Robust Control for Singular Systems Based on the Uncertainty and Disturbance Estimator

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ABSTRACT This paper is devoted to investigating the robust control problem for a class of singular systems with model structure uncertainty and external disturbance. Firstly, the uncertainty and disturbance estimator (UDE)-based robust control law is established for uncertain singular systems. Secondly, the two-degree-of-freedom (2DOF) nature of singular systems under the UDE-based robust controller is revealed, which shows that asymptotic reference tracking and disturbance rejection are decoupled. Additionally, on the basis of the small-gain theorem, sufficient conditions are established to ensure robust stability of the closed-loop system and to achieve asymptotic reference tracking and disturbance rejection. Finally, three numerical examples and a practical application to the multi-agent supporting systems are provided to illustrate the validity of the methods proposed.

INDEX TERMS Robust control, singular systems, two-degree-of-freedom, uncertainty and disturbance estimator.

I. INTRODUCTION

Model uncertainty, external disturbance and parameter perturbation commonly exist in many practical applications, which bring negative effects on the performance of the control system. Therefore, robust control theory has gained significant attention to deal with such uncertainties. For classical state-space systems, a number of different techniques have been presented to investigate the robust control problem; see [1] for sliding mode control (SMC), [2] for the adaptive control, [3] for the model predictive control and [4] for the H_∞ control. Since uncertainties are usually unknown and unmeasurable, another class of approaches has been proposed to estimate or compensate for the influence of uncertainties by using the measurable states and known dynamics of systems. This kind of method can be found in [5] for the active disturbance rejection control (ADRC), [6] for the disturbance-observer-based control (DOBC), [7] for the equivalent input disturbance (EID), [8] for the extended state observer (ESO) and [9] for the unknown input observer (UIO).

Singular systems are also referred to as descriptor systems, differential algebraic systems, generalized systems and

constrained systems. Singular systems consist of differential and algebraic equations, which are different from state-space systems, the solutions of singular systems always contain impulses. In other words, state-space systems are a special class of singular systems. Therefore, singular systems have been widely researched over the last few decades because they can describe more complex dynamical models [10]–[12] in many practical applications, such as circuit systems, power systems, robot systems, multi-agent systems, chemical process, aircraft modeling and so on. However, the analysis and design of singular systems are more complicated due to the existence of impulse and algebraic modes. Finding regular and impulse-free conditions and designing the controller so that the closed-loop systems are regular, impulse-free and stable are the focus of the study of singular systems, where regularity and impulse-freeness ensure that the singular systems have a unique solution without impulse.

It is noteworthy that many meaningful disturbance attenuation control methods of uncertain state-space systems are still effective for singular systems. Robust control conservatively considers the worst-case scenario of model uncertainty, and in detail, stability for uncertain singular systems was reported by [13], the parameter uncertainty is assumed to be time-varying and norm-bounded, the necessary and sufficient conditions for quadratic stability and stabilization are

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proposed by using some matrix inequalities. The parameter uncertainty is also considered to belong to a convex bounded domain (polytopic type), and the robust stability condition is proposed through the parameter-dependent Lyapunov functional [14]. Robust H_∞ control has been considered by [15], [16], where the controlled system is quadratic stable or exponential stable and satisfies an H_∞ performance, the uncertainty and disturbance considered herein are norm-bounded. The robustness of the above robust control methods is usually obtained at the expense of transient performance. Moreover, adaptive control has been shown to ensure the robust stability of singular systems by [17], [18], and such methods usually rely on the identification of time-varying model parameters, which cannot be used when they are difficult to identify or estimate online. See also [19]–[21] for the SMC, which can fully compensate for the matching uncertainties when the closed-loop system enters a sliding mode, however, discontinuous control tends to lead to high-frequency chattering in the system. Follow-up research on the uncertain singular systems has been carried out in [22]–[24] by the observer-based control method, and in [25], [26] for the filtering methods, in these methods, the boundary assumptions of uncertainties and disturbances are necessary. In [27], the problem of event-triggered H_∞ control for the singular systems with randomly occurring uncertainty and nonlinearity was investigated. However, all of the above papers always assume that the uncertainties are known or bounded, and the robust control problem was solved by the time domain method, the robust stability conditions were proposed in terms of linear matrix inequalities (LMIs). In practical systems, the norm-bounded condition or convex bounded condition may not be applicable and unknown external disturbance inevitably appear in the controlled process. In this situation, the LMI approach in the time domain may be invalid.

The above traditional anti-disturbance methods suppress disturbance through feedback control rather than feedforward compensation, and cannot respond directly or quickly enough in the presence of strong disturbance. In order to overcome the limitations of traditional anti-disturbance methods, a class of methods so-called active anti-disturbance control (AADC) have been proposed [28]. In [29], the disturbance observer (DOB) based control was used to estimate the disturbance by the nominal system model and a low-pass filter, which implies that the system model needs to be known exactly, however, it is sometimes not available. Very recently, the EID-based control method was considered for singular system in [30], which can reduce the influence of uncertainties and disturbances, however, the disturbances must be present on the same channel as the control input. The ADRC method used in [5] was that total disturbance were treated as a new state of extended system, and a state observer was designed for the extended system to estimate the disturbances in real time and reasonably compensate for the effects caused by the disturbances. However, the derivatives of disturbance needs to be bounded, which is difficult

to satisfy. Therefore, it is necessary to find a new method to address the robust control problem for singular systems, which inspires this study.

In recent years, a new active anti-disturbance control method based on uncertainty and disturbance estimator (UDE) proposed by [31] has received much attention due to its fast response and accurate estimation. The basic idea is that the controller compensates for the influence of uncertainties by estimating uncertainties from measurable variables. One of the primary benefits of this method is that the uncertainty can be quickly estimated on the basis of a strictly-proper low pass filter. On the other hand, what we need to know is only the frequency spectrum (bandwidth) information of the sum term of uncertainties and disturbances that can be measured in engineering, while the upper bound of uncertainties is not essential. In addition, the UDE-based control approach also does not require an accurate system model and informations on the derivatives of disturbances, which overcomes the drawbacks of DOBC and ADRC. The UDE-based control approach has been applied in many publications due to its excellent performance, such as nonlinear systems [32], the nonlinear functions were considered as an additional unknown uncertainty term of the system, which were estimated by the estimator. In [33], UDE-based approach is used for the SMC, which overcomes the chattering phenomenon in the design of SMC. In [34], [35], by combining with SMC method or dynamic feedback, the modified UDE-based robust control laws were proposed for chaotic systems; As reported by [36], the UDE-based control law has also been extended to the stabilization of partial differential equation systems which are more complex than ODEs. In the light of the above statements, the UDE-based control strategy has good robust control performance. However, to the best of our knowledge, the UDE-based control has not been given for singular systems, which also motivates our research. Motivated by the above discussions, this article studies the robust control problem for singular systems under the UDE-based control strategy. There are still some issues that need to be overcome: (1) How to design an UDE-based controller for the singular system subject to model uncertainty and external disturbance? (2) Whether the closed-loop singular systems still have two-degree-of-freedom nature in the frequency domain. (3) How to obtain admissibility conditions and how to develop an algorithm to select the appropriate error feedback gain K? To address the above challenges, the main contributions of this work are summarized as follows:

1) The UDE-based control law for singular systems with model uncertainty and external disturbance is presented. The boundary assumption is removed when the matching condition is satisfied, which is different from norm-bounded uncertainties and polytopic type uncertainties discussed in the existing literature [13], [14]. In addition, when the matching condition is not satisfied but the bounded assumption is satisfied, the proposed UDE-based robust controller can still be designed effectively.

2) The 2DOF nature of controlled singular systems is presented under the UDE-based controller. Asymptotic reference tracking and disturbance rejection are decoupled in frequency domain, meanwhile, the lumped disturbance can be attenuated by two decoupled designable filters. Compared with uncertain systems addressed in other work [37], the 2DOF nature considered in this paper is more general.

3) The sufficient conditions are derived to guarantee the robust stability and asymptotic performance of the singular systems under the robust controller. Besides, a novel design algorithm for error feedback gain K is also presented to derive the trade-off between stability conditions and attenuation filters.

4) Compared with the DOB-based robust H_∞ control [29], the UDE-based control approach does not require an exact system model, only a suitable filter is needed, which means that the UDE-based method is easier to implement and has a wider range of applications.

The rest of this work is organized in what follows. Section II formulates the robust control problem of singular systems and presents essential preliminaries. The main results of UDE-based control for singular systems are presented in Section III, which includes the controller design, two-degree-of-freedom nature analysis, stability analysis, and error feedback gain design. In Section IV, illustrative examples along with numerical and simulation results are provided. The conclusions are presented in Section V.

Notations: $\mathbb{R}^{n \times m}$, $\mathbb{R}^n$, $\mathbb{C}$ and $\mathbb{C}^-$ denote the set of all real $n \times m$ matrices, n-dimensional real Euclidean space, the set of all complex numbers and the open left half complex plane, respectively. $det(\cdot)$ and $deg(\cdot)$ stand for the determinant of a matrix and degree of a polynomial. For the column full rank matrix B, $B^+ = (B^T B)^{-1} B^T$ is the Moore–Penrose inverse of B. I and 0 represent, respectively, the identity matrix and zero matrix with appropriate dimensions. The symbol "$*$" is the convolution operator. $L^{-1}\{\cdot\}$ is the inverse Laplace transform operator. H_∞ norm of transfer matrix $G(s)$ is defined in the frequency domain as $\|G(s)\|_\infty = \sup_\omega \sigma_{max}\{G(j\omega)\}$, $\rho(M)$ is the spectral radius of $M \in \mathbb{C}^{n \times n}$.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a class of time-invariant singular systems with uncertainty and disturbance as follows

$$E\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + w(t), \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is state vector, $u(t) \in \mathbb{R}^m$ is control input and $w(t) \in \mathbb{R}^n$ is unpredictable external disturbance. $E \in \mathbb{R}^{n \times n}$ is a singular matrix with $rank(E) = p < n$, $A \in \mathbb{R}^{n \times n}$, $\Delta A \in \mathbb{R}^{n \times n}$ is unknown matrix, and $B \in \mathbb{R}^{n \times m}$ is of full column rank.

Remark 1: In this work, the uncertainty ΔA and disturbance $w(t)$ are not required to be bounded, only their frequency range (bandwidth) needs to be known, which is different from the assumptions of norm-bounded uncertainty,

convex bounded uncertainty and bounded disturbance discussed in [13], [14]. In addition, the controller parameter perturbation is omitted to simplify the complexity of system, which can be regarded as part of the external disturbance, i.e., $w(t) = \Delta Bu(t) + d(t)$.

Definition 1 [10], [12]: Singular system $E\dot{x}(t) = Ax(t)$ (or the pair (E, A)) is said to be

(1) regular if the polynomial $det(sE - A)$ is not identically zero, $s \in \mathbb{C}$.

(2) impulse-free if it is regular and $deg(det(sE - A)) = rank(E)$, $s \in \mathbb{C}$.

(3) stable if $\sigma(E, A) \subset \mathbb{C}^-$, where $\sigma(E, A) = \{s|s \in \mathbb{C}, det(sE - A) = 0\}$.

(4) admissible if it is regular, impulse-free and stable.

Definition 2 [12]: Singular system (1) with $u(t) = 0$ is said to be robustly stable if it is regular, impulse-free and stable for all uncertainties and disturbances.

Assumption 1: The pair (E, A, B) is impulse controllable and R-controllable, i.e., there exists a control input formed with $u(t) = Kx(t)$ such that closed-loop system (1) under $\Delta A = 0$ and $w(t) = 0$ is regular, impulse-free and stable.

Remark 2: Different from linear systems, singular systems need to consider regularity and non-impulsiveness besides stability, where regularity guarantees the existence and uniqueness of solution of the system, and non-impulsiveness guarantees that the unique solution contains no impulse terms. In order to establish the solvable conditions of robust stability problem for system (1), Assumption 1 is necessary to guarantee that an admissible closed-loop system (1) can be obtained.

Assumption 2: The spectrum information of the lumped of uncertainties $\Delta Ax(t)$ and external disturbance $w(t)$ are available in the frequency domain.

Remark 3: Different from the traditional methods in the time domain, the boundary assumption of disturbance is unnecessary and only their bandwidth information needs to be known for the design. In fact, it is easy to be measured in engineering. Assumption 2 always holds in this paper.

The reference model of singular systems is given as

$$E\dot{x}_r(t) = A_r x_r(t) + B_r r(t), \qquad (2)$$

where $x_r(t) \in \mathbb{R}^n$ is the state of reference system, $r(t) \in \mathbb{R}^m$ is the given reference input. $A_r \in \mathbb{R}^{n \times n}$ and $B_r \in \mathbb{R}^{n \times m}$ are selected to satisfy the specified performance characteristics. Obviously, (E, A_r) is chosen to be admissible, which ensures that desired performance could be achieved.

The purpose of this work is to develop a physically implementable controller $u(t)$ such that system (1) asymptotically tracks reference system (2), that is, the tracking error between reference system (2) and system (1)

$$\theta(t) = x_r(t) - x(t), \qquad (3)$$

is asymptotically stable, to this end, let $\theta(t)$ satisfy the following error dynamic system form directly

$$E\dot{\theta}(t) = (A_r + K)\theta(t), \qquad (4)$$

where $K \in \mathbb{R}^{n \times n}$ is an error feedback gain matrix which needs to be designed so that $(E, A_r + K)$ is admissible.

Remark 4: The purpose of this work is to develop a physically implementable controller $u(t)$ to make system (1) possess the same dynamic performance as reference model (2), which has the desired performance such as admissibility, overshoot and settling time. To achieve above control objective, an appropriate controller $u(t)$ can be designed such that the state of system (1) asymptotically tracks the state of reference system (2), i.e., $\lim_{t \to \infty} \| \theta(t) \| = 0$. For convenience, the error dynamics system can be chosen as the form of (4) to meet the control objective. Since (E, A_r) is admissible, it is easy to find an appropriate K such that the error system (4) is admissible, which ensures that error state $\theta(t)$ asymptotically converges to 0. In addition, K can be used to adjust the performance of the error system.

III. MAIN RESULTS

In this section, first of all, an UDE-based control law is presented to ensure that the uncertain singular system (1) is stable, and the state $x(t)$ asymptotically tracks the state $x_r(t)$ of reference model. It is then shown that 2DOF nature of singular systems under the UDE-based controller is proposed. Finally, stability conditions and a design algorithm of error feedback gain are given.

A. UDE-BASED CONTROL LAW OF SINGULAR SYSTEMS

From (1)-(3), it is clear that

$$E\dot{\theta}(t) = A_r\theta(t) + A_rx(t) + B_rr(t)$$
$$-Ax(t) - \Delta Ax(t) - Bu(t) - w(t), \quad (5)$$

together with (4), one has that

$$A_rx(t) + B_rr(t) - Ax(t) - Bu(t) - \Delta Ax(t) - w(t) = K\theta(t). \quad (6)$$

Further, based on (6), the controller $u(t)$ is derived as

$$u(t) = B^+ (A_rx(t) + B_rr(t) - Ax(t) - \Delta Ax(t)$$
$$-w(t) - K\theta(t)), \quad (7)$$

substituting (7) into (6), we can obtain the following structural constraint

$$(I - BB^+)(A_rx(t) + B_rr(t) - Ax(t) - K\theta(t)$$
$$-w(t) - \Delta Ax(t)) = 0. \quad (8)$$

Clearly, if condition (8) is satisfied, (7) is the exact solution of equation (6). Otherwise, it is just the least squares approximate solution of equation (6). In particular, it must satisfy structural constraint (8) if B is invertible. As reported by [38], structural constraint (8) can be considered as a matching condition, which means that all uncertainties and external disturbances exist on the identical channels as the control input.

In this paper, the sum of uncertainty $\Delta Ax(t)$ and external disturbance $w(t)$ can be regarded as a generalized disturbance,

which can be described as

$$u_w(t) = \Delta Ax(t) + w(t), \quad (9)$$

then from (1), it can be obtained that

$$u_w(t) = E\dot{x}(t) - Ax(t) - Bu(t). \quad (10)$$

According to [31], there exists a strictly-proper filter $G_f(s)$ with unit steady-state gain ($G_f(0) = 1$) and a sufficiently large bandwidth such that $u_w(t)$ can be estimated by $\hat{u}_w(t)$ described as follows

$$\hat{u}_w(t) = u_w(t) * g_f(t) = (E\dot{x}(t) - Ax(t) - Bu(t)) * g_f(t), \quad (11)$$

where $g_f(t) = L^{-1} \{G_f(s)\}$.

Remark 5: Generally, on the basis of [39], [40], for the most commonly used reference input signals and disturbance signals such as unit step functions ($1(t)$) and sine functions ($A \sin \omega t$), the filter $G_f(s)$ can be selected as follows:

1) Both the reference input signals and the disturbance are step signals, the first-order filter is

$$G_{f1}(s) = \frac{a}{s + a}.$$

2) Both the reference input signals and the disturbance are sine signals with frequency ω_0, the second-order filter is

$$G_{f2}(s) = \frac{a_1s + a_2 - \omega_0^2}{s^2 + a_1s + a_2}.$$

3) One of the reference input signal and the disturbance is a sine signal with the measurable frequency ω_0 and the other is a step signal, the form of filter is

$$G_{f3}(s) = \frac{(a + a_1)s^2 + (a_2 + aa_1 - \omega_0^2)s + aa_2}{(s + a)(s^2 + a_1s + a_2)}.$$

Thus, the uncertainties $u_w(t)$ in (7) can be replaced by the estimation $\hat{u}_w(t)$. Taking (11) into consideration, the modified controller $u(t)$ can be rewritten as

$$u(t) = B^+ (A_rx(t) + B_rr(t) - Ax(t) - K\theta(t)$$
$$- (E\dot{x}(t) - Ax(t) - Bu(t)) * g_f(t)), \quad (12)$$

similar to [31], from (12), the UDE-based controller of singular system (1) is derived as

$$u(t) = B^+ \left(-L^{-1}\left\{\frac{sG_f(s)}{1 - G_f(s)}\right\} * Ex(t) - Ax(t) \right.$$
$$\left. + L^{-1}\left\{\frac{1}{1 - G_f(s)}\right\} * (A_rx(t) + B_rr(t) - K\theta(t))\right). \quad (13)$$

Obviously, the design of controller (13) is only related to the reference system, the feedback gain K and the filter $G_f(s)$, it is independent of uncertainty and disturbance.

Since there is no unknown dynamics in controller (13), it is not change if the uncertainty ΔA or external disturbance $w(t)$ is zero. Specially, when $\Delta A = 0$ and $w(t) = 0$, the robust control problem translates directly into the tracking problem and controller (13) can be reduced to

$$u(t) = B^+ (A_rx(t) + B_rr(t) - Ax(t) - K\theta(t)),$$

where the error feedback gain K is the unique parameter to be designed. Moreover, the problem is transformed into the stabilization of singular systems if the tracking problem is not considered. The controller is given that

$$u(t) = B^+ \left(-Ax(t) + Kx(t) \right), \tag{14}$$

which is a state feedback controller, and we can derive the closed-loop system as

$$E\dot{x}(t) = (I - BB^+)Ax(t) + BB^+Kx(t). \tag{15}$$

Remark 6: For controller $u(t)$ given in (14), we have the following interpretations:

(1) If B is invertible, controller (14) is an exact solution.
(2) If B is not invertible, controller (14) is an exact solution if the structural constraint

$$\left(I - BB^+ \right) \left(-Ax(t) + Kx(t) \right) = 0 \tag{16}$$

is satisfied. In this case, the controlled system (15) is reduced to

$$E\dot{x}(t) = Kx(t), \tag{17}$$

and the design of K in (17) is simpler than the general state feedback matrix.
(3) If B is not invertible and structural constraint (16) is not satisfied, controller (14) is only an approximate solution. In this situation, the control performance of $u(t)$ can be improved by adjusting the feedback gain K.

B. TWO-DEGREE-OF-FREEDOM NATURE OF SINGULAR SYSTEMS

In the following, the 2DOF nature analysis of singular systems under UDE-based robust controller is discussed.

Assuming that constraint condition (8) is satisfied, substituting (8)-(12) into system (1) yields

$$E\dot{x}(t) = A_r x(t) + B_r r(t) - K\theta(t) \\ + BB^+ u_w(t) - BB^+ u_w(t) * g_f(t), \tag{18}$$

when $Ex(0) = 0$, applying the Laplace transform to (18), it can be obtained that

$$sEX(s) = A_r X(s) + B_r R(s) - K\Theta(s) \\ + BB^+ \left(1 - G_f(s) \right) U_w(s). \tag{19}$$

When $Ex_r(0) = 0$, the reference model can be represented as

$$sEX_r(s) = A_r X_r(s) + B_r R(s), \tag{20}$$

it follows from (20) that

$$sEX_r(s) = A_r X_r(s) + B_r R(s), \tag{21}$$
$$H_{RX_r}(s) = (sE - A_r)^{-1} B_r, \tag{22}$$

in which, the transfer matrix from $R(s)$ to $X_r(s)$ denoted by H_{RX_r}. Since (E, A_r) is regular and impulse-free, $(sE - A_r)^{-1}$ exists and is proper.

Substituting (3) and (21) into (19), one has

$$X(s) = (sE - (A_r + K))^{-1} \left(I - K(sE - A_r)^{-1} \right) B_r R(s) \\ + (sE - (A_r + K))^{-1} BB^+ \left(1 - G_f(s) \right) U_w(s), \tag{23}$$

where $(E, A_r + K)$ is admissible so that $(sE - (A_r + K))^{-1}$ exists and is proper. Let $H_r(s)$ denotes the transfer matrix from $R(s)$ to $X(s)$, and $H_w(s)$ denotes the transfer matrix $U_w(s)$ to $X(s)$, it gives that

$$H_r(s) = (sE - (A_r + K))^{-1} \left(I - K(sE - A_r)^{-1} \right) B_r \\ = (sE - (A_r + K))^{-1} \\ \times (sE - A_r - K) (sE - A_r)^{-1} B_r \\ = (sE - A_r)^{-1} B_r, \\ H_w(s) = (sE - (A_r + K))^{-1} BB^+ \left(1 - G_f(s) \right).$$

It is obvious that $H_r(s) = H_{RX_r}$, which indicates that they have the same transfer matrix from $R(s)$ to $X(s)$ and $X_r(s)$, thus, for a reference input $R(s)$, $X(s)$ can asymptotically track $X_r(s)$ without all uncertainties and external disturbances. Moreover, the design of the reference system is independent of the gain K. It can be concluded from (23) that disturbance rejection and asymptotic reference tracking of uncertain singular systems are decoupled.

Substituting (13) into (5) and combining with (3), the actual error dynamics is obtained as

$$E\dot{\theta} - (A_r + K)\theta(t) \\ = A_r x(t) - Ax(t) + B_r r(t) - Bu(t) - K\theta(t) - u_w(t) \\ = -BB^+ u_w(t) + BB^+ u_w(t) * g_f(t). \tag{24}$$

Applying the Laplace transform to (24), in the frequency domain, the actual error dynamics can be described as

$$sE\Theta(s) = (A_r + K)\Theta(s) + BB^+ \left[G_f(s) - 1 \right] U_w(s),$$

where the actual error state is derived

$$\Theta(s) = -H_w(s)U_w(s) \\ = -(sE - (A_r + K))^{-1} BB^+ \left(1 - G_f(s) \right) U_w(s), \tag{25}$$
$$H_k(s) = (sE - (A_r + K))^{-1}, \tag{26}$$
$$H_f(s) = BB^+ \left(1 - G_f(s) \right), \tag{27}$$

as a consequence, the transfer matrix from $-U_w(s)$ to $\Theta(s)$ can be written as

$$H_w(s) = H_k(s) \cdot H_f(s).$$

It is worth noting that the transfer matrix from $-U_w(s)$ to $\Theta(s)$ is same as that from $U_w(s)$ to $X(s)$. Therefore, the control goal is to eliminate the influence of $U_w(s)$ on system state and error state by adjusting the transfer matrix $H_w(s)$ approaches to 0. When $U_w(s)$ is unknown and nonzero, $H_k(s)$ and $H_f(s)$ can be chosen to converge to 0 that can satisfy above condition. As a matter of fact, signal $U_w(s)$ contains two parts: high frequency signal and low frequency signal. The low-frequency signal of $U_w(s)$ are attenuated by $H_f(s)$, and the high-frequency signal of $U_w(s)$ are attenuated by

$H_k(s)$ with a low-pass filter form such that the gain of $H_k(s)$ is designed to be small. Hence, K and $G_f(s)$ are designed to satisfy the above conditions to make the tracking error converges to 0 and suppress disturbance. Algorithm 1 for designing K is given in the next subsection.

Remark 7: It should be noted that the design of filter $G_f(s)$ does not need to consider the amplitude range of disturbance signal, only need to know the generalized disturbance frequency range (bandwidth) and it is generally measurable. It is a very important feature of UDE-based controller design.

C. STABILITY ANALYSIS

In general, structural constraint (8) is not easily satisfied, which means that controller (13) is the least squares solution of (7) and matching condition is not satisfied. In view of this, several sufficient conditions for robust stability are given.

Theorem 1: Under Assumption 1, the closed-loop system formed by singular system (1) with bounded ΔA and controller (13) is regular, impulse-free and robustly stable if

(1) $(E, A_r + K)$ is admissible.
(2)

$$\|H_k(s) \cdot M_1(s)\|_\infty < 1, \tag{28}$$

where $M_1(s) = BB^+ (1 - G_f(s)) \Delta A - (I - BB^+)(A_r + K - A - \Delta A)$, $H_k(s)$ is shown in (26).

Proof: Applying the Laplace transform to (1), (9), (12), we have that

$$sEX(s) = (A + \Delta A)X(s) + BU(s) + W(s), \tag{29}$$

$$U_w(s) = \Delta A X(s) + W(s), \tag{30}$$

$$U(s) = B^+ (A_r X(s) + B_r R(s) - AX(s) \\ - K\Theta(s) - U_w(s)G_f(s)). \tag{31}$$

Substituting (30) and (31) into (29), the closed-loop system is obtained that

$$sEX(s) = AX(s) + BB^+ (A_r X(s) + B_r R(s) - AX(s) \\ - K\Theta(s) - G_f(s)U_w(s)) + U_w(s). \tag{32}$$

Since $\Theta(s) = X_r(s) - X(s)$ and $sEX_r(s) = A_r X_r(s) + B_r R(s)$ from (3) and (2), the closed-loop system of the actual error is

$$sE\Theta(s) = A_r X_r(s) + B_r R(s) - AX(s) - U_w(s) \\ - BB^+ (A_r X(s) + B_r R(s) - AX(s) \\ - K\Theta(s) - G_f(s)U_w(s)) \\ = (A_r + K)\Theta(s) - BB^+ (1 - G_f(s)) U_w(s) \\ + (I - BB^+)(A_r X(s) + B_r R(s) - AX(s) \\ - U_w(s) - K\Theta(s)). \tag{33}$$

Therefore, the dynamics of actual error system from (33) can be rewritten as

$$\left(I - (sE - (A_r + K))^{-1} \left(BB^+ (1 - G_f(s))\Delta A \\ - (I - BB^+)(A_r + K - A - \Delta A))\right)X(s) \\ = (sE - (A_r + K))^{-1} BB^+ (B_r R(s) - KX_r(s)) \\ + (sE - (A_r + K))^{-1} (I - BB^+ G_f(s)) W(s), \tag{34}$$

that is

$$X(s) = (I - H_k(s)M_1(s))^{-1} H_k(s)U_e(s) \\ + (I - H_k(s)M_1(s))^{-1} H_k(s)D_e(s), \tag{35}$$

where $U_e(s) = BB^+ (B_r R(s) - KX_r(s))$ and $D_e(s) = (I - BB^+ G_f(s)) W(s)$ denote the equivalent input and the equivalent disturbance, respectively. The equivalent structure diagram of system (35) is illustrated in Figure 1.
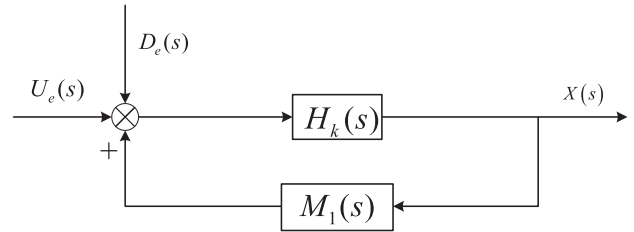

FIGURE 1. The equivalent structure diagram of system (35).

From (28), one has

$$\rho (H_k(\infty)M_1(\infty)) \le \|H_k(\infty)M_1(\infty)\|_\infty < 1,$$

then, it can be shown that $I - H_k(\infty)M_1(\infty)$ is invertible. According to Lemma 5.1 in [4], the closed-loop system is well-posed, which is equivalent to the condition that $(I - H_k(s)M_1(s))^{-1}$ exists and is proper. Since $(E, A_r + K)$ is admissible, $H_k(s)$ is proper, closed-loop transfer function $(I - H_k(s)M_1(s))^{-1} H_k(s)$ is proper, closed-loop system (35) is regular and impulse-free [10], [11].

If uncertain ΔA is bounded, it is easily obtained that $\|M_1(s)\|_\infty < \infty$. On the basis of the small-gain theorem [4], the closed-loop system is stable if the condition (28) holds. Therefore, closed-loop system (35) is robustly stable. This completes the proof. ∎

Remark 8: For a singular system, regularity and non-impulsiveness are essential to ensure the existence and uniqueness of solution of system and the solution does not contain impulse terms. The conditions are developed in the frequency domain, which reduce the computational complexity and avoid some of the numerical problems caused by decomposing singular systems instead of the dynamic decomposition form as literature [10], [11]. It should be noted that $(E, A_r + K)$ is easily designed to be admissible, because (E, A_r) is admissible. Furthermore, admissibility condition (28) does not depend on external disturbance $w(t)$, but depends on the feedback gain K, filter $G_f(s)$ and model uncertainty ΔA. Indeed, $H_k(s)$ is the transfer matrix of $(E, A_r + K)$, $M_1(s)$ is related to ΔA, $G_f(s)$ and K, if ΔA is bounded, it is easily obtained that the gain of $M_1(s)$ is norm bounded. Therefore, the closed-loop system (35) remains stable so long as the total gain of the closed-loop is less than 1. A design method of feedback gain K is given in Algorithm 1.

In system (1), if $\Delta A = 0$, from Theorem 1, it can be obtained the following result.

Corollary 1: Singular system (1) without uncertainty ($\Delta A = 0$) satisfies Assumption 1 is regular, impulse-free and robustly stable under the controller (13) if $(E, A_r + K)$ is admissible and

$$\|H_k(s) \cdot M_2\|_\infty < 1, \tag{36}$$

where $M_2 = (I - BB^+)(A_r + K - A)$.

Remark 9: When $\Delta A = 0$, the stability conditions are only dependent on feedback gain K. The robust control problem for system (1) can be solved by H_∞ control [12], H_∞ filtering [25] and unknown input observer [22]. However, In [12], [25], the external disturbance $w(t)$ was assumed to belong to $L_2[0, \infty)$ and the problem was mainly solved by finding equivalent feasibility conditions of linear matrix inequality (LMI). In [22], the conditions of observability or detectability were necessary for the existence of observers for singular systems, and some matching conditions are assumed to be satisfied. Compared with above approaches, the result of UDE-based control method is less conservative and computationally complex since the estimator could achieve accurate estimation without the upper bound information.

The following theorem shows that if constraint (8) is satisfied, i.e., controller (13) is the exact solution of (7), the uncertainties and external disturbances satisfy the matching condition. The asymptotic performance of uncertain singular system (1) under controller (13) can be guaranteed.

Theorem 2: For the uncertain singular system (1) satisfies Assumption 1, if constraint condition (8) is satisfied, then the closed-loop system formed by system (1) and controller (13) is regular, impulse-free and robustly stable, and achieves asymptotic reference tracking and disturbance rejection if

(1) $(E, A_r + K)$ is admissible.
(2) The state $x_r(t)$ of reference system (2) asymptotically tracks the command signal $r(t)$.
(3)

$$\|H_k(s) \cdot M_3(s)\|_\infty < 1, \tag{37}$$

where $M_3(s) = BB^+ \left(1 - G_f(s)\right) \Delta A$, $H_k(s)$ is shown in (26).

Proof: Applying the Laplace transform to (8) and substituting it into (33), the actual tracking error dynamics is given as

$$sE\Theta(s) = (A_r + K)\Theta(s) - BB^+ \left(1 - G_f(s)\right) U_w(s), \tag{38}$$

and the closed-loop system from (38) is

$$\left(I - (sE - (A_r + K))^{-1} \left(BB^+ \left(1 - G_f(s)\right) \Delta A\right)\right) X(s)$$
$$= (sE - (A_r + K))^{-1} (B_r R(s) - KX_r(s))$$
$$+ (sE - (A_r + K))^{-1} BB^+ \left(1 - G_f(s)\right) W(s). \tag{39}$$

which can be rewritten as

$$X(s) = (I - H_k(s)M_3(s))^{-1} H_k(s) (B_r R(s) - KX_r(s))$$
$$+ (I - H_k(s)M_3(s))^{-1} H_k(s)BB^+ \left(1 - G_f(s)\right) W(s). \tag{40}$$

Similar to the proof of Theorem 2, it follows that $(I - H_k(s)M_3(s))^{-1}$ is proper. Thus, the closed-loop transfer function $(I - H_k(s)M_3(s))^{-1} H_k(s)$ is proper, closed-loop system (40) is regular and impulse-free. On the basis of the small-gain theorem, the controlled system (40) is robustly stable if the condition (37) holds.

Because of the filter $G_f(s)$ with unit steady-state gain is strictly-proper, and covers the spectrum of the generalized disturbance, we have that

$$\lim_{s \to 0} s \left(1 - G_f(s)\right) U_w(s) = 0. \tag{41}$$

Closed-loop system (40) achieves disturbance rejection, as stated in the previous subsection that (38) is equivalent to (25), according to (25), (41), and the final value theorem, it follows that

$$\lim_{t \to \infty} \theta(t) = \lim_{s \to 0} s\Theta(s) = 0, \tag{42}$$

that is, $x(t)$ asymptotically tracks $x_r(t)$, $x_r(t)$ asymptotically tracks the reference input signal by the internal model principle [41]. $x(t)$ asymptotically tracks $r(t)$ achieves asymptotic tracking, since they have the same transfer matrices from $R(s)$ to $X(s)$ and $X_r(s)$, i.e.,

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x_r(t) = \lim_{t \to \infty} r(t). \tag{43}$$

Therefore, closed-loop system (40) achieves asymptotic reference tracking. This completes the proof. ∎

Remark 10: It should be mentioned that Theorem 2 guarantees the stability of closed-loop system (40) and the static-free tracking (simultaneously achieves asymptotic tracking and disturbance rejection) for the given reference input, which is very significant in practical applications. Due to the influence of mismatched disturbances, the two-degree-of-freedom nature of closed-loop system (32) is vanished, the state of system (32) cannot asymptotically track the input signal $r(t)$, but it can still suppress disturbances and ensure that the closed-loop system (32) is stable. In other words, Theorem 1 can guarantee the anti-disturbance performance but cannot realize the tracking of the reference input signals.

Remark 11: The robust problem for uncertain singular system (1) can be resolved by generalized quadratic stability problem [13] and robust H_∞ control [15]. The uncertain parameter ΔA is considered to be norm-bounded with the form $\Delta A = MF(\sigma)N$, where the norm-bounded uncertain matrix $F(\sigma)$ is required to satisfy $F(\sigma)^T F(\sigma) < I$ and the transfer matrix needs to satisfy a disturbance attenuation constant γ for all uncertainties. Additionally, the uncertain parameter is also considered to belong to a convex bounded domain $\mathcal{A} = \left\{A(t)|A(t) = \sum_{i=1}^m \alpha_i A_i; \sum_{i=1}^m \alpha_i = 1\right\}$ (polytopic type) [14]. The external disturbance $w(t)$ is always assumed to belong to $L_2[0, \infty)$ (the set of signals whose L_2 norm is bounded). The stability conditions are given by parameter-independent or parameter-dependent Lyapunov matrix, and all of the stability conditions are presented in terms of LMIs. As we all know, the introduction of LMI brings an increase in computational complexity, and the

results obtained increase conservativeness or even have no solution. Compared with above control methods, the UDE-based robust control approach does not require knowledge of the boundary of uncertain matrix, and achieves robust stability with less conservatism.

In this paper, for simplicity of presentation, K can be decided by using the algorithm sketch below. The core idea of the algorithm is to find an appropriate K so that $(E, A_r + K)$ is admissible and the gain value of H_k is as small as possible, that is, $\|H_k(s)\|_\infty < \varepsilon$, where ε is a sufficiently small positive number. Meanwhile, the conditions of theorems should be satisfied.

Remark 12: Algorithm 1 is designed to be simple to understand and has low computational complexity. Specially, when $E = I$, $(E, A_r + K)$ can be reduced to $(I, A_r + K)$, which implies that the equivalent transformation is not necessary and $H_k(s) = (sI - (A_r + K))^{-1}$. In Theorem 2, K can be designed to place the poles of $H_k(s)$ at the open left half complex plane and the condition $\|H_k(s)\|_\infty < \varepsilon$ is satisfied. Actually, for a linear system, the above conditions can be easily satisfied by designing the poles of $H_k(s)$ are away from the imaginary axis.

IV. SIMULATION EXAMPLES

In this section, three numerical examples and a practical application are presented. The first example illustrates the effectiveness of Theorem 1 with mismatched uncertainties and is compared with the robust H_∞ control method. The second example illustrates the validity of Theorem 2 with matched different types of uncertainties and disturbances. The third example compares UDE-based method with the DOB-based robust H_∞ control method. The fourth example shows the application to the multi-agent supporting systems with unbounded disturbances and is compared with the SMC method.

Example 1: Consider the unstable system (1) with model mismatch ΔA and the system parameters are given as follows

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.6 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_1(t) \\ 0 \end{bmatrix}.$$

$w_1(t)$ is unknown external disturbance, the admissible reference system (2) is chosen as

$$A_r = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where the compatible initial values are $Ex(0) = [0 \ \ 0]^T$, $Ex_r(0) = [0 \ \ 0]^T$. For comparison, assume that the command signal $r(t) = 0$, the mismatched ΔA always exist in the simulation process, and the bounded disturbance $w_1(t) = \sin(3t)$ is imposed on the system (1). The filter would be selected as

$$G_{f1}(s) = \frac{1}{Ts + 1},$$

Algorithm 1

1) Ensure that $(E, A_r + K)$ is admissible. There exists a nonsingular matrix Q and an orthogonal matrix V such that

$$\bar{E} = QEV = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \bar{A}_r = QA_rV = \begin{bmatrix} A_{r1} & 0 \\ A_{r3} & A_{r4} \end{bmatrix},$$

$$\bar{K} = QKV = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix},$$

where the partitions are in appropriate dimensions, it is obtained that

$$\bar{A}_r + \bar{K} = \begin{bmatrix} A_{r1} + K_1 & K_2 \\ A_{r3} + K_3 & A_{r4} + K_4 \end{bmatrix},$$

without loss of generality, let

$$K_2 = 0, K_3 = -A_{r3}.$$

Since (E, A_r) is admissible, it is easy to find K_1, K_4 such that

$$\sigma(E, A_r + K) = \sigma(A_{r1} + K_1) \subset \mathbb{C}^-, \quad (44)$$

and

$$det(A_{r4} + K_4) \neq 0. \quad (45)$$

Therefore, $(E, A_r + K)$ is stable, impulse-free and admissible.

2) Let $\|H_k(s)\|_\infty$ as small as possible and satisfy the conditions of Theorem 1-2. It is obtained that

$$\|H_k(s)\|_\infty$$
$$= \left\| (sE - (A + K))^{-1} \right\|_\infty$$
$$= \left\| \begin{bmatrix} (sI_p - (A_{r1} + K_1))^{-1} & 0 \\ 0 & (A_{r4} + K_4)^{-1} \end{bmatrix} \right\|_\infty \quad (46)$$

(a) In Theorem 1, $\|H_k(s)\|_\infty$ is not only small, but also satisfies condition (28). In order to have a smaller $\|H_k(s)\|_\infty$, K_1 can be chosen so that the poles of $(sI_p - (A_{r1} + K_1))^{-1}$ are away from the imaginary axis. And K_4 can be selected to make $\sigma_{min}(A_{r4} + K_4)$ large enough due to $\sigma_{max}(A_{r4} + K_4)^{-1} = (\sigma_{min}(A_{r4} + K_4))^{-1}$. However, the selection method will result in a larger $\|M_1(s)\|_\infty$. Thus, the choice of K_1 and K_4 is a trade-off between $\|H_k(s)\|_\infty$ and $\|M_1(s)\|_\infty$ such that the condition (28) is satisfied.

(b) In Theorem 2, $H_k(s)$ and $M_3(s)$ are decoupled. The choice of K does not affect the $M_3(s)$. Therefore, K_1 can be selected so that the poles of $(sI_p - (A_{r1} + K_1))^{-1}$ are away from the imaginary axis and K_4 can be selected to make $\sigma_{min}(A_{r4} + K_4)$ large enough, while the condition (37) is easily satisfied.

3) Give the gain matrix K. The gain K can be acquired in the following ways

$$K = Q^{-1}\bar{K}V^{-1} = Q^{-1} \begin{bmatrix} K_1 & 0 \\ -A_{r3} & K_4 \end{bmatrix} V^{-1}.$$

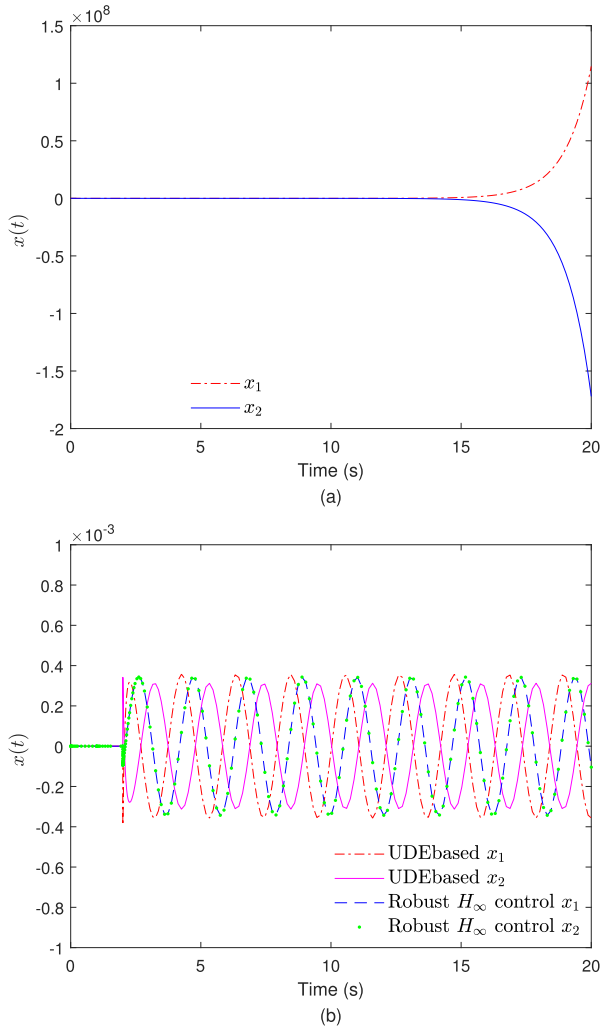where K_1, K_4 satisfy (44), (45) and Step 2.

FIGURE 2. Simulation results with model mismatch. (a) The state trajectories without control input. (b)The state trajectories under UDE-based control and Robust H_∞ control with $\gamma = 0.05$.
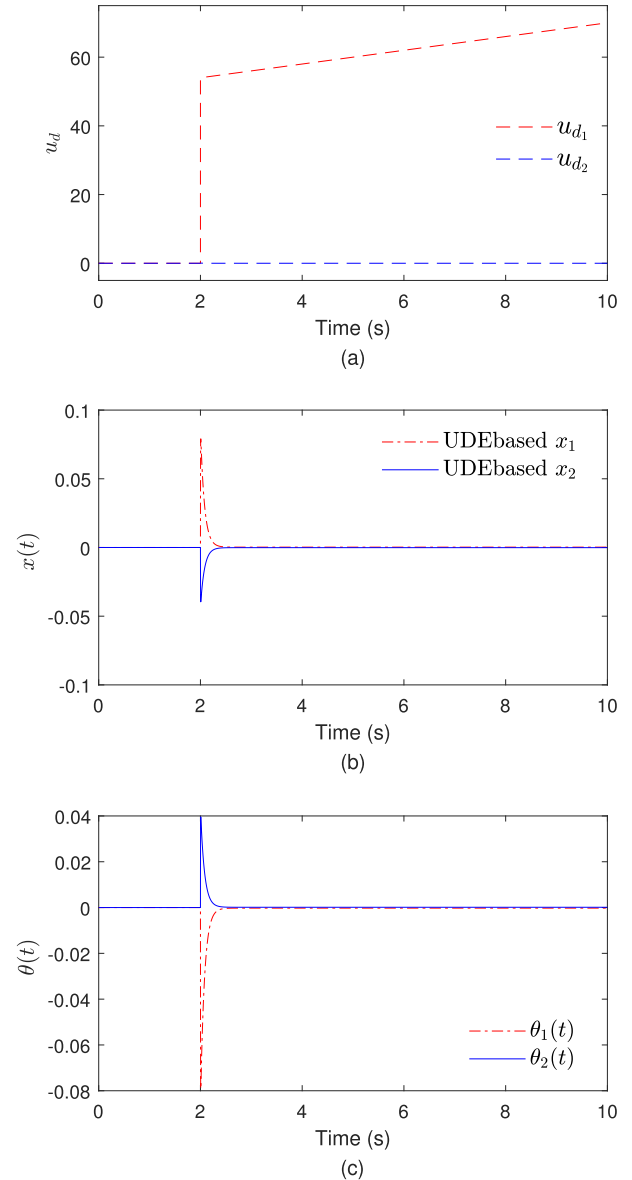

FIGURE 3. Simulation results with unbounded disturbance under UDE-based control. (a)The generalized disturbance with unbounded $w_1(t)$. (b)The state trajectories. (c)The tracking error $\theta(t)$.

where $T = 1/200\pi$ is small enough such that the bandwidth of filter can cover the spectrum of the generalized disturbance. The stability of singular system (1) under the UDE-based controller is discussed by choosing the choose feedback gain matrix as $K = \begin{bmatrix} -12 & 0 \\ -1 & 10 \end{bmatrix}$, which yields that $(E, A_r + K)$ is admissible. Moreover, conditon (28) is satisfied. Thus, from Theorem 1, it is clear that the closed-loop system is robustly stable.

The Robust H_∞ control method with $\gamma = 0.05$ proposed in [15] is taken as the comparison. The simulation results are depicted in Figure 2. It can be observed from Figure 2(a) that the open-loop system is unstable. As can be seen in Figure 2(b), a bounded external disturbance is imposed at $t = 2s$, $x(t)$ rapidly converges to 0, which means that both of these methods can be effective.

When unbounded disturbance $w_1(t) = 10 + 5t$ is imposed on the system (1), the simulation results under UDE-based control are shown in Figure 3. The generalized disturbance u_{d_1} is unbounded, the tracking error tends to zero, and the

controlled system is robustly stable, which means Theorem 1 is effective.

Compared with the work of [13], [15], for the norm-bounded uncertainties and unknown external disturbances, the UDE-based control method has the same robustness as the robust H_∞ method. However, when the unknown disturbances is added, the robust stability problem cannot be solved by the work of [13], [15], so that the method is also invalid for unbounded disturbance.

Example 2: Consider unstable singular system (1) with the following parameters:

$$E = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 6 & 7 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_2(t) \\ 0 \end{bmatrix}.$$

$w_2(t)$ is unknown disturbance, admissible reference model (2) is chosen as

$$A_r = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_r = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

where the compatible initial values are $Ex(0) = [0 \ \ 0]^T$, $Ex_r(0) = [0 \ \ 0]^T$. K is chosen as $\begin{bmatrix} 10 & 0 \\ 0 & 0.5 \end{bmatrix}$, it can be checked that $(E, A_r + K)$ is admissible.

Case 1: External disturbance and reference input are step signals

In this case, the uncertainty ΔA always exists in the simulation process, the disturbance signal $w_2(t) = 100$ is added at time $t = 2s$, command signal $r(t)$ steps from 0 to 1 at time $t = 4s$. The filter is selected as

$$G_{f1}(s) = \frac{1}{Ts + 1},$$

where $T = 1/200\pi$ is small enough such that the bandwidth of filter can cover the spectrum of the generalized disturbance. Hence, condition (37) is satisfied. By Theorem 2, the controlled system is robustly stable, and achieves asymptotic reference tracking and disturbance rejection.

The simulation results with matched model and tracking a step signal are shown in Figure 4. From Figure 4(a) and Figure 4(b), for $2s < t < 4s$, the state $x_1(t)$ and the tracking error e_1 are suddenly changed due to step disturbance, state $x_1(t)$ asymptotically tracks state $x_{r1}(t)$ and quickly converges to zero. At $t = 4s$, a step input $r(t)$ is put on the reference model, the states $x_1(t)$ and $x_{r1}(t)$ can track the reference signal. From Figure 4(c), the UDE estimator can accurately estimate the generalized disturbance, and the UDE-based controller can keep good robust performance.

Case 2: External disturbance and reference input are sine signals

In such case, in addition to the step signal, the sine signal with obvious periodic characteristics is the commonly used signal. In this case, the uncertainty ΔA and external disturbance $w_2(t) = 1.5 \sin(1.9\pi t + \pi/3) + 2t + 10$ always exist in the simulation process, the command signal $r(t)$ is a sine signal with a period of 2π and an amplitude of 1 added at time $t = 4s$. The filter is selected as a second-order filter

$$G_{f2}(s) = \frac{a_1 s + a_2 - \omega_0^2}{s^2 + a_1 s + a_2},$$

where $a_1 = 100\omega_0$, $a_2 = 100\omega_0^2$, $\omega_0 = 2\pi$. Therefore, from Theorem 2, it follows that condition (37) is satisfied.

The simulation results with matched model and tracking a sine signal are illustrated in Figure 5. As shown in Figure 5(a) and Figure 5(b), a sine input $r(t)$ is imposed on the reference model at $t = 4s$, $x(t)$ can still track the state $x_r(t)$ and reference input $r(t)$ accurately, and the tracking error closes to 0. Figure 5(c) shows that the uncertainties and external disturbances have always existed and the UDE estimator can accurately estimate them from the beginning to the end, meanwhile, the controller effectively suppresses
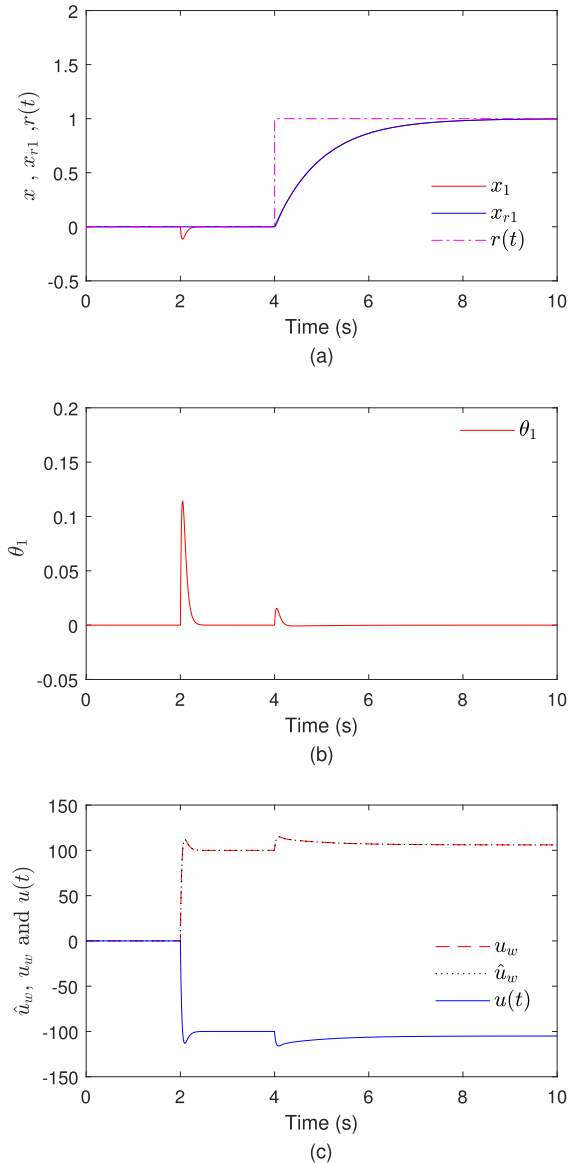

FIGURE 4. Simulation results with matched model and tracking a step signal. (a) State trajectories of x_1, x_{r1} and r. (b) Tracking error θ_1. (c) Control input u, the generalized disturbance u_w, and its estimation $\hat{u}_w$.

disturbances. Hence, for matched uncertainties, the proposed approach is effective.

Although the uncertainty ΔA in this example is bounded, its norm is relatively large, a suitable controller can be chosen so that closed-loop system is robustly stable. The conditions of Theorem 2 have no requirement on the norm of ΔA, which is different from the work of [13], [15].

Example 3: Singular system (1) with the following parameters from the literature [29]:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta A = 0, w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ 0 \end{bmatrix},$$
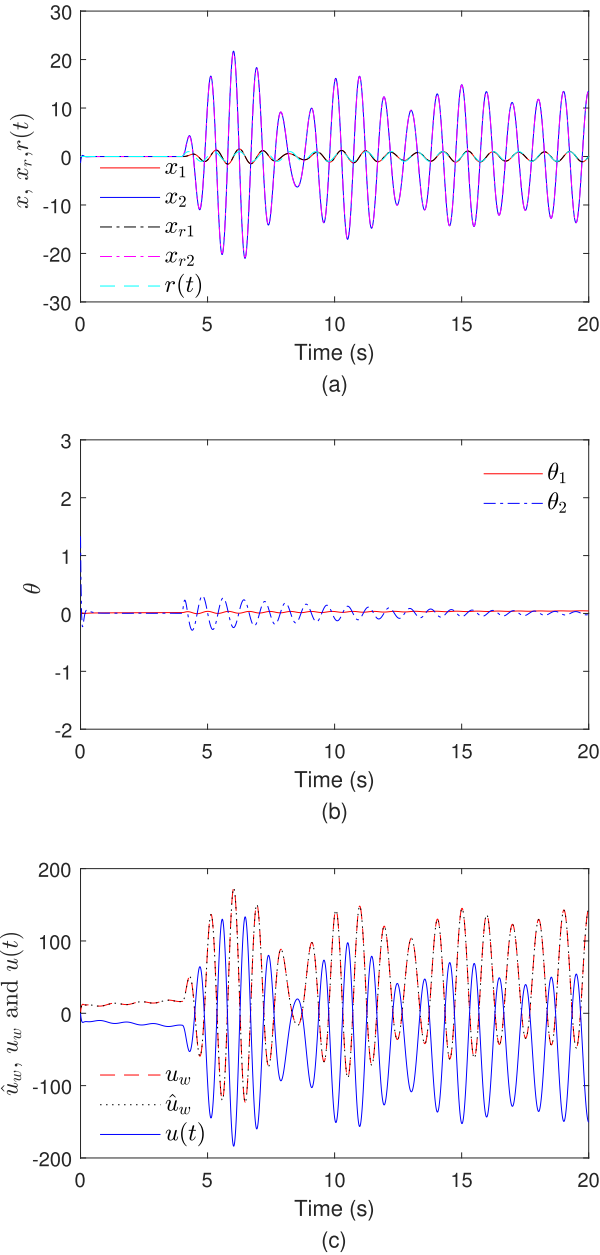
FIGURE 5. Simulation results with matched model and tracking a sine signal. (a) State trajectories of x and x_r. (b) Tracking error θ. (c) Control input u, the generalized disturbance u_w, and its estimation $\hat{u}_w$.

for comparison, the reference system (2) is chosen with the same form as the nominal system (1) as

$$A_r = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the command signal $r(t) = 0$. The disturbances at the initial time are $w_1(t) = 1 + 0.02\sin(0.1t) + 0.03\cos(0.1t)$ and $w_2(t) = -1 + 0.05\cos(0.1t)$, after 10 seconds, the disturbances changed to $w_1(t) = 2 + 0.02\sin(0.1t) + 0.03\cos(0.1t)$ and $w_2(t) = -2 + 0.05\cos(0.1t)$. The filter would be selected as
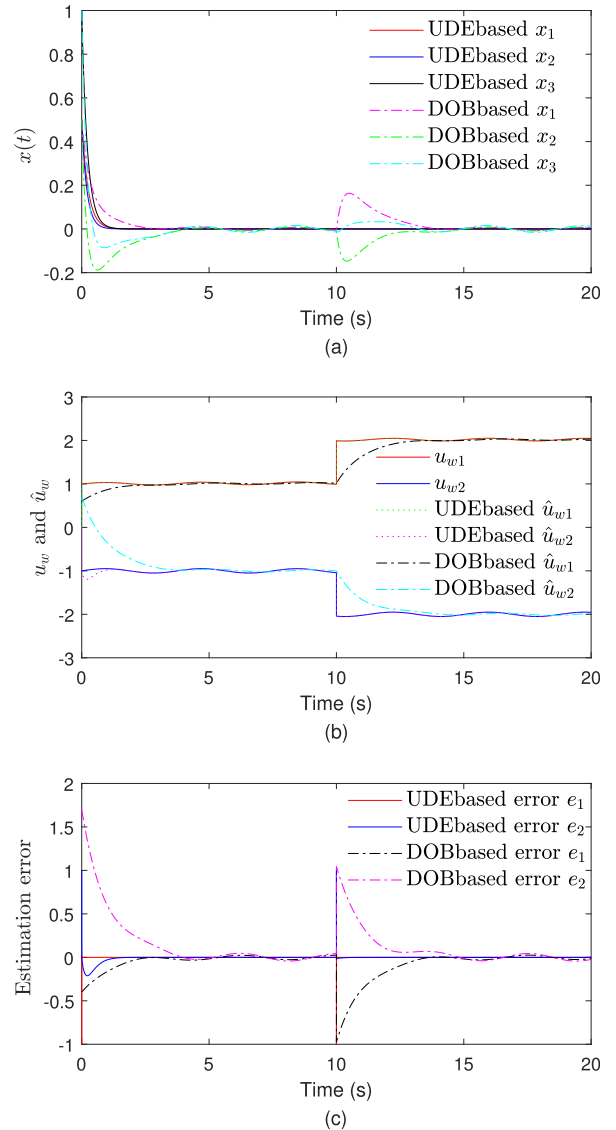
$$G_{f1}(s) = \frac{1}{Ts+1},$$

FIGURE 6. Simulation results based on UDE and DOB. (a) State trajectories. (b) the disturbance u_w and its estimation. (c) Estimation error.

where $T = 1/200\pi$ and K is chosen as 0, which yields that $(E, A_r + K)$ is admissible. Moreover, conditon (36) is satisfied. Thus, from Corollary 1, it is clear that the closed-loop system is robustly stable.

The DOB-based robust H_∞ control approach proposed in [29] is taken as the comparison, and system (1) takes the same initial value as [29], i.e., $Ex(0) = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix}^T$, $Ex_r(0) = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix}^T$. The simulation results based on UDE and DOB are shown in Figure 6. From Figure 6(a), it can be seen that the system states converge quickly under the both control methods, and the disturbances change at 10 seconds, the UDE-based method is less affected. As shown in Figure 6(b) and Figure 6(c), the UDE-based estimation is faster and the estimation error is smaller. Compared with the DOB-based robust H_∞ control approach, the UDE-based method has better anti-disturbance performance.
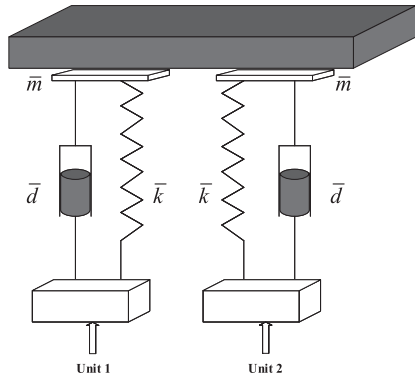
FIGURE 7. The model of each block in an MASS.

Example 4: In order to illustrate the applications of the UDE-based control approach, a practical application to the multi-agent supporting systems (MASSs) is given, which is from the work of [20]. It was shown that the MASSs can be applied to large phased array radar, earthquake damage-preventing buildings and aperture spherical radio telescope. An MASS is composed of a number of separate blocks, each supported by two pillars which can be referred to as Unit 1 and Unit 2, as illustrated in Figure 7. The damping coefficient, stiffness coefficient, and mass are denoted by $\bar{d}$, $\bar{k}$ and $\bar{m}$, respectively. On the ground of the above discussions, each block can be described by a singular system. Furthermore, it is essential to consider the uncertainty and external disturbance in such systems. The block model can be described by

$$E\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + w(t), \qquad (47)$$

where $x(t) = [x_{i1}(t) \quad v_{i1}(t) \quad x_{i2}(t) \quad v_{i2}(t)]^T$ is the state vector, $x_{i1}(t), v_{i1}(t), x_{i2}(t), v_{i2}(t)$ denote the heights and velocities of each Unit, respectively. $u(t)$ denotes control input, ΔA is model uncertainty, $w(t)$ is unknown disturbance. A and B denote the state matrix and control matrix, such as

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{\bar{k}}{\bar{m}} & -\dfrac{\bar{d}}{\bar{m}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$w(t) = \begin{bmatrix} 0 \\ w_3(t) \\ 0 \\ 0 \end{bmatrix}.$$

The system parameters are selected as $\bar{k} = 12$, $\bar{d} = -25$ and $\bar{m} = 16$. In the beginning, the block is disturbed by model internal uncertainty but is stable, after 1 seconds, the unbounded external disturbance
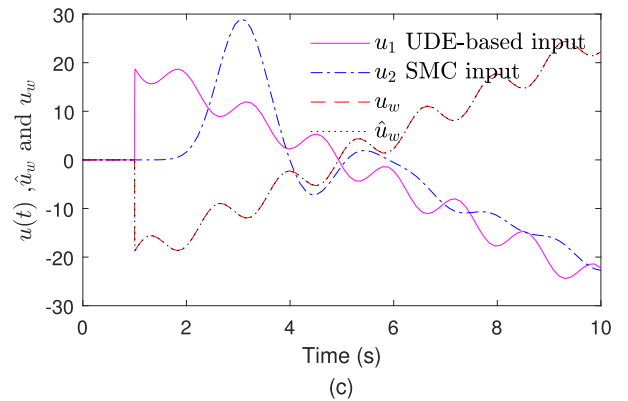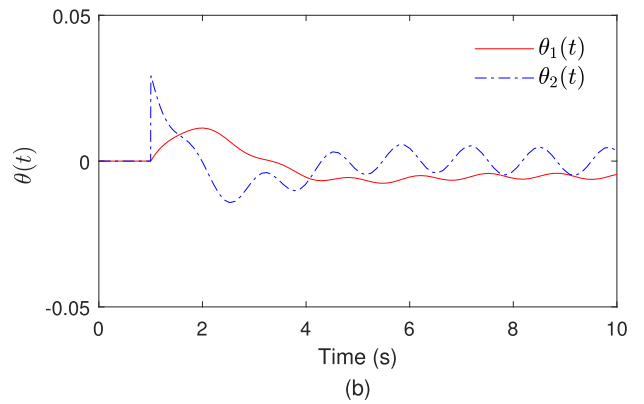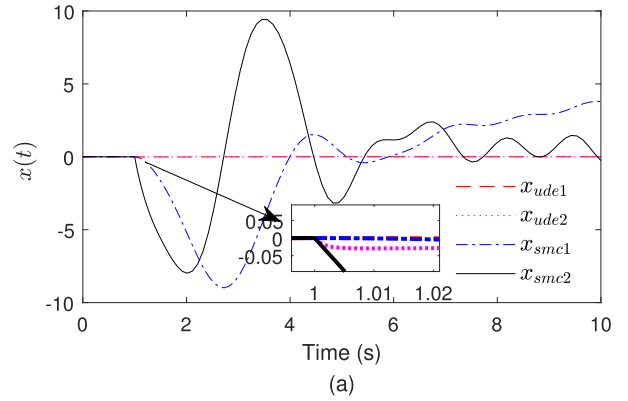

FIGURE 8. Simulation results. (a)State trajectories under the UDE-based controller and SMC controller. (b)Tracking error $\theta(t)$ under the UDE-based controller. (c)Control input, the generalized disturbance u_w and its estimation $\hat{u}_w$.

$w_3(t) = 3sin(1.5\pi t + 2) + 5(t - 5)$ is added. The reference system is given as

$$E\dot{x}_r(t) = A_r x_r(t) + B_r r(t), \qquad (48)$$

where A_r and B_r are given as

$$A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{3}{2} & -\dfrac{3}{2} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

On the basis of the actual requirements, the command input signal is chosen as $r(t) = 0$, and the asymptotic stability of the reference model can be ensured easily. K is chosen as 0, it follows that $(E, A_r + K)$ is admissible. Based on all coefficient matrices, obviously, constraint condition (8) is satisfied. The choice of filter is crucial to verify that stability condition (3) in Theorem 2, which can be selected as

$$G_{f1}(s) = \frac{1}{Ts + 1},$$

where $T = 1/200\pi$. By Theorem 2, condition (37) is satisfied. Given the compatible initial value $Ex(0) = [0\ 0\ 0\ 0]^T$ and $Ex_r(0) = [0\ 0\ 0\ 0]^T$.

The control approach proposed in [20] is taken as the comparison. As is illustrated in Figure 8(a), after the unbounded external disturbance is put into the system, the uncertain system can also be stabilized quickly under the UDE-based controller, while sliding mode controller is failed. From Figure 8(b), the tracking error $\theta(t)$ is very small. It is shown that the filter estimator presented in this paper can accurately estimate the generalized disturbance in Figure 8(c). It also demonstrates that the controller u_1 can effectively offset the impact of all uncertainties, which implies the effectiveness of the UDE-based control method for the unbounded disturbance.

Compared with the SMC method in [20], the uncertainties are required to be bounded and the external disturbances are not considered. However, when the unknown disturbances are added, the robust stability problem cannot be solved by the work of [20].

V. CONCLUSION

In this work, we adopt UDE-based robust control strategy to stabilize the singular systems with the structure uncertainty and external disturbance. The control law is proposed by using the spectrum information of sum term of all uncertainties and unknown disturbances to construct an estimator. The reference tracking and disturbance rejection are decoupled based on the 2DOF nature. Besides, it is then shown that sufficient conditions are presented to ensure that the uncertain singular systems are stable and achieve reference tracking and disturbance rejection. Moreover, an algorithm is identified for the error feedback control gain K. Finally, three numerical examples and a practical application are provided to show the effectiveness of the proposed methods.

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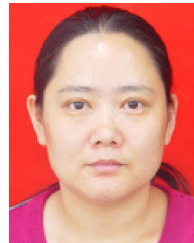
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