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New Exact Nematicon Solutions of Liquid Crystal Model With Different Types of Nonlinearities

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ABSTRACT The aim of this work is to explore and find new closed-form nematicon solutions for different nonlinearities which occur in nematic liquid crystals (NLC) along with proposing optical system application that utilizes NLC nonlinearities. In particular, Lie point symmetry method is employed to scrupulously inspect and acquire solutions for some interesting cases of nonlinearities which have not been fully examined in literature such as quadratic, generalized dual-power law, and eighth-order nonlinearities. A variety of different nematicon dynamics are observed, including bright solitons, dark solitons and periodic behaviors. The explicit solution form for each dynamical behavior is obtained and the solution dependence on model parameters is investigated. The proposed optical system enables the flexible realization of different types of NLC nonlinearities. To the best of our knowledge, this is the first time to attain explicit exact nontrivial solutions for the particular cases of generalized dual-power law and eighth-order nonlinearities.

INDEX TERMS Jacobi elliptic function, lie point symmetry, liquid crystals, nematicons.

I. INTRODUCTION

Analytical techniques are employed as highly reliable tools to scrutinize and thoroughly understand the possible qualitative and quantitative dynamical behaviors of nonlinear systems. These invaluable tools are successfully utilized in different engineering, physical, biological and economical disciplines. For example, theories of chaotic systems and applied bifurcation [1]–[8] are widely used in analyzing and interpreting the dynamics of nonlinear models, opening the door for many promising applications. Oscillatory behavior is also an interesting dynamical behavior which has been thoroughly investigated in literature (see [9]–[11] and references therein). Furthermore, the Lie-symmetry analysis methods [12]–[20] can be arguably considered the most powerful tool in solving ordinary and partial differential equations.

Among the fascinating nonlinear phenomena which can be observed in real world nonlinear systems, the soliton is an interesting type of dynamical behaviors that attracted much interest from scientists. It is defined as a nonlinear localized wave, with self-reinforcing and permanent shape, that propagates uniformly at a constant velocity. From a physical point of view, solitons can be observed in nonlinear optical fibers, biological models, metamaterials, plasma and nuclear physics, and many other fields (e.g., see [21]–[24] and references therein).

From application perspectives, the liquid crystals (LCs) have found its place in a plethora of technological applications such as electrooptic control, display and sensing devices and biological applications [25], [26]. LCs are composed of materials with an intermediate phase or mesophases between the solid and fluid state. Their phases can be categorized into two classes, namely, thermotropic, such as nematic LC, and lyotropic LCs. The degree of material nonlinearities determines the possibility of experimental observations of optical solitons. NLCs exhibit a large optical nonlinearity and support the existence of optical spatial solitons via the nonlinearity of the director reorientation i.e. molecular response to light intensity [25], [26].

Nematicons, the spatial solitary waves in NLC, have witnessed a considerable interest in the last two decades. Recently, NLC has been examined for several forms of nonlinearities, e.g., quadratic power, parabolic, and dual-power laws, see [27] and references therein. According to the form of nonlinearity and the parameters of the model, various types of solutions can be observed including bright, dark and singular solitons. However, the nontrivial exact solution corresponding to some types of nonlinearity, e.g., the dual-power nonlinearity, is yet to be determined [27]. On the other hand, only the quadratic nonlinearity naturally exists in NLC according to the Kerr nonlinearity; that the

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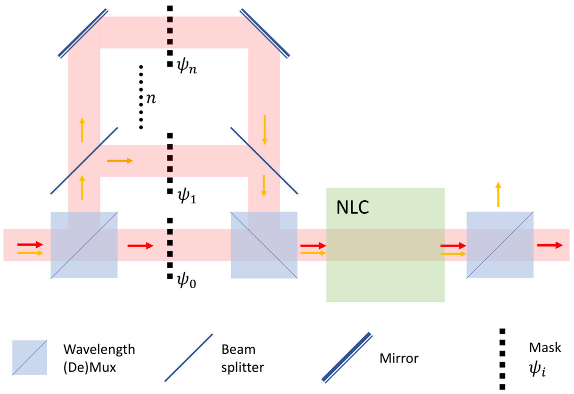

FIGURE 1. Schematic of the proposed setup. The beam is split spectrally using a wavelength (de)mux into a main (red arrows) and ancillary (orange arrows) bands. The ancillary portion is divided into several arms of heavily unbalanced lengths relative to each other and relative to the arm of the main band, so that their recombination is an incoherent process yielding intensity accumulation. Each arm includes a spatial mask of a transmission pattern proportional to the slowly-varying transverse profile of the field or higher orders of it. The intensity of the recombined beams are represented by the functional *F*.

induced refractive-index change is proportional to the square of the field. The nonlinearities other than the quadratic form, although offers new promising operation-regions and control of nematicons, seems hypothetical without a tackled practical realization, which is to be carried out in this work.

In this paper, exact nematicon solutions along with the practical realization for several forms of nonlinearity are presented. We examine a nontrivial exact solution for the dual-power nonlinearity case. Moreover, a solution for a new form including eighth-order nonlinearity is introduced. More importantly, a proposal for an optical system that allows realizing all suggested nonlinearities is presented. Using Lie point symmetry method [28]–[31], novel exact solutions in the form of soliton and periodic solutions are obtained in each case which are then joined to the mathematical model's parameters. The rest of the paper is organized as follows: The mathematical model and its realization is introduced in section 2. The application of Lie point symmetry method to determine the explicit exact solutions for the mathematical model at different cases of nonlinearities is presented in section 3. Section 4 contains summary of results and the conclusion.

II. PROPOSED SYSTEM AND MATHEMATICAL MODEL

Consider the optical setup in Fig. 1. A linearly polarized beam of a transverse profile $q(x,t)$ is split spectrally into two portions; one of them has a spectral band above wavelength λ_c while the band of the other is under λ_c. The two beam portions are then of the same polarization and transverse characteristics. As will be seen later, this spectral distinction is essential to the filtration process by the end of the system.

One of the two bands (henceforth, termed ancillary band) is split by a series of beam splitters into several portions; each of them is spatially modulated by a transmission mask before being incoherently recombined again with each other as well as with the other band. To verify that the recombination process is incoherent, the path differences between all portions of the beam are made greater than the coherence length of the used laser beam. In this sense, because the relative phases of the recombined beams are statistically random, no interference takes place and the recombination process is described by intensity accumulation.

The transverse patterns of the transmission masks are proportional to the transverse profile of the beam or higher orders of it, which reads $\psi_i(x,t) \propto |q(x,t)|^{m_i}$ with an integer $m_i \geq 1$. All of these incoherently superimposed components are then delivered with extraordinary polarization to an NLC to be able to collectively engineer the distribution of the molecular orientations. After the NLC, similar spectral splitting is used to pass only the main band after filtering the ancillary band. Although, the splitting and recombination processes can, in principle, be realized using any photonic degree of freedom (DOF): polarization, spatial, or spectral, only the splitting and recombination in the spectral DOF can readily produce copropagating and copolarized beams to excite the NLC.

The dynamics of nematicons for the main band, in the dimensionless form, is then generally governed by the following coupled system of equations [27]

$$\begin{aligned} iq_t + aq_{xx} + b\theta q &= 0, \\ c\theta_{xx} + \lambda\theta + \alpha F(|q|^2) &= 0, \end{aligned} \tag{1}$$

where the second dependent variable $\theta(x,t)$ is the reorientation angle of the director of the NLC molecules, a, b, c, α and λ are all real-valued constants. The functional F is the type of nonlinearity introduced by the incoherent superposition of main and ancillary beams which will be studied for different forms. While the nonlinear Schrodinger (NLS) equation (appearing first in 1) describes the main-band wave, there is a corresponding NLS equation for each ancillary portion which we do not consider here. However, we notice that the accumulative effect of the multiple incoherent beams appears in the director reorientation equation (appearing second in 1).

In [32], Eq. (1) is solved using the tan $(\Phi/2)$-expansion method for the Kerr, parabolic and dual-power law nonlinearity forms. The obtained solutions are in the forms of tan and exponential functions. Here, we will obtain the more complete and general set of solutions in the forms of elliptic, Weierstrass and hyperbolic functions.

In [33], Eq. (1) is solved using modified simple equation method for Anti quadratic nonlinearity $F(s) = b_8/|q|^2 + b_9|q| + b_{10}|q|^2$ and triple power law $F = b_8|q|^2 + b_9|q|^4 + b_{10}|q|^6$. The obtained solutions in case of anti quadratic nonlinearity are in the form of exponential function. The author fails to obtain the solutions in case of triple power law, but here, we will successfully obtain exact solution in the more general case of $F = C_3|q|^2 + C_4|q|^4 + C_5|q|^6 + C_6|q|^8$.

Also, the cases of Kerr and parabolic law nonlinearity in Eq.(1) is solved using extended sinh Gordon equation expansion method in [34] and using Exp$(\varphi(\xi))$-expansion method in [35]. The obtained solutions are in the form of hyperbolic and exponential functions, and have constraints on the parameters in Eq. (1). In this work, the complete and exact set of solutions for this cases will be obtained without imposing any constraints on the parameters of Eq. (1).

Now, let

$$q(x,t) = P(z)e^{i(\kappa_s x - \omega_s t + \sigma)},$$
$$\theta(x,t) = Q(z), \tag{2}$$

where $z = k(x - \nu t)$, κ_s denotes the wave number of the soliton, ω_s is the soliton frequency, σ is a constant phase and k, ν are constants.

Substituting (2) into (1) and then equating real and imaginary parts to zero, we obtain two equations. The imaginary part gives

$$\nu = 2a\kappa_s, \tag{3}$$

whereas the real part gives

$$ak^2P''(z) - P(z)\left(a\kappa_s^2 - \omega_s\right) + bP(z)Q(z) = 0,$$
$$ck^2Q''(z) + \alpha F\left(P(z)^2\right) + \lambda Q(z) = 0. \tag{4}$$

In the next section, we will use Lie point symmetry method to obtain the solutions of the system (4) for different forms of the function $F(P(z)^2)$.

III. LIE POINT SYMMETRY METHOD

Lie point symmetry analysis method is a general tool that can be used to find exact solutions of ordinary differential equations (ODEs) or to reduce the number of the independent variables of partial differential equations. The theory and applications of Lie point symmetry method is well known and can be found in [12]–[20].

Following [12], [13], the invariance condition of the system (4) is given by

$$X^{[2]}(ak^2P''(z) - P(z)\left(a\kappa_s^2 - \omega_s\right) + bP(z)Q(z)) = 0,$$
$$X^{[2]}(ck^2Q''(z) + \alpha F\left(P(z)^2\right) + \lambda Q(z)) = 0. \tag{5}$$

where $X^{[2]}$ is the second extension of the infinitesimal generator

$$X = \xi(z,P,Q)\partial_z + \eta(z,P,Q)\partial_P + \tau(z,P,Q)\partial_Q \tag{6}$$

which is given by

$$X^{[2]} = \xi\partial_z + \eta\partial_P + \tau\partial_Q + \eta^{[1]}\partial_{P'} + \tau^{[1]}\partial_{Q'} + \eta^{[2]}\partial_{P''} + \tau^{[2]}\partial_{Q''} \tag{7}$$

The definitions of the extended coefficients ($\eta^{[1]}$, $\tau^{[1]}$, $\eta^{[2]}$, $\tau^{[2]}$) are standard and well-known (see for example [12], [13]). When we substitute (7) into (5) and taking (4) into account, we obtain some determining equations. Solving the obtained determining equations, we obtain the infinitesimals ξ, η and τ as follows

$$\xi = 1, \quad \eta = 0, \quad \tau = 0, \tag{8}$$

Hence, the Lie point symmetry algebra is spanned by

$$X = \partial_z \tag{9}$$

The canonical variables associated to the generator (9) are determined as given in [12], [13], and [20]

$$r = P(z), \quad u(r) = Q(z), \quad V(r) = z \tag{10}$$

and they can be prolonged to

$$V'(r) = \frac{1}{P'(z)}, \quad u'(r) = \frac{Q'(z)}{P'(z)}, \quad V''(r) = -\frac{P''(z)}{P'(z)^3},$$
$$u''(r) = \frac{P'(z)Q''(z) - P''(z)Q'(z)}{P'(z)^3}. \tag{11}$$

Substituting (10) and (11) into (4), we obtain

$$ak^2V''(r) + r\left(a\kappa_s^2 - bu(r) - \omega_s\right)V'(r)^3 = 0,$$
$$ck^2\left(u''(r)V'(r) - u'(r)V''(r)\right) + \left(\alpha F\left(r^2\right) + \lambda u(r)\right)V'(r)^3 = 0. \tag{12}$$

Let

$$V'(r) = \frac{1}{\sqrt{J(r)}}, \tag{13}$$

hence the system (12) becomes

$$-ak^2J'(r) + 2r\left(a\kappa_s^2 - \omega_s\right) - 2bru(r) = 0,$$
$$ck^2\left(J'(r)u'(r) + 2J(r)u''(r)\right) + 2\alpha F\left(r^2\right) + 2\lambda u(r) = 0. \tag{14}$$

Now, different forms of nonlinearities of $F\left(r^2\right)$ will be considered in the following subsections.

A. SOLUTION OF QUADRATIC NONLINEARITY FORM OF F

The quadratic nonlinearity function is given by [27]

$$F\left(r^2\right) = \gamma r^2,$$

where γ is a constant. In this case, there is no ancillary beams in the optical setup in Fig. 1 and the beam is directly delivered to the NLC. Now, system (14) becomes

$$-ak^2J'(r) + 2r\left(a\kappa_s^2 - \omega_s\right) - 2bru(r) = 0,$$
$$ck^2\left(J'(r)u'(r) + 2J(r)u''(r)\right) + 2\alpha\gamma r^2 + 2\lambda u(r) = 0. \tag{15}$$

System (15) has the following solutions

$$J(r) = \gamma_1 - \frac{\lambda}{ck^2}r^2 + \frac{2\sqrt{\alpha b\gamma}}{3\sqrt{ack^2}}r^3, \tag{16}$$

$$u(r) = -\frac{\sqrt{\alpha\gamma a}}{\sqrt{bc}}r, \tag{17}$$

where γ_1 is a constant and $\omega_s = \frac{a\lambda}{c} + a\kappa_s^2$. From (10-11), (13) and (16-17), we can obtain

$$P'(z)^2 = \frac{2\sqrt{\gamma\alpha b}}{3\sqrt{ack^2}}P(z)^3 - \frac{\lambda}{ck^2}P(z)^2 + \gamma_1, \tag{18}$$

$$Q(z) = -\frac{\sqrt{\alpha\gamma a}}{\sqrt{bc}}P(z). \tag{19}$$

Then Eq. (18) has the solution [16], [28]

$$P(z) = \frac{6\sqrt{ac}k^2}{\sqrt{\alpha b\gamma}}\left(\frac{\lambda}{12ck^2} + \wp(C+z; g_2, g_3)\right), \quad (20)$$

where $\wp(C+z; g_2, g_3)$ is Weierstrass elliptic function, C is a constant,

$$g_2 = \frac{\lambda^2}{12\, c^2\, k^4}, \quad g_3 = \frac{\lambda^3}{216\, c^3\, k^6} - \frac{\alpha b\gamma\gamma_1}{36\, ack^4}.$$

Substituting from (20) into (19), we obtain

$$Q(z) = -\frac{a\lambda}{2bc} - \frac{6ak^2}{b}\wp(C+z; g_2, g_3). \quad (21)$$

From (20) and (2) it implies that

$$q(x,t) = \frac{6\sqrt{ac}k^2}{\sqrt{\alpha b\gamma}}\left(\frac{\lambda}{12ck^2} + \wp(C+z; g_2, g_3)\right) \times e^{i(\kappa_s x - \omega_s t + \sigma)}. \quad (22)$$

and

$$|q| = \frac{6\sqrt{ac}k^2}{\sqrt{\alpha b\gamma}}\left|\frac{\lambda}{12ck^2} + \wp(C+z; g_2, g_3)\right|. \quad (23)$$

For $\gamma_1 = \frac{2\, ad\lambda^2}{\alpha bc\gamma}$ and $k = \frac{\sqrt{\lambda}}{\sqrt{6\, cd}}$, Eq. (23) can be rewritten in the form [28]

$$|q| = \frac{\sqrt{a}}{2\sqrt{\alpha bc}}\left|\frac{\lambda}{\gamma}\right|\left|3\coth^2\left(C_0 + \frac{1}{2}\sqrt{\frac{\lambda}{c}}\,(x - 2a\kappa_s t)\right) - 1\right|, \quad (24)$$

where C_0 and d are constants.

The solution (24) can be classified as follows:

1) CASE 1

For $\frac{\lambda}{c} > 0$ and $C_0 = 0$, we obtain singular soliton solution.

For example, for $\lambda = a = b = c = \gamma = \alpha = \kappa_s = 1$, we get Fig.2 (a).

2) CASE 2

For $\frac{\lambda}{c} < 0$ and $C_0 = 0$, we obtain singular periodic solution.

$$|q| = \frac{\sqrt{a}}{2\sqrt{\alpha bc}}\left|\frac{\lambda}{\gamma}\right|\left|3\cot^2\left(\frac{1}{2}\sqrt{\frac{-\lambda}{c}}\,(x - 2a\kappa_s t)\right) + 1\right|.$$

For example, for $a = b = c = \gamma = \alpha = \kappa_s = 1$ and $\lambda = -1$, we obtain Fig.2 (b).

3) CASE 3

For $\frac{\lambda}{c} > 0$ and $C_0 = \imath\pi/2$, we obtain the soliton solution

$$|q| = \frac{\sqrt{a}}{2\sqrt{\alpha bc}}\left|\frac{\lambda}{\gamma}\right|\left|3\tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{c}}\,(x - 2a\kappa_s t)\right) - 1\right|.$$

For example, for $\lambda = a = b = c = \gamma = \alpha = \kappa_s = 1$, we get Fig.2 (c).

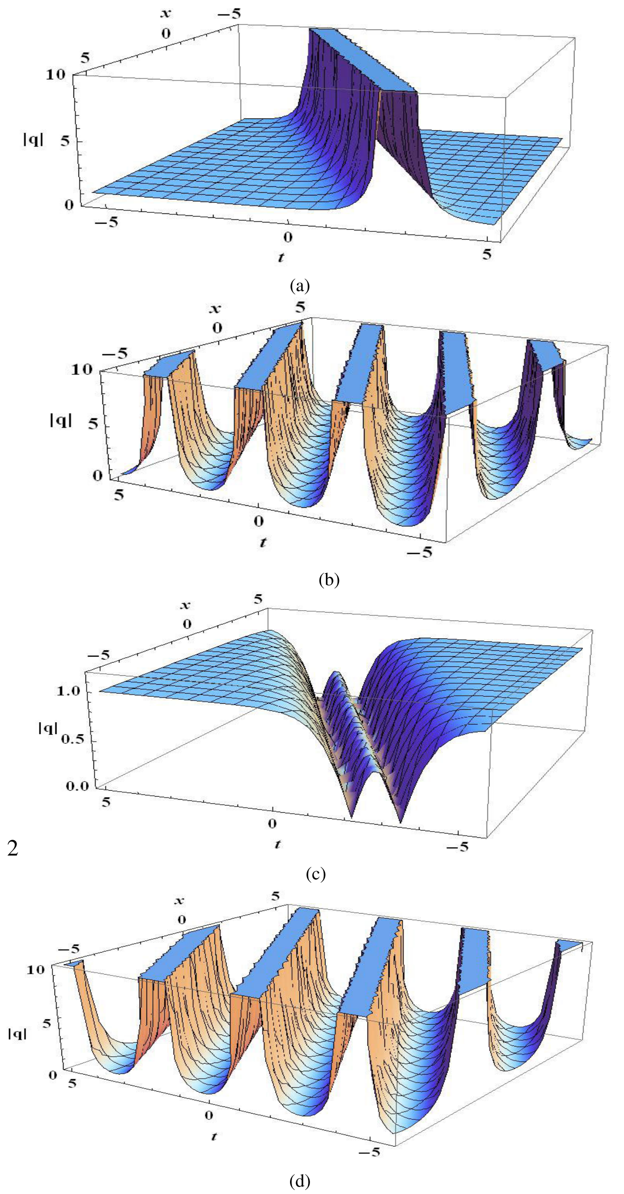

FIGURE 2. Examples of solutions obtained from first form of $F(|q|^2) = \gamma|q|^2$ at (a) $\lambda = a = b = c = \gamma = \alpha = \kappa_s = 1$, (b) $a = b = c = \gamma = \alpha = \kappa_s = 1$ and $\lambda = -1$, (c) $\lambda = a = b = c = \gamma = \alpha = \kappa_s = 1$ and (d) $a = b = c = \kappa_s = \gamma = \alpha = 1$ and $\lambda = -1$.

4) CASE 4

For $\frac{\lambda}{c} < 0$ and $C_0 = \imath\pi/2$, we obtain singular periodic solution

$$|q| = \frac{\sqrt{a}}{2\sqrt{\alpha bc}}\left|\frac{\lambda}{\gamma}\right|\left|3\tan^2\left(\frac{1}{2}\sqrt{\frac{-\lambda}{c}}\,(x - 2a\kappa_s t)\right) + 1\right|.$$

For example, for $a = b = c = \kappa_s = \gamma = \alpha = 1$ and $\lambda = -1$, we attain Fig.2 (d).

Figure 2 shows soliton solution and singular periodic and soliton solutions of (1) which are obtained in case of the quadratic form of $F(|q|^2)$.

B. SOLUTION OF DUAL-POWER NONLINEARITY FORM OF F

The nonlinear function (dual-power nonlinearity) is given by [27]

$$F\left(r^2\right) = C_1\, r^{2\,n} + C_2\, r^{4\,n},$$

where C_1, C_2 and n are constants. Although, this form can be realized by the optical setup in Fig. 1 only for $n = 1$, we are interested here to solve system (14) for the hypothetical case with arbitrary value n. The system (14) becomes

$$\begin{aligned} &-ak^2 J'(r) + 2r\left(a\kappa_s^2 - \omega_s\right) - 2bru(r) = 0, \\ &ck^2\left(J'(r)u'(r) + 2J(r)u''(r)\right) + 2\alpha\left(C_1 r^{2n} + C_2 r^{4n}\right) \\ &\qquad + 2\lambda u(r) = 0. \end{aligned} \tag{25}$$

The solution of system (25) can be expressed as

$$J(r) = -Ar^{2n+2} + Br^2, \tag{26}$$

$$u(r) = \frac{a}{b} Ak^2(n+1)r^{2n}, \tag{27}$$

where

$$A = \frac{1}{k^2\, n}\sqrt{\frac{\alpha b C_2}{6\, ac(n+1)}},$$

$$B = \frac{-3\, AC_1}{2\, C_2} - \frac{\lambda}{4\, ck^2\, n^2},$$

$$\omega_s = a\kappa_s^2 - aBk^2.$$

Using (10-11) and (13), Eq. (26) and Eq. (27) become

$$P'(z)^2 = -AP(z)^{2n+2} + BP(z)^2, \tag{28}$$

$$Q(z) = \frac{a}{b} Ak^2(n+1)P(z)^{2n}. \tag{29}$$

It can be verified that Eq.(28) has the following solution

$$P(z) = \left(\sqrt{\beta}\operatorname{sech}\left(\sqrt{B}nz + c_3\right)\right)^{1/n}, \tag{30}$$

where c_3 is an arbitrary constant and $\beta = \frac{-\lambda}{4\, Ack^2\, n^2} - \frac{3\, C_1}{2\, C_2}$. Substituting (30) into (29), we obtain

$$Q(z) = \frac{a}{b} A\beta k^2(n+1)\operatorname{sech}^2\left(\sqrt{B}nz + c_3\right). \tag{31}$$

Also, substituting (30) into (2) to get

$$q(x,t) = \left(\sqrt{\beta}\operatorname{sech}\left(\sqrt{B}nz + c_3\right)\right)^{1/n} e^{i(\kappa x - \omega t + \sigma)}. \tag{32}$$

Taking the absolute value, we obtain

$$|q| = \left|\sqrt{\beta}\operatorname{sech}\left(\sqrt{B}nk\,(x - 2a\kappa_s t) + c_3\right)\right|^{1/n}. \tag{33}$$

Now, the solution (33) can be classified as follows:

1) CASE 1

For $n > 0$ and $B > 0$, we obtain bright solitons solutions.

For example, for $n = \beta = 1$, $B = k = a = \kappa_s = 1$ and $c_3 = 0$, we can procure Fig.3 (a).

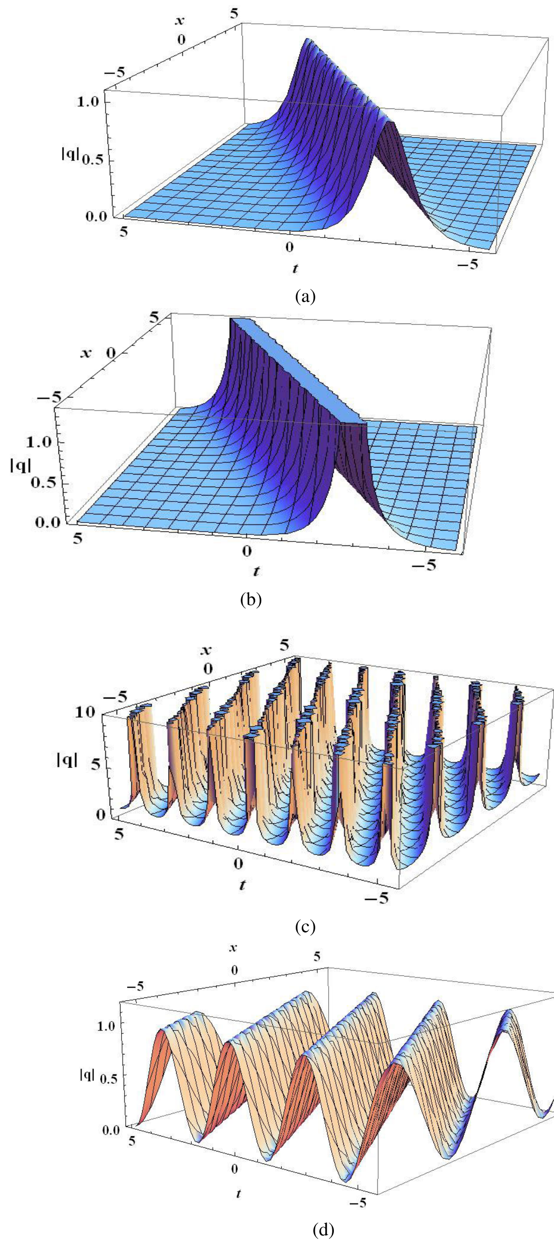

FIGURE 3. Examples of solutions obtained from second form of $F(|q|^2) = C_1|q|^{2n} + C_2|q|^{4n}$ at (a) $n = \beta = 1, B = k = a = \kappa_s = 1$ and $c_3 = 0$, (b) $n = 1, \beta = 1$ and $B = k = a = \kappa_s = 1$, (c) $n = \beta = k = a = \kappa_s = 1, B = -1$ and $c_4 = 0$, and (d) $n = -0.5, B = -1, \beta = k = a = \kappa_s = 1$ and $c_4 = 0$.

2) CASE 2

For $n > 0$ and $B > 0$ and $c_3 = \frac{i\pi}{2}$, we obtain singular solitons solutions

$$|q| = \left|\sqrt{\beta}\operatorname{csch}\left(\sqrt{B}nk\,(x - 2a\kappa_s t)\right)\right|^{1/n}.$$

For example, taking $n = 1, \beta = 1$ and $B = k = a = \kappa_s = 1$, Fig.3 (b) is acquired.

3) CASE 3

For $n > 0$ and $B < 0$, we obtain singular periodic solutions

$$|q| = \left|\sqrt{\beta}\sec\left(\sqrt{-B}nk\,(x - 2a\kappa_s t) + c_4\right)\right|^{1/n}, \tag{34}$$

where, c_4 is a constant. For example, taking $n = \beta = k = a = \kappa_s = 1$, $B = -1$ and $c_4 = 0$, Fig.3 (c) is obtained.

4) CASE 4

For $n < 0$ and $B < 0$, we obtain periodic solutions

$$|q| = \left|\sqrt{\beta}\cos\left(\sqrt{-B}nk\,(x - 2a\kappa_s t) + c_4\right)\right|^{-1/n}.$$

For example, taking $n = -0.5$, $B = -1$, $\beta = k = a = \kappa_s = 1$ and $c_4 = 0$, we get Fig.3 (d)

Figure 3 depicts bright, singular soliton, periodic and singular periodic solutions of (1) which are obtained in case of the second form of $F(|q|^2)$.

C. SOLUTION OF THIRD NONLINEARITY FORM OF $F\left(r^2\right)$

Assume that the nonlinear function takes the form

$$F\left(r^2\right) = C_6 r^8 + C_5 r^6 + C_4 r^4 + C_3 r^2, \tag{35}$$

where C_3, C_4, C_5 and C_6 are constants. The realization of this case requires the ancillary wave to be split into three portions that pass masks of transmissions proportional to $|q(x,t)|$, $|q(x,t)|^2$, and $|q(x,t)|^3$ with respective amplitudes $\sqrt{C_4}$, $\sqrt{C_5}$ and $\sqrt{C_6}$. Hence, the system (14) becomes

$$\begin{aligned}
&-ak^2 J'(r) + 2r\left(a\kappa_s^2 - \omega_s\right) - 2bru(r) = 0,\\
&ck^2\left(J'(r)u'(r) + 2J(r)u''(r)\right)\\
&+ 2\alpha\left(C_6 r^8 + C_5 r^6 + C_4 r^4 + C_3 r^2\right) + 2\lambda u(r) = 0. \qquad (36)
\end{aligned}$$

It can be demonstrated that system (36) has the solution

$$J(r) = A_4 r^6 + A_3 r^4 + A_2 r^2 + A_1 \tag{37}$$

$$u(r) = \frac{-aA_2k^2 + a\kappa_s^2 - \omega_s}{b} - \frac{2a}{b}A_3k^2r^2 - \frac{3a}{b}A_4k^2r^4, \tag{38}$$

where

$$\begin{aligned}
A_1 &= \frac{\left(-162C_5^3 + 1083C_4C_6C_5 - 6859C_3C_6^2\right)}{20577C_6^{5/2}k^2}\sqrt{\frac{\alpha b}{2ac}}\\
&\quad - \frac{3C_5\lambda}{76cC_6k^2},\\
A_2 &= \frac{\left(54C_5^2 - 361C_4C_6\right)}{1444C_6^{3/2}k^2}\sqrt{\frac{\alpha b}{2ac}} - \frac{\lambda}{16ck^2},\\
A_3 &= \frac{-3C_5}{19\sqrt{C_6}k^2}\sqrt{\frac{\alpha b}{2ac}},\\
A_4 &= -\frac{\sqrt{C_6}}{6k^2}\sqrt{\frac{\alpha b}{2ac}},\\
\omega_s &= \frac{a\left(361C_4C_6 - 90C_5^2\right)}{1444C_6^{3/2}}\sqrt{\frac{\alpha b}{2ac}} + \frac{a\left(16c\kappa_s^2 + \lambda\right)}{16c}\\
&= -\frac{\left(162C_5^3 - 1083C_4C_6C_5 + 6859C_3C_6^2\right)2\alpha bC_5}{130321C_6^3\lambda}.
\end{aligned} \tag{39}$$

Using (10), (11) and (13), Eq.(37) and Eq. (38) become

$$P'(z)^2 = A_4P(z)^6 + A_3P(z)^4 + A_2P(z)^2 + A_1, \tag{40}$$

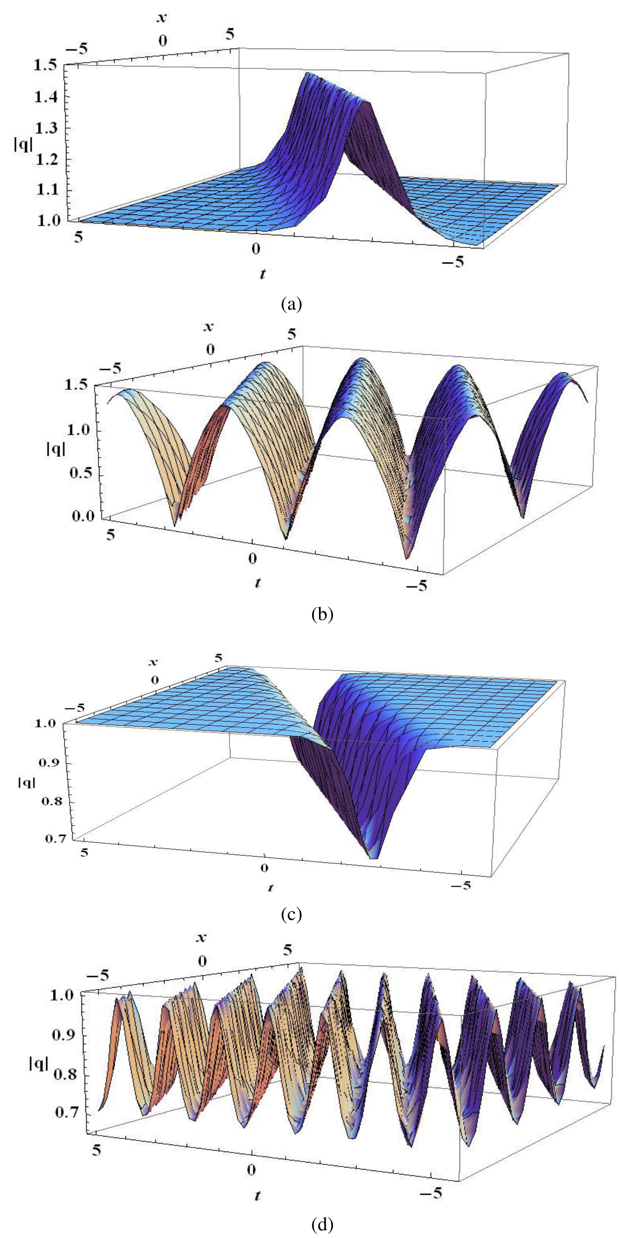

FIGURE 4. Examples of first and second groups of solutions result from the third form of $F\left(|q|^2\right) = C_6|q|^8 + C_5|q|^6 + C_4|q|^4 + C_3|q|^2$ exist for (a) $\kappa_s = k = a = 1$, (b) $m = 0.5$ and $\kappa_s = k = a = 1$, (c) $\kappa_s = k = a = 1$ and (d) $\kappa_s = k = a = 1$.

$$\begin{aligned}
Q(z) = &\frac{-aA_2k^2 + a\kappa_s^2 - \omega_s}{b} - \frac{2a}{b}A_3k^2P(z)^2\\
&- \frac{3a}{b}A_4k^2P(z)^4.
\end{aligned} \tag{41}$$

The first equation, i.e., Eq. (40), has many solutions, see [29]–[31], which are investigated in the following subsections

1) THE FIRST GROUP OF SOLUTIONS

The first group of solutions is

$$P(z) = \sqrt{1 + \mathrm{cn}(z,m)}, \tag{42}$$

where cn is Jacobi elliptic cosine function and m is its modulus. Solution (42) is satisfied when

$$C_3 = \frac{ak^2m^2}{2\alpha b}\left(4\lambda - ck^2\left(16m^2+13\right)\right),$$
$$C_4 = \frac{3ak^2m^2}{4\alpha b}\left(4ck^2\left(8m^2+1\right)-\lambda\right),$$
$$C_5 = \frac{-19ack^4m^4}{\alpha b},$$
$$C_6 = \frac{9ack^4m^4}{2\alpha b},$$
$$\omega_s = a\kappa_s^2 + ak^2m^2 + \frac{ak^2}{4} - \frac{2ack^4m^2}{\lambda}.$$

Substituting from (42) into (2), we have

$$q(x,t) = \sqrt{1+\mathrm{cn}(k\,(x-2a\kappa_s t)\,,m)}e^{i(\kappa_s x-\omega_s t+\sigma)}. \quad (43)$$

Taking the absolute value to get

$$|q| = \sqrt{1+\mathrm{cn}(k\,(x-2a\kappa_s t)\,,m)}. \quad (44)$$

Now we investigate the following cases of solution (44):

a: CASE 1

When $m = 1$, solution (44) degenerates to

$$|q| = \sqrt{\mathrm{sech}(k\,(x-2a\kappa_s t))+1}. \quad (45)$$

which is bright soliton solution of (1). For example, for $\kappa_s = k = a = 1$, we obtain Fig. 4 (a).

b: CASE 2

when $m \neq 1$, solution (44) becomes periodic solution. For example when $m = 0.5$ and $\kappa_s = k = a = 1$, we get the periodic solution illustrated in Fig.4 (b).

2) THE SECOND GROUP OF SOLUTIONS

The second group of solutions can be expressed as

$$P(z) = \frac{1}{\sqrt{\mathrm{cn}(z,m)^2+1}}, \quad (46)$$

Solution (46) is satisfied when

$$C_3 = \frac{-2ak^2}{\alpha b}\left(4\,ck^2\left(58\,m^4-32\,m^2+3\right)+\lambda\left(8\,m^2-3\right)\right),$$
$$C_4 = \frac{6ak^2}{\alpha b}\left(2\,ck^2\left(144\,m^4-104\,m^2+17\right)+\lambda\left(2\,m^2-1\right)\right),$$
$$C_5 = \frac{-152\,ack^4}{\alpha b}\left(16\,m^4-14\,m^2+3\right),$$
$$C_6 = \frac{288\,ack^4}{\alpha b}\left(1-2\,m^2\right)^2,$$
$$\omega_s = \frac{a}{\lambda}\left(4\,ck^4\,m^2\left(3-8\,m^2\right)+\lambda\left(\kappa_s^2+k^2\left(1-5\,m^2\right)\right)\right).$$

Substituting from (46) into (2), we can secure

$$q(x,t) = \frac{1}{\sqrt{\mathrm{cn}(z,m)^2+1}}e^{i(\kappa_s x-\omega_s t+\sigma)}. \quad (47)$$

After taking the absolute value it yields

$$|q| = \frac{1}{\sqrt{\mathrm{cn}(k\,(x-2a\kappa_s t)\,,m)^2+1}}. \quad (48)$$

Now, we will consider some cases for solution (46).

a: CASE 1

When $m = 1$, solution (48) degenerates to

$$|q| = \frac{1}{\sqrt{\mathrm{sech}^2(k\,(x-2a\kappa_s t))+1}}. \quad (49)$$

which is a dark soliton solution of (1). For example, for $\kappa_s = k = a = 1$, we obtain Fig. 4 (c).

b: CASE 2

When $m = 0$, solution (48) degenerates to

$$|q| = \frac{1}{\sqrt{\cos^2(k\,(x-2a\kappa_s t))+1}}. \quad (50)$$

which represents a periodic solution of (1). For example, for $\kappa_s = k = a = 1$, we obtain Fig. 4 (d).

Fig. 4 shows the dark and bright solitons solutions and periodic solutions which are obtained in the cases of third form of F.

IV. CONCLUSION

The dynamics of nematic liquid crystals have been scrutinized for new distinct intrinsic nonlinearities. Using the travelling wave transformation (2), the nonlinear second order system (1) is transformed into the second order system (4). Therefore, we use Lie symmetry method to obtain some new exact traveling wave solutions of Eq. (1). Several kinds of new solutions have been identified for the cases nonlinearities, which involve quadratic, generalized dual-power law and eighth order power. These kinds involve periodic solutions, bright solitons, dark solitons and singular solitons. Also, some new doubly periodic solutions in the form of Jacobi elliptic functions and Weierstrass elliptic function are obtained in the two cases of quadratic and eighth order nonlinearities. The Lie point symmetry method was used efficiently to derive doubly periodic solutions of the problem. The work justified the reliability of the Lie point symmetry method in handling identical problems. The future work can include applications the solution technique in this work to the more realistic 2D and 3D structures of liquid crystals. Moreover, the occurrence and stability of oscillatory behaviors in this model can be examined.

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