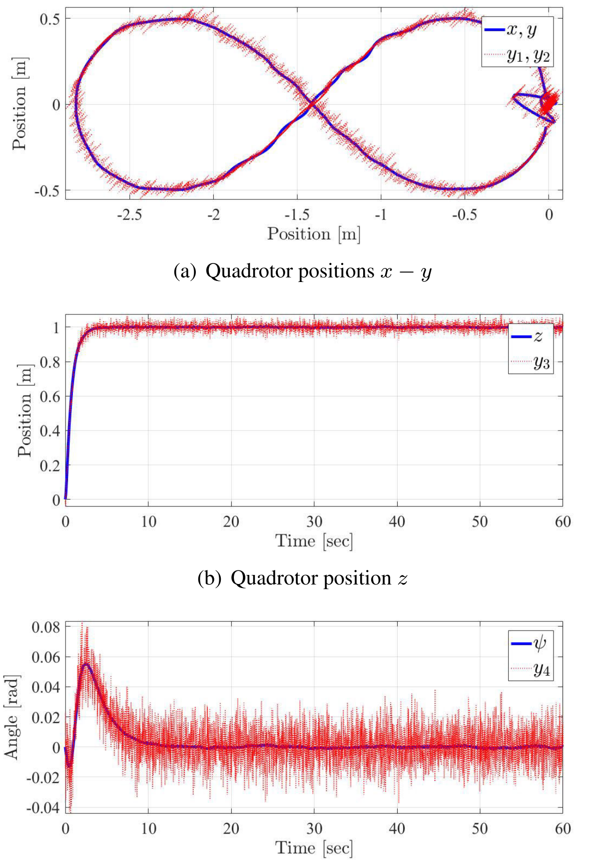
$$\lim_{\Theta\to\infty}\frac{\alpha(\Theta)}{\Theta^{\frac{\eta}{2q-1}}} = \lim_{\mu\to\infty}\frac{1}{\mu} = 0. \tag{70}$$

That allows us to consider that $0 < \xi < 1$, thus:

$$1 - \sqrt{\frac{2}{\omega(\Theta)\rho}}\sqrt{\frac{1}{\mu}}\left(c_1 + c_0\sqrt{\alpha(\Theta)}\right) > \xi \tag{71}$$

Likewise, one has $\Theta\omega(\Theta) \leq \frac{\Theta\mu\rho}{2}$ which implies that

$$\Theta\omega(\Theta)V \leq \bar{V} \leq \frac{\Theta\mu\rho}{2}V \tag{72}$$



(a) Quadrotor positions $x - y$

(b) Quadrotor position z

(c) Quadrotor attitude ψ

FIGURE 2. Evaluation of the trajectories with noisy output measurements (Test 1).

Hence, from (69), (71) and (72), one has:

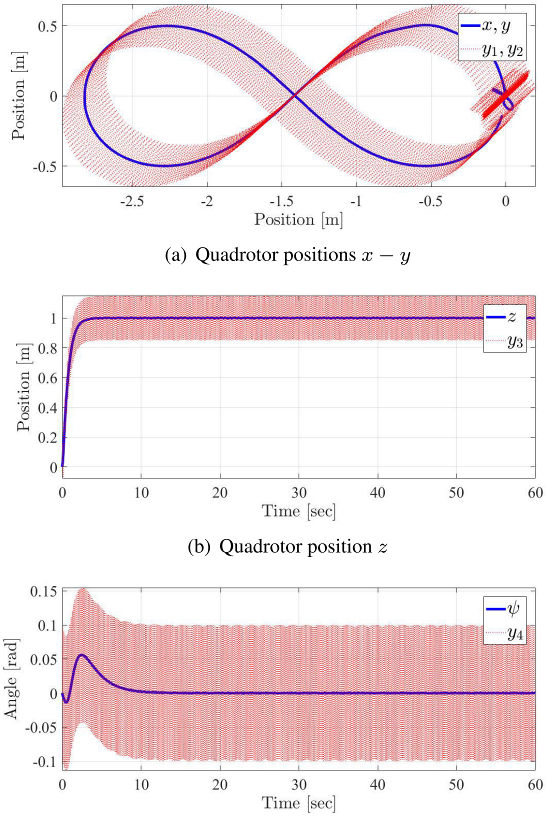
$$\begin{aligned}\dot{V}(\bar{e},\bar{z},t) &= -\xi\Theta\omega(\Theta)V \\ &\quad + 2\left(q\sqrt{\lambda_M(S)}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}} + \Theta\mu\bar{w}\right)\sqrt{\bar{V}}\end{aligned} \tag{73}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\sqrt{V(\bar{e},\bar{z},t)} &= -\xi\Theta\omega(\Theta)\sqrt{V} \\ &\quad + 2\left(q\sqrt{\lambda_M(S)}\frac{\bar{\varepsilon}}{\Theta^{(\lambda_1-1)\delta_1}} + \Theta\mu\bar{w}\right)\end{aligned} \tag{74}$$

According to (64), (70) and (71), allowing us to show that for large sufficiently values of Θ and μ, $V(\bar{e},\bar{z},t)$ exponentially converges to a neighbourhood around zero, as it is shown in (74), the rate of convergence depends on the upper bounds of the uncertainties and the noise signals $\bar{\varepsilon}$ and $\bar{w}$. This concludes the proof of Theorem 1.

Remark 1: The complexity of the proposed algorithm depends on the size of the uncertainty, which can be reduced to a certain value through the parameter Θ. Nevertheless, there is a trade-off between the sensitivity of the measurement noise and the convergence error. The sensitivity can be compensated by the observer immersed in the controller by means

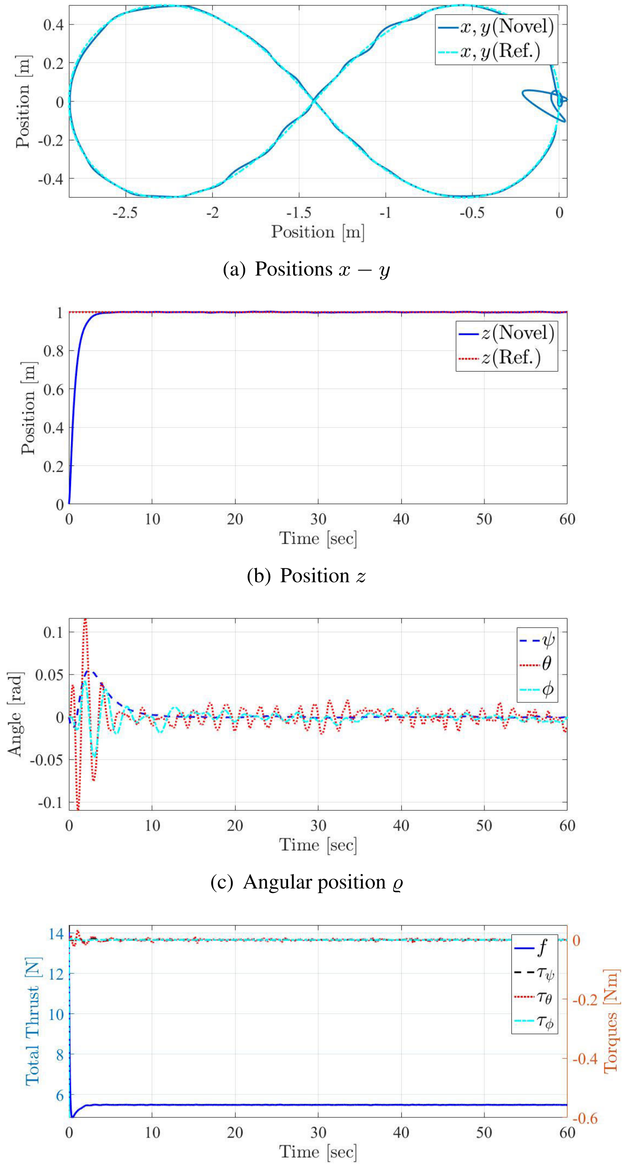


(a) Quadrotor positions $x - y$

(b) Quadrotor position z

(c) Quadrotor attitude ψ

FIGURE 3. Evaluation of the trajectories with noisy output measurements (Test 2).



(a) Positions $x - y$

(b) Position z

(c) Angular position ϱ

(d) Control inputs

FIGURE 4. Evaluation of the proposed observer based on IDA-PBC control with noisy output measurements (Test 1).

of the adjustment parameter μ, however, the uncertainty is limited to a certain value.

V. SIMULATIONS

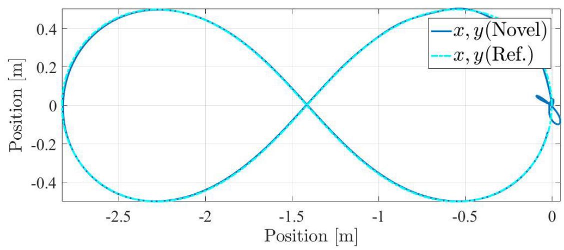
To validate the performance of the novel observer-based IDA-PBC, some simulation tests have been performed. The observer given by (25)-(27) is applied to the quadrotor dynamical model (18). An IDA-PBC controller with gravity compensation is used to stabilize the vehicle's position and attitude dynamics.

In simulation tests, the model parameters have been taken close to real aerial platforms, such parameters are given by: $m = 0.56Kg$ $I_x = I_y = 14.2e^{-3}Kgm^2$, $d = 0.21m$, $g = 9.81m/s^2$ $I_z = 2I_x$. From the block structure (18), it has $p = 4$, since the available outputs are $[x\ y\ z\ \psi]$, one has four subsystems (16) and (17), thus $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = \lambda_4 = 2$, therefore, according (20) one obtains $\delta_1 = 5$, $\delta_2 = \delta_3 = \delta_4 = 1$. The tuning parameters are taken as $\Theta = 1.2$ and $\mu = 7$. The output vector is $y(t) = [x,\ y,\ z,\ \psi]^T + w(t)$ and represents the noisy output measurements.

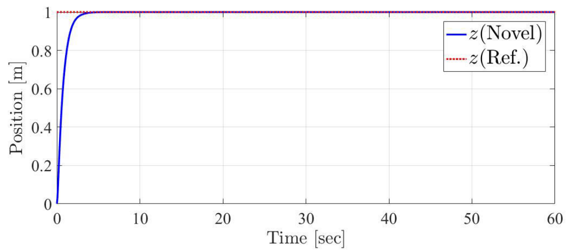
A. PERFORMANCE IN PRESENCE OF MEASUREMENT NOISE

In order to highlight the performance of the proposed control law in the presence of the different noise signals, two experimental results are provided. In the first test, the noise signal $w(t)$ is a Gaussian noise with a variance equal to 0.0005, this results in the noisy output measurements plotted in Figure 2. In the second test, a noise signal $w(t)$ is given by $0.155\sin(20t)$, the perturbed trajectories are plotted in Figure 3. In both simulations, the uncertainty is $\varepsilon(t) = \varepsilon^1 = \varepsilon^2 = \varepsilon^3 = \varepsilon^4 = 0.1\cos(20t)$, this uncertainties can be attached to unmodeled dynamics, such as wind frictions and structural vibrations, as [25].

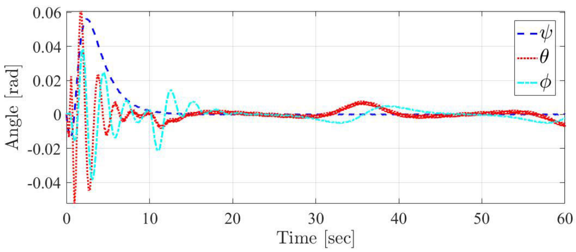
The performance of the novel observer-based IDA-PBC control in the presence of the Gaussian noise is presented in Figure 4, in addition, the behavior of the design in the presence of the sinusoidal noise signal is illustrated in Figure 5. Note that in both experiments the proposed



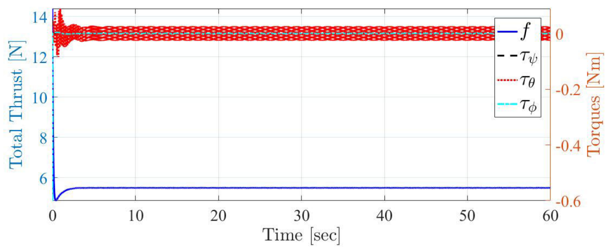
(a) Positions $x - y$



(b) Position z



(c) Angular position ϱ



(d) Control inputs

FIGURE 5. Evaluation of the proposed observer based on IDA-PBC control with noisy output measurements (Test 2).

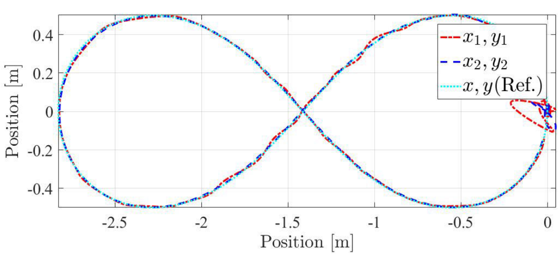
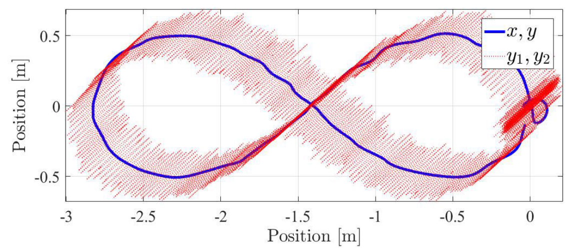
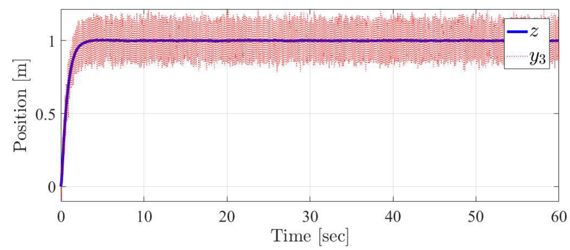


FIGURE 6. Comparison of the position with different noise signals.

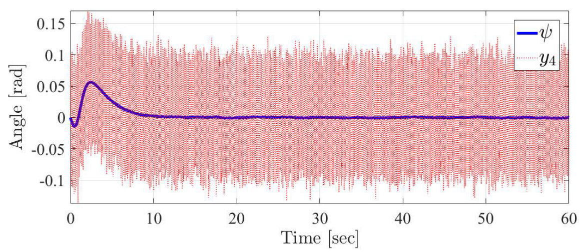
observer-based control provides a good trajectory tracking in presence of uncertainties and disturbances as the measurement noise (Gaussian noise and Sinusoidal noise).



(a) Quadrotor positions $x - y$



(b) Quadrotor position z



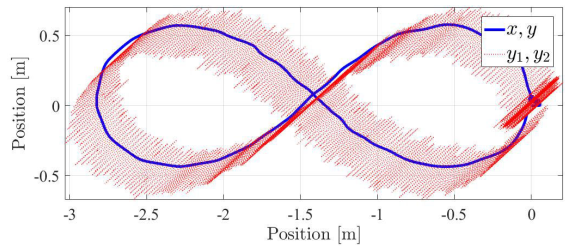
(c) Quadrotor attitude ψ

FIGURE 7. Trajectories of the observer based on IDA-PBC control with high amplitude noise.

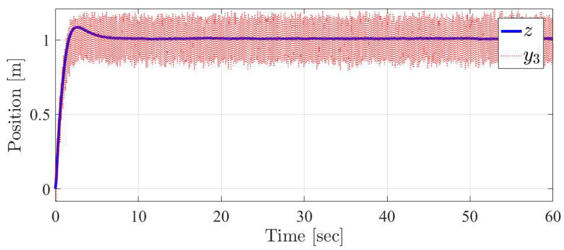
Considering all these non-negligible effects, the novel approach will make easier the practical control implementation. It is also important to highlight that the performance of the proposed design has a slight performance degradation in the presence of the Gaussian noise, as shown in Figure 6, this degradation relates in particular to high-frequency measurement noise, in order to reduce this effect, some works add a low-pass filter of high order [29]. However, this may be impractical to compute for high-order systems as the quadrotor. Furthermore, these designs address only the SISO uniformly observable systems, this is not the case for quadrotor dynamics (MIMO non-uniformly observable system).

B. COMPARISON TO A STANDARD DESIGN

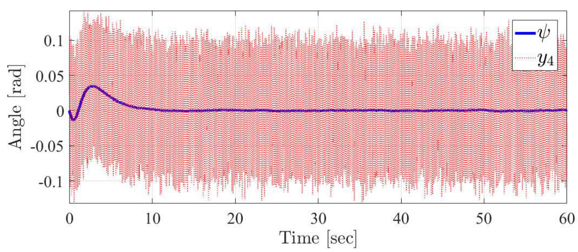
In order to compare the behavior of the standard and proposed control design in the presence of the high amplitude noise signals, a comparison test is performed. For this test, the noise signal $w(t)$ is a sum of a Gaussian noise with a variance equal to 0.001 and a signal given by $0.25\sin(20t)$, this results in the noisy output measurements plotted in Figure 7 (proposed design) and Figure 8 (standard design). Moreover, this test considers the same uncertainty of the previous tests. Figure 9 shows that the proposed controller can reach the reference in a very short time in the presence of the high-amplitude



(a) Quadrotor positions $x - y$

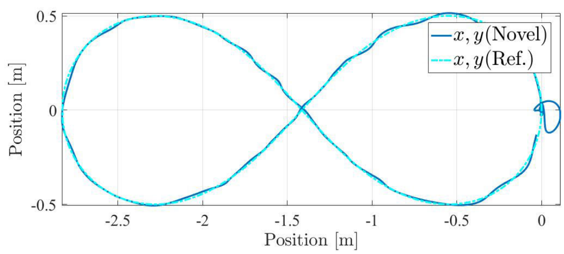


(b) Quadrotor position z

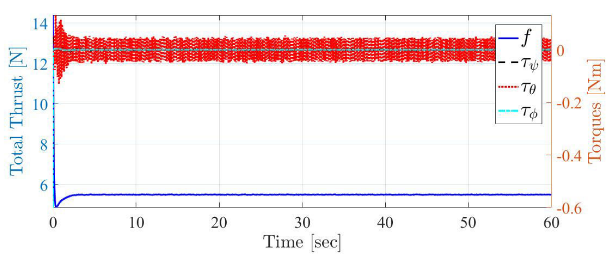


(c) Quadrotor attitude ψ

FIGURE 8. Trajectories of the observer based on PD control with high amplitude noise.

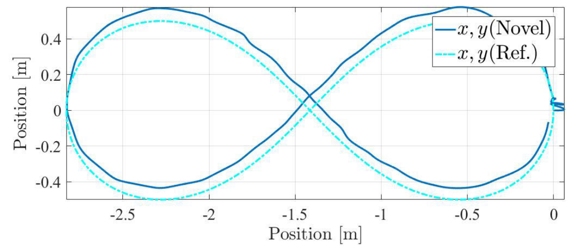


(a) Positions $x - y$

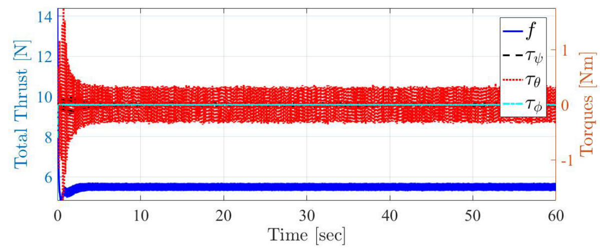


(b) Control inputs

FIGURE 9. Evaluation of the proposed observer based on IDA-PBC control with high amplitude noise.



(a) Positions $x - y$



(b) Control inputs

FIGURE 10. Evaluation of the proposed observer based on PD control with high amplitude noise.

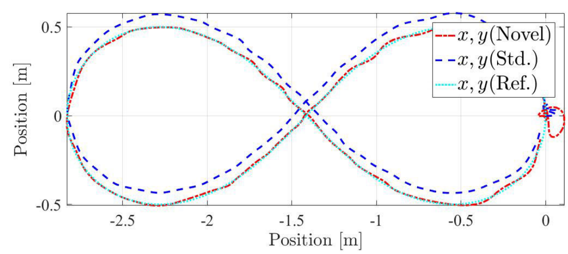


FIGURE 11. Comparison of the novel observer based on IDA-PBC control versus the standard observer based on PD control.

noise signal, even in the mixed noise (Gaussian noise and Sinusoidal noise). Moreover, as it can be seen in Figure 10, the proposed algorithm outperforms the standard approach of [14], which assumes that the output measurements are noise-free. On the other hand, Figure 11 makes evident that the quadrotor cannot reach a near-zero steady-state error with the traditional observer-based PD control. Furthermore, it is clear that there is high chattering phenomenon in the standard control law, in Figure 10.

VI. CONCLUSION

In order to avoid the heavy design effort and ensure the robust control strategy in the presence of adverse effects as disturbance, uncertainty, and noisy measurements, it has been presented a new observer-based IDA-PBC control for a quadrotor. The original idea of the proposed strategy of the observer-based control is that it is able to guarantee a near-zero steady-state error, in the presence of relatively large uncertainties and disturbances, as measurement noise and unmodeled dynamics of the quadrotor (wind frictions and structural vibrations). The features of the proposed strategy were further validated with simulation tests for a quadrotor, which show a remarkable improvement of the sensitivity of the estimator subject to measurement noise and thereby achieving a reduction of the steady-state error. The behavior of the proposed observer-based IDA-PBC control for small uncertainties and disturbances tends to be similar to the behavior of the typical approach, i.e., high-gain observer-based PD control without uncertainties and disturbances. Future work includes extending the scheme for the case of sampled output measurements and delayed output measurements, as well as the combination of both. Moreover, it will consider an IDA-PBC design with total energy-shaping.

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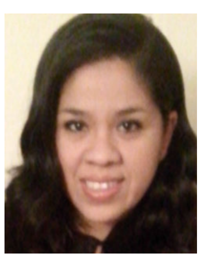

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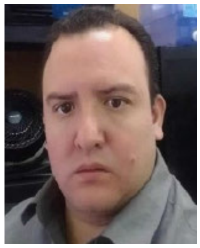

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