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A New Generalized Θ-Inverse vs. Moore-Penrose Structure: A Comparative Control-Oriented Practical Investigation

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ABSTRACT A new non-unique Θ-inverse of non-square polynomial matrices is presented in this paper. It is shown that the above inverse specializes to the unique Moore-Penrose one under several specific assumptions. Due to the existence of some degrees of freedom, the Θ-inverse outperforms the Moore-Penrose inverse in various inverse-related problems covering the multivariable control theory. In many scenarios the former inverse can stabilize a closed-loop control plant being unstable by application of the Moore-Penrose formula. Practical and theoretical simulation examples confirm the correctness of the proposed method.

INDEX TERMS Generalized inverses, polynomial matrices, non-square matrices, Moore-Penrose inverse, optimal control, perfect control, stability, MIMO, practical implementation.

LIST OF SYMBOLS

$\mathbf{A}(q)$	polynomial matrix in q
$\mathbf{a_0}, \mathbf{a_1}, \ldots, \mathbf{a_l}$	parameter matrix coefficients (left-invertible plants)
$\mathbf{a_0}, \mathbf{a_1}, \ldots, \mathbf{a_m}$	parameter matrix coefficients (right-invertible plants)
$\mathbf{I}_m, \mathbf{I}_n$	identity m- and n-matrix, respectively
q^{-1}	backward shift operator
s	arbitrary order
z	complex operator
$\underline{\beta}(q^{-1}), \beta(z)$	degrees of freedom-oriented polynomial matrices
$(.)^{\mathrm{L}}$	any left inverse
$(.)^{\mathrm{L}}_0$	left T-inverse
$(.)^{\mathrm{R}}$	any right inverse
$(.)^{\mathrm{R}}_0$	right T-inverse
$(.)^{\mathrm{T}}$	transpose symbol
$\Vert.\Vert$	norm operator

I. MOTIVATION

The inverses of the parameter and polynomial matrices play an important role in the modern mathematics and practice [1]–[7]. The well-known Moore-Penrose inverse has often been applied in a variety of engineering tasks, including the branches of the control theory, signal processing, electrical networks, pattern recognition, etc. [2], [7]–[14]. Moreover, the original accepted algorithms in the Maple environment providing the Moore-Penrose inverse for any defined non-square matrices, have resulted in the interesting starting points for the world multivariable system analysis [15]–[17]. Notwithstanding, this inverse may turn out to be imperfect, which has been proved in a number of examples (for instance see [18]). Below the main achievement of these studies is presented, in particular the new generalized Θ-inverse is proposed. Henceforth, the Θ-inverse surpasses the Moore-Penrose one giving a clear contribution to the control theory and practice. Unlike the Moore-Penrose technique, the suggested solution clearly impacts the stability behavior of the inverse model control-originated multivariable procedures. In order to obtain a compact inverse-related control stability tool, some drawbacks, performed in the past, should be carefully resolved first.

In [19] a new definition of so-called σ-inverse strictly dedicated to non-square polynomial matrices is provided. On the basis of the theorem expressed there as

Theorem 1: Let the polynomial matrix $\underline{\mathbf{A}}(q^{-1}) = \underline{\mathbf{a}}_\mathbf{0} + \underline{\mathbf{a}}_\mathbf{1} q^{-1} + \ldots + \underline{\mathbf{a}}_\mathbf{m} q^{-m}$. *Then, a general σ-inverse of* $\underline{\mathbf{A}}(q^{-1})$ *can be defined as*

$$\underline{\mathbf{A}}^{\mathrm{R}}(q^{-1}) = \left\{\mathbf{I}_n + \underline{\beta}^{\mathrm{R}}_0(q^{-1})\left[\underline{\mathbf{A}}(q^{-1}) - \underline{\beta}(q^{-1})\right]\right\}^{-1} \underline{\beta}^{\mathrm{R}}_0(q^{-1}), \quad (1)$$

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where $z^s\underline{\beta}(z^{-1}) = \beta(z) \in \mathfrak{R}^{m\times n}[z]$ is arbitrary, including an arbitrary order s, and $[\cdot]_0^{\mathrm{R}}$ stands for the T-inverse,
we have concluded that the non-unique right σ-inverse of full normal rank polynomial matrix $\mathbf{A} \in \mathfrak{R}^{m\times n}[q]$ takes the following form

$$\mathbf{A}^{\mathrm{R}}(q) = \left\{\mathbf{I}_n + \beta_0^{\mathrm{R}}(q)\left[\mathbf{A}(q) - \beta(q)\right]\right\}^{-1} \beta_0^{\mathrm{R}}(q), \tag{2}$$

where $\beta_0^{\mathrm{R}}(q) = \beta^{\mathrm{T}}(q)\left[\beta(q)\beta^{\mathrm{T}}(q)\right]^{-1}$ whilst $\beta \in \mathfrak{R}^{m\times n}[q]$ constitutes the so-called degree of freedom in the form of matrix polynomial in q-operator domain, whereas $\mathbf{I}_n$ denotes the identity n-matrix. A defective proof of this, shown in [20], should be given in a new structure

Proof: Immediately, the formula $\mathbf{X} = \{\mathbf{I}_n + \beta_0^{\mathrm{R}}(q)\left[\mathbf{A}(q) - \beta(q)\right]\}^{-1}\beta_0^{\mathrm{R}}(q)$ with $\beta_0^{\mathrm{R}}(q) = \beta^{\mathrm{T}}(q)[\beta(q)\beta^{\mathrm{T}}(q)]^{-1}$ holds for $\mathbf{X} = \beta^{\mathrm{T}}(q)\left[\mathbf{A}(q)\beta^{\mathrm{T}}(q)\right]^{-1}$ or equivalently for $\mathbf{X} = \beta_0^{\mathrm{R}}(q)\left[\mathbf{A}(q)\beta_0^{\mathrm{R}}(q)\right]^{-1}$, both representing the new forms of non-unique polynomial matrix right σ-inverse. □

Remark 1: For $\beta(q) = \mathbf{A}(q)$ the right σ-inverse of the full normal rank $\mathbf{A}(q)$ specializes to the unique minimum-norm one as follows $\mathbf{A}_0^{\mathrm{R}}(q) = \mathbf{A}^{\mathrm{T}}(q)\left[\mathbf{A}(q)\mathbf{A}^{\mathrm{T}}(q)\right]^{-1}$.

In the next section a parallel definition of a non-unique left σ-inverse of a non-square matrix polynomial is proposed.

II. A NEW LEFT σ-INVERSE

Similarly to the Eq. (2) we can formulate a theorem related to the new left σ-inverse in the following manner

Theorem 2: Let the full normal rank polynomial matrix $\mathbf{A}(q) = \mathbf{a_0}+\mathbf{a_1}q+\ldots+\mathbf{a_l}q^l$ be left-invertible. Then, a general left σ-inverse of $\mathbf{A}(q)$ can be defined as

$$\mathbf{A}^{\mathrm{L}}(q) = \left\{\mathbf{I}_n + \beta_0^{\mathrm{L}}(q)\left[\mathbf{A}(q) - \beta(q)\right]\right\}^{-1} \beta_0^{\mathrm{L}}(q), \tag{3}$$

where $\beta(q) \in \mathfrak{R}^{m\times n}[q]$ is arbitrary and $\beta_0^{\mathrm{L}}(q) = \left[\beta^{\mathrm{T}}(q)\beta(q)\right]^{-1}\beta^{\mathrm{T}}(q)$.

Proof: Immediately, after the operation $\beta_0^{\mathrm{L}}(q)\beta(q) = \mathbf{I}_n$, which leads to the new forms of Eq. (3) as follows $\mathbf{A}^{\mathrm{L}}(q) = \left[\beta_0^{\mathrm{L}}(q)\mathbf{A}(q)\right]^{-1}\beta_0^{\mathrm{L}}(q)$ or $\mathbf{A}^{\mathrm{L}}(q) = \left[\beta^{\mathrm{T}}(q)\mathbf{A}(q)\right]^{-1}\beta^{\mathrm{T}}(q)$. □

Remark 2: For $\beta(q) = \mathbf{A}(q)$ the left σ-inverse of the full normal rank $\mathbf{A}(q)$ specializes to the unique least-squares one as follows $\mathbf{A}_0^{\mathrm{L}}(q) = \left[\mathbf{A}^{\mathrm{T}}(q)\mathbf{A}(q)\right]^{-1}\mathbf{A}^{\mathrm{T}}(q)$.

Remark 3: It is worth mentioning that the minimum-norm right and least-squares left unique inverses of $\beta(q)$ in forms of $\beta_0^{\mathrm{R}}(q)$ and $\beta_0^{\mathrm{L}}(q)$, respectively, fulfill the four Moore-Penrose conditions [21]–[23].

Having the new elements of the new proposal, in the next section we can formulate a theorem concerning a new non-unique left Θ-inverse of non-square polynomial matrix, which constitutes a generalization of the σ-inverse.

III. A NEW Θ-INVERSE

We start our considerations with non-unique left Θ-inverse.

Theorem 3: Let the full normal rank polynomial matrix $\mathbf{A}(q) = \mathbf{a_0}+\mathbf{a_1}q+\ldots+\mathbf{a_l}q^l$ be left-invertible. Then, a general left Θ-inverse of $\mathbf{A}(q)$ can be defined as

$$\mathbf{A}^{\mathrm{L}}(q) = \left[\beta^{\mathrm{L}}(q)\mathbf{A}(q)\right]^{-1} \beta^{\mathrm{L}}(q), \tag{4}$$

where $\beta(q) \in \mathfrak{R}^{m\times n}[q]$ is arbitrary and $\beta^{\mathrm{L}}(q)$ denotes any left inverse including the regular one.

Proof: Immediately, after the operation $\mathbf{A}^{\mathrm{L}}(q)\mathbf{A}(q) = \left[\beta^{\mathrm{L}}(q)\mathbf{A}(q)\right]^{-1}\beta^{\mathrm{L}}(q)\mathbf{A}(q) = \mathbf{I}_n$. □

Remark 4: Of course, based on the Eq. (4) a corresponding form of Θ-inverse can be given as follows

$$\mathbf{A}^{\mathrm{L}}(q) = \left\{\mathbf{I}_n + \beta^{\mathrm{L}}(q)\left[\mathbf{A}(q) - \beta(q)\right]\right\}^{-1} \beta^{\mathrm{L}}(q). \tag{5}$$

Next, the right-invertible version of the non-unique Θ-inverse is given.

IV. RIGHT Θ-INVERSE

A new right σ-inverse was introduced in [20] in the following form

$$\mathbf{A}^{\mathrm{R}}(q) = \beta^{\mathrm{T}}(q)\left[\mathbf{A}(q)\beta^{\mathrm{T}}(q)\right]^{-1}, \tag{6}$$

where $\beta(q) \in \mathfrak{R}^{m\times n}[q]$ is arbitrary. An incorrect proof leading the Eq. (2) to Eq. (6) has additionally been given there and successfully improved in the Section I of this paper. Based on the related considerations we can formulate a notion of a new generalized right Θ-inverse as follows

Theorem 4: Let the full normal rank polynomial matrix $\mathbf{A}(q) = \mathbf{a_0} + \mathbf{a_1}q + \ldots + \mathbf{a_l}q^m$ be right-invertible. Then, a general right Θ-inverse of $\mathbf{A}(q)$ can be defined as

$$\mathbf{A}^{\mathrm{R}}(q) = \beta^{\mathrm{R}}(q)\left[\mathbf{A}(q)\beta^{\mathrm{R}}(q)\right]^{-1}, \tag{7}$$

where $\beta(q) \in \mathfrak{R}^{m\times n}[q]$ is arbitrary and $\beta^{\mathrm{R}}(q)$ denotes any right inverse including the regular one.

Proof: Immediately, after the operation $\mathbf{A}(q)\mathbf{A}^{\mathrm{R}}(q) = \mathbf{A}(q)\beta^{\mathrm{R}}(q)\left[\mathbf{A}(q)\beta^{\mathrm{R}}(q)\right]^{-1} = \mathbf{I}_m$. □

Remark 5: Of course, based on the Eq. (7) a corresponding form of Θ-inverse can be expressed in the following manner

$$\mathbf{A}^{\mathrm{R}}(q) = \left\{\mathbf{I}_n + \beta^{\mathrm{R}}(q)\left[\mathbf{A}(q) - \beta(q)\right]\right\}^{-1} \beta^{\mathrm{R}}(q). \tag{8}$$

Remark 6: Note that, as before, the formulas (4) and (7) reduce to the well-known Moore-Penrose inverses of $\mathbf{A}(q)$, i.e., the least-squares and minimum-norm ones, respectively. In order to achieve those inverses we can, for instance, substitute $\beta^{\mathrm{L}}(q) = \mathbf{A}^{\mathrm{T}}(q)$ and $\beta^{\mathrm{R}}(q) = \mathbf{A}^{\mathrm{T}}(q)$. The mentioned subject is worth of further investigation.

In order to certify the entire theory presented throughout the manuscript, the simulation examples are shown below. The representative instances certainly verify the new inverses-originated approach, especially in terms of the multivariable control theory. The last scenario of the following section covering the inverse model control-related perfect control case clearly manifests the favorable practical-oriented peculiarities derived from the new generalized Θ-inverse.

V. NUMERICAL EXAMPLES

A. LEFT INVERSES

Let us consider the left-invertible matrix

$$\mathbf{A}(q)=\begin{bmatrix} 0.5-0.1q & -2 \\ 0 & 4-0.2q \\ 2 & -6 \end{bmatrix}. \tag{9}$$

For $\beta(q) = \begin{bmatrix} -q & 2 \\ -4 & 0 \\ 7 & -0.8q \end{bmatrix}$ we obtain the polynomial matrix left σ-inverse in (10), as shown at the bottom of the next page.

On the other hand, a new Θ-inverse of $\mathbf{A}(q)$ can be calculated employing the non-unique $\beta^{\mathrm{L}}(q)$, e.g., in the form: $\beta^{\mathrm{L}}(q) = \left[\alpha^{\mathrm{T}}(q)\beta(q)\right]^{-1}\alpha^{\mathrm{T}}(q)$ with arbitrary $\alpha(q) = \begin{bmatrix} 3q & 0 \\ 2 & 0 \\ -3+q & 2.5q \end{bmatrix}$, ultimately yielding

$$\mathbf{A}_{\Theta}^{\mathrm{L}} = \frac{\begin{bmatrix} -90q & -60 & 32q-40 \\ -30q & -20 & -1.5q^2+7.5q \end{bmatrix}}{9q^2+19q-80}. \tag{11}$$

B. RIGHT INVERSES

Let us study the right-invertible matrix

$$\mathbf{A}(q)=\begin{bmatrix} -0.2+0.3q & 2 & -0.25 \\ -0.4 & 0.1-q^2 & 1.6 \end{bmatrix}. \tag{12}$$

For $\beta(q) = \begin{bmatrix} -q & 2-0.1q^2 & 4 \\ 5 & 0 & 8 \end{bmatrix}$ we receive the right σ-inverse in the form of (13), as shown at the bottom of the next page, in opposite to the classical minimum-norm one as follows

$$\mathbf{A}_0^{\mathrm{R}} = \frac{\begin{bmatrix} -12000q^5+8000q^4+2400q^3+30400q^2-30840q+23760 \\ 4800q^3+12800q^2-480q-218880 \\ 10000q^4-130000q^2-7680q+19620 \end{bmatrix}}{-3600q^6+4800q^5-3380q^4+18240q^3+} \quad \frac{\begin{matrix} -24000q^3+16000q^2-2400q+66600 \\ 3600q^4-4800q^3+3740q^2-9120q-26010 \\ 14240q^2+8880q-261360 \end{matrix}}{42768q^2+14256q-447417}. \tag{14}$$

C. Θ-INVERSE VERSUS THE MOORE-PENROSE ONE

1) PARAMETER CASE

Take a look at the following matrix

$$\mathbf{A}=\begin{bmatrix} 0.7637 & 0.0511 & -2.6329 & 0.5880 \\ 0.6251 & 0.0473 & -3.0179 & 0.6671 \\ -2.6024 & -0.0880 & -4.6170 & 0.9224 \\ 0.9530 & 0.0337 & 1.4598 & -0.2881 \end{bmatrix}, \tag{15}$$

representing the product of $\mathbf{A} = \left[\mathbf{I}-\mathbf{C}(\mathbf{DC})_0^{\mathrm{R}}\mathbf{D}\right]\mathbf{B}$ with

$$\mathbf{B}=\begin{bmatrix} 0.5 & 0.38 & -3.5 & 0.75 \\ 1.8 & -1.3 & 2.8 & -0.06 \\ -2.25 & -0.4 & -1.35 & 0.7 \\ 0.8 & 0.35 & 3.03 & -0.2 \end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix} -0.12 & 0.67 & 0.4 \\ 1.45 & -1.2 & 1.03 \\ 1.4 & 0.7 & 0.72 \\ 1.4 & 1.63 & -0.30 \end{bmatrix},$$

and

$$\mathbf{D}=\begin{bmatrix} 0.2 & -0.35 & 1 & 2.8 \\ -2.3 & 2.2 & 0 & 0.4 \end{bmatrix}.$$

Observe, that the application of the unique Moore-Penrose inverse directly to the product of **DC** provides eigenvalues of **A** equal to $z_1 = -6.0321$, $z_2 = 1.9380$ and $z_{3,4} = 0$, with $\|z_1\| > 1$ and $\|z_2\| > 1$.

On the other hand, the employment of the new generalized Θ-inverse with respect to the full rank matrix

$$\mathbf{DC}=\begin{bmatrix} 4.7885 & 5.8180 & -0.4005 \\ 4.0260 & -3.5290 & 1.2260 \end{bmatrix}, \tag{16}$$

with special selected degrees of freedom in forms of $\alpha = \begin{bmatrix} -1.46 & -0.38 & -0.67 \\ -0.64 & -0.94 & -1.92 \end{bmatrix}$ and $\beta = \begin{bmatrix} -0.11 & 0.54 & 1.07 \\ -0.52 & 0.06 & 1.62 \end{bmatrix}$, entails the use of eigenvalues of $\mathbf{A} = [\mathbf{I}-\mathbf{C}(\beta_\alpha^{\mathrm{R}}\mathbf{DC}\beta_\alpha^{\mathrm{R}})\mathbf{D}]\mathbf{B}$ in the following structure

$$\mathbf{A}=\begin{bmatrix} 1.6773 & -1.0896 & 0.3559 & 0.0270 \\ 1.6405 & -1.2203 & 0.3035 & 0.0436 \\ -1.5032 & -1.4603 & -1.0212 & 0.2474 \\ 0.6221 & 0.4468 & 0.3772 & -0.0848 \end{bmatrix}, \tag{17}$$

as follows $z_{1,2} = -0.3245 \pm 0.2550i$ and $z_{3,4} = 0$. Let us note that in this scenario we obtain the eigenvalues with modules less than one. This behavior is followed e.g. in control and systems theory branch related to the discrete-time stability issues.

2) POLYNOMIAL CASE

Consider the full normal rank polynomial matrix defined in the q-domain in the subsequent form

$$\mathbf{A}(q) = \begin{bmatrix} -2 & q \end{bmatrix}. \tag{18}$$

After using the Moore-Penrose inverse directly to the matrix **A**, we obtain the unique result as follows

$$\mathbf{A}_0^{\mathrm{R}}(q) = \frac{\begin{bmatrix} -2 \\ q \end{bmatrix}}{q^2+4}. \tag{19}$$

Observe, that the characteristic roots of $\mathbf{A}_0^{\mathrm{R}}(q)$ are equal to $z_{1,2} = \pm 2i$.

However, the special chosen degrees of freedom expressed by $\beta^{\mathrm{T}}(q) = \begin{bmatrix} q+0.2 \\ 1 \end{bmatrix}$ provide

$$\mathbf{A}_{\sigma}^{\mathrm{R}}(q) = \frac{\begin{bmatrix} -5q-1 \\ -5 \end{bmatrix}}{5q+2}, \tag{20}$$

with accompanying root equals $z = -0.4$. Again, the new approach brings a root characterized by $\|z\| < 1$.

D. MULTIVARIABLE PERFECT CONTROL STRATEGY

1) INPUT-OUTPUT SYSTEM DESCRIPTION

Let us consider the complex linear time-invariant fourth-input and three-output unstable (eigs.: $z_{1,2} = -0.7060 \pm 2.0791i$, $z_{3,4} = 1.9049 \pm 0.5336i$, $z_5 = -0.3866$, $z_6 = 0.3888$) second-order plant defined by the following deterministic transfer-function-oriented discrete-time model

$$\underline{\mathbf{A}}(q^{-1})\mathbf{u}(t) = \underline{\mathbf{B}}(q^{-1})q^{-d}\mathbf{y}(t), \tag{21}$$

with

$$\underline{\mathbf{A}}(q^{-1}) = \frac{\begin{bmatrix} 10q^2-5q+1 & -9q-13 & 8q+14 \\ 3q-4 & 10q^2-17q+7 & -q-26 \\ 23q-6 & -21q+3 & 10q^2-2q+29 \end{bmatrix}}{10q^2},$$

$$\underline{\mathbf{B}}(q^{-1}) = \frac{\begin{bmatrix} 30q^2-2q+1 & 20q^2+4q-7 & \ldots \\ 4q^2-5q+12 & -7q^2+6q-9 & \ldots \\ 12q^2+7q-3 & 17q^2-5q+21 & \ldots \end{bmatrix}}{10q^2}$$

$$\frac{\begin{bmatrix} 25q^2-3q+0.1 & -7q^2-4q+2 \\ 12.5q^2-4q+2 & 8q^2-24q+15 \\ 21q^2+15q+8 & -27q^2+6q-12 \end{bmatrix}}{10q^2}$$

and $d = 1$, having $\mathbf{u}(t)$-input and $\mathbf{y}(t)$-output vectors of the corresponding dimensions n_u and n_y in the discrete time t. The perfect control formula dedicated to the right-invertible one-delayed system, minimizing the performance index

$$J|_{d=1} = \min_{\mathbf{u}(t)}\{[\mathbf{y}(t+d) - \mathbf{y_{ref}}(t+d)]^{\mathrm{T}} \times [\mathbf{y}(t+d) - \mathbf{y_{ref}}(t+d)]\}, \tag{22}$$

where the parts $\mathbf{y}(t+1)$ and $\mathbf{y_{ref}}(t+1)$ deal with the deterministic one-step output predictor and reference output/setpoint, respectively, sounds now as follows

$$\mathbf{u}(t) = \underline{\mathbf{B}}^{\mathrm{R}}(q^{-1})\underline{\mathbf{y}}(t), \tag{23}$$

with regard to $\underline{\mathbf{y}}(t) = \underline{\mathbf{F}}^{-1}(q^{-1})[\mathbf{y_{ref}}(t+1) - \underline{\mathbf{H}}(q^{-1})\mathbf{y}(t)]$, whilst symbol $(.)^{\mathrm{R}}$ denotes any right inverse. On the other hand, the calculation of the respective matrix polynomials in the general forms of

$$\underline{\mathbf{F}}(q^{-1}) = \mathbf{I}_{n_y} + \hat{\mathbf{f}}_{\mathbf{1}}q^{-1} + \ldots + \hat{\mathbf{f}}_{\mathbf{d-1}}q^{-d+1}, \tag{24}$$

and

$$\underline{\mathbf{H}}(q^{-1}) = \hat{\mathbf{h}}_{\mathbf{0}} + \hat{\mathbf{h}}_{\mathbf{1}}q^{-1} + \ldots + \hat{\mathbf{h}}_{\mathbf{n-1}}q^{-n+1}, \tag{25}$$

under order of the system n, can be provided according to the polynomial matrix identity (called the Diophantine equation)

$$\mathbf{I}_{n_y} = \underline{\mathbf{F}}(q^{-1})\underline{\mathbf{A}}(q^{-1}) + q^{-d}\underline{\mathbf{H}}(q^{-1}), \tag{26}$$

to finally obtain

$$\underline{\mathbf{F}}(q^{-1}) = \mathbf{I}_3, \tag{27}$$

and

$$\underline{\mathbf{H}}(q^{-1}) = \hat{\mathbf{h}}_{\mathbf{0}} + \hat{\mathbf{h}}_{\mathbf{1}}q^{-1}, \tag{28}$$

employing $\hat{\mathbf{h}}_{\mathbf{0}} = \begin{bmatrix} 0.5 & 0.9 & -0.8 \\ -0.3 & 1.7 & 0.1 \\ -2.3 & 2.1 & 0.2 \end{bmatrix}$ and $\hat{\mathbf{h}}_{\mathbf{1}} = \begin{bmatrix} -0.1 & 1.3 & -1.4 \\ 0.4 & -0.7 & 2.6 \\ 0.6 & -0.3 & -2.9 \end{bmatrix}$.

Thus, for application of the Moore-Penrose to the polynomial matrix $\underline{\mathbf{B}}(q^{-1})$, we obtain the unstable perfect control scenario associated with the unstable inverse model control-originated signal runs depicted in Fig. 1. This is caused by the unstable so-called control zeros [19].

However, the new approach in the form of Θ-inverse as in formula (7) with $\beta^{\mathrm{R}} = \alpha_0^{\mathrm{R}}$ and the special selected degrees of freedom providing:

$$\alpha = \begin{bmatrix} -2.2 & -0.8 & 0.9 & -2.4 \\ 0.4 & 0.4 & 0.1 & -0.3 \\ 0 & -0.8 & -1.6 & 1.1 \end{bmatrix} \text{ and}$$

$$\mathbf{A}_{\sigma}^{\mathrm{L}} = \frac{\begin{bmatrix} -60q^2-20q+1450 & -240q+200 & 28q^2-160q-350 \\ -20q^2+350 & -90q+50 & -q^3+5q^2+17.5q-87.5 \end{bmatrix}}{6q^3+28q^2-475q+25}. \tag{10}$$

$$\mathbf{A}_{\sigma}^{\mathrm{R}} = \frac{\begin{bmatrix} 100q^4-2010q^2+2560q+6600 & 200q^2+400q-3000 \\ 216q^2-4320 & -30q^3+60q^2+600q-1200 \\ 160q^4-3216q^2+640q+1920 & 800q^2+880q-7200 \end{bmatrix}}{30q^5-60q^4-603q^3+2406q^2+1308q-10440}, \tag{13}$$

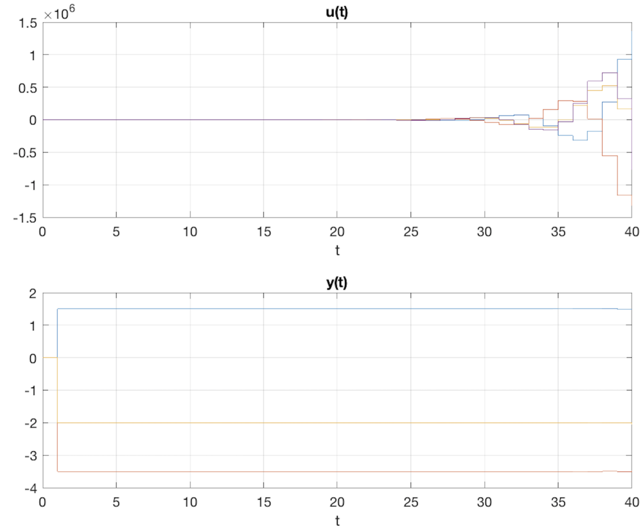

FIGURE 1. **Perfect control: system's signals, scenario – *T*-inverse application.**

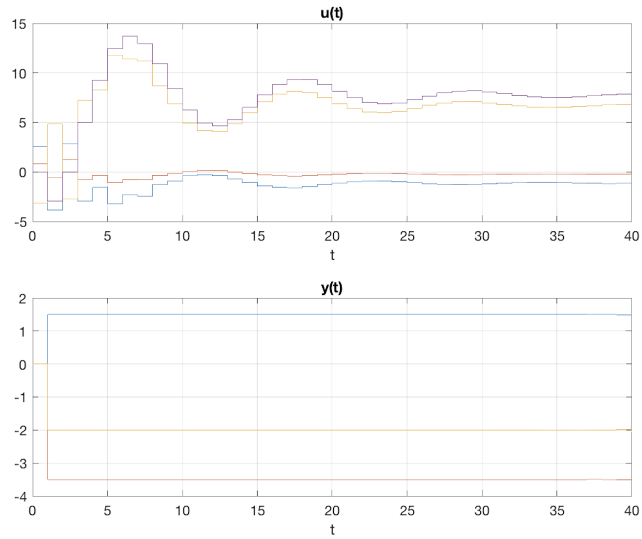

FIGURE 2. **Perfect control: system's signals, scenario – Θ-inverse application.**

$$\beta = \begin{bmatrix} 1.8 & -0.8 & -0.1 & 1.0 \\ 0.9 & -0.5 & -1.6 & -0.8 \\ 0.8 & 0.2 & -1.5 & 2.1 \end{bmatrix},$$

immediately stabilizes the entire closed-loop perfect control scheme. This fact is proven by the Fig. 2. Observe that in two cases, we receive the assumed output reference $\mathbf{y_{ref}}(t+1) = \begin{bmatrix} 1.5 & -3.5 & -2.0 \end{bmatrix}$ after $d \geq 1$, which is in the relation with the inverse model control design paradigm, in general.

2) STATE-SPACE SYSTEM REPRESENTATION

Let us into consideration the linear time-invariant fourth-input and two-output stable three-order plant defined by the well-known deterministic state-space-oriented discrete-time model as follows

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{Ax}(t) + \mathbf{Bu}(t)q^{-d+1}, \quad \mathbf{x}(0) = \mathbf{x_0}, \\ \mathbf{y}(t) &= \mathbf{Cx}(t), \end{aligned} \tag{29}$$

with $\mathbf{A} = \begin{bmatrix} 0.5 & -0.3 & 0.4 \\ -1.2 & 2 & 3.2 \\ -1.5 & 1.4 & 0.7 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1.2 & 3.1 & -0.8 & 1.4 \\ 3.2 & 1.1 & 1.2 & -0.7 \\ 0.2 & 0.5 & 1 & -0.9 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1.2 & 0.5 & -0.7 \\ 0.4 & 1.3 & 0.6 \end{bmatrix}$, $d = 3$ and the initial state condition $\mathbf{x_0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

After using the performance index (22), with application of the d-step output predictor $\mathbf{y}(t+3)$ derived from the iterative process combining the plant formulas (29), the author perfect control strategy sounds in the following manner

$$\mathbf{u}(t) = (\mathbf{CB})^{\mathrm{R}}[\mathbf{y_{ref}}(t+d) - \mathbf{C}(\sum_{p=1}^{d-1} \mathbf{A}^p\mathbf{Bu}(t-p) + \mathbf{A}^d\mathbf{x}(t))], \tag{30}$$

or rather for our system

$$\begin{aligned} \mathbf{u}(t) = (\mathbf{CB})^{\mathrm{R}}[\mathbf{y_{ref}}(t+3) - \mathbf{CABu}(t-1) \\ - \mathbf{CA}^2\mathbf{Bu}(t-2) - \mathbf{CA}^3\mathbf{x}(t)]. \end{aligned} \tag{31}$$

Thus, for application of the minimum-norm right Moore-Penrose inverse to the product of **CB**, the undesirable perfect control properties in terms of the instability of the crucial plant signals have occurred, for instance see Fig. 3.

On the other hand, our generalized Θ-inverse (7), where $\beta^{\mathrm{R}} = \alpha^{\mathrm{T}}[\alpha\alpha^{\mathrm{T}}]^{-1}$, with the special selected degrees of freedom $\beta = \begin{bmatrix} 1 & 1.5 & -2.7 & 0.1 \\ -0.9 & 0.5 & 2 & -3 \end{bmatrix}$ and $\alpha = \begin{bmatrix} 0.7 & -0.6 & 2.1 & 3.5 \\ 1.2 & 0.8 & 2.5 & -0.3 \end{bmatrix}$, brings us the expected stable perfect control strategy confirmed by Fig. 4 (the presence of stable control zeros).

3) PRACTICAL SCENARIO

In the third example, the real-life two-input one-output system is investigated in the context of application of the new approach proposed throughout the manuscript. For this reason, the two-wheels mobile robot presented in [24] is controlled, in order to maintain the inverse model control-related performance index (22). According to the continuous-time-type formula (7) of the above-mentioned reference, the discrete-time linear model supported by the Euler method has been created and successfully implemented in the Matlab/Simulink environment. Thus, the aim of the perfect control action is to achieve a zero-error between the mobile robot position (x, y) in the Cartesian space and the orientation θ, with regard to the arbitrary selected setpoint. Again, the Moore-Penrose inverse provides the instability, in contrary to the application of the generalized Θ-inverse, which

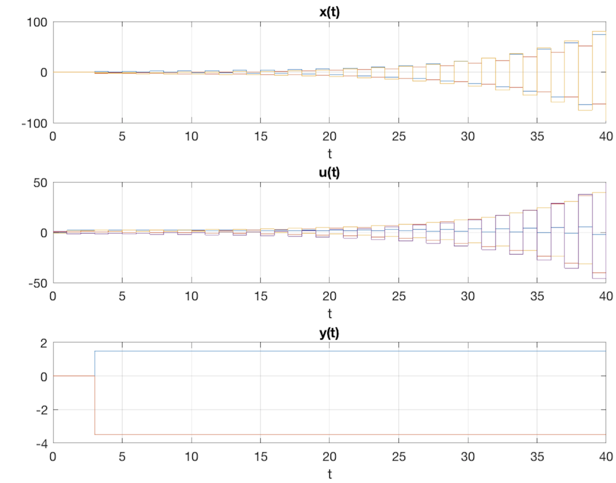

FIGURE 3. **Perfect control: system's signals, scenario – T-inverse application, $y_{ref}(t+1) = [1.5 \; -3.5]$.**

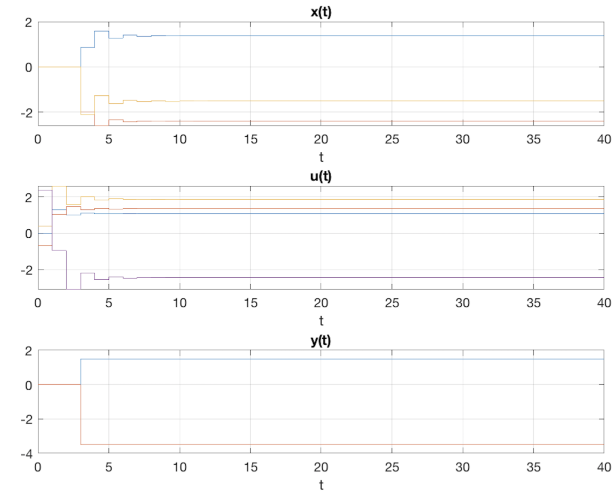

FIGURE 4. **Perfect control: system's signals, scenario – Θ-inverse application, $y_{ref}(t+1) = [1.5 \; -3.5]$.**

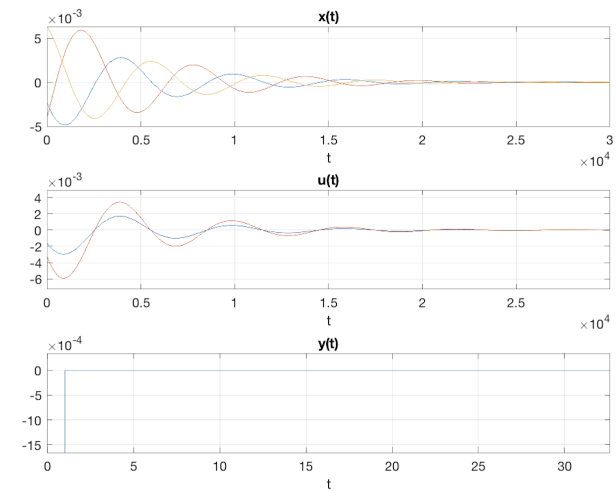

FIGURE 5. **Perfect control: system's signals, scenario – Θ-inverse application.**

clearly outperforms the former structure in the best way. The confirmation assuming the non-zero initial state vector $\mathbf{x_0} = \left[-0.003|_x \; -0.004|_y \; 0.005|_\theta\right]$ and $\beta = \left[-30 \; 40\right]$ as well as $\alpha = \left[10 \; 20\right]$, with step discretization $T_s = 0.001s$, is manifested by the Fig. 5. Of course, the robustness of discussed control strategy can be improved through the manipulation of our degrees of freedom, which should be subjected to the different optimization procedures.

VI. CONCLUSION

The new generalized Θ-inverses strictly dedicated to the right- and left-invertible matrices have been introduced in this paper. The new forms can be used in a plethora of engineering tasks related to the inverse problem. As a result of the application of a number of degrees of freedom, we can impact the robustification of various processes in the spate of employment of the unique Moore-Penrose inverse. The future works will be focused on the synthesis of the proposed Θ-inverses in the different symbolic-originated computing environments. The analytical criteria for selecting the appropriate degrees of freedom in regard to the respective type of applied inverse really constitute a research challenge. These interesting issues are worth of intense investigation shortly.

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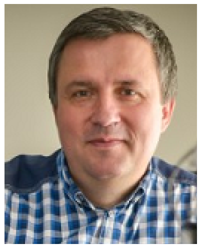

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