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Correlation Coefficients for Cubic Bipolar Fuzzy Sets With Applications to Pattern Recognition and Clustering Analysis

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ABSTRACT Cubic bipolar fuzzy set (CBFS) is a powerful model for dealing with bipolarity and vagueness altogether because it contains bipolar fuzzy information and interval-valued bipolar fuzzy information simultaneously. In this article, we define some new notions such as concentration, dilation, support and core of a CBFS. We introduce cubic bipolar fuzzy relations (CBFRs) and some of their types. As in statistics with real variables, we define variance and covariance between two CBFSs. Then, we propose correlation coefficients and their weighted extensions on the basis of variance and covariance of CBFSs. Later on, some properties of these correlation coefficients are discussed. We explore that their values lie in $[-1, 1]$. Moreover, we discuss the applications of the proposed correlation coefficients in pattern recognition and clustering analysis. Numerical examples are provided for better understanding of the applicability and efficiency of proposed correlation coefficients.

INDEX TERMS Cubic bipolar fuzzy sets, correlation coefficients, pattern recognition, clustering algorithm.

I. INTRODUCTION

Zadeh [1] initiated the concept of fuzzy set (FS) theory which is a generalization of crisp set theory. This theory was developed to address the inexactness and uncertainty that arise in decision-making problems as a result of ambiguities in data and human judgments. Till now, this theory has been successfully applied in various fields including medical science, economics, computer science, physics, and chemistry. Later on, many researchers inaugurated different extensions of fuzzy sets such as interval-valued fuzzy set (IVFS) [2], intuitionistic fuzzy set (IFS) [3], hesitant fuzzy set (HFS) [26], pythagorean fuzzy set (PFS) [5], [6], q-rung orthopair fuzzy set (q-ROPFS) [7], neutrosophic set (NS) [27], and single-valued neutrosophic set (SVNS) [28], etc.

In many real life circumstances, human perception is based on bipolar or dual-sided thoughts. For instance, effects

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and side effects, friendship and enmity, profit and loss are some examples of two-sided features of decision analysis. Zhang [8], [9] proposed the idea of bipolar fuzzy set (BFS) which unifies bipolarity and fuzziness. This set assigns each element a positive membership degree from $[0, 1]$ and a negative membership degree from $[-1, 0]$. These degrees indicate the extent to which an element satisfies a property as well as its counter-property. Lee [10] gave the operations on bipolar-valued fuzzy sets. Lee and Hur [43] defined bipolar fuzzy relations. Wei *et al.* [11] introduced the idea of interval-valued bipolar fuzzy set (IVBFS) and discussed multi-criteria decision-making (MCDM) under interval-valued bipolar fuzzy information. Deli *et al.* [29] gave the abstraction of bipolar neutrosophic set (BNS) and applied it in MCDM problems. Ulucay *et al.* [30] explored some similarity measures on BNSs and applied them in multi-criteria decision making. Basset *et al.* [31] defined cosine similarity measures on BNSs and used them in the diagnosis of bipolar disorders.

Jun et al. [4] brought out the concept of cubic set (CS) (hybrid of IVFS and FS) and defined the notions of P-union, P-intersection, R-union and R-intersection. They also defined internal cubic sets (ICSs) and external cubic sets (ECSs) and derived some useful results by taking into consideration both the ICSs and ECSs.

Correlation coefficient, an important notion in statistics, measures the linear relationship between two random variables. It is widely used in statistical analysis and engineering sciences. Since crisp set theory cannot tackle the ambiguities and uncertainties, therefore, the idea of correlation coefficient has been extended to FS theory for better results [45]. Later on, Gerstenkorn and Manko proposed a correlation coefficient for intuitionistic fuzzy sets whose values lie in [0,1]. Hung [33] adopted the statistical viewpoint to define a correlation coefficient for IFSs. Garg [16] proposed novel correlation coefficients for Pythagorean fuzzy sets and applied them in pattern recognition and medical diagnosis. Garg and Kaur [19] developed correlation coefficients for cubic intuitionistic fuzzy environment and discussed MCDM problems. Pramanik et al. [35] proposed a novel correlation coefficient for interval-valued bipolar neutrosophic set and applied it in multi-attribute decision making (MADM) problem. For more about pattern recognition and MADM, we refer to [36]–[40].

Riaz and Tehrim [14], Peng et al. [15], Basset et al. [31] initiated a novel model named as cubic bipolar fuzzy set (CBFS) which is a hybrid set of BFS and IVBFS. This model gives more precision and pliability as compared to the existing models because it accommodates bipolar and interval-valued bipolar fuzzy information simultaneously. As a result, this model offers maximum details about the occurrence of ratings, inexactness and bipolarity. They proposed some aggregation operators like cubic bipolar fuzzy weighted geometric (CBFWG) aggregation operators and cubic bipolar fuzzy weighted averaging (CBFWA) aggregation operators under P(R)-order and applied these operators in some multi-criteria group decision making (MCGDM) problems.

The main objectives and advantages of this article are listed below:

- 1) The first objective of this article is to handle vagueness and ambiguities more efficiently with the help of cubic bipolar fuzzy sets (CBFSs).
- 2) The second objective is to define new notions like concentration, dilation, support, core and binary relations for CBFSs.
- 3) The third objective is to develop correlation coefficients and their weighted versions for CBFSs.
- 4) The fourth objective is to propose new algorithms with the help of suggested correlation coefficients to solve complex problems of pattern recognition and clustering analysis under CBF environment. The usability and effectiveness of these algorithms is determined by numerical illustrations.

The rest of the article is structured as follows: In section 2, we review some basic definitions of fuzzy sets, interval-valued fuzzy sets, bipolar fuzzy sets, interval-valued

bipolar fuzzy sets. Then, we recall the definition and operations of CBFSs. In section 3, we propose concentration, dilation, support and core of a CBFS. Moreover, we inaugurate cubic bipolar fuzzy relations and some of their types. In section 4, we propose correlation measures and their weighted extensions and discuss their properties. In section 5, novel algorithms are presented for pattern recognition and clustering analysis on the basis of suggested correlation coefficients and the applicability of these algorithms is substantiated through numerical illustrations. Section 6 contains concluding remarks.

II. PRELIMINARIES

We devote this section to discuss some fundamental concepts related to cubic bipolar fuzzy sets that are helpful throughout this article.

Definition 1: [1] Let \mathfrak{M} be an initial universe. A fuzzy set \mathfrak{F} on \mathfrak{M} is defined as

$$\mathfrak{F} = \{ \langle m, \mu_{\mathfrak{F}}(m) \rangle : m \in \mathfrak{M} \}$$

where $\mu_{\mathfrak{F}} : \mathfrak{M} \rightarrow [0, 1]$ is described as membership function and $\mu_{\mathfrak{F}}(m)$ is called membership degree of an element $m \in \mathfrak{M}$.

Definition 2: [2] Let $I([0, 1])$ be the collection of all closed sub-intervals of [0,1]. An interval-valued fuzzy set \mathfrak{J} on initial universe \mathfrak{M} is an object having form

$$\mathfrak{J} = \{ \langle m, \mu_{\mathfrak{J}}(m) \rangle : m \in \mathfrak{M} \}$$

where $\mu_{\mathfrak{J}} : \mathfrak{M} \rightarrow I([0, 1])$ is described as interval-valued membership function and $\mu_{\mathfrak{J}}(m)$ is called interval-valued membership degree of an element $m \in \mathfrak{M}$.

Definition 3: [4] Let \mathfrak{M} be an initial universe. A cubic set \mathfrak{C} on \mathfrak{M} can be defined as

$$\mathfrak{C} = \{ \langle m, \mathfrak{J}(m), \mathfrak{F}(m) \rangle : m \in \mathfrak{M} \}$$

where \mathfrak{J} is an IVFS on \mathfrak{M} and \mathfrak{F} is a FS on \mathfrak{M} .

Definition 4: [8] A bipolar fuzzy set \mathfrak{B} on \mathfrak{M} takes the following form

$$\mathfrak{B} = \{ \langle m, \mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{B}}^-(m) \rangle : m \in \mathfrak{M} \}$$

Here, each element $m \in \mathfrak{M}$ is assigned with a positive membership degree $\mu_{\mathfrak{B}}^+(m) \in [0, 1]$ and a negative membership degree $\mu_{\mathfrak{B}}^-(m) \in [-1, 0]$. For convenience, an ordered pair $\langle \mu_{\mathfrak{B}}^+, \mu_{\mathfrak{B}}^- \rangle$ is termed as bipolar fuzzy number (BFN).

Definition 5: [11] Let $I([0, 1])$ be the collection of all closed sub-intervals of [0,1] and $I^*([-1, 0])$ be the collection of all closed sub-intervals of [-1,0]. Then, an interval-valued bipolar fuzzy set \mathfrak{S} is expressed as

$$\mathfrak{S} = \{ \langle m, \mu_{\mathfrak{S}}^+(m), \mu_{\mathfrak{S}}^-(m) \rangle : m \in \mathfrak{M} \}$$

where $\mu_{\mathfrak{S}}^+(m) \in I([0, 1])$ depicts interval-valued positive membership degree and $\mu_{\mathfrak{S}}^-(m) \in I^*([-1, 0])$ depicts interval-valued negative membership degree of an element $m \in \mathfrak{M}$. An interval-valued bipolar fuzzy number (IVBFN) can be represented as $\langle [\mu_{\ell\mathfrak{S}}^+, \mu_{u\mathfrak{S}}^+], [\mu_{\ell\mathfrak{S}}^-, \mu_{u\mathfrak{S}}^-] \rangle$.

Definition 6: [13] A cubic bipolar fuzzy set \mathfrak{A} over the initial universe \mathfrak{M} can be defined as

$$\mathfrak{A} = \{ \langle m, \mathfrak{S}(m), \mathfrak{B}(m) \rangle : m \in \mathfrak{M} \}$$

where \mathfrak{S} is an IVBFS and \mathfrak{B} is a BFS on \mathfrak{M} . Thus, cubic bipolar fuzzy set can also be written as

$$\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m) \rangle : m \in \mathfrak{M} \}$$

where the intervals $[\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] \in I([0, 1])$ and $[\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] \in I^*([-1, 0])$ represent the interval-valued positive and negative membership degrees, respectively and $\mu_{\mathfrak{A}}^+(m) \in [0, 1]$ and $\mu_{\mathfrak{A}}^-(m) \in [-1, 0]$ represent the positive and negative membership, respectively, of an element $m \in \mathfrak{M}$. A cubic bipolar fuzzy number can be written as $\langle [\mu_{\ell\mathfrak{A}}^+, \mu_{u\mathfrak{A}}^+], [\mu_{\ell\mathfrak{A}}^-, \mu_{u\mathfrak{A}}^-], \mu_{\mathfrak{A}}^+, \mu_{\mathfrak{A}}^- \rangle$.

Definition 7: [13] Consider an initial universe \mathfrak{M} . Let $\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m) \rangle : m \in \mathfrak{M} \}$ and $\mathfrak{B} = \{ \langle m, [\mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{B}}^+(m)], [\mu_{\ell\mathfrak{B}}^-(m), \mu_{u\mathfrak{B}}^-(m)], \mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{B}}^-(m) \rangle : m \in \mathfrak{M} \}$ be two CBFs on \mathfrak{M} and $\lambda > 0$. Then, the operations on CBFs under P-order are given below:

- i. $\mathfrak{A} \cup_P \mathfrak{B} = \{ \langle m, [\max\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \max\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}], [\min\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, \min\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\}], \{\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \}$
- ii. $\mathfrak{A} \cap_P \mathfrak{B} = \{ \langle m, [\min\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \min\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}], [\max\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, \max\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\}], \{\min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \}$
- iii. $\mathfrak{A} \oplus_P \mathfrak{B} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m) + \mu_{\ell\mathfrak{B}}^+(m) - \mu_{\ell\mathfrak{A}}^-(m) \mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{A}}^+(m) + \mu_{u\mathfrak{B}}^+(m) - \mu_{u\mathfrak{A}}^-(m) \mu_{u\mathfrak{B}}^+(m)], [-(\mu_{\ell\mathfrak{A}}^-(m) \mu_{\ell\mathfrak{B}}^-(m)), -(\mu_{u\mathfrak{A}}^-(m) \mu_{u\mathfrak{B}}^-(m))], \mu_{\mathfrak{A}}^+(m) + \mu_{\mathfrak{B}}^+(m) - \mu_{\mathfrak{A}}^-(m) \mu_{\mathfrak{B}}^+(m), -(\mu_{\mathfrak{A}}^-(m) \mu_{\mathfrak{B}}^-(m)) \rangle : m \in \mathfrak{M} \}$
- iv. $\mathfrak{A} \otimes_P \mathfrak{B} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m) \mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{A}}^+(m) \mu_{u\mathfrak{B}}^+(m)], [-(\mu_{\ell\mathfrak{A}}^-(m) - \mu_{\ell\mathfrak{B}}^-(m) - (\mu_{\ell\mathfrak{A}}^-(m) \mu_{\ell\mathfrak{B}}^-(m))), -(\mu_{u\mathfrak{A}}^-(m) - \mu_{u\mathfrak{B}}^-(m) - (\mu_{u\mathfrak{A}}^-(m) \mu_{u\mathfrak{B}}^-(m))], \mu_{\mathfrak{A}}^+(m) \mu_{\mathfrak{B}}^+(m), -(\mu_{\mathfrak{A}}^-(m) - \mu_{\mathfrak{B}}^-(m) - (\mu_{\mathfrak{A}}^-(m) \mu_{\mathfrak{B}}^-(m))) \rangle : m \in \mathfrak{M} \}$
- v. $\mathfrak{A}^\lambda = \{ \langle m, [(\mu_{\ell\mathfrak{A}}^+(m))^\lambda, (\mu_{u\mathfrak{A}}^+(m))^\lambda], [-(1 - (1 - (\mu_{\ell\mathfrak{A}}^-(m)))^\lambda), -(1 - (1 - (\mu_{u\mathfrak{A}}^-(m)))^\lambda)], \{(\mu_{\mathfrak{A}}^+(m))^\lambda, -(1 - (1 - (\mu_{\mathfrak{A}}^-(m)))^\lambda)\} \rangle : m \in \mathfrak{M} \}$
- vi. $\lambda\mathfrak{A} = \{ \langle m, [1 - (1 - \mu_{\ell\mathfrak{A}}^+(m))^\lambda, 1 - (1 - \mu_{u\mathfrak{A}}^+(m))^\lambda], [-(\mu_{\ell\mathfrak{A}}^-(m))^\lambda, -(\mu_{u\mathfrak{A}}^-(m))^\lambda], \{1 - (1 - \mu_{\mathfrak{A}}^+(m))^\lambda, -(1 - (1 - \mu_{\mathfrak{A}}^-(m))^\lambda)\} \rangle : m \in \mathfrak{M} \}$
- vii. $\mathfrak{A} \subseteq_P \mathfrak{B}$ if $[\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] \leq [\mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{B}}^+(m)]$ and $[\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] \geq [\mu_{\ell\mathfrak{B}}^-(m), \mu_{u\mathfrak{B}}^-(m)]$ and

$$\mu_{\mathfrak{A}}^+(m) \leq \mu_{\mathfrak{B}}^+(m) \text{ and } \mu_{\mathfrak{A}}^-(m) \geq \mu_{\mathfrak{B}}^-(m), \forall m \in \mathfrak{M}.$$

Definition 8: [13] Let $\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m) \rangle : m \in \mathfrak{M} \}$ and $\mathfrak{B} = \{ \langle m, [\mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{B}}^+(m)], [\mu_{\ell\mathfrak{B}}^-(m), \mu_{u\mathfrak{B}}^-(m)], \mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{B}}^-(m) \rangle : m \in \mathfrak{M} \}$ be two CBFs on \mathfrak{M} and $\lambda > 0$. Then, the operations on CBFs under R-order are given below:

- i. $\mathfrak{A} \cup_R \mathfrak{B} = \{ \langle m, [\max\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \max\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}], [\min\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, \min\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\}], \{\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \}$
- ii. $\mathfrak{A} \cap_R \mathfrak{B} = \{ \langle m, [\min\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \min\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}], [\max\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, \max\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\}], \{\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \}$
- iii. $\mathfrak{A} \oplus_R \mathfrak{B} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m) + \mu_{\ell\mathfrak{B}}^+(m) - \mu_{\ell\mathfrak{A}}^-(m) \mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{A}}^+(m) + \mu_{u\mathfrak{B}}^+(m) - \mu_{u\mathfrak{A}}^-(m) \mu_{u\mathfrak{B}}^+(m)], [-(\mu_{\ell\mathfrak{A}}^-(m) \mu_{\ell\mathfrak{B}}^-(m)), -(\mu_{u\mathfrak{A}}^-(m) \mu_{u\mathfrak{B}}^-(m))], \mu_{\mathfrak{A}}^+(m) \mu_{\mathfrak{B}}^+(m), -(\mu_{\mathfrak{A}}^-(m) - \mu_{\mathfrak{B}}^-(m) - (\mu_{\mathfrak{A}}^-(m) \mu_{\mathfrak{B}}^-(m))) \rangle : m \in \mathfrak{M} \}$
- iv. $\mathfrak{A} \otimes_R \mathfrak{B} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m) \mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{A}}^+(m) \mu_{u\mathfrak{B}}^+(m)], [-(\mu_{\ell\mathfrak{A}}^-(m) - \mu_{\ell\mathfrak{B}}^-(m) - (\mu_{\ell\mathfrak{A}}^-(m) \mu_{\ell\mathfrak{B}}^-(m))), -(\mu_{u\mathfrak{A}}^-(m) - \mu_{u\mathfrak{B}}^-(m) - (\mu_{u\mathfrak{A}}^-(m) \mu_{u\mathfrak{B}}^-(m))], \mu_{\mathfrak{A}}^+(m) + \mu_{\mathfrak{B}}^+(m) - \mu_{\mathfrak{A}}^-(m) \mu_{\mathfrak{B}}^+(m), -(\mu_{\mathfrak{A}}^-(m) \mu_{\mathfrak{B}}^-(m)) \rangle : m \in \mathfrak{M} \}$
- v. $\mathfrak{A}^\lambda = \{ \langle m, [(\mu_{\ell\mathfrak{A}}^+(m))^\lambda, (\mu_{u\mathfrak{A}}^+(m))^\lambda], [-(1 - (1 - (\mu_{\ell\mathfrak{A}}^-(m)))^\lambda), -(1 - (1 - (\mu_{u\mathfrak{A}}^-(m)))^\lambda)], \{1 - (1 - \mu_{\mathfrak{A}}^+(m))^\lambda, -(1 - (1 - (\mu_{\mathfrak{A}}^-(m)))^\lambda)\} \rangle : m \in \mathfrak{M} \}$
- vi. $\lambda\mathfrak{A} = \{ \langle m, [1 - (1 - \mu_{\ell\mathfrak{A}}^+(m))^\lambda, 1 - (1 - \mu_{u\mathfrak{A}}^+(m))^\lambda], [-(\mu_{\ell\mathfrak{A}}^-(m))^\lambda, -(\mu_{u\mathfrak{A}}^-(m))^\lambda], \{(\mu_{\mathfrak{A}}^+(m))^\lambda, -(1 - (1 - (\mu_{\mathfrak{A}}^-(m)))^\lambda)\} \rangle : m \in \mathfrak{M} \}$
- vii. $\mathfrak{A} \subseteq_R \mathfrak{B}$ if $[\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] \leq [\mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{B}}^+(m)]$ and $[\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] \geq [\mu_{\ell\mathfrak{B}}^-(m), \mu_{u\mathfrak{B}}^-(m)]$ and $\mu_{\mathfrak{A}}^+(m) \geq \mu_{\mathfrak{B}}^+(m)$ and $\mu_{\mathfrak{A}}^-(m) \leq \mu_{\mathfrak{B}}^-(m) \forall m \in \mathfrak{M}$.

Definition 9: [13] Let $\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m) \rangle : m \in \mathfrak{M} \}$ be a CBFs on \mathfrak{M} . The complement of \mathfrak{A} can be defined as $\mathfrak{A}^c = \{ \langle m, [1 - \mu_{\ell\mathfrak{A}}^+(m), 1 - \mu_{u\mathfrak{A}}^+(m)], [-1 + (\mu_{\ell\mathfrak{A}}^-(m)), -1 + (\mu_{u\mathfrak{A}}^-(m))], [1 - \mu_{\mathfrak{A}}^+(m), -1 + (\mu_{\mathfrak{A}}^-(m)) \rangle : m \in \mathfrak{M} \}$

III. SOME NOVEL CONCEPTS OF CBFs

In this section, we propose some important concepts including concentration, dilation, support and core of a cubic bipolar fuzzy set. Moreover, we introduce cubic bipolar fuzzy relations and discuss some of their types.

Definition 10: Let $\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m) \rangle : m \in \mathfrak{M} \}$ be a CBFs on

initial universe \mathfrak{M} . Referring to definition 7(part v) and taking $\lambda = 2$ in it, we obtain

$$\mathfrak{A}^2 = \{ \langle m, [(\mu_{\ell\mathfrak{A}}^+(m))^2, (\mu_{u\mathfrak{A}}^+(m))^2], [-(1 - (1 - (-\mu_{\ell\mathfrak{A}}^-(m)))^2), -(1 - (1 - (-\mu_{u\mathfrak{A}}^-(m)))^2)], \{(\mu_{\mathfrak{A}}^+(m))^2, -(1 - (1 - (-\mu_{\mathfrak{A}}^-(m)))^2)\} : m \in \mathfrak{M} \}$$

Then, \mathfrak{A}^2 is termed as P-concentration of \mathfrak{A} and denoted by $CON_P(\mathfrak{A})$.

Example 1: Let $\mathfrak{M} = \{m_1, m_2, m_3\}$ be initial universe and let

$$\mathfrak{A} = \left\{ \begin{array}{l} \langle m_1, [0.16, 0.28], [-0.69, -0.53], \{0.36, -0.61\} \rangle, \\ \langle m_2, [0.32, 0.46], [-0.71, -0.63], \{0.42, -0.56\} \rangle, \\ \langle m_3, [0.66, 0.78], [-0.36, -0.22], \{0.51, -0.44\} \rangle \end{array} \right\}$$

be a CBFS on \mathfrak{M} . Then, P-concentration of \mathfrak{A} is given as follows

$$CON_P(\mathfrak{A}) = \left\{ \begin{array}{l} \langle m_1, [0.025, 0.078], [-0.903, -0.779], \\ \quad \{0.129, -0.847\} \rangle, \\ \langle m_2, [0.102, 0.211], [-0.915, -0.863], \\ \quad \{0.176, -0.806\} \rangle, \\ \langle m_3, [0.435, 0.608], [-0.590, -0.391], \\ \quad \{0.260, -0.686\} \rangle \end{array} \right\}$$

Definition 11: Let $\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m)\} : m \in \mathfrak{M} \}$ be a CBFS on initial universe \mathfrak{M} . Referring to definition 8(part v) and taking $\lambda = 2$ in it, we obtain

$$\mathfrak{A}^2 = \{ \langle m, [(\mu_{\ell\mathfrak{A}}^+(m))^2, (\mu_{u\mathfrak{A}}^+(m))^2], [-(1 - (1 - (-\mu_{\ell\mathfrak{A}}^-(m)))^2), -(1 - (1 - (-\mu_{u\mathfrak{A}}^-(m)))^2)], \{1 - (1 - \mu_{\mathfrak{A}}^+(m))^2, -(-\mu_{\mathfrak{A}}^-(m))^2\} : m \in \mathfrak{M} \}$$

Then, \mathfrak{A}^2 is termed as R-concentration of \mathfrak{A} and denoted by $CON_R(\mathfrak{A})$.

Example 2: For the CBFS \mathfrak{A} given in example 1, the R-concentration is given as follows

$$CON_R(\mathfrak{A}) = \left\{ \begin{array}{l} \langle m_1, [0.025, 0.078], [-0.903, -0.779], \\ \quad \{0.590, -0.372\} \rangle, \\ \langle m_2, [0.102, 0.211], [-0.915, -0.863], \\ \quad \{0.663, -0.313\} \rangle, \\ \langle m_3, [0.435, 0.608], [-0.590, -0.391], \\ \quad \{0.759, -0.193\} \rangle \end{array} \right\}$$

Definition 12: Let $\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m)\} : m \in \mathfrak{M} \}$ be a CBFS on initial universe \mathfrak{M} . If we take $\lambda = \frac{1}{2}$ in definition 7(part v), we get

$$\mathfrak{A}^{\frac{1}{2}} = \{ \langle m, [(\mu_{\ell\mathfrak{A}}^+(m))^{\frac{1}{2}}, (\mu_{u\mathfrak{A}}^+(m))^{\frac{1}{2}}], [-(1 - (1 - (-\mu_{\ell\mathfrak{A}}^-(m)))^{\frac{1}{2}}), -(1 - (1 - (-\mu_{u\mathfrak{A}}^-(m)))^{\frac{1}{2}})], \{(\mu_{\mathfrak{A}}^+(m))^{\frac{1}{2}}, -(1 - (1 - (-\mu_{\mathfrak{A}}^-(m)))^{\frac{1}{2}})\} : m \in \mathfrak{M} \}$$

Then, $\mathfrak{A}^{\frac{1}{2}}$ is called P-dilation of \mathfrak{A} and denoted by $DIL_P(\mathfrak{A})$.

Example 3: Consider the same CBFS \mathfrak{A} given in example 1. Then, P-dilation of \mathfrak{A} is given as follows

$$DIL_P(\mathfrak{A}) = \left\{ \begin{array}{l} \langle m_1, [0.4, 0.529], [-0.443, -0.314], \\ \quad \{0.6, -0.375\} \rangle, \\ \langle m_2, [0.565, 0.678], [-0.461, -0.391], \\ \quad \{0.648, -0.336\} \rangle, \\ \langle m_3, [0.812, 0.883], [-0.2, -0.116], \\ \quad \{0.714, -0.251\} \rangle \end{array} \right\}$$

Definition 13: Let $\mathfrak{A} = \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m)\} : m \in \mathfrak{M} \}$ be a CBFS on initial universe \mathfrak{M} . If we take $\lambda = \frac{1}{2}$ in definition 8(part v), we get

$$\mathfrak{A}^{\frac{1}{2}} = \{ \langle m, [(\mu_{\ell\mathfrak{A}}^+(m))^{\frac{1}{2}}, (\mu_{u\mathfrak{A}}^+(m))^{\frac{1}{2}}], [-(1 - (1 - (-\mu_{\ell\mathfrak{A}}^-(m)))^{\frac{1}{2}}), -(1 - (1 - (-\mu_{u\mathfrak{A}}^-(m)))^{\frac{1}{2}})], \{1 - (1 - \mu_{\mathfrak{A}}^+(m))^{\frac{1}{2}}, -(-\mu_{\mathfrak{A}}^-(m))^{\frac{1}{2}}\} : m \in \mathfrak{M} \}$$

Then, $\mathfrak{A}^{\frac{1}{2}}$ is called R-dilation of \mathfrak{A} and denoted by $DIL_R(\mathfrak{A})$.

Example 4: The R-dilation of CBFS \mathfrak{A} taken from example 1 is given as follows

$$DIL_R(\mathfrak{A}) = \left\{ \begin{array}{l} \langle m_1, [0.4, 0.529], [-0.443, -0.314], \\ \quad \{0.2, -0.781\} \rangle, \\ \langle m_2, [0.565, 0.678], [-0.461, -0.391], \\ \quad \{0.238, -0.748\} \rangle, \\ \langle m_3, [0.812, 0.883], [-0.2, -0.116], \\ \quad \{0.3, -0.663\} \rangle \end{array} \right\}$$

Definition 14: Let \mathfrak{A} be a CBFS on \mathfrak{M} . The support of \mathfrak{A} , denoted by $Supp(\mathfrak{A})$, can be defined by taking into consideration both the positive and negative membership degrees separately. Therefore, it is expressed as union of positive support ($Supp^+(\mathfrak{A})$) and negative support ($Supp^-(\mathfrak{A})$), i.e.,

$$Supp(\mathfrak{A}) = Supp^+(\mathfrak{A}) \cup Supp^-(\mathfrak{A})$$

where

$$Supp^+(\mathfrak{A}) = \{ m \in \mathfrak{M} : [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] \neq [0, 0] \text{ and } \mu_{\mathfrak{A}}^+(m) \neq 0 \}$$

$$Supp^-(\mathfrak{A}) = \{ m \in \mathfrak{M} : [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] \neq [0, 0] \text{ and } \mu_{\mathfrak{A}}^-(m) \neq 0 \}$$

Clearly, $Supp(\mathfrak{A})$ is a crisp set.

Example 5: Let

$$\mathfrak{A} = \left\{ \begin{array}{l} \langle m_1, [0.12, 0.29], [-0.78, -0.61], \{0, -0.59\} \rangle, \\ \langle m_2, [0, 0], [-0.89, -0.73], \{0.67, 0\} \rangle, \\ \langle m_3, [0.68, 0.74], [-0.23, -0.12], \{0.66, -0.38\} \rangle, \\ \langle m_4, [0.19, 0.27], [0, 0], \{0.88, -1\} \rangle, \\ \langle m_5, [0, 0.91], [-0.57, 0], \{0.62, -0.78\} \rangle \end{array} \right\}$$

be a CBFS on \mathfrak{M} . Then, $Supp^+(\mathfrak{A}) = \{c, d, e\}$ and $Supp^-(\mathfrak{A}) = \{a, c, e\}$ so, $Supp(\mathfrak{A}) = \{a, c, d, e\}$.

Definition 15: Let \mathfrak{A} be a CBFS on \mathfrak{M} . Then, the upper core of \mathfrak{A} is defined as the huddle of all those elements of \mathfrak{M} for which $[\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] = [1, 1]$ and $\mu_{\mathfrak{A}}^+(m) = 1$. Mathematically,

$$c^U = \{ m \in \mathfrak{M} : [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] = [1, 1] \text{ and } \mu_{\mathfrak{A}}^+(m) = 1 \}$$

Likewise, the lower core is the huddle of all those elements of \mathfrak{M} for which $[\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] = [-1, -1]$ and $\mu_{\mathfrak{A}}^-(m) = -1$, i.e.,

$$c^L = \{ m \in \mathfrak{M} : [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] = [-1, -1] \text{ and } \mu_{\mathfrak{A}}^-(m) = -1 \}$$

Example 6: Let

$$\mathfrak{A} = \left\{ \begin{array}{l} \langle m_1, [0.67, 0.73], [-1, -1], \{0.59, -1\} \rangle, \\ \langle m_2, [1, 1], [-0.73, -0.57], \{1, -0.88\} \rangle, \\ \langle m_3, [0.17, 0.39], [-1, -1], \{0.12, -0.59\} \rangle, \\ \langle m_4, [0.9, 1], [-0.69, -0.52], \{1, -1\} \rangle, \\ \langle m_5, [1, 1], [-1, -1], \{0.92, -1\} \rangle \end{array} \right\}$$

be a CBFS on \mathfrak{M} . Then, $c^U = \{b\}$ and $c^L = \{a, e\}$.

Definition 16: Let \mathfrak{M} be initial universe. A cubic bipolar fuzzy set on \mathfrak{M} in which $[\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] = [0, 0]$, $[\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] = [0, 0]$, and $\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m)\} = \{0, 0\}$ for all $m \in \mathfrak{M}$ is called null cubic bipolar fuzzy set. It is denoted by Φ .

Definition 17: Let \mathfrak{M} be initial universe. A cubic bipolar fuzzy set on \mathfrak{M} in which $[\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)] = [1, 1]$, $[\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)] = [-1, -1]$, and $\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m)\} = \{1, -1\}$ for all $m \in \mathfrak{M}$ is called absolute cubic bipolar fuzzy set. It is denoted by \tilde{A} .

Remark 1: The two well-known laws of crisp set theory, named as law of contradiction and law of excluded middle, do not hold in cubic bipolar fuzzy theory. That is,

- (i) $\mathfrak{A} \cap_P \mathfrak{A}^c \neq \Phi$
- (ii) $\mathfrak{A} \cup_P \mathfrak{A}^c \neq \tilde{A}$
- (iii) $\mathfrak{A} \cap_R \mathfrak{A}^c \neq \Phi$
- (iv) $\mathfrak{A} \cup_R \mathfrak{A}^c \neq \tilde{A}$

Example 7: Let

$$\mathfrak{A} = \left\{ \begin{array}{l} \langle m_1, [0.23, 0.47], [-0.12, -0.06], \{0.08, -0.29\} \rangle, \\ \langle m_2, [0.59, 0.67], [-0.48, -0.31], \{0.26, -0.31\} \rangle, \\ \langle m_3, [0, 0.9], [-1, -0.8], \{0, -0.2\} \rangle \end{array} \right\}$$

be a CBFS on $\mathfrak{M} = \{m_1, m_2, m_3\}$. Then,

$$\mathfrak{A}^c = \left\{ \begin{array}{l} \langle m_1, [0.53, 0.77], [-0.94, -0.88], \{0.92, -0.71\} \rangle, \\ \langle m_2, [0.33, 0.41], [-0.69, -0.52], \{0.74, -0.69\} \rangle, \\ \langle m_3, [0.1, 1], [-0.2, 0], \{1, -0.8\} \rangle \end{array} \right\}$$

Now,

$$\mathfrak{A} \cap_P \mathfrak{A}^c = \left\{ \begin{array}{l} \langle m_1, [0.23, 0.47], [-0.12, -0.06], \\ \quad \{0.08, -0.29\} \rangle, \\ \langle m_2, [0.33, 0.41], [-0.48, -0.31], \\ \quad \{0.26, -0.31\} \rangle, \\ \langle m_3, [0, 0.9], [-0.2, 0], \\ \quad \{0, -0.2\} \rangle \end{array} \right\} \neq \Phi$$

and

$$\mathfrak{A} \cup_P \mathfrak{A}^c = \left\{ \begin{array}{l} \langle m_1, [0.53, 0.77], [-0.94, -0.88], \\ \quad \{0.92, -0.71\} \rangle, \\ \langle m_2, [0.59, 0.67], [-0.69, -0.52], \\ \quad \{0.74, -0.69\} \rangle, \\ \langle m_3, [0.1, 1], [-1, -0.8], \\ \quad \{1, -0.8\} \rangle \end{array} \right\} \neq \tilde{A}$$

If we replace P-order by R-order in the above example, we can see that law of contradiction and excluded middle still do not hold.

Theorem 1: Let $\mathfrak{A} = \{\langle m, [\mu_{\ell\mathfrak{A}}^+(m), \mu_{u\mathfrak{A}}^+(m)], [\mu_{\ell\mathfrak{A}}^-(m), \mu_{u\mathfrak{A}}^-(m)], \{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m)\} : m \in \mathfrak{M}\}$ and $\mathfrak{B} = \{\langle m, [\mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{B}}^+(m)], [\mu_{\ell\mathfrak{B}}^-(m), \mu_{u\mathfrak{B}}^-(m)], \{\mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{B}}^-(m)\} : m \in \mathfrak{M}\}$ be two CBFSs on \mathfrak{M} . Then,

- i. $(\mathfrak{A} \cup_P \mathfrak{B}) \oplus_P (\mathfrak{A} \cap_P \mathfrak{B}) = \mathfrak{A} \oplus_P \mathfrak{B}$;
- ii. $(\mathfrak{A} \cup_P \mathfrak{B}) \otimes_P (\mathfrak{A} \cap_P \mathfrak{B}) = \mathfrak{A} \otimes_P \mathfrak{B}$;
- iii. $(\mathfrak{A} \cup_R \mathfrak{B}) \oplus_R (\mathfrak{A} \cap_R \mathfrak{B}) = \mathfrak{A} \oplus_R \mathfrak{B}$;
- iv. $(\mathfrak{A} \cup_R \mathfrak{B}) \otimes_R (\mathfrak{A} \cap_R \mathfrak{B}) = \mathfrak{A} \otimes_R \mathfrak{B}$.

Proof: We prove (i)-(ii), and (iii)-(iv) can be proved analogously.

- (i). $(\mathfrak{A} \cup_P \mathfrak{B}) \oplus_P (\mathfrak{A} \cap_P \mathfrak{B}) = \{ \langle m, [\max\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \max\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}] \{ \max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} \} : m \in \mathfrak{M} \} \oplus \{ \langle m, [\min\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \min\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}] \{ \max\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, \max\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\} \} : m \in \mathfrak{M} \} \}$
 $= \{ \langle m, [\max\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\} + \min\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\} - \max\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\} \min\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \max\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\} + \min\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\} - \max\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\} \min\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}] \{ -\min\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\} \max\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, -\min\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\} \max\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\} \} \{ \max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\} + \min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\} - \max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\} \min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, -\min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} \max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} \} : m \in \mathfrak{M} \}$
 $= \{ \langle m, [\mu_{\ell\mathfrak{A}}^+(m) + \mu_{\ell\mathfrak{B}}^+(m) - \mu_{\ell\mathfrak{A}}^+(m)\mu_{\ell\mathfrak{B}}^+(m), \mu_{u\mathfrak{A}}^+(m) + \mu_{u\mathfrak{B}}^+(m) - \mu_{u\mathfrak{A}}^+(m)\mu_{u\mathfrak{B}}^+(m)], [-\mu_{\ell\mathfrak{A}}^-(m)\mu_{\ell\mathfrak{B}}^-(m), -(\mu_{u\mathfrak{A}}^-(m)\mu_{u\mathfrak{B}}^-(m))] \{ \mu_{\mathfrak{A}}^+(m) + -\mu_{\mathfrak{A}}^+(m)\mu_{\mathfrak{B}}^+(m), -(\mu_{\mathfrak{A}}^-(m)\mu_{\mathfrak{B}}^-(m)) \} : m \in \mathfrak{M} \} = \mathfrak{A} \oplus_P \mathfrak{B}$
- (ii). $(\mathfrak{A} \cup_P \mathfrak{B}) \otimes_P (\mathfrak{A} \cap_P \mathfrak{B}) = \{ \langle m, [\max\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \max\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}] \{ \min\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, \min\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\} \} : m \in \mathfrak{M} \} \otimes_P \{ \langle m, [\min\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \min\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}] \{ \max\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\}, \max\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\} \} : m \in \mathfrak{M} \} \}$
 $= \{ \langle m, [\max\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\} \min\{\mu_{\ell\mathfrak{A}}^+(m), \mu_{\ell\mathfrak{B}}^+(m)\}, \max\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\} \min\{\mu_{u\mathfrak{A}}^+(m), \mu_{u\mathfrak{B}}^+(m)\}] \{ -(-\min\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\} - \max\{\mu_{\ell\mathfrak{A}}^-(m), \mu_{\ell\mathfrak{B}}^-(m)\} - \min\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\} - \max\{\mu_{u\mathfrak{A}}^-(m), \mu_{u\mathfrak{B}}^-(m)\}) \} : m \in \mathfrak{M} \}$

$$\begin{aligned} & \max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}[\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m) \\ & \min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, -(-\min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m) \\ & - \max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\} - \min\{\mu_{\mathfrak{A}}^-(m), \\ & \mu_{\mathfrak{B}}^-(m)\} \max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\})] : m \in \mathfrak{M}\} \\ = & \{ \langle m, [\mu_{\mathfrak{A}}^+(m)\mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{A}}^+(m)\mu_{\mathfrak{B}}^+(m)], \\ & [-(-\mu_{\mathfrak{A}}^-(m) - \mu_{\mathfrak{B}}^-(m) - \mu_{\mathfrak{A}}^-(m)\mu_{\mathfrak{B}}^-(m)), \\ & -(\mu_{\mathfrak{A}}^-(m)\mu_{\mathfrak{B}}^-(m))], \{\mu_{\mathfrak{A}}^+(m)\mu_{\mathfrak{B}}^+(m), \\ & -(-\mu_{\mathfrak{A}}^-(m) - \mu_{\mathfrak{B}}^-(m) - \mu_{\mathfrak{A}}^-(m)\mu_{\mathfrak{B}}^-(m))\} \rangle : m \in \mathfrak{M} \} \\ = & \mathfrak{A} \otimes_{\mathfrak{P}} \mathfrak{B} \end{aligned}$$

Theorem 2: Let $\mathfrak{A} = \{ \langle m, [\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{A}}^-(m)], [\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{A}}^+(m)] \rangle : m \in \mathfrak{M} \}$ and $\mathfrak{B} = \{ \langle m, [\mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{B}}^-(m)], [\mu_{\mathfrak{B}}^-(m), \mu_{\mathfrak{B}}^+(m)], \{\mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \}$ be two CBFSSs on \mathfrak{M} . Then,

- i. $(\mathfrak{A} \cup_{\mathfrak{P}} \mathfrak{B}) \cap_{\mathfrak{P}} \mathfrak{B} = \mathfrak{B}$;
- ii. $(\mathfrak{A} \cap_{\mathfrak{P}} \mathfrak{B}) \cup_{\mathfrak{P}} \mathfrak{B} = \mathfrak{B}$;
- iii. $(\mathfrak{A} \cup_{\mathfrak{R}} \mathfrak{B}) \cap_{\mathfrak{R}} \mathfrak{B} = \mathfrak{B}$;
- iv. $(\mathfrak{A} \cap_{\mathfrak{R}} \mathfrak{B}) \cup_{\mathfrak{R}} \mathfrak{B} = \mathfrak{B}$.

Proof: We prove (i) & (ii), and (iii) & (iv) can be proved similarly.

- (i). $(\mathfrak{A} \cup_{\mathfrak{P}} \mathfrak{B}) \cap_{\mathfrak{P}} \mathfrak{B} = \{ \langle m, [\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}], [\min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}], \{\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}\} \rangle : m \in \mathfrak{M} \} \cap_{\mathfrak{P}} \{ \langle m, [\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)], [\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)], \{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \} = \{ \langle m, [\min(\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \mu_{\mathfrak{A}}^+(m)), \min(\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \mu_{\mathfrak{A}}^+(m))], [\max(\min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \mu_{\mathfrak{A}}^-(m)), \max(\min\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \mu_{\mathfrak{A}}^-(m))], \{\min(\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \mu_{\mathfrak{A}}^+(m)), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \} = \{ \langle m, [\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)], [\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)], \{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \} = \mathfrak{B}$
- (ii). $(\mathfrak{A} \cap_{\mathfrak{P}} \mathfrak{B}) \cup_{\mathfrak{P}} \mathfrak{B} = \{ \langle m, [\min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}], [\max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}], \{\min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}\} \rangle : m \in \mathfrak{M} \} \cup_{\mathfrak{P}} \{ \langle m, [\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)], [\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)], \{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \} = \{ \langle m, [\max(\min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \mu_{\mathfrak{A}}^+(m)), \max(\min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \mu_{\mathfrak{A}}^+(m))], [\min(\max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \mu_{\mathfrak{A}}^-(m)), \min(\max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \mu_{\mathfrak{A}}^-(m))], \{\min(\max\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \mu_{\mathfrak{A}}^+(m)), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \}$

$$\begin{aligned} & \mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \{\max(\min\{\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)\}, \\ & \mu_{\mathfrak{B}}^+(m), \min(\max\{\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)\}, \mu_{\mathfrak{B}}^-(m))\} \\ & : m \in \mathfrak{M} \} \\ = & \{ \langle m, [\mu_{\mathfrak{A}}^+(m), \mu_{\mathfrak{B}}^+(m)], [\mu_{\mathfrak{A}}^-(m), \mu_{\mathfrak{B}}^-(m)], \\ & \{\mu_{\mathfrak{B}}^+(m), \mu_{\mathfrak{B}}^-(m)\} \rangle : m \in \mathfrak{M} \} = \mathfrak{B} \end{aligned}$$

A. CUBIC BIPOLAR FUZZY RELATIONS

Definition 18: Let $\mathfrak{M} \times \mathfrak{N}$ be cartesian product of two initial universes \mathfrak{M} and \mathfrak{N} . A cubic bipolar fuzzy relation R from \mathfrak{M} to \mathfrak{N} (in short, $R : \mathfrak{M} \rightarrow \mathfrak{N}$) is a cubic bipolar fuzzy set on $\mathfrak{M} \times \mathfrak{N}$, i.e.,

$$R = \{ \langle (m, n), [\mu_{\mathfrak{R}}^+(m, n), \mu_{\mathfrak{R}}^-(m, n)], [\mu_{\mathfrak{R}}^-(m, n), \mu_{\mathfrak{R}}^+(m, n)], \{\mu_{\mathfrak{R}}^+(m, n), \mu_{\mathfrak{R}}^-(m, n)\} \rangle : (m, n) \in \mathfrak{M} \times \mathfrak{N} \}$$

In particular, a cubic bipolar fuzzy relation (CBFR) from \mathfrak{M} to \mathfrak{M} is called CBFR on \mathfrak{M} . The collection of all CBFRs on \mathfrak{M} (resp. from \mathfrak{M} to \mathfrak{N}) is denoted by $CBFR(\mathfrak{M})$ (resp. $CBFR(\mathfrak{M} \times \mathfrak{N})$).

Example 8: Let $\mathfrak{M} = \{m_1, m_2, m_3\}$ be initial universe. A CBFR on \mathfrak{M} can be represented in the matrix form in Table 1.

Now, we discuss some types of CBFRs.

Definition 19: Let $R \in CBFR(\mathfrak{M} \times \mathfrak{N})$. Then, the inverse of R is a CBFR from \mathfrak{N} to \mathfrak{M} ($R^{-1} : \mathfrak{N} \rightarrow \mathfrak{M}$) and is defined as $R^{-1} = \{ \langle (n, m), [\mu_{\mathfrak{R}^{-1}}^+(n, m), \mu_{\mathfrak{R}^{-1}}^-(n, m)], [\mu_{\mathfrak{R}^{-1}}^-(n, m), \mu_{\mathfrak{R}^{-1}}^+(n, m)], \{\mu_{\mathfrak{R}^{-1}}^+(n, m), \mu_{\mathfrak{R}^{-1}}^-(n, m)\} \rangle : (n, m) \in \mathfrak{N} \times \mathfrak{M} \}$, where for each $(n, m) \in \mathfrak{N} \times \mathfrak{M}$, $[\mu_{\mathfrak{R}^{-1}}^+(n, m), \mu_{\mathfrak{R}^{-1}}^-(n, m)] = [\mu_{\mathfrak{R}}^+(m, n), \mu_{\mathfrak{R}}^-(m, n)]$, $[\mu_{\mathfrak{R}^{-1}}^-(n, m), \mu_{\mathfrak{R}^{-1}}^+(n, m)] = [\mu_{\mathfrak{R}}^-(m, n), \mu_{\mathfrak{R}}^+(m, n)]$ and $\mu_{\mathfrak{R}^{-1}}^+(n, m) = \mu_{\mathfrak{R}}^+(m, n)$, $\mu_{\mathfrak{R}^{-1}}^-(n, m) = \mu_{\mathfrak{R}}^-(m, n)$.

Example 9: Consider the CBFR given in example 8. Then, R^{-1} is given as in Table 1.

Definition 20: A CBFR $R \in CBFR(\mathfrak{M})$ is said to be reflexive if for each $m \in \mathfrak{M}$, $[\mu_{\mathfrak{R}}^+(m, m), \mu_{\mathfrak{R}}^-(m, m)] = [1, 1]$, $[\mu_{\mathfrak{R}}^-(m, m), \mu_{\mathfrak{R}}^+(m, m)] = [-1, -1]$ and $\mu_{\mathfrak{R}}^+(m, m) = 1$, $\mu_{\mathfrak{R}}^-(m, m) = -1$.

Definition 21: A CBFR $R \in CBFR(\mathfrak{M})$ is said to be symmetric if for each $m, m' \in \mathfrak{M}$, $[\mu_{\mathfrak{R}}^+(m, m'), \mu_{\mathfrak{R}}^-(m, m')] = [\mu_{\mathfrak{R}}^+(m', m), \mu_{\mathfrak{R}}^-(m', m)]$, $[\mu_{\mathfrak{R}}^-(m, m'), \mu_{\mathfrak{R}}^+(m, m')] = [\mu_{\mathfrak{R}}^-(m', m), \mu_{\mathfrak{R}}^+(m', m)]$ and $\mu_{\mathfrak{R}}^+(m, m') = \mu_{\mathfrak{R}}^+(m', m)$, $\mu_{\mathfrak{R}}^-(m, m') = \mu_{\mathfrak{R}}^-(m', m)$.

Definition 22: A CBFR $R \in CBFR(\mathfrak{M})$ is said to be transitive if for $(m, \mathfrak{M}) \in \mathfrak{M} \times \mathfrak{M}$, $[\mu_{\mathfrak{R}}^+(m, \mathfrak{M}), \mu_{\mathfrak{R}}^-(m, \mathfrak{M})] \geq \bigvee_{\mathfrak{M} \in \mathfrak{M}} ([\mu_{\mathfrak{R}}^+(m, \mathfrak{M}), \mu_{\mathfrak{R}}^-(m, \mathfrak{M})] \wedge [\mu_{\mathfrak{R}}^+(\mathfrak{M}, \mathfrak{M}), \mu_{\mathfrak{R}}^-(\mathfrak{M}, \mathfrak{M})])$, $[\mu_{\mathfrak{R}}^-(m, \mathfrak{M}), \mu_{\mathfrak{R}}^+(m, \mathfrak{M})] \leq \bigwedge_{\mathfrak{M} \in \mathfrak{M}} ([\mu_{\mathfrak{R}}^-(m, \mathfrak{M}), \mu_{\mathfrak{R}}^+(m, \mathfrak{M})] \wedge [\mu_{\mathfrak{R}}^-(\mathfrak{M}, \mathfrak{M}), \mu_{\mathfrak{R}}^+(\mathfrak{M}, \mathfrak{M})])$ and $\mu_{\mathfrak{R}}^+(m, \mathfrak{M}) \geq \bigvee_{\mathfrak{M} \in \mathfrak{M}} (\mu_{\mathfrak{R}}^+(m, \mathfrak{M}) \wedge \mu_{\mathfrak{R}}^+(\mathfrak{M}, \mathfrak{M}))$, $\mu_{\mathfrak{R}}^-(m, \mathfrak{M}) \leq \bigwedge_{\mathfrak{M} \in \mathfrak{M}} (\mu_{\mathfrak{R}}^-(m, \mathfrak{M}) \wedge \mu_{\mathfrak{R}}^-(\mathfrak{M}, \mathfrak{M}))$.

Definition 23: A CBFR $R \in CBFR(\mathfrak{M})$ is called an equivalence relation on \mathfrak{M} if it is reflexive, symmetric and transitive.

TABLE 1. CBFR for examples 8, 9 and 10.

CBFR for Example 8		
$\langle [0.67, 0.78], [-0.26, -0.13], \{0.37, -0.32\} \rangle$	$\langle [0.73, 0.86], [-0.47, -0.38], \{0.57, -0.51\} \rangle$	$\langle [0.23, 0.36], [-0.47, -0.39], \{0.29, -0.46\} \rangle$
$\langle [0.81, 0.93], [-0.21, -0.12], \{0.71, -0.67\} \rangle$	$\langle [0.59, 0.73], [-0.36, -0.24], \{0.38, -0.47\} \rangle$	$\langle [0.39, 0.48], [-0.76, -0.62], \{0.42, -0.71\} \rangle$
$\langle [0, 0.71], [-0.56, -0.38], \{0.62, -0.57\} \rangle$	$\langle [0.21, 0.39], [-0.82, -0.67], \{0.33, -0.73\} \rangle$	$\langle [0.87, 0.92], [-0.23, -0.17], \{0.89, -0.19\} \rangle$
CBFR for Example 9		
$\langle [0.67, 0.78], [-0.26, -0.13], \{0.37, -0.32\} \rangle$	$\langle [0.81, 0.93], [-0.21, -0.12], \{0.71, -0.67\} \rangle$	$\langle [0, 0.71], [-0.56, -0.38], \{0.62, -0.57\} \rangle$
$\langle [0.73, 0.86], [-0.47, -0.38], \{0.57, -0.51\} \rangle$	$\langle [0.59, 0.73], [-0.36, -0.24], \{0.38, -0.47\} \rangle$	$\langle [0.21, 0.39], [-0.82, -0.67], \{0.33, -0.73\} \rangle$
$\langle [0.23, 0.36], [-0.47, -0.39], \{0.29, -0.46\} \rangle$	$\langle [0.39, 0.48], [-0.76, -0.62], \{0.42, -0.71\} \rangle$	$\langle [0.87, 0.92], [-0.23, -0.17], \{0.89, -0.19\} \rangle$
CBFR for Example 10		
$\langle [1, 1], [-1, -1], \{1, -1\} \rangle$	$\langle [0.36, 0.49], [-0.63, -0.52], \{0.17, -0.56\} \rangle$	$\langle [0.49, 0.57], [-0.59, -0.47], \{0.36, -0.72\} \rangle$
$\langle [0.36, 0.49], [-0.63, -0.52], \{0.17, -0.56\} \rangle$	$\langle [1, 1], [-1, -1], \{1, -1\} \rangle$	$\langle [0.61, 0.72], [-0.32, -0.26], \{0.29, -0.66\} \rangle$
$\langle [0.49, 0.57], [-0.59, -0.47], \{0.36, -0.72\} \rangle$	$\langle [0.61, 0.72], [-0.32, -0.26], \{0.29, -0.66\} \rangle$	$\langle [1, 1], [-1, -1], \{1, -1\} \rangle$

Example 10: A CBFR on $\mathfrak{M} = \{m_1, m_2, m_3\}$ is given in Table 1. It is easy to check that this CBFR is an equivalence relation.

IV. CORRELATION COEFFICIENTS OF CBFs

In this section, we propose some correlation coefficients for any two cubic bipolar fuzzy sets (CBFSs), which determine the strength of relationship between them. Moreover, an advanced feature of these correlation coefficients is their ability to determine whether two CBFSs are positively or negatively correlated.

Definition 24: Let $\tilde{\alpha} = \langle [\mu_{\ell\tilde{\alpha}}^+, \mu_{u\tilde{\alpha}}^+], [\mu_{\ell\tilde{\alpha}}^-, \mu_{u\tilde{\alpha}}^-], \{\mu_{\tilde{\alpha}}^+, \mu_{\tilde{\alpha}}^-\} \rangle$ and $\tilde{\beta} = \langle [\mu_{\ell\tilde{\beta}}^+, \mu_{u\tilde{\beta}}^+], [\mu_{\ell\tilde{\beta}}^-, \mu_{u\tilde{\beta}}^-], \{\mu_{\tilde{\beta}}^+, \mu_{\tilde{\beta}}^-\} \rangle$ be two CBFNs, then the deviation of $\tilde{\alpha}$ and $\tilde{\beta}$ can be computed by using the following expression

$$\begin{aligned} \tilde{d}(\tilde{\alpha}, \tilde{\beta}) = & \left(\frac{\mu_{\ell\tilde{\alpha}}^+ - \mu_{\ell\tilde{\beta}}^+ + \mu_{u\tilde{\alpha}}^+ - \mu_{u\tilde{\beta}}^+}{2} \right) \\ & + \left(\frac{(-\mu_{\ell\tilde{\alpha}}^-) - (-\mu_{\ell\tilde{\beta}}^-) + (-\mu_{u\tilde{\alpha}}^-) - (-\mu_{u\tilde{\beta}}^-)}{2} \right) \\ & + (\mu_{\tilde{\alpha}}^+ - \mu_{\tilde{\beta}}^+) + ((-\mu_{\tilde{\alpha}}^-) - (-\mu_{\tilde{\beta}}^-)) \end{aligned}$$

The deviation consists of two parts: the deviation of positive membership degrees and that of negative membership degrees. Since, cubic bipolar fuzzy information deals with a property as well as its counter-property so it sounds reasonable to add up the deviation of positive and negative membership degrees.

The deviation of CBFNs satisfies the following properties

Theorem 3: Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be three CBFNs, then

- i. $-4 \leq \tilde{d}(\tilde{\alpha}, \tilde{\beta}) \leq 4$
- ii. $\tilde{d}(\tilde{\alpha}, \tilde{\beta}) = -\tilde{d}(\tilde{\beta}, \tilde{\alpha})$
- iii. $\tilde{d}(\tilde{\alpha}, \tilde{\beta}) + \tilde{d}(\tilde{\beta}, \tilde{\gamma}) = \tilde{d}(\tilde{\alpha}, \tilde{\gamma})$

Proof: Straightforward. ■

Example 11: Let $\tilde{\alpha} = \langle [0.14, 0.53], [-0.37, -0.26], \{0.49, -0.55\} \rangle$ and $\tilde{\beta} = \langle [0.48, 0.63], [-0.44, -0.31], \{0.71, -0.68\} \rangle$ be two CBFNs. Then,

$$\begin{aligned} \tilde{d}(\tilde{\alpha}, \tilde{\beta}) = & \left(\frac{0.14 - 0.48 + 0.53 - 0.63}{2} \right) \\ & + \left(\frac{0.37 - 0.44 + 0.26 - 0.31}{2} \right) \\ & + (0.49 - 0.71) + (0.55 - 0.68) = -0.63 \end{aligned}$$

Definition 25: Let $\mathfrak{M} = \{m_1, m_2, \dots, m_n\}$ be initial universe and $\mathfrak{A} = \langle \langle m_i, [\mu_{\ell\mathfrak{A}}^+(m_i), \mu_{u\mathfrak{A}}^+(m_i)], [\mu_{\ell\mathfrak{A}}^-(m_i), \mu_{u\mathfrak{A}}^-(m_i)], \{\mu_{\mathfrak{A}}^+(m_i), \mu_{\mathfrak{A}}^-(m_i)\} \rangle : m_i \in \mathfrak{M} \rangle$ be a CBFS on \mathfrak{M} . The mean value of \mathfrak{A} is given by

$$\begin{aligned} E(\mathfrak{A}) = & \langle [\bar{\mu}_{\ell\mathfrak{A}}^+, \bar{\mu}_{u\mathfrak{A}}^+], [\bar{\mu}_{\ell\mathfrak{A}}^-, \bar{\mu}_{u\mathfrak{A}}^-], \{\bar{\mu}_{\mathfrak{A}}^+, \bar{\mu}_{\mathfrak{A}}^-\} \rangle \\ = & \left\langle \left[\frac{1}{n} \sum_{i=1}^n \mu_{\ell\mathfrak{A}}^+(m_i), \frac{1}{n} \sum_{i=1}^n \mu_{u\mathfrak{A}}^+(m_i) \right], \right. \\ & \times \left[\frac{1}{n} \sum_{i=1}^n \mu_{\ell\mathfrak{A}}^-(m_i), \frac{1}{n} \sum_{i=1}^n \mu_{u\mathfrak{A}}^-(m_i) \right], \\ & \left. \times \left\{ \frac{1}{n} \sum_{i=1}^n \mu_{\mathfrak{A}}^+(m_i), \frac{1}{n} \sum_{i=1}^n \mu_{\mathfrak{A}}^-(m_i) \right\} \right\rangle \end{aligned}$$

$E(\mathfrak{A})$ is again a CBFN.

Let $\mathfrak{M} = \{m_1, m_2, \dots, m_n\}$ be initial universe and $\mathfrak{A} = \langle \langle m_i, [\mu_{\ell\mathfrak{A}}^+(m_i), \mu_{u\mathfrak{A}}^+(m_i)], \mu_{\ell\mathfrak{A}}^-(m_i), \mu_{u\mathfrak{A}}^-(m_i) \rangle, \{\mu_{\mathfrak{A}}^+(m_i), \mu_{\mathfrak{A}}^-(m_i)\} \rangle : m_i \in \mathfrak{M} \rangle$ and $\mathfrak{B} = \langle \langle m_i, [\mu_{\ell\mathfrak{B}}^+(m_i), \mu_{u\mathfrak{B}}^+(m_i)], [\mu_{\ell\mathfrak{B}}^-(m_i), \mu_{u\mathfrak{B}}^-(m_i)], \{\mu_{\mathfrak{B}}^+(m_i), \mu_{\mathfrak{B}}^-(m_i)\} \rangle : m_i \in \mathfrak{M} \rangle$ be two CBFSs on \mathfrak{M} . We define

$$\begin{aligned} d_i(\mathfrak{A}) = & \left(\frac{\mu_{\ell\mathfrak{A}}^+(m_i) - \bar{\mu}_{\ell\mathfrak{A}}^+ + \mu_{u\mathfrak{A}}^+(m_i) - \bar{\mu}_{u\mathfrak{A}}^+}{2} \right) \\ & + \left(\frac{(-\mu_{\ell\mathfrak{A}}^-(m_i)) - (-\bar{\mu}_{\ell\mathfrak{A}}^-) + (-\mu_{u\mathfrak{A}}^-(m_i)) - (-\bar{\mu}_{u\mathfrak{A}}^-)}{2} \right) \\ & + (\mu_{\mathfrak{A}}^+(m_i) - \bar{\mu}_{\mathfrak{A}}^+) + ((-\mu_{\mathfrak{A}}^-(m_i)) - (-\bar{\mu}_{\mathfrak{A}}^-)) \\ d_i(\mathfrak{B}) = & \left(\frac{\mu_{\ell\mathfrak{B}}^+(m_i) - \bar{\mu}_{\ell\mathfrak{B}}^+ + \mu_{u\mathfrak{B}}^+(m_i) - \bar{\mu}_{u\mathfrak{B}}^+}{2} \right) \\ & + \left(\frac{(-\mu_{\ell\mathfrak{B}}^-(m_i)) - (-\bar{\mu}_{\ell\mathfrak{B}}^-) + (-\mu_{u\mathfrak{B}}^-(m_i)) - (-\bar{\mu}_{u\mathfrak{B}}^-)}{2} \right) \\ & + (\mu_{\mathfrak{B}}^+(m_i) - \bar{\mu}_{\mathfrak{B}}^+) + ((-\mu_{\mathfrak{B}}^-(m_i)) - (-\bar{\mu}_{\mathfrak{B}}^-)) \end{aligned}$$

Definition 26: Let $\mathfrak{A} = \langle \langle m_i, [\mu_{\ell\mathfrak{A}}^+(m_i), \mu_{u\mathfrak{A}}^+(m_i)], [\mu_{\ell\mathfrak{A}}^-(m_i), \mu_{u\mathfrak{A}}^-(m_i)], \{\mu_{\mathfrak{A}}^+(m_i), \mu_{\mathfrak{A}}^-(m_i)\} \rangle : m_i \in \mathfrak{M} \rangle$ be a CBFS on \mathfrak{M} . The variance of \mathfrak{A} can be defined as

$$D(\mathfrak{A}) = \frac{1}{n-1} \sum_{i=1}^n (d_i(\mathfrak{A}))^2 \tag{1}$$

Definition 27: Let $\mathfrak{A} = \langle \langle m_i, [\mu_{\ell\mathfrak{A}}^+(m_i), \mu_{u\mathfrak{A}}^+(m_i)], [\mu_{\ell\mathfrak{A}}^-(m_i), \mu_{u\mathfrak{A}}^-(m_i)], \{\mu_{\mathfrak{A}}^+(m_i), \mu_{\mathfrak{A}}^-(m_i)\} \rangle : m_i \in \mathfrak{M} \rangle$ and $\mathfrak{B} = \langle \langle m_i, [\mu_{\ell\mathfrak{B}}^+(m_i), \mu_{u\mathfrak{B}}^+(m_i)], [\mu_{\ell\mathfrak{B}}^-(m_i), \mu_{u\mathfrak{B}}^-(m_i)], \{\mu_{\mathfrak{B}}^+(m_i), \mu_{\mathfrak{B}}^-(m_i)\} \rangle : m_i \in \mathfrak{M} \rangle$

$\{\mu_{\mathfrak{B}}^+(m_i), \mu_{\mathfrak{B}}^-(m_i)\} : m_i \in \mathfrak{M}\}$ be two CBFSSs on \mathfrak{M} . The covariance between \mathfrak{A} and \mathfrak{B} is given by

$$Cov(\mathfrak{A}, \mathfrak{B}) = \frac{1}{n-1} \sum_{i=1}^n d_i(\mathfrak{A})d_i(\mathfrak{B}) \quad (2)$$

Clearly, covariance satisfies the following properties:

- i. $Cov(\mathfrak{A}, \mathfrak{B}) = Cov(\mathfrak{B}, \mathfrak{A})$
- ii. $Cov(\mathfrak{A}, \mathfrak{A}) = D(\mathfrak{A})$

Definition 28: For two CBFSSs \mathfrak{A} and \mathfrak{B} , the correlation coefficient is defined by

$$\theta_1(\mathfrak{A}, \mathfrak{B}) = \frac{Cov(\mathfrak{A}, \mathfrak{B})}{\sqrt{D(\mathfrak{A})D(\mathfrak{B})}} \quad (3)$$

Theorem 4: Let \mathfrak{A} and \mathfrak{B} be two CBFSSs on initial universe $\mathfrak{M} = \{m_1, m_2, \dots, m_n\}$. Then, the correlation coefficient given in Eq.(3) satisfies the following conditions:

- i. $\theta_1(\mathfrak{A}, \mathfrak{B}) = \theta_1(\mathfrak{B}, \mathfrak{A})$
- ii. $-1 \leq \theta_1(\mathfrak{A}, \mathfrak{B}) \leq 1$
- iii. $\theta_1(\mathfrak{A}, \mathfrak{B}) = 1$ if $\mathfrak{A} = \mathfrak{B}$
- iv. $\theta_1(\mathfrak{A}, \mathfrak{A}^c) = -1$.

Proof: (i) Straightforward.

(ii) To prove this, we utilize Cauchy-Schwarz inequality which states that $(\sum_{j=1}^m t_j s_j)^2 \leq (\sum_{j=1}^m t_j^2)(\sum_{j=1}^m s_j^2)$ for all $t = (t_1, t_2, \dots, t_m)$, $s = (s_1, s_2, \dots, s_m) \in \mathbb{R}^m$. Now,

$$\begin{aligned} (Cov(\mathfrak{A}, \mathfrak{B}))^2 &= \left\{ \frac{1}{n-1} \sum_{i=1}^n d_i(\mathfrak{A})d_i(\mathfrak{B}) \right\}^2 \\ &\leq \frac{1}{n-1} \sum_{i=1}^n (d_i(\mathfrak{A}))^2 \times \frac{1}{n-1} \sum_{i=1}^n (d_i(\mathfrak{B}))^2 \\ &= D(\mathfrak{A})D(\mathfrak{B}) \\ &\Rightarrow |Cov(\mathfrak{A}, \mathfrak{B})| \leq \sqrt{D(\mathfrak{A})D(\mathfrak{B})} \\ &\Rightarrow -\sqrt{D(\mathfrak{A})D(\mathfrak{B})} \leq Cov(\mathfrak{A}, \mathfrak{B}) \\ &\leq \sqrt{D(\mathfrak{A})D(\mathfrak{B})} \end{aligned}$$

$$\Rightarrow -1 \leq \frac{Cov(\mathfrak{A}, \mathfrak{B})}{\sqrt{D(\mathfrak{A})D(\mathfrak{B})}} \leq 1$$

Hence, $-1 \leq \theta_1(\mathfrak{A}, \mathfrak{B}) \leq 1$.

(iii) If $\mathfrak{A} = \mathfrak{B}$, then $[\mu_{\ell\mathfrak{A}}^+(m_i), \mu_{u\mathfrak{A}}^+(m_i)] = [\mu_{\ell\mathfrak{B}}^+(m_i), \mu_{u\mathfrak{B}}^+(m_i)]$, $[\mu_{\ell\mathfrak{A}}^-(m_i), \mu_{u\mathfrak{A}}^-(m_i)] = [\mu_{\ell\mathfrak{B}}^-(m_i), \mu_{u\mathfrak{B}}^-(m_i)]$, $\mu_{\mathfrak{A}}^+(m_i) = \mu_{\mathfrak{B}}^+(m_i)$ and $\mu_{\mathfrak{A}}^-(m_i) = \mu_{\mathfrak{B}}^-(m_i)$ for all $m_i \in \mathfrak{M}$. Clearly, $D(\mathfrak{A}) = D(\mathfrak{B})$ and $Cov(\mathfrak{A}, \mathfrak{B}) = D(\mathfrak{A})$. Then,

$$\begin{aligned} \theta_1(\mathfrak{A}, \mathfrak{B}) &= \frac{Cov(\mathfrak{A}, \mathfrak{B})}{\sqrt{D(\mathfrak{A})D(\mathfrak{B})}} \\ &= \frac{D(\mathfrak{A})}{\sqrt{D(\mathfrak{A})D(\mathfrak{A})}} = 1. \end{aligned}$$

(iv) For a CBFSS \mathfrak{A} , the complement of \mathfrak{A} is given by $\mathfrak{A}^c = \{ \langle m_i, [1 - \mu_{u\mathfrak{A}}^+(m_i), 1 - \mu_{\ell\mathfrak{A}}^+(m_i)], [-1 + (-\mu_{u\mathfrak{A}}^-(m_i)), -1 + (-\mu_{\ell\mathfrak{A}}^-(m_i))] \rangle : m_i \in \mathfrak{M} \}$

Now, as shown at the bottom of the page.

Thus,

$$\begin{aligned} D(\mathfrak{A}^c) &= \frac{1}{n-1} \sum_{i=1}^n (d_i(\mathfrak{A}^c))^2 = \frac{1}{n-1} \sum_{i=1}^n (-d_i(\mathfrak{A}))^2 = \frac{1}{n-1} \sum_{i=1}^n (d_i(\mathfrak{A}))^2 = D(\mathfrak{A}). \text{ Further, we have } Cov(\mathfrak{A}, \mathfrak{A}^c) = \frac{1}{n-1} \sum_{i=1}^n d_i(\mathfrak{A})d_i(\mathfrak{A}^c) \\ &= \frac{1}{n-1} \sum_{i=1}^n -(d_i(\mathfrak{A}))^2 = -D(\mathfrak{A}). \text{ Hence,} \end{aligned}$$

$$\begin{aligned} \theta_1(\mathfrak{A}, \mathfrak{A}^c) &= \frac{Cov(\mathfrak{A}, \mathfrak{A}^c)}{\sqrt{D(\mathfrak{A})D(\mathfrak{A}^c)}} \\ &= \frac{-D(\mathfrak{A})}{\sqrt{D(\mathfrak{A})D(\mathfrak{A})}} = -1 \end{aligned}$$

Example 12: Let

$$\mathfrak{A} = \left\{ \begin{aligned} &\langle m_1, [0.23, 0.36], [-0.47, -0.31], \{0.41, -0.26\} \rangle, \\ &\langle m_2, [0.51, 0.69], [-0.34, -0.27], \{0.21, -0.39\} \rangle, \\ &\langle m_3, [0.47, 0.63], [-0.58, -0.43], \{0.49, -0.61\} \rangle \end{aligned} \right\}$$

$$\begin{aligned} d_i(\mathfrak{A}^c) &= \left(\frac{\mu_{\ell\mathfrak{A}^c}^+(m_i) - \bar{\mu}_{\ell\mathfrak{A}^c}^+ + \mu_{u\mathfrak{A}^c}^+(m_i) - \bar{\mu}_{u\mathfrak{A}^c}^+}{2} \right) \\ &\quad + \left(\frac{(-\mu_{\ell\mathfrak{A}^c}^-(m_i)) - (-\bar{\mu}_{\ell\mathfrak{A}^c}^-) + (-\mu_{u\mathfrak{A}^c}^-(m_i)) - (-\bar{\mu}_{u\mathfrak{A}^c}^-)}{2} \right) \\ &\quad + (\mu_{\mathfrak{A}^c}^+(m_i) - \bar{\mu}_{\mathfrak{A}^c}^+) + ((-\mu_{\mathfrak{A}^c}^-(m_i)) - (-\bar{\mu}_{\mathfrak{A}^c}^-)) \\ &= \left(\frac{(1 - \mu_{u\mathfrak{A}}^+(m_i)) - (1 - \bar{\mu}_{u\mathfrak{A}}^+) + (1 - \mu_{\ell\mathfrak{A}}^+(m_i)) - (1 - \bar{\mu}_{\ell\mathfrak{A}}^+)}{2} \right) \\ &\quad + \left(\frac{(1 - (-\mu_{u\mathfrak{A}}^-(m_i))) - (1 - (-\bar{\mu}_{u\mathfrak{A}}^-)) + (1 - (-\mu_{\ell\mathfrak{A}}^-(m_i))) - (1 - (-\bar{\mu}_{\ell\mathfrak{A}}^-))}{2} \right) \\ &\quad + ((1 - \mu_{\mathfrak{A}}^+(m_i)) - (1 - \bar{\mu}_{\mathfrak{A}}^+)) + ((1 - (-\mu_{\mathfrak{A}}^-(m_i))) - (1 - (-\bar{\mu}_{\mathfrak{A}}^-))) \\ &= - \left\{ \left(\frac{\mu_{\ell\mathfrak{A}}^+(m_i) - \bar{\mu}_{\ell\mathfrak{A}}^+ + \mu_{u\mathfrak{A}}^+(m_i) - \bar{\mu}_{u\mathfrak{A}}^+}{2} \right) \right. \\ &\quad \left. + \left(\frac{(-\mu_{\ell\mathfrak{A}}^-(m_i)) - (-\bar{\mu}_{\ell\mathfrak{A}}^-) + (-\mu_{u\mathfrak{A}}^-(m_i)) - (-\bar{\mu}_{u\mathfrak{A}}^-)}{2} \right) + (\mu_{\mathfrak{A}}^+(m_i) - \bar{\mu}_{\mathfrak{A}}^+) + ((-\mu_{\mathfrak{A}}^-(m_i)) - (-\bar{\mu}_{\mathfrak{A}}^-)) \right\} = -d_i(\mathfrak{A}) \end{aligned}$$

and

$$\mathfrak{B} = \left\{ \langle \mathfrak{m}_1, [0.36, 0.49], [-0.61, -0.47], \{0.41, -0.56\} \rangle, \right. \\ \left. \langle \mathfrak{m}_2, [0.56, 0.71], [-0.43, -0.36], \{0.54, -0.44\} \rangle, \right. \\ \left. \langle \mathfrak{m}_3, [0.18, 0.32], [-0.51, -0.29], \{0.36, -0.39\} \rangle \right\}$$

be two CBFs on \mathfrak{M} . At first, we calculate the mean values of \mathfrak{A} and \mathfrak{B} as follows

$$E(\mathfrak{A}) = \langle [0.40, 0.56], [-0.46, -0.34], \{0.37, -0.42\} \rangle \\ E(\mathfrak{B}) = \langle [0.37, 0.51], [-0.52, -0.37], \{0.44, -0.46\} \rangle$$

The variance and covariance of \mathfrak{A} and \mathfrak{B} can be computed by finding the values of $d_i(\mathfrak{A})$ and $d_i(\mathfrak{B})$ for $i = 1, 2, 3$. After calculations, we have

$$d_1(\mathfrak{A}) = -0.315, d_2(\mathfrak{A}) = -0.165, \\ d_3(\mathfrak{A}) = 0.485, d_1(\mathfrak{B}) = 0.15, \\ d_2(\mathfrak{B}) = 0.225, d_3(\mathfrak{B}) = -0.385.$$

By using Eq.(1), the variance of \mathfrak{A} and \mathfrak{B} is given below

$$D(\mathfrak{A}) = \frac{1}{2} [(-0.315)^2 + (-0.165)^2 + (0.485)^2] = 0.1808 \\ D(\mathfrak{B}) = \frac{1}{2} [(0.15)^2 + (0.225)^2 + (-0.385)^2] = 0.1107$$

By using Eq.(2), the covariance between \mathfrak{A} and \mathfrak{B} is given as

$$Cov(\mathfrak{A}, \mathfrak{B}) = \frac{1}{2} [(-0.315)(0.15) \\ + (0.485)(-0.385)] = -0.1356$$

The correlation coefficient using Eq.(3) can be calculated as

$$\theta_1(\mathfrak{A}, \mathfrak{B}) = \frac{-0.1356}{\sqrt{0.1808 \times 0.1107}} = -0.9585.$$

Definition 29: Let \mathfrak{A} and \mathfrak{B} be two CBFs on initial universe $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n\}$, then another correlation coefficient is defined by

$$\theta_2(\mathfrak{A}, \mathfrak{B}) = \frac{Cov(\mathfrak{A}, \mathfrak{B})}{\max\{D(\mathfrak{A}), D(\mathfrak{B})\}} \quad (4)$$

Theorem 5: Let \mathfrak{A} and \mathfrak{B} be two CBFs on initial universe $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n\}$. Then, the correlation coefficient given in Eq.(4) satisfies the following conditions:

- i. $\theta_2(\mathfrak{A}, \mathfrak{B}) = \theta_2(\mathfrak{B}, \mathfrak{A})$
- ii. $-1 \leq \theta_2(\mathfrak{A}, \mathfrak{B}) \leq 1$
- iii. $\theta_2(\mathfrak{A}, \mathfrak{B}) = 1$ if $\mathfrak{A} = \mathfrak{B}$
- iv. $\theta_2(\mathfrak{A}, \mathfrak{A}^c) = -1$.

Proof: We omit the proof. ■

In various multi-attribute decision making (MADM) scenarios, different attributes are assigned different weights by the decision experts. Therefore, weights of the elements $\mathfrak{m}_i \in \mathfrak{M}$ ($i = 1, 2, \dots, n$) should be taken into consideration. For this purpose, the above-defined correlation coefficients $\theta_1(\mathfrak{A}, \mathfrak{B})$ and $\theta_2(\mathfrak{A}, \mathfrak{B})$ can be extended to weighted correlation coefficients as follows:

Definition 30: Let $w = (w_1, w_2, \dots, w_n)$ be the weight vector of the elements $\mathfrak{m}_i \in \mathfrak{M}$ ($i = 1, 2, \dots, n$) with the conditions that $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Then, for any two CBFs \mathfrak{A} and \mathfrak{B} on \mathfrak{M} , the weighted correlation coefficient is defined by

$$\theta_3(\mathfrak{A}, \mathfrak{B}) = \frac{Cov_w(\mathfrak{A}, \mathfrak{B})}{\sqrt{D_w(\mathfrak{A})D_w(\mathfrak{B})}} \quad (5)$$

where

$$D_w(\mathfrak{A}) = \frac{1}{n-1} \sum_{i=1}^n w_i (d_i(\mathfrak{A}))^2 \\ Cov_w(\mathfrak{A}, \mathfrak{B}) = \frac{1}{n-1} \sum_{i=1}^n w_i d_i(\mathfrak{A}) d_i(\mathfrak{B})$$

Theorem 6: For two CBFs \mathfrak{A} and \mathfrak{B} on $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n\}$, the weighted correlation coefficient proposed in Eq. (5) fulfills the following properties:

- i. $\theta_3(\mathfrak{A}, \mathfrak{B}) = \theta_3(\mathfrak{B}, \mathfrak{A})$
- ii. $-1 \leq \theta_3(\mathfrak{A}, \mathfrak{B}) \leq 1$
- iii. $\theta_3(\mathfrak{A}, \mathfrak{B}) = 1$ if $\mathfrak{A} = \mathfrak{B}$
- iv. $\theta_3(\mathfrak{A}, \mathfrak{A}^c) = -1$.

Proof: We only prove (ii) and the remaining parts are straightforward.

(ii) By using Cauchy-Schwarz inequality, we have

$$(Cov_w(\mathfrak{A}, \mathfrak{B}))^2 = \left\{ \frac{1}{n-1} \sum_{i=1}^n w_i d_i(\mathfrak{A}) d_i(\mathfrak{B}) \right\}^2 \\ = \left\{ \frac{1}{n-1} \sum_{i=1}^n \sqrt{w_i} d_i(\mathfrak{A}) \sqrt{w_i} d_i(\mathfrak{B}) \right\}^2 \\ \leq \frac{1}{n-1} \sum_{i=1}^n w_i (d_i(\mathfrak{A}))^2 \\ \times \frac{1}{n-1} \sum_{i=1}^n w_i (d_i(\mathfrak{B}))^2 \\ = D_w(\mathfrak{A}) D_w(\mathfrak{B}) \\ \Rightarrow |Cov_w(\mathfrak{A}, \mathfrak{B})| \leq \sqrt{D_w(\mathfrak{A}) D_w(\mathfrak{B})} \\ \Rightarrow -\sqrt{D_w(\mathfrak{A}) D_w(\mathfrak{B})} \leq Cov_w(\mathfrak{A}, \mathfrak{B}) \\ \leq \sqrt{D_w(\mathfrak{A}) D_w(\mathfrak{B})} \\ \Rightarrow -1 \leq \frac{Cov_w(\mathfrak{A}, \mathfrak{B})}{\sqrt{D_w(\mathfrak{A}) D_w(\mathfrak{B})}} \leq 1$$

Hence, $-1 \leq \theta_3(\mathfrak{A}, \mathfrak{B}) \leq 1$. ■

Definition 31: Let $w = (w_1, w_2, \dots, w_n)$ be the weight vector of the elements $\mathfrak{m}_i \in \mathfrak{M}$ ($i = 1, 2, \dots, n$) with the conditions that $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Then, for any two CBFs \mathfrak{A} and \mathfrak{B} on \mathfrak{M} , the weighted correlation coefficient is defined by

$$\theta_4(\mathfrak{A}, \mathfrak{B}) = \frac{Cov_w(\mathfrak{A}, \mathfrak{B})}{\max\{D_w(\mathfrak{A}), D_w(\mathfrak{B})\}} \quad (6)$$

Algorithm 1

Step 1. Consider some known patterns $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m$ in the form of CBFSSs in a finite initial universe \mathfrak{M} .

Step 2. Construct an unknown pattern \mathcal{Q} in the form of CBFSS in \mathfrak{M} , this pattern is to be recognized.

Step 3. Find the correlation coefficients of \mathcal{Q} and \mathcal{L}_j ($j = 1, 2, \dots, m$) by using Eqs.(3)or (4). If the elements of the initial universe \mathfrak{M} own some weights, then weighted correlation coefficients given in Eqs. (5) or (6) can be utilized.

Step 4. The pattern \mathcal{Q} belongs to the pattern \mathcal{L}_j for which the value of correlation coefficient is maximum.

Theorem 7: For two CBFSSs \mathcal{A} and \mathcal{B} on $\mathfrak{M} = \{m_1, m_2, \dots, m_n\}$, the weighted correlation coefficient proposed in Eq. (6) fulfills the following properties:

- i. $\theta_4(\mathcal{A}, \mathcal{B}) = \theta_4(\mathcal{B}, \mathcal{A})$
- ii. $-1 \leq \theta_4(\mathcal{A}, \mathcal{B}) \leq 1$
- iii. $\theta_4(\mathcal{A}, \mathcal{B}) = 1$ if $\mathcal{A} = \mathcal{B}$
- iv. $\theta_4(\mathcal{A}, \mathcal{A}^c) = -1$.

Proof: We omit the proof. ■

Remark 2: If we take $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then the weighted correlation coefficients given in Eqs. (5) and (6) reduce to unweighted correlation coefficients presented in Eqs. (3) and (4).

V. APPLICATIONS IN PATTERN RECOGNITION AND CLUSTERING ANALYSIS

This section provides applications of our proposed correlation coefficients to pattern recognition and clustering analysis under cubic bipolar fuzzy environment.

A. PATTERN RECOGNITION

Pattern recognition is a data analysis technique that employs machine learning algorithms to identify patterns and regularities in data. Pattern recognition has a broad range of applications, including image processing, aerial photo interpretation, speech and fingerprint recognition, optical character recognition in scanned documents such as contracts and photographs, and even medical imaging and diagnosis.

To identify an unknown pattern from the known ones under cubic bipolar fuzzy data, we adopt the following steps:

The flow diagram of the proposed algorithm 1 is presented in Figure 1.

1) NUMERICAL EXAMPLE

Consider three known patterns which are given in the form of CBFSSs in initial universe $\mathfrak{M} = \{m_1, m_2, m_3\}$ as

$$\mathcal{L}_1 = \left\{ \begin{aligned} &\langle m_1, [0.37, 0.48], [-0.51, -0.40], \{0.46, -0.32\} \rangle, \\ &\langle m_2, [0.22, 0.33], [-0.36, -0.25], \{0.39, -0.24\} \rangle, \\ &\langle m_3, [0.41, 0.52], [-0.62, -0.51], \{0.53, -0.67\} \rangle \end{aligned} \right\}$$

$$\mathcal{L}_2 = \left\{ \begin{aligned} &\langle m_1, [0.21, 0.34], [-0.47, -0.34], \{0.48, -0.59\} \rangle, \\ &\langle m_2, [0.52, 0.65], [-0.36, -0.23], \{0.67, -0.44\} \rangle, \\ &\langle m_3, [0.47, 0.60], [-0.29, -0.16], \{0.41, -0.33\} \rangle \end{aligned} \right\}$$

$$\mathcal{L}_3 = \left\{ \begin{aligned} &\langle m_1, [0.29, 0.44], [-0.33, -0.18], \{0.56, -0.26\} \rangle, \\ &\langle m_2, [0.52, 0.67], [-0.36, -0.21], \{0.69, -0.41\} \rangle, \\ &\langle m_3, [0.36, 0.51], [-0.28, -0.13], \{0.44, -0.31\} \rangle \end{aligned} \right\}$$

The unknown pattern is given as follows:

$$\mathcal{Q} = \left\{ \begin{aligned} &\langle m_1, [0.17, 0.36], [-0.41, -0.22], \{0.38, -0.43\} \rangle, \\ &\langle m_2, [0.43, 0.69], [-0.33, -0.18], \{0.71, -0.52\} \rangle, \\ &\langle m_3, [0.26, 0.49], [-0.52, -0.31], \{0.22, -0.67\} \rangle \end{aligned} \right\}$$

We want to know which pattern \mathcal{Q} belongs to? For this purpose, we determine the correlation coefficients of \mathcal{Q} and \mathcal{L}_j , $j = 1, 2, 3$ by using Eqs. (3) and (4). The results are presented in Table 2.

TABLE 2. Results of correlation coefficients between unknown and known patterns.

WCCs	$\theta_k(\mathcal{L}_1, \mathcal{Q})$	$\theta_k(\mathcal{L}_2, \mathcal{Q})$	$\theta_k(\mathcal{L}_3, \mathcal{Q})$
$k = 1$	-0.5211	0.5461	0.8607
$k = 2$	-0.3350	0.4076	0.8579

It is evident from the above table that \mathcal{Q} belongs to the pattern \mathcal{L}_3 . Now, suppose that the elements of \mathfrak{M} carry weights and their weights are 0.37, 0.28 and 0.35, respectively. Then, the weighted correlation coefficients by using Eqs. (5) and (6) can be calculated between unknown and known patterns. The results are summarized in Table 3.

TABLE 3. Results of weighted correlation coefficients between unknown and known patterns.

WCCs	$\theta_k(\mathcal{L}_1, \mathcal{Q})$	$\theta_k(\mathcal{L}_2, \mathcal{Q})$	$\theta_k(\mathcal{L}_3, \mathcal{Q})$
$k = 3$	-0.4615	0.4888	0.8430
$k = 4$	-0.2983	0.3613	0.8208

Again, the pattern \mathcal{Q} belongs to the pattern \mathcal{L}_3 .

B. CLUSTERING ANALYSIS

Clustering refers to a process that divides a set of data points into clusters such that the data points in the same cluster have more similar traits than those in different clusters. In what follows, we propose a novel clustering algorithm under cubic bipolar fuzzy environment. Before this, we discuss some basic ideas.

Definition 32: Let \mathcal{L}_j be m CBFSSs, then $\mathcal{F} = (f_{st})_{m \times m}$ is called correlation matrix, where $f_{st} = \theta(\mathcal{L}_s, \mathcal{L}_t)$ denotes the correlation coefficient between \mathcal{L}_s and \mathcal{L}_t and satisfies:

- i. $-1 \leq f_{st} \leq 1, s, t = 1, 2, \dots, m;$
- ii. $f_{ss} = 1, s = 1, 2, \dots, m;$
- iii. $f_{st} = f_{ts}, s, t = 1, 2, \dots, m.$

Definition 33: Let $\mathcal{F} = (f_{st})_{m \times m}$ be a correlation matrix, then $\mathcal{F}^2 = \mathcal{F} \circ \mathcal{F} = (\tilde{f}_{st})_{m \times m}$ is called composition matrix of \mathcal{F} , where

$$\tilde{f}_{st} = \max_u \{ \min \{ f_{su}, f_{ut} \} \}, s, t = 1, 2, \dots, m.$$

Definition 34: Let $\mathcal{F} = (f_{st})_{m \times m}$ be a correlation matrix, then after finite number of compositions: $\mathcal{F} \rightarrow \mathcal{F}^2 \rightarrow \mathcal{F}^4 \rightarrow \dots \rightarrow \mathcal{F}^{2^\kappa} \rightarrow \dots$, there exists a positive integer κ such that

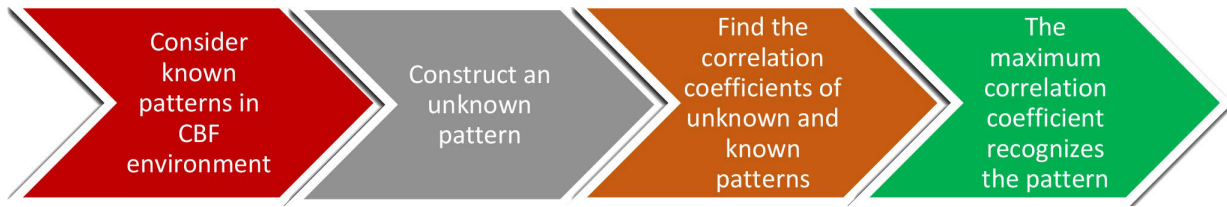


FIGURE 1. Flow diagram of algorithm 1.

$\mathcal{F}^{2^k} = \mathcal{F}^{2^{(k+1)}}$, and \mathcal{F}^{2^k} is termed as equivalent correlation matrix.

Definition 35: Let $\mathcal{F} = (f_{st})_{m \times m}$ be an equivalent correlation matrix, then $\mathcal{F}_\rho = (f_{st}^\rho)_{m \times m}$ is a Boolean matrix, called ρ -cutting matrix of \mathcal{F} , where

$$f_{st}^\rho = \begin{cases} 0 & \text{if } f_{st} < \rho, \\ 1 & \text{if } f_{st} \geq \rho, \end{cases}$$

$s, t = 1, 2, \dots, m$ and ρ is confidence level with $\rho \in [0, 1]$. We now develop a novel CBF clustering algorithm as follows:

Algorithm 2

Step 1. Let $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m\}$ be a huddle of CBFs on \mathcal{M} and $\{\xi_1, \xi_2, \dots, \xi_n\}$ be a set of attributes. First, construct the correlation matrix $\mathcal{F} = (f_{st})_{m \times m}$, where f_{st} can be computed by utilizing Eq.(5), i.e.,

$$f_{st} = \theta_3(\mathcal{L}_s, \mathcal{L}_t) = \sum_{i=1}^n w_i \frac{Cov_w(\mathcal{L}_s, \mathcal{L}_t)}{\sqrt{D_w(\mathcal{L}_s)D_w(\mathcal{L}_t)}}$$

where $w_i, i = 1, 2, \dots, n$ are the weights assigned to the attributes by decision experts.

Step 2. Find \mathcal{F}^2 and check whether $\mathcal{F}^2 \subseteq \mathcal{F}$. If this holds, then \mathcal{F} is the equivalent correlation matrix, otherwise, construct the equivalent correlation matrix \mathcal{F}^{2^k} :

$$\mathcal{F} \rightarrow \mathcal{F}^2 \rightarrow \mathcal{F}^4 \rightarrow \dots \rightarrow \mathcal{F}^{2^k} \rightarrow \dots, \text{ until } \mathcal{F}^{2^k} = \mathcal{F}^{2^{(k+1)}}.$$

Step 3. For a confidence level ρ , find a ρ -cutting matrix $\mathcal{F}_\rho = (f_{st}^\rho)_{m \times m}$, where f_{st}^ρ is defined by using definition 5.4.

Step 4. Classify the CBFs by the principle: If all the entries of sth row(column) in \mathcal{F}_ρ are the same as the corresponding entries of tth row(column) in \mathcal{F}_ρ , then the CBFs \mathcal{L}_s and \mathcal{L}_t belong to the same cluster, otherwise not.

The flow chart diagram of the proposed algorithm 2 is given in Figure 2.

1) NUMERICAL EXAMPLE

A robot is a re programmable multi-functional manipulator that can move material, parts, equipment, or specialized devices using variable programmed motion to perform a wide range of tasks. Due to recent development in information technology and engineering sciences, the utilization of robots has been increased in different advanced manufacturing systems. Robots are capable of performing repetitive,

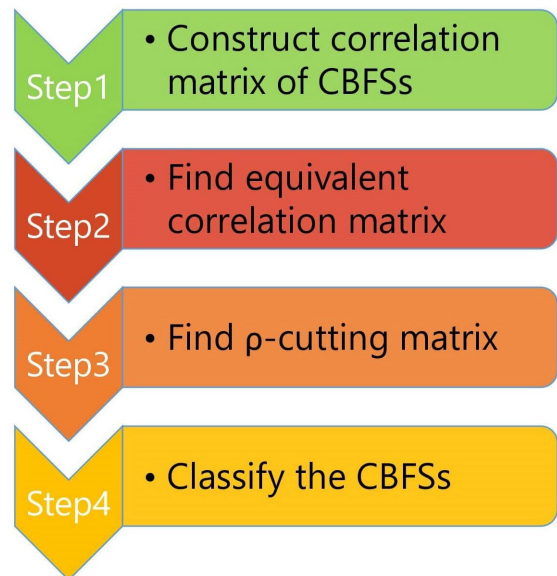


FIGURE 2. Flowchart diagram of algorithm 2.

challenging, and dangerous tasks with great accuracy. Therefore, a variety of industrial applications such as automated assembly, material handling, machine loading, spray painting and welding, are proficiently performed by the robots.

Suppose that there are five robots $\mathcal{L}_j, j = 1, 2, \dots, 5$, which are to be evaluated by a committee of technical experts on the basis of five attributes: ξ_1 (cost), ξ_2 (load capacity), ξ_3 (positioning accuracy), ξ_4 (repeatability) and ξ_5 (programming flexibility). The weight vector of these attributes is $w = (0.19, 0.20, 0.17, 0.21, 0.23)$.

To demonstrate the differences of the opinions of different experts, we present the evaluation information in the form of CBFs which is given in Table 4.

Now, we apply the clustering algorithm to cluster the robots.

Step 1: Calculate the correlation coefficients of $\mathcal{L}_j (j = 1, 2, \dots, 5)$ by using Eq.(5) and construct the correlation matrix:

$$\mathcal{F} = \begin{pmatrix} 1.000 & 0.505 & -0.474 & 0.174 & -0.222 \\ 0.505 & 1.000 & 0.236 & -0.343 & 0.079 \\ -0.474 & 0.236 & 1.000 & 0.218 & 0.341 \\ 0.174 & -0.343 & 0.218 & 1.000 & -0.125 \\ -0.222 & 0.079 & 0.341 & -0.125 & 1.000 \end{pmatrix}$$

TABLE 4. Evaluation information of 5 robots under 5 attributes.

	ξ_1	ξ_2
\mathcal{L}_1	$\langle\langle 0.37, 0.49 \rangle, [-0.44, -0.32], \{0.27, -0.46\}\rangle$	$\langle\langle 0.31, 0.43 \rangle, [-0.51, -0.39], \{0.42, -0.33\}\rangle$
\mathcal{L}_2	$\langle\langle 0.23, 0.35 \rangle, [-0.33, -0.21], \{0.39, -0.52\}\rangle$	$\langle\langle 0.42, 0.54 \rangle, [-0.41, -0.29], \{0.59, -0.49\}\rangle$
\mathcal{L}_3	$\langle\langle 0.44, 0.56 \rangle, [-0.24, -0.12], \{0.43, -0.37\}\rangle$	$\langle\langle 0.61, 0.73 \rangle, [-0.54, -0.42], \{0.29, -0.37\}\rangle$
\mathcal{L}_4	$\langle\langle 0.52, 0.64 \rangle, [-0.39, -0.27], \{0.56, -0.26\}\rangle$	$\langle\langle 0.27, 0.39 \rangle, [-0.39, -0.27], \{0.44, -0.46\}\rangle$
\mathcal{L}_5	$\langle\langle 0.39, 0.51 \rangle, [-0.29, -0.17], \{0.46, -0.47\}\rangle$	$\langle\langle 0.43, 0.55 \rangle, [-0.42, -0.30], \{0.58, -0.47\}\rangle$
	ξ_3	ξ_4
\mathcal{L}_1	$\langle\langle 0.59, 0.71 \rangle, [-0.39, -0.27], \{0.37, -0.21\}\rangle$	$\langle\langle 0.29, 0.41 \rangle, [-0.52, -0.40], \{0.41, -0.57\}\rangle$
\mathcal{L}_2	$\langle\langle 0.33, 0.45 \rangle, [-0.47, -0.35], \{0.47, -0.31\}\rangle$	$\langle\langle 0.57, 0.69 \rangle, [-0.39, -0.27], \{0.52, -0.27\}\rangle$
\mathcal{L}_3	$\langle\langle 0.27, 0.39 \rangle, [-0.56, -0.44], \{0.43, -0.53\}\rangle$	$\langle\langle 0.38, 0.50 \rangle, [-0.46, -0.34], \{0.46, -0.32\}\rangle$
\mathcal{L}_4	$\langle\langle 0.56, 0.68 \rangle, [-0.47, -0.35], \{0.66, -0.49\}\rangle$	$\langle\langle 0.43, 0.55 \rangle, [-0.63, -0.51], \{0.57, -0.66\}\rangle$
\mathcal{L}_5	$\langle\langle 0.61, 0.73 \rangle, [-0.55, -0.43], \{0.72, -0.63\}\rangle$	$\langle\langle 0.29, 0.41 \rangle, [-0.38, -0.26], \{0.39, -0.36\}\rangle$
	ξ_5	
\mathcal{L}_1	$\langle\langle 0.46, 0.58 \rangle, [-0.29, -0.17], \{0.58, -0.48\}\rangle$	
\mathcal{L}_2	$\langle\langle 0.49, 0.61 \rangle, [-0.27, -0.15], \{0.67, -0.43\}\rangle$	
\mathcal{L}_3	$\langle\langle 0.53, 0.65 \rangle, [-0.37, -0.25], \{0.36, -0.21\}\rangle$	
\mathcal{L}_4	$\langle\langle 0.29, 0.41 \rangle, [-0.49, -0.37], \{0.46, -0.38\}\rangle$	
\mathcal{L}_5	$\langle\langle 0.56, 0.68 \rangle, [-0.39, -0.27], \{0.69, -0.46\}\rangle$	

TABLE 5. Clustering results of 5 robots.

Class	Confidence level	Clusters
5	$0.5059 < \rho \leq 1$	$\{\mathcal{L}_1\}, \{\mathcal{L}_2\}, \{\mathcal{L}_3\}, \{\mathcal{L}_4\}, \{\mathcal{L}_5\}$
4	$0.3417 < \rho \leq 0.5059$	$\{\mathcal{L}_1, \mathcal{L}_2\}, \{\mathcal{L}_3\}, \{\mathcal{L}_4\}, \{\mathcal{L}_5\}$
3	$0.2367 < \rho \leq 0.3417$	$\{\mathcal{L}_1, \mathcal{L}_2\}, \{\mathcal{L}_3, \mathcal{L}_5\}, \{\mathcal{L}_4\}$
2	$0.2188 < \rho \leq 0.2367$	$\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_5\}, \{\mathcal{L}_4\}$
1	$0 \leq \rho \leq 0.2188$	$\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5\}$

Step 2: Calculate

$$\mathcal{F}^2 = \mathcal{F} \circ \mathcal{F} = \begin{pmatrix} 1.000 & 0.505 & 0.236 & 0.174 & 0.079 \\ 0.505 & 1.000 & 0.236 & 0.218 & 0.236 \\ 0.236 & 0.236 & 1.000 & 0.218 & 0.341 \\ 0.174 & 0.218 & 0.218 & 1.000 & 0.218 \\ 0.079 & 0.341 & 0.341 & 0.218 & 1.000 \end{pmatrix}$$

Since $\mathcal{F}^2 \notin \mathcal{F}$, so we construct the equivalent correlation matrix by calculating further compositions as follows:

$$\mathcal{F}^4 = \mathcal{F}^2 \circ \mathcal{F}^2 = \begin{pmatrix} 1.000 & 0.505 & 0.236 & 0.218 & 0.236 \\ 0.505 & 1.000 & 0.236 & 0.218 & 0.236 \\ 0.236 & 0.341 & 1.000 & 0.218 & 0.341 \\ 0.218 & 0.218 & 0.218 & 1.000 & 0.218 \\ 0.341 & 0.341 & 0.341 & 0.218 & 1.000 \end{pmatrix}$$

$$\mathcal{F}^8 = \mathcal{F}^4 \circ \mathcal{F}^4 = \begin{pmatrix} 1.000 & 0.505 & 0.236 & 0.218 & 0.236 \\ 0.505 & 1.000 & 0.236 & 0.218 & 0.236 \\ 0.341 & 0.341 & 1.000 & 0.218 & 0.341 \\ 0.218 & 0.218 & 0.236 & 1.000 & 0.218 \\ 0.341 & 0.341 & 0.341 & 0.218 & 1.000 \end{pmatrix}$$

$$\mathcal{F}^{16} = \mathcal{F}^8 \circ \mathcal{F}^8 = \begin{pmatrix} 1.000 & 0.505 & 0.236 & 0.218 & 0.236 \\ 0.505 & 1.000 & 0.236 & 0.218 & 0.236 \\ 0.341 & 0.341 & 1.000 & 0.218 & 0.341 \\ 0.236 & 0.236 & 0.236 & 1.000 & 0.236 \\ 0.341 & 0.341 & 0.341 & 0.218 & 1.000 \end{pmatrix}$$

$$\mathcal{F}^{32} = \mathcal{F}^{16} \circ \mathcal{F}^{16} = \begin{pmatrix} 1.000 & 0.505 & 0.236 & 0.218 & 0.236 \\ 0.505 & 1.000 & 0.236 & 0.218 & 0.236 \\ 0.341 & 0.341 & 1.000 & 0.218 & 0.341 \\ 0.236 & 0.236 & 0.236 & 1.000 & 0.236 \\ 0.341 & 0.341 & 0.341 & 0.218 & 1.000 \end{pmatrix}$$

Thus, \mathcal{F}^{16} is an equivalent correlation matrix.

Step 3: For a confidence level ρ , apply the definition 35 on the entries of equivalent correlation matrix \mathcal{F}^{16} to obtain a

ρ -cutting matrix. It is obvious that different values of ρ will give different ρ -cutting matrices.

Step 4: We discuss a sensitivity analysis on the basis of different values of confidence level ρ , and acquire all possible clusters of the 5 robots. The results are summarized in Table 5.

VI. CONCLUSION

CBFS is a generalization of bipolar fuzzy set which handles the two-sided approach of decision analysis and inexactness of the data by taking into consideration both IVBFS and BFS simultaneously. In this article, we defined some well-known terminologies for CBFSs which include concentration, dilation, support and core of a CBFS. We developed cubic bipolar fuzzy relations along with their certain types. In statistics, correlation coefficient examines the strength of relationship between two variables. It also determines whether two variables tend to behave in the same manner (positive correlation) or in the opposite manner (negative correlation). The correlation coefficients for CBFSs should also possess these characteristics. Therefore, as in statistics with real variables, we defined variance and covariance for two CBFSs and then, developed correlation coefficients and their weighted extensions on the basis of variance and covariance of CBFSs. These correlation coefficients range in $[-1, 1]$ which make them superior to some of the existing correlation coefficients that lie in $[0, 1]$. We investigated some properties of these correlation coefficients. We applied these correlation coefficients to pattern recognition and clustering analysis and demonstrated their usability and effectiveness through numerical illustrations. In the future, we will study distance and similarity measures [36] and knowledge measures [40] for CBFS.

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