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Exponential Stability for a Class of Linear Delay Differential Systems Under Logic Impulsive Control

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ABSTRACT This paper is concerned with the exponential stability for a class of linear delay differential systems under logic impulsive control. Based on logic impulsive control, the strong constraint that the coefficient functions of this kind of linear delay differential system need to be non-negative, which is required in most of the previous studies, is reduced. By establishing the relationship between the logic impulsive system and a corresponding non-impulsive system, some exponential stability criteria for linear delay differential systems under logic impulsive control are presented in the following two cases: *Case A*–The coefficient functions are non-negative, *Case B*–The coefficient functions can be negative. Two numerical examples are discussed to verify the results.

INDEX TERMS Exponential stability, impulsive control, logic choice, linear delay differential system.

I. INTRODUCTION

In recent years, Berezansky L. *et al.* have made a lot of in-depth researches on the oscillation and stability of the following linear differential systems:

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \geq t_0 \tag{1}$$

where $a_k(t)$ is Lebesgue measurable essentially bounded function, $h_k(t)$ is Lebesgue measurable function satisfies $h_k(t) \leq t$ and $\limsup_{t\to\infty} h_k(t) = \infty$, $k = 1, \ldots, m$, see [1]–[3] and references therein. Note that, almost all the results in above previous studies are based on a strong constraint requirement, that is the coefficient functions need to be non-negative, namely $a_k(t) \geq 0$, $k = 1, 2, \ldots, m$. Even in subsequent studies of higher dimensional systems:

$$\dot{x}_i(t) + \sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)) = 0, \quad t \geq t_0, \quad i = 1, \ldots, n$$

The diagonal terms condition $a_{ii}^k(t) \geq 0$, $i = 1, 2, \ldots, n$ is always assumed, which means the dominance of diagonal elements (see [4], [5]). This situation was improved in [6] and [7]. In [6], the dominance of diagonal terms was not required, but after careful derivation, the author found that the stability results under the weakening diagonal elements dominance condition have higher requirements for the delay conditions, and if the stability result is applied to the scalar case (1-dimension), the condition $a_k(t) \geq 0$ is still required. In [7], though each $a_k(t) \geq 0$ is not required, but $\sum_{k=1}^{m} a_k(t) > 0$ is implicated in the main stable results.

On the other hand, due to the importance of theory and application, impulsive system has always been the focus of many researchers (see [8]–[33] and references therein). From the perspective of the research topics on impulsive systems, in addition to periodic solutions [8], oscillation [9], noise [10], etc., various kinds of stability have also been studied extensively. For example, exponential stability [11], practical stability [12], interval stability [13], finite-time stability [14], numerical stability [15] and so on. From the perspective of the classification of impulsive systems, many research results have been reported for linear systems [9], [11], [13], [16], [17], nonlinear systems [18], [19], functional differential systems [12], [20], [21], integro-differential systems [22], fractional differential systems [23], [24] and so on. From the perspective of the

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model composition of the impulsive system, due to the needs of practical problems, time-delay and stochastic effects are often taken into account in the impulsive model. Therefore, the delay impulsive systems such as [9], [11], [13], [15], [16], [18], [19], [20], [25], [26], stochastic impulsive systems such as [27], [28], and more complex stochastic delay impulsive systems such as [21], [29] should be fully considered.

In some practical situations, impulsive control can make the system more realistic or make the system model have better properties, such as the unstable system under impulsive control becomes stable. Recently, logic impulses, in other words, impulsive control influenced by logic choice, have been proposed in [30]. Because of the input of logic effects, logic impulsive control may have better control effect than general impulsive control. So far, the research on logic impulsive system (LIS) has yielded some results. For example, Suo J. *et al.* studied asymptotic stability of differential LIS in [30], Zhang J. *et al.* gave the finite-time stability results of nonlinear LIS in [31], He Z. *et al.* considered the stability of discrete LIS in [32], and Li C. obtained some stability criteria of stochastic LIS in [33].

Meanwhile, the author note that for system (1), Yan J. *et al.* studies its stability under general linear impulsive control in [9] and Li C. studies its stability under logic impulsive control with stochastic effects in [33], both of which are studied by establishing the relationship between the impulsive system and a corresponding non-impulsive system.

Inspired by the above analysis, the main purpose and work of this paper are as follows: (i) By constructing logic impulsive control to system (1), break through the limitation of the requirement that the coefficient function of system (1) is non-negative. (ii) By establishing the relationship between the system (1) under logic impulsive control and a corresponding non-impulsive system, some exponential stability criteria for linear delay differential systems under logic impulsive control are presented in the following two cases: *Case A*–The coefficient functions are non-negative, *Case B*–The coefficient functions can be negative. It's worth noting that, some criteria extend the results in [9], [11] and [15]. (iii) Two numerical examples are discussed to verify the results.

This paper is organized as follows: Some basic notations, lemmas and the model introduction are collected in Section II; The main results are given in Section III, in which stability criteria for the case that the coefficient functions are non-negative are given in subsection A, and stability criteria for the case that the coefficient functions can be negative are given in subsection B; Two numerical examples are discussed in Section IV; Concluding remark and future work are given in Section V.

II. PRELIMINARIES

At the beginning, some necessary notations and lemmas are given which are used throughout the whole paper.

For a vector A, the transpose is denoted by A^T. Let $[\cdot]$ denotes the greatest integer function, namely, for a real number a, $[a]$ represents the greatest integer which is not exceeding a. Let $\Delta_2 = \{\delta_2^i | i = 1, 2\}$, δ_2^i is the ith column of the identity matrix I_2, and identify logical values with equivalent vectors as: $T = 1 \sim \delta_2^1$, $F = 0 \sim \delta_2^2$.

The following lemmas are some sufficient conditions for the exponential stability of linear differential system (1).

Lemma 2.1 (See [2], Corollary 2.2; [3], Lemma 2.8): Suppose $a_k \geq 0$, $\liminf_{t\to\infty} \sum_{k=1}^m a_k(t) > 0$, $\limsup_{t\to\infty}(t-h_k(t)) < \infty$, $k = 1, \ldots, m$, and there exists $r(t) \leq t$ such that for sufficiently large t, $\int_{r(t)}^t \sum_{k=1}^m a_k(s)ds \leq \frac{1}{e}$. If

$$\limsup_{t\to\infty} \sum_{k=1}^m \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \left| \int_{h_k(t)}^{r(t)} \sum_{i=1}^m a_i(s)ds \right| < 1.$$

Then (1) is exponentially stable.

Lemma 2.2 (See [3], Lemma 3.1): Suppose $a_k \geq 0$, $\liminf_{t\to\infty} \sum_{k=1}^m a_k(t) > 0$, $\limsup_{t\to\infty}(t - h_k(t)) < \infty$, $k = 1, \ldots, m$, if

$$\limsup_{t\to\infty} \sum_{k=1}^m \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \int_{h_k(t)}^t \sum_{i=1}^m a_i(s)ds < 1 + \frac{1}{e}.$$

Then (1) is exponentially stable.

Lemma 2.3 (See [3], Corollary 3.2): Suppose $a_k \geq 0$, $\liminf_{t\to\infty} \sum_{k=1}^m a_k(t) > 0$, $\limsup_{t\to\infty}(t - h_k(t)) < \infty$, $k = 1, \ldots, m$, if

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^t \sum_{i=1}^m a_i(s)ds < 1 + \frac{1}{e}.$$

Then (1) is exponentially stable.

Lemma 2.4 (See [3], Theorem 3.3): Suppose $a_k \geq 0$, $\sum_{k=1}^m a_k(t) > 0$, $\int_0^\infty \sum_{k=1}^m a_k(t)dt = \infty$, $\limsup_{t\to\infty} \int_{h_k(t)}^t \sum_{i=1}^m a_i(s)ds < \infty$, $\limsup_{t\to\infty}(t - h_k(t)) < \infty$, $k = 1, \ldots, m$, and

$$\limsup_{t\to\infty} \sum_{k=1}^m \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \int_{h_k(t)}^t \sum_{i=1}^m a_i(s)ds < 1 + \frac{1}{e}.$$

If there exists a constant $R > 0$ such that

$$\liminf_{t\to\infty} \int_t^{t+R} \sum_{i=1}^m a_i(s)ds > 0.$$

Then (1) is exponentially stable.

A. LIS WITH INFINITE DELAYS

Consider the following infinite delay differential systems under logic impulsive control:

$$\begin{cases} \dot{y}(t) + \sum_{k=1}^m a_k(t)y(h_k(t)) = 0, & t \neq t_i,\ t \geq \sigma \\ y(t_i^+) - y(t_i) = \Phi_i(y(t_i)), & t = t_i,\ i \in N \\ y(t) = \varphi(t), & t \in [r_\sigma,\ \sigma] \end{cases} \tag{2}$$

with the following assumptions:

(A1) (see [9]) $a_k(t) : [t_0, \infty) \to R$ are locally summable functions;

(A2) (see [9]) $h_k(t) : [t_0, \infty) \to R$ are Lebesgue measurable functions and $h_k(t) \le t, \lim_{t\to\infty} h_k(t) = \infty$.

(A3) (see [9]) $0 \le t_0 < t_1 < \ldots < t_i < \ldots$ are fixed impulsive points and $\lim_{i\to\infty} t_i = \infty$.

(A4) (see [33], subtle improvement) $\Phi_i(y(t_i))$ are the logic impulsive control in the system, which affected by p logical index related to $y(t_i)$, is denoted by:

$$\Phi_i(y(t_i)) = \sum_{l=1}^{p}[I_i^l(y(t_i)), J_i^l(y(t_i))]g_l(y(t_i))$$

where $I_i^l, J_i^l \in C(R, R)$ satisfies $I_i^l(0) = J_i^l(0) = 0$, $l = 1, 2, \ldots, p$, $i \in N$. $g_l : R \to \{\delta_2^1, \delta_2^2\}$ is a piecewise logical function as follows:

$$g_l(u) = \begin{cases} \delta_2^2, & |f_l(u)| \ge c_l \\ \delta_2^1, & |f_l(u)| < c_l \end{cases}$$

$f_l \in C(R, R)$, $c_l > 0$ is the threshold, for $l = 1, 2, \ldots, p$. Then, there are p^2 choices for each Φ_i at $t = t_i$, which can also be described as:

$$\Phi_i = [I_i^1, J_i^1, \ldots, I_i^p, J_i^p][g_1^T, \ldots, g_p^T]^T$$

(A5) (see [9]) $\varphi(t) : [r_\sigma, \sigma] \to R$ is real-valued absolutely continuous in $[t_i, t_{i+1}) \bigcap (r_\sigma, \sigma)$ and may have discontinuity of the first kind at t_i situated in $(r_\sigma, \sigma]$. Where, for any $\sigma \ge t_0$, r_σ denoted

$$r_\sigma = \min_{1\le k\le m} \inf_{t\ge\sigma} h_k(t).$$

A function $y : [r_\sigma, \infty) \to R$ is called a solution of (2) (see [9]), if: (i) $y(t)$ is absolutely continuous on each interval $(t_i, t_{i+1}] \subset [r_\sigma, \infty)$. (ii) For any $t_i \in [\sigma, \infty)$, $y(t_i^+)$ and $y(t_i^-)$ exist, $y(t_i^-) = y(t_i)$. (iii) $y(t)$ satisfies the differential equation in (2) almost everywhere on $[\sigma, \infty)$ and may have discontinuity of the first kind at t_i situated in $[\sigma, \infty)$, and satisfies the initial condition on $[r_\sigma, \sigma]$.

Easy to see that, $y(t) = 0$ is a solution of (2), which is called the zero solution. Throughout this paper, we also assume that $y(t_i) \neq 0$ and $\Phi_i(y(t_i)) \neq -y(t_i)$.

Hypothesis 2.1 (See [33]): Suppose there exists $\alpha(t)$ satisfies: (i) $\alpha(t)$ is continuous differential on each interval $(t_i, t_{i+1}] \subset [\sigma, \infty)$.

(ii) For any $t_i \in [\sigma, \infty)$, $\alpha(t_i) = \left(1 + \frac{\Phi_i(y(t_i))}{y(t_i)}\right)\alpha(t_i^+)$.

(iii) $\alpha(t) \neq 0, \forall t > \sigma$ and $\alpha(t) = 1, \forall t \le \sigma$.

Remark 2.1: According to the characteristics of the given differential equations, many different forms of function $\alpha(t)$ can be selected, see [9], [11], [15] and [33].

In this paper, we denote $\alpha_k^t := \dfrac{\alpha(t)}{\alpha(h_k(t))}$.

Now, a non-impulsive delay differential system can be proposed:

$$\begin{cases} \dot{x}(t) - \dfrac{\dot{\alpha}(t)}{\alpha(t)}x(t) + \displaystyle\sum_{k=1}^{m} \alpha_k^t a_k(t)x(h_k(t)) = 0, & t \ge \sigma \\ x(t) = \varphi(t), & t \in [r_\sigma, \sigma] \end{cases} \tag{3}$$

An absolutely continuous function $x(t)$ is said to be a solution of system (3) (see [9]), if: $x(t)$ satisfies the differential equation in (3) almost everywhere and the initial condition on $[r_\sigma, \sigma]$.

B. A CLASS OF LIS WITH FINITE DELAYS

In this part, based on system (2), a class of LIS with finite time delay is constructed and its exponential stability is studied in the following.

Firstly, a necessary assumption for system (2), which shows that the time delay is finite, is given as follows:

(A6) (see [4]–[6]) There exist constants $\tau_k \ge 0$ such that $t - h_k(t) \le \tau_k$, $k = 1, 2, \ldots, m$, and $\tau := \max_{1\le k\le m}\{\tau_k\}$.

Remark 2.2: Under assumption (A6), one can infer that

$$\limsup_{t\to\infty}(t - h_k(t)) < \infty, \quad k = 1, 2, \ldots, m$$

This condition is frequently encountered in the stability analysis of system (1) (see [1]–[7]).

Next, a linear logic impulsive control is constructed. Let $I_i^l(y(t_i)) = \lambda_i^l \cdot y(t_i)$, $J_i^l(y(t_i)) = \mu_i^l \cdot y(t_i)$, where λ_i^l, μ_i^l are real number, $l = 1, 2, \ldots, p$, $i \in N$, then

$$\begin{aligned} \Phi_i(y(t_i)) &= \sum_{l=1}^{p} y(t_i)[\lambda_i^l, \mu_i^l]g_l(y(t_i)) \\ &= \left(\sum_{l=1}^{p}[\lambda_i^l, \mu_i^l]g_l(y(t_i))\right) \cdot y(t_i) \\ &:= \phi_i \cdot y(t_i). \end{aligned}$$

In addition, for brevity, set the initial time to $\sigma = t_0$.

By now, a class of linear finite delay differential systems under logic impulsive control is constructed:

$$\begin{cases} \dot{y}(t) + \displaystyle\sum_{k=1}^{m} a_k(t)y(h_k(t)) = 0, & t \neq t_i,\ t \ge t_0 \\ y(t_i^+) - y(t_i) = \phi_i y(t_i), & t = t_i,\ k \in N \\ y(t) = \varphi(t), & t \in [t_0 - \tau,\ t_0] \end{cases} \tag{4}$$

On the other hand, set the function $\alpha(t)$ as follows:

$$\alpha(t) := \begin{cases} \displaystyle\prod_{t_0 \le t_j < t} \frac{1}{1 + \phi_j} & t > t_1 \\ 1 & t_0 - \tau \le t \le t_1 \end{cases} \tag{5}$$

thus,

$$\alpha_k^t = \begin{cases} \displaystyle\prod_{h_k(t) \le t_j < t} \frac{1}{1 + \phi_j} & t > t_1 \\ 1 & t_0 - \tau \le t \le t_1 \end{cases}$$

It is easy to verify that $\alpha(t)$ defined by (5) satisfies Hypothesis 2.1, and it is clearly a piecewise constant function, so $\dot{\alpha}(t) = 0$.

Accordingly, based on system (3), a non-impulsive delay differential system is proposed as follows:

$$\begin{cases} \dot{x}(t) + \sum_{k=1}^{m} \alpha_k^t a_k(t) x(h_k(t)) = 0, & t \geq t_0 \\ x(t) = \varphi(t), & t \in [t_0 - \tau,\ t_0] \end{cases} \tag{6}$$

Definition 2.1 (See [11]): The zero solution of (4) is said to be exponentially stable, if there exist a pair of positive constants λ and K such that, for any initial function $\varphi(t)$,

$$|y(t)| \leq K\|\varphi\| e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

where $\|\varphi\| = \sup_{t_0-\tau \leq t \leq t_0} |\varphi(t)|$.

The exponential stability of system (6) can be similarly defined, omitted here.

III. STABILITY CRITERIA

Lemma 3.1: Assume that (A1)–(A5) hold, and $\alpha(t)$ satisfies Hypothesis 2.1.

(i) if $x(t)$ is a solution of (3), then $y(t) = \alpha^{-1}(t)x(t)$ is a solution of (2) on $[r_\sigma, +\infty)$.

(ii) if $y(t)$ is a solution of (2), then $x(t) = \alpha(t)y(t)$ is a solution of (3) on $[r_\sigma, +\infty)$.

Proof: The proof is similar to Lemma 3.1 in [33], omitted here.

Remark 3.1: It's worth noting that, system (2) has infinite time delays and logic impulses, so Lemma 2.1 generalizes Theorem 1 in [9], Theorem 3.1 in [11], Theorem 2.2 in [15]. Moreover, based on our assumptions, it is not difficult to generalize Lemma 3.1 in [33] to the systems with infinite time delay.

Lemma 3.2: Assume that (A1)–(A6) hold, and $\alpha(t)$ is defined by (5).

(i) Suppose $\alpha^{-1}(t)$ is bounded on the interval $[t_0 - \tau,\ \infty)$, then if the zero solution of (6) is exponentially stable, the zero solution of (4) is also exponentially stable.

(ii) Suppose $\alpha(t)$ is bounded on the interval $[t_0 - \tau,\ \infty)$, then if the zero solution of (4) is exponentially stable, the zero solution of (6) is also exponentially stable.

(iii) Suppose both $\alpha^{-1}(t)$ and $\alpha(t)$ are bounded on the interval $[t_0 - \tau,\ \infty)$, then the zero solution of (4) is exponentially stable if and only if the zero solution of (6) is exponentially stable.

Proof: The proof is similar to Theorem 3.1 in [33], omitted here.

A. THE COEFFICIENT FUNCTIONS ARE NON-NEGATIVE

In this part, some exponential stability criteria are proposed for system (4) where all the coefficient functions $a_k(t)$ are non-negative.

Put forward the following assumption:

(A7) $\lambda_i^l,\ \mu_i^l > -\frac{1}{p}$, $\liminf_{i\to\infty} \lambda_i^l > -\frac{1}{p}$, $\liminf_{i\to\infty} \mu_i^l > -\frac{1}{p}$, $\limsup_{i\to\infty} \lambda_i^l < \infty$, $\limsup_{i\to\infty} \mu_i^l < \infty$, $l = 1, 2, \ldots, p$, and $\alpha(t)$ is defined by (5).

Remark 3.2: Under assumptions (A6)–(A7), there exist constants $\varpi_1 \geq \varpi_2 > 0$ such that $\varpi_2 \leq \alpha_k^t \leq \varpi_1$.

The reasons for Remark 3.2 are as follows:

Under assumption (A7), it's easy to verify that $\phi_i = \sum_{l=1}^{p} [\lambda_i^l, \mu_i^l] g_i(y(t_i)) > -1$, $\liminf_{i\to\infty} \phi_i > -1$ and $\limsup_{i\to\infty} \phi_i < \infty$. Note that $\lim_{i\to\infty} t_i = \infty$ and (A6) is hold, we may assume that there are at most q impulsive points on the interval $[h_k(t), t)$, $\forall t \geq t_0$, $k = 1, 2, \ldots, m$. Then, the values of ϖ_1 and ϖ_2 are as follows:

$$\varpi_1 = \max_{\substack{j\in N,\\ k=1,\ldots,q}} \left\{ 1, \prod_{i=j,\ldots,j+k-1} \frac{1}{1+\phi_i} \right\}$$

$$\varpi_2 = \min_{\substack{j\in N,\\ k=1,\ldots,q}} \left\{ 1, \prod_{i=j,\ldots,j+k-1} \frac{1}{1+\phi_i} \right\}$$

For example, if $t_{i+1} - t_i \geq \tau$, then there exists at most one impulsive point on the interval $[h_k(t), t)$, $\forall t \geq t_0$, $k = 1, 2, \ldots, m$, thus

$$\min_{i\in N} \left\{ \frac{1}{1+\phi_i}, 1 \right\} \leq \alpha_k^t \leq \max_{i\in N} \left\{ \frac{1}{1+\phi_i}, 1 \right\}.$$

The following theorems (Theorems 3.1-3.4) are the main results of this paper. The meanings of ϖ_1 and ϖ_2 in Theorems 3.1-3.4 are as mentioned in Remark 3.2, which will not be emphasized later for brevity.

Theorem 3.1: Assume that (A1)–(A7) hold, and

(i) $a_k(t) \geq 0$, $k = 1, 2, \ldots, m$.

(ii) $\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t) > 0$.

(iii) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \leq \Xi$.

(iv) There exists $r(t) \leq t$ such that for sufficiently large t

$$\int_{r(t)}^{t} \sum_{k=1}^{m} a_k(s)ds \leq \frac{1}{e\varpi_1}.$$

(v)

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \left| \int_{h_k(t)}^{r(t)} \sum_{i=1}^{m} a_i(s)ds \right| < \frac{\varpi_2}{\varpi_1^2}.$$

Then, the zero solution of (4) is exponentially stable.

Proof: Step 1. According to (A7), conditions (i) and (ii), we have $\alpha_k^t a_k(t) \geq \varpi_2\ a_k(t) \geq 0$, $k = 1, 2, \ldots, m$, and

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \alpha_k^t a_k(t) \geq \liminf_{t\to\infty} \sum_{k=1}^{m} \varpi_2 a_k(t)$$
$$= \varpi_2 \liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t) > 0$$

Meanwhile, $\limsup_{t\to\infty}(t - h_k(t)) < \infty$ holds due to (A6).

Step 2. From condition (iv), there exists $r(t) \le t$ such that for sufficiently large t,

$$\int_{r(t)}^{t} \sum_{k=1}^{m} \alpha_k^t a_k(s)ds \le \int_{r(t)}^{t} \sum_{k=1}^{m} \varpi_1 a_k(s)ds$$
$$\le \varpi_1 \int_{r(t)}^{t} \sum_{k=1}^{m} a_k(s)ds$$
$$\le \varpi_1 \frac{1}{e\varpi_1} = \frac{1}{e}$$

Step 3. From condition (v), we have

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{\alpha_k^t a_k(t)}{\sum_{i=1}^{m} \alpha_i^t a_i(t)} \left| \int_{h_k(t)}^{r(t)} \sum_{i=1}^{m} \alpha_i^s a_i(s)ds \right|$$
$$\le \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{\varpi_1 a_k(t)}{\sum_{i=1}^{m} \varpi_2 a_i(t)} \left| \int_{h_k(t)}^{r(t)} \sum_{i=1}^{m} \varpi_1 a_i(s)ds \right|$$
$$\le \frac{\varpi_1^2}{\varpi_2} \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \left| \int_{h_k(t)}^{r(t)} \sum_{i=1}^{m} a_i(s)ds \right|$$
$$< 1$$

Step 4. From Lemma 2.1, (6) is exponentially stable. Then according to condition (iii) and Lemma 3.2, (4) is also exponentially stable.

Theorem 3.2: Assume that (A1)–(A7) hold, and

(i) $a_k(t) \ge 0, k = 1, 2, \dots, m$.

(ii) $\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t) > 0$.

(iii) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \le \Xi$.

(iv)

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < \left(1 + \frac{1}{e}\right) \frac{\varpi_2}{\varpi_1^2}.$$

Then, the zero solution of (4) is exponentially stable.

Proof: From condition (iv), we have

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{\alpha_k^t a_k(t)}{\sum_{i=1}^{m} \alpha_i^t a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} \alpha_i^s a_i(s)ds$$
$$\le \frac{\varpi_1^2}{\varpi_2} \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds$$
$$< 1 + \frac{1}{e}$$

The verification of the remaining conditions is the same as the proof in the step 1 of Theorem 3.1, from Lemma 2.2, system (6) is exponentially stable. Then according to condition (iii) and Lemma 3.2, (4) is also exponentially stable.

Theorem 3.3: Assume that (A1)–(A7) hold, and

(i) $a_k(t) \ge 0, k = 1, 2, \dots, m$.

(ii) $\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t) > 0$.

(iii) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \le \Xi$.

(iv)

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{i=1}^{m} a_i(s)ds < \left(1 + \frac{1}{e}\right) \frac{1}{\varpi_1}.$$

Then, the zero solution of (4) is exponentially stable.

Proof: From condition (iv), we have

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{i=1}^{m} \alpha_i^s a_i(s)ds$$
$$\le \varpi_1 \limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{i=1}^{m} a_i(s)ds$$
$$< 1 + \frac{1}{e}$$

The verification of the remaining conditions is the same as the proof in the step 1 of Theorem 3.1, from Lemma 2.3, system (6) is exponentially stable. Then according to condition (iii) and Lemma 3.2, (4) is also exponentially stable.

In the following stability result, $\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t) > 0$ is no longer required.

Theorem 3.4: Assume that (A1)–(A7) hold, and

(i) $a_k(t) \ge 0$, $k = 1, 2, \dots, m$, and $\sum_{k=1}^{m} a_k(t) > 0$.

(ii) $\int_0^\infty \sum_{k=1}^{m} a_k(t)dt = \infty$

(iii) $\limsup_{t\to\infty} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < \infty$.

(iv) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \le \Xi$.

(v) There exists a constant $R > 0$ such that

$$\liminf_{t\to\infty} \int_{t}^{t+R} \sum_{i=1}^{m} a_i(s)ds > 0.$$

(vi)

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < \left(1 + \frac{1}{e}\right) \frac{\varpi_2}{\varpi_1^2}.$$

Then, the zero solution of (4) is exponentially stable.

Proof: Note that $0 < \varpi_2 \le \alpha_k^t \le \varpi_1$, $\alpha_k^t a_k(t) \ge 0$, and

$$\sum_{k=1}^{m} \alpha_k^t a_k(t) \ge \varpi_2 \sum_{k=1}^{m} a_k(t) > 0.$$

From condition (ii), we have

$$\int_0^\infty \sum_{k=1}^{m} \alpha_k^t a_k(t)dt = \infty$$

From condition (iii), we have

$$\limsup_{t\to\infty} \int_{h_k(t)}^{t} \sum_{i=1}^{m} \alpha_i^s a_i(s)ds$$
$$\le \varpi_1 \limsup_{t\to\infty} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds$$
$$< \infty$$

From condition (v), there exists a constant $R > 0$ such that

$$\liminf_{t\to\infty} \int_{t}^{t+R} \sum_{i=1}^{m} \alpha_i^s a_i(s)ds$$
$$\ge \varpi_2 \liminf_{t\to\infty} \int_{t}^{t+R} \sum_{i=1}^{m} a_i(s)ds$$
$$> 0$$

From condition (vi), the same proof as in Theorem 3.2, we have

$$\limsup_{t\to\infty}\sum_{k=1}^{m}\frac{\alpha_k^t a_k(t)}{\sum_{i=1}^m \alpha_i^t a_i(t)}\int_{h_k(t)}^{t}\sum_{i=1}^{m}\alpha_i^s a_i(s)ds < 1+\frac{1}{e}$$

Then, according to Lemma 2.4, (6) is exponentially stable. From condition (iv) and Lemma 3.2, (4) is also exponentially stable.

B. THE COEFFICIENT FUNCTIONS CAN BE NEGATIVE

In this part, the exponential stability of a class of linear delay systems under logic impulsive control is discussed. For this kind of system, the requirement that the coefficient function be non-negative, which exists in most of the previous results, is reduced. That is, the coefficient function can be negative.

Consider the following linear delay differential systems under logic impulsive control:

$$\begin{cases} \dot{y}(t)+\sum_{k=1}^{m}\delta_k(t)a_k(t)y(t-k\tau)=0, & t\neq t_i,\ t\geq t_0 \\ y(t_i^+)-y(t_i)=\phi_i y(t_i), & t=t_i,\ k\in N \\ y(t)=\varphi(t), & t\in[t_0-m\tau,\ t_0] \end{cases} \tag{7}$$

where $\tau > 0$ is a constant, $t_{i+1}-t_i=\tau, i=0,1,2,\cdots$, the logic impulsive control $\phi_i y(t_i)$ is the same as in system (4), $\delta_k(t)$ is defined by:

$$\delta_k(t) := \begin{cases} (-1)^{j+1} & t\in(t_j,t_{j+1}], j=0,1,\ldots,k-1. \\ (-1)^{k+1} & t\in(t_k,\infty) \end{cases}$$

Meanwhile, $\alpha(t)$ is still defined by (5), namely

$$\alpha(t)=\begin{cases} 1 & t\in[t_0-m\tau,t_1] \\ \prod_{t_0\leq t_j<t}\frac{1}{1+\phi_j} & t>t_1 \end{cases}$$

It's obvious that, $\alpha_k^t=\dfrac{\alpha(t)}{\alpha(t-k\tau)}$.

Accordingly, the non-impulsive delay differential system is as follows:

$$\begin{cases} \dot{x}(t)+\sum_{k=1}^{m}\delta_k(t)\alpha_k^t a_k(t)y(t-k\tau)=0, & t\geq t_0 \\ x(t)=\varphi(t), & t\in[t_0-m\tau,t_0] \end{cases} \tag{8}$$

Remark 3.3: If $\lambda_i^l,\ \mu_i^l < -\frac{1}{p}, l=1,2,\ldots,p, i\in N$, then $\phi_i < -1$ for all $i\in N$, and $\delta_k(t)\alpha_k^t < 0, k=1,2,\ldots,m$.

Now, let's give a check on Remark 3.3.

It's easy to see that $\phi_i=\sum_{l=1}^{p}[\lambda_i^l,\mu_i^l]g_i(x(t_i)) < -1$ due to $\lambda_i^l,\mu_i^l < -\frac{1}{p}$. By derivation, α_k^t is of the following form:

$$\alpha_k^t=\frac{\alpha(t)}{\alpha(t-k\tau)}$$

$$=\begin{cases} 1 & t\in(t_0,t_1] \\ \frac{1}{1+\phi_1} & t\in(t_1,t_2] \\ \frac{1}{1+\phi_1}\frac{1}{1+\phi_2} & t\in(t_2,t_3] \\ \vdots & \vdots \\ \frac{1}{1+\phi_1}\frac{1}{1+\phi_2}\cdots\frac{1}{1+\phi_{k-1}} & t\in(t_k-1,t_k] \\ \frac{1}{1+\phi_1}\frac{1}{1+\phi_2}\cdots\frac{1}{1+\phi_k} & t\in(t_k,t_{k+1}] \\ \frac{1}{1+\phi_2}\frac{1}{1+\phi_3}\cdots\frac{1}{1+\phi_{k+1}} & t\in(t_{k+1},t_{k+2}] \\ \vdots & \vdots \\ \frac{1}{1+\phi_{j-k+1}}\cdots\frac{1}{1+\phi_{j-1}}\frac{1}{1+\phi_j} & t\in(t_j,t_{j+1}] \\ \vdots & \vdots \end{cases}$$

that is,

$$\alpha_k^t=\begin{cases} 1 & t\in(t_0,t_1] \\ \prod_{i=1,\ldots,j}\frac{1}{1+\phi_i} & t\in(t_j,t_{j+1}] \\ \prod_{i=i^*,\ldots,i^*+k-1}\frac{1}{1+\phi_i} & t\in(t_k,\infty) \end{cases}$$

where, $j=1,2\ldots,k-1$, $i^*=[\frac{t}{\tau}]-k+1$. So, we have

$$\delta_k(t)\alpha_k^t=\begin{cases} -1 & t\in(t_0,t_1] \\ (-1)^{j+1}\prod_{i=1,\ldots,j}\frac{1}{1+\phi_i} & t\in(t_j,t_{j+1}] \\ (-1)^{k+1}\prod_{i=i^*,\ldots,i^*+k-1}\frac{1}{1+\phi_i} & t\in(t_k,\infty) \end{cases}$$

for $j=1,2\ldots,k-1$, $i^*=[\frac{t}{\tau}]-k+1$.

Thus, we have come to a conclusion that $\delta_k(t)\alpha_k^t < 0$, for all $i\in N, k=1,2,\ldots,m$.

Put forward the following assumption:

(A8) $\lambda_i^l,\ \mu_i^l < -\frac{1}{p}$, $\limsup_{i\to\infty}\lambda_i^l < -\frac{1}{p}$, $\limsup_{i\to\infty}\mu_i^l < -\frac{1}{p}$, $\liminf_{i\to\infty}\lambda_i^l > -\infty$, $\liminf_{i\to\infty}\mu_i^l > -\infty$, $l=1,2,\ldots,p$, and $\alpha(t)$ is defined by (5).

Remark 3.4: Under assumption (A8), it is easy to verify that $\limsup_{i\to\infty}\phi_i < -1$ and $\liminf_{i\to\infty}\phi_i > -\infty$, then there exist constants $\varrho_1\geq\varrho_2 > 0$ such that $-\varrho_1\leq\delta_k(t)\alpha_k^t\leq-\varrho_2$.

The values of ϱ_1 and ϱ_2 are as follows:

$$\varrho_1=\max_{\substack{j\in N,\\ k=1,\ldots,m}}\left\{1,\ \prod_{i=j,\ldots,j+k-1}\left|\frac{1}{1+\phi_i}\right|\right\}$$

$$\varrho_2=\min_{\substack{j\in N,\\ k=1,\ldots,m}}\left\{1,\ \prod_{i=j,\ldots,j+k-1}\left|\frac{1}{1+\phi_i}\right|\right\}$$

The following theorems (Theorems 3.5-3.8), are the main results of this paper. The meanings of ϱ_1 and ϱ_2 in

Theorems 3.5-3.8 are as mentioned in Remark 3.4, which will not be emphasized later for brevity.

Theorem 3.5: Assume that (A1)–(A5), (A8) hold, and

(i) $a_k(t) \le 0, k = 1, 2, \ldots, m$.

(ii) $\limsup_{t\to\infty} \sum_{k=1}^m a_k(t) < 0$.

(iii) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \le \Xi$.

(iv) There exists $r(t) \le t$ such that for sufficiently large t

$$\int_{r(t)}^{t} \sum_{k=1}^{m} a_k(s)ds \ge -\frac{1}{e\varrho_1}.$$

(v)

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \left| \int_{t-k\tau}^{r(t)} \sum_{i=1}^{m} a_i(s)ds \right| < \frac{\varrho_2}{\varrho_1^2}.$$

Then, the zero solution of (7) is exponentially stable.

Proof: Step 1. From conditions (i) and (ii), we have $\delta_k(t)\alpha_k^t a_k(t) \ge -\varrho_2\, a_k(t) \ge 0, k = 1, 2, \ldots, m$, and

$$\begin{aligned}\liminf_{t\to\infty} \sum_{k=1}^{m} \delta_k(t)\alpha_k^t a_k(t) &= \liminf_{t\to\infty} \sum_{k=1}^{m} (-\delta_k(t)\alpha_k^t)(-a_k(t)) \\ &\ge \varrho_2 \liminf_{t\to\infty} \sum_{k=1}^{m} (-a_k(t)) \\ &= -\varrho_2 \limsup_{t\to\infty} \sum_{k=1}^{m} a_k(t) \\ &> 0\end{aligned}$$

Obviously, $\limsup_{t\to\infty}(t - h_k(t)) < \infty$ here, due to $h_k(t) = t - k\tau, k = 1, 2, \ldots, m$.

Step 2. From condition (iv), there exists $r(t) \le t$ such that for sufficiently large t,

$$\begin{aligned}\int_{r(t)}^{t} \sum_{k=1}^{m} \delta_k(s)\alpha_k^s a_k(s)ds &= \int_{r(t)}^{t} \sum_{k=1}^{m} (-\delta_k(s)\alpha_k^s)(-a_k(s))ds \\ &\le -\varrho_1 \int_{r(t)}^{t} \sum_{k=1}^{m} a_k(s)ds \\ &\le \varrho_1 \frac{1}{e\varrho_1} = \frac{1}{e}\end{aligned}$$

Step 3. From condition (v), we have

$$\begin{aligned}&\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{\delta_k(t)\alpha_k^t a_k(t)}{\sum_{i=1}^m \delta_i(t)\alpha_i^t a_i(t)} \left| \int_{t-k\tau}^{r(t)} \sum_{i=1}^{m} \delta_i(s)\alpha_i^s a_i(s)ds \right| \\ &\le \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{\varrho_1 a_k(t)}{\sum_{i=1}^m \varrho_2 a_i(t)} \left| \int_{t-k\tau}^{r(t)} \sum_{i=1}^{m} \varrho_1 a_i(s)ds \right| \\ &\le \frac{\varrho_1^2}{\varrho_2} \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \left| \int_{t-k\tau}^{r(t)} \sum_{i=1}^{m} a_i(s)ds \right| \\ &< 1\end{aligned}$$

Step 4. From Lemma 2.1, (8) is exponentially stable. Then according to condition (iii) and Lemma 3.2, (7) is also exponentially stable.

Theorem 3.6: Assume that (A1)–(A5), (A8) hold, and

(i) $a_k(t) \le 0, k = 1, 2, \ldots, m$.

(ii) $\limsup_{t\to\infty} \sum_{k=1}^m a_k(t) < 0$.

(iii) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \le \Xi$.

(iv)

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \int_{t-k\tau}^{t} \sum_{i=1}^{m} a_i(s)ds > -\left(1 + \frac{1}{e}\right)\frac{\varrho_2}{\varrho_1^2}.$$

Then, the zero solution of (7) is exponentially stable.

Proof: From condition (iv), we have

$$\begin{aligned}&\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{\delta_k(t)\alpha_k^t a_k(t)}{\sum_{i=1}^m \delta_i(t)\alpha_i^t a_i(t)} \int_{t-k\tau}^{t} \sum_{i=1}^{m} \delta_i(s)\alpha_i^s a_i(s)ds \\ &\le \frac{\varrho_1^2}{\varrho_2} \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \int_{t-k\tau}^{t} \sum_{i=1}^{m} (-a_i(s))ds \\ &= -\frac{\varrho_1^2}{\varrho_2} \liminf_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^m a_i(t)} \int_{t-k\tau}^{t} \sum_{i=1}^{m} a_i(s)ds \\ &< 1 + \frac{1}{e}\end{aligned}$$

The verification of the remaining conditions is the same as the proof in the step 1 of Theorem 3.5, from Lemma 2.2, (8) is exponentially stable. Then according to condition (iii) and Lemma 3.1, (7) is also exponentially stable.

Theorem 3.7: Assume that (A1)–(A5), (A8) hold, and

(i) $a_k(t) \le 0, k = 1, 2, \ldots, m$.

(ii) $\limsup_{t\to\infty} \sum_{k=1}^m a_k(t) < 0$.

(iii) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \le \Xi$.

(iv)

$$\liminf_{t\to\infty} \int_{t-m\tau}^{t} \sum_{i=1}^{m} a_i(s)ds > -\left(1 + \frac{1}{e}\right)\frac{1}{\varrho_1}.$$

Then, the zero solution of (7) is exponentially stable.

Proof: From condition (iv), we have

$$\begin{aligned}&\limsup_{t\to\infty} \int_{\min_k\{t-k\tau\}}^{t} \sum_{i=1}^{m} \delta_i(s)\alpha_i^s a_i(s)ds \\ &\le \varrho_1 \limsup_{t\to\infty} \int_{t-m\tau}^{t} \sum_{i=1}^{m} (-a_i(s))ds \\ &= -\varrho_1 \liminf_{t\to\infty} \int_{t-m\tau}^{t} \sum_{i=1}^{m} a_i(s)ds \\ &< 1 + \frac{1}{e}\end{aligned}$$

The verification of the remaining conditions is the same as the proof in the step 1 of Theorem 3.5, from Lemma 2.3, system (8) is exponentially stable. Then according to condition (iii) and Lemma 3.2, (7) is also exponentially stable.

In the following stability results, $\limsup_{t\to\infty} \sum_{k=1}^m a_k(t) < 0$ is no longer required.

Theorem 3.8: Assume that (A1)–(A5), (A8) hold, and
(i) $a_k(t) \leq 0, k = 1, 2, \ldots, m$.
(ii) $\sum_{k=1}^{m} a_k(t) < 0$ and $\int_0^\infty \sum_{k=1}^{m} a_k(t)dt = -\infty$.
(iii) $\liminf_{t\to\infty} \int_{t-k\tau}^{t} \sum_{i=1}^{m} a_i(s)ds > -\infty$.
(iv) There exists a constant $\Xi > 0$ such that $\left|\alpha^{-1}(t)\right| \leq \Xi$.
(v) There exists a constant $R > 0$ such that

$$\limsup_{t\to\infty} \int_t^{t+R} \sum_{i=1}^{m} a_i(s)ds < 0.$$

(vi)

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{t-k\tau}^{t} \sum_{i=1}^{m} a_i(s)ds > -\left(1+\frac{1}{e}\right)\frac{\varrho_2}{\varrho_1^2}.$$

Then, the zero solution of (7) is exponentially stable.

Proof: Due to Remark 3.4, $\delta_k(t)\alpha_k^t a_k(t) \geq 0$, and

$$\sum_{k=1}^{m} \delta_k(t)\alpha_k^t a_k(t) \geq -\varrho_2 \sum_{k=1}^{m} a_k(t) > 0$$

$$\int_0^\infty \sum_{k=1}^{m} \delta_k(t)\alpha_k^t a_k(t)dt = \infty$$

From condition (iii), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{t-k\tau}^{t} \sum_{i=1}^{m} \delta_i(s)\alpha_i^s a_i(s)ds \\
&\leq \varrho_1 \limsup_{t\to\infty} \int_{t-k\tau}^{t} \sum_{i=1}^{m} (-a_i(s))ds \\
&= -\varrho_1 \liminf_{t\to\infty} \int_{t-k\tau}^{t} \sum_{i=1}^{m} a_i(s)ds \\
&< \infty
\end{aligned}$$

From condition (v), we have

$$\begin{aligned}
&\liminf_{t\to\infty} \int_t^{t+R} \sum_{i=1}^{m} \delta_i(s)\alpha_i^s a_i(s)ds \\
&\geq \varrho_2 \liminf_{t\to\infty} \int_t^{t+R} \sum_{i=1}^{m} (-a_i(s))ds \\
&= -\varrho_2 \limsup_{t\to\infty} \int_t^{t+R} \sum_{i=1}^{m} a_i(s)ds \\
&> 0
\end{aligned}$$

From condition (vi), the same proof as in Theorem 3.6, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{\delta_k(t)\alpha_k^t a_k(t)}{\sum_{i=1}^{m} \delta_i(t)\alpha_i^t a_i(t)} \int_{t-k\tau}^{t} \sum_{i=1}^{m} \delta_i(s)\alpha_i^s a_i(s)ds \\
&\leq -\frac{\varrho_1^2}{\varrho_2} \liminf_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{t-k\tau}^{t} \sum_{i=1}^{m} a_i(s)ds \\
&< 1 + \frac{1}{e}
\end{aligned}$$

Then, from Lemma 2.4, (8) is exponentially stable. Then according to condition (iv) and Lemma 3.2, (7) is also exponentially stable.

Remark 3.5: In most of previous studies on the stability of system (1), the coefficient function $\alpha_k(t)$ is required to be non-negative. In this part, we propose a class of delay differential system:

$$\dot{y}(t) + \sum_{k=1}^{m} \delta_k(t)a_k(t)y(t-k\tau) = 0, \quad t \geq t_0$$

in which the coefficient function $\delta_k(t)a_k(t)$ can be negative. Based on logic impulsive control, the condition that the coefficient function is non-negative is no longer required, and $\sum_{k=1}^{m} \delta_k(t)a_k(t) > 0$ is not required too, some stability results are obtained. As far as the author knows, there are no such stable results for this class of systems.

IV. EXAMPLES

Example 4.1: Consider the following delay differential system under logic impulsive control:

$$\begin{cases} \dot{y}(t)+\beta(|\cos t|-\cos t+\frac{4}{e\pi})y(h(t))=0, & t \neq t_i \\ y(t_i^+) - y(t_i)=\phi_i y(t_i), & t=t_i \\ y(t) = 0.5, & t \in [-\pi, 0] \end{cases} \tag{9}$$

where $\beta > 0$ is a costant, $h(t)$ is a Lebesgue measurable function such that $0 \leq t - h(t) \leq \pi$, $t_0 = 0$, $t_{i+1} - t_i = 4$, and the logic impulsive control $\phi_i y(t_i)$ is denoted by:

$$\begin{aligned}
\phi_i y(t_i) &= \Big\{\sum_{l=1}^{3}[\lambda_i^l, \mu_i^l]g_l(y(t_i))\Big\}y(t_i) \\
&= \Big\{\Big[\frac{1}{2^i}, \frac{1}{2^{i+1}}\Big]g_1(y(t_i)) + \Big[\frac{1}{3^i}, \frac{1}{3^{i+1}}\Big]g_2(y(t_i)) \\
&\quad +\Big[\frac{1}{6^i}, \frac{1}{6^{i+1}}\Big]g_3(y(t_i))\Big\}y(t_i)
\end{aligned}$$

The rules for piecewise logical functions g_l are as follows:

$$g_l(u) = \begin{cases} \delta_2^2, & |u| \geq c_l \\ \delta_2^1, & |u| < c_l \end{cases}$$

$c_1 = 0.3,\ c_2 = 0.1,\ c_3 = 0.05$ are the thresholds.

Now, we verify system (9) according to the condition requirements in Theorem 3.3.

(i) $a(t) = \beta(|\cos t| - \cos t + \frac{4}{e\pi}) \geq 0$.
$\lambda_i^l,\ \mu_i^l \in \{\frac{1}{2^i}, \frac{1}{3^i}, \frac{1}{6^i}\} > -\frac{1}{3}, l = 1, 2, 3, i \in N$.
$-\frac{1}{3} < \lim_{i\to\infty} \lambda_i^l = \lim_{i\to\infty} \mu_i^l = 0 < \infty$.
(ii) $\liminf_{t\to\infty} \beta(|\sin t| - \sin t + \frac{4}{e\pi}) = \frac{4\beta}{e\pi} > 0$.
(iii) There exists a constant $\Xi > 0$ such that

$$\left|\alpha^{-1}(t)\right| = \prod_{0\leq t_j<t} (1+\phi_j) \leq \prod_{i\in N}\left(1+\frac{1}{2^i}+\frac{1}{3^i}+\frac{1}{6^i}\right) \leq \Xi.$$

(iv) Due to $t - h(t) \leq \pi$ and $t_{i+1} - t_i = 4$, there exists at most one impulsive point on the interval $[h(t), t)$,

$\forall t > 0$, then

$$\frac{1}{2} = \min_{i\in N}\left\{\frac{1}{1+\phi_i}, 1\right\} \le \alpha_k^t \le \max_{i\in N}\left\{\frac{1}{1+\phi_i}, 1\right\} = 1.$$

so let $\varpi_1 = 1$ and $\varpi_2 = \frac{1}{2}$.

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{h(t)}^{t}\beta(|\cos s| - \cos s + \frac{4}{e\pi})ds \\
&\quad \le \beta\limsup_{t\to\infty}\int_{t-\pi}^{t}(|\cos s| - \cos s + \frac{4}{e\pi})ds \\
&\quad = \beta\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(|\cos s| - \cos s)ds + \frac{4\beta}{e} \\
&\quad = 4\beta\left(1+\frac{1}{e}\right)
\end{aligned}$$

Then, according to Theorem 3.3, if $4\beta(1+\frac{1}{e}) < 1 + \frac{1}{e}$, i.e., $\beta < \frac{1}{4}$, the zero solution of (9) is exponentially stable. For example, let β in system (9) be $\beta = 0.2$, and $h(t)$ in system (9) be $h(t) = t - \pi|\sin t|$, then the zero solution of (9) is exponentially stable, as shown in Figure 1.

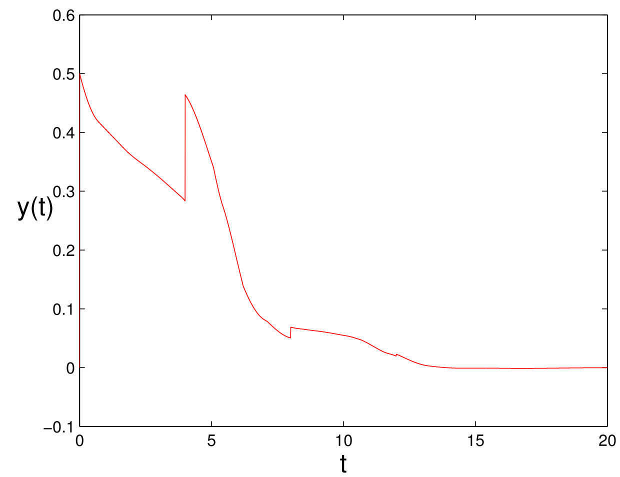

FIGURE 1. Dynamical behavior of $y(t)$ in system (9) with respect to time. Here, $\beta = 0.2$ and $h(t) = t - \pi|\sin t|$.

Example 4.2: Consider the following delay differential system under logic impulsive control:

$$\begin{cases}\dot{y}(t) - \beta\delta_1(t)(|\sin t| - \sin t)y(t-3) \\ \qquad -\dfrac{\delta_2(t)}{t+10}y(t-6) = 0, \quad t \ne t_i \\ y(t_i^+) - y(t_i) = \phi_i y(t_i), \qquad t = t_i \\ y(t) = 0.5, \qquad t\in[-6,\ 0]\end{cases} \tag{10}$$

where $\beta > 0$ is a constant, $t_0 = 0$, $t_{i+1} - t_i = 3$, and the logic impulsive control $\phi_i y(t_i)$ is denoted by:

$$\begin{aligned}
\phi_i y(t_i) &= \left\{\sum_{l=1}^{2}[\lambda_i^l, \mu_i^l]g_l(y(t_i))\right\}y(t_i) \\
&= \left\{\left[-1-\frac{1}{2^i+1}, -1-\frac{1}{2^{i+1}}\right]g_1(y(t_i)) \right.\\
&\quad \left.+\left[-1-\frac{1}{3^i+1}, -1-\frac{1}{3^{i+1}}\right]g_2(y(t_i))\right\}y(t_i)
\end{aligned}$$

The rules for piecewise logical function g_l is as follows:

$$g_1(u) = \begin{cases}\delta_2^2, & |u-0.1| \ge 0.3 \\ \delta_2^1, & |u-0.1| < 0.3\end{cases}$$

and

$$g_2(u) = \begin{cases}\delta_2^2, & |u-0.05| \ge 0.1 \\ \delta_2^1, & |u-0.05| < 0.1\end{cases}$$

Now, we verify system (10) according to the condition requirements in Theorem 3.8.

Easy to see that $a_1(t) = -\beta(|\sin t| - \sin t)$, $a_2(t) = -\frac{1}{t+10}$ in (10), then

(i) For $t > 0$, $a_1(t) \le 0$, and $a_2(t) < 0$.

$\lambda_i^l,\ \mu_i^l \in \{-1-\frac{1}{2^i+1}, -1-\frac{1}{2^{i+1}}, -1-\frac{1}{3^i+1}, -1-\frac{1}{3^{i+1}}\} < -\frac{1}{2}$, for $l = 1, 2$, $i \in N$.

$-\infty < \lim_{i\to\infty}\lambda_i^l = \lim_{i\to\infty}\mu_i^l = -1 < -\frac{1}{2}$.

(ii) Obviously $\sum_{k=1}^{2} a_k(t) < 0$ for $t > 0$, and easy to prove that $\int_0^\infty \sum_{k=1}^{2} a_k(t)dt = -\infty$.

(iii)

$$\begin{aligned}
&\liminf_{t\to\infty}\int_{t-3k}^{t}(a_1(s)+a_2(s))ds \\
&\quad \ge -\limsup_{t\to\infty}\int_{t-6}^{t}(\beta(|\sin s| - \sin s) + \frac{1}{s+10})ds \\
&\quad \ge -\beta\int_0^{2\pi}(|\sin s| - \sin s)ds - \limsup_{t\to\infty}\ln(1+\frac{6}{t+4}) \\
&\quad = -4\beta > -\infty
\end{aligned}$$

(iv) There exists a constant $\Xi > 0$ such that

$$\left|\alpha^{-1}(t)\right| = \prod_{0\le t_j<t}|1+\phi_j| \le \prod_{i\in N}\left(1+\frac{1}{2^i+1}+\frac{1}{3^i+1}\right) \le \Xi.$$

(v) There exists a constant $R = 2\pi > 0$ such that

$$\limsup_{t\to\infty}\int_t^{t+2\pi}\sum_{i=1}^{2}a_i(s)ds < 0.$$

(vi) The values of ϱ_1 and ϱ_2 are as follows:

$$\begin{aligned}
\varrho_1 &= \max_{\substack{j\in N,\\ k=1,2}}\left\{1, \prod_{i=j,\dots,j+k-1}\left|\frac{1}{1+\phi_i}\right|\right\} \\
&= \max_{j\in N}\left\{1, \left|\frac{1}{1+\phi_j}\right|, \prod_{i=j,j+1}\left|\frac{1}{1+\phi_i}\right|\right\} \\
&= 1 \\
\varrho_2 &= \min_{\substack{j\in N,\\ k=1,2}}\left\{1, \prod_{i=j,\dots,j+k-1}\left|\frac{1}{1+\phi_i}\right|\right\} \\
&= \min_{j\in N}\left\{1, \left|\frac{1}{1+\phi_j}\right|, \prod_{i=j,j+1}\left|\frac{1}{1+\phi_i}\right|\right\} \\
&= \frac{120}{247} \approx 0.4858
\end{aligned}$$

Note that,

$$\begin{aligned}
&\liminf_{t\to\infty}\sum_{k=1}^{2}\frac{a_k(t)}{\sum_{i=1}^{2}a_i(t)}\int_{t-3k}^{t}\sum_{i=1}^{2}a_i(s)ds\\
&=-\limsup_{t\to\infty}\sum_{k=1}^{2}\frac{a_k(t)}{\sum_{i=1}^{2}a_i(t)}\int_{t-3k}^{t}(-a_1(s)-a_2(s))ds\\
&\geq-\limsup_{t\to\infty}\int_{t-3k}^{t}\Big(\beta(|\sin s|-\sin s)+\frac{1}{s+10}\Big)ds\\
&=-4\beta
\end{aligned}$$

Then, according to Theorem 3.8, if $4\beta<\left(1+\frac{1}{e}\right)\frac{120}{247}$, i.e., $\beta_1<\left(1+\frac{1}{e}\right)\frac{30}{247}\approx 0.1661$, the zero solution of (10) is exponentially stable. For example, let β in system (10) be $\beta=0.12$, then the zero solution of (10) is exponentially stable, as shown in Figure 2.

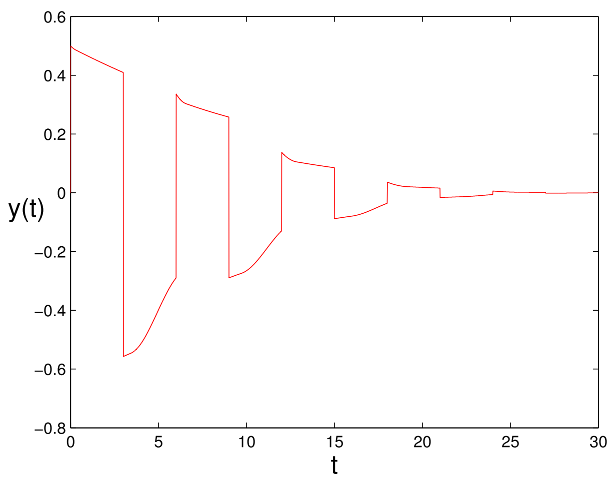

FIGURE 2. Dynamical behavior of $y(t)$ in system (10) with respect to time. Here, $\beta = 0.12$.

Remark 4.1: Note that, condition (C1): $\liminf_{t\to\infty}\sum_{k=1}^{m}a_k(t)>0$, is required in Theorems 3.1-3.3 and is not required in Theorem 3.4; Condition (C2): $\limsup_{t\to\infty}\sum_{k=1}^{m}a_k(t)<0$, is required in Theorems 3.5-3.7 and is not required in Theorem 3.8. In this section, we give two examples: Example 4.1 is a typical case satisfying (C1); Example 4.2 can be considered as a typical case that (C2) is not satisfied. Similarly, examples of other theorems (Theorems 3.1-3.2, 3.4-3.7) can be given, which are not discussed here.

V. CONCLUSION

In recent years, many significant results have been obtained on the stability of a class of linear delay equations like system (1). It is noted that almost all of these results are studied under the condition that the coefficient function $a_k(t)$ is non-negative. In this paper, based on logic impulsive control, the strong constraint that the coefficient functions need to be non-negative has been reduced. By establishing the relationship between the delay differential system under logic impulsive control and a corresponding non-impulsive system, some exponential stability criteria are presented from the following two aspects: Stability criteria for system (4), in which the coefficient functions are non-negative; Stability criteria for system (7), in which the coefficient functions can be negative. Moreover, Lemma 3.1 extend the results in [9], [11] and [15] by comparison. Two numerical examples are discussed to verify the stability results.

Future Work: One of the main works of this paper is to discuss the exponential stability of system (7), which is a class of system (1) under logic impulse control. Although, by constructing logic impulsive control of system (1), system (7) reduces the requirement that the coefficient functions of system (1) need to be non-negative in most previous studies and $\sum_{k=1}^{m}\delta_k(t)a_k(t)>0$ is not required too. However, the delay function $h_k(t)$ in system (7) is set to the form of $h_k(t)=t-k\tau$. In the future work, we hope to study system (1) with a more general form of delay function by logic impulsive control.

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