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Resilient Observer-Based H_∞ Control of Uncertain Nonlinear Networked Control Systems

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ABSTRACT This paper considers the resilient H_∞ dynamic control of nonlinear uncertain systems over a network assuming random communication packet dropouts, a subject which is seldom considered in the current literature for such uncertain systems. The uncertainties in the system and the controller are real, time-varying and norm bounded. Bernoulli distribution with white sequence is used to model the random packet losses with assumed conditions on the probability distribution. The resilient controller designed is an observer-based dynamic. The resulted closed-loop system is exponentially mean square stable and the H_∞ performance is less than a prescribed level γ for all admissible uncertainties. New sufficient conditions for the existence of such a controller are presented and proved based on the linear matrix inequalities (LMIs) approach. A numerical example is presented to demonstrate and show the effectiveness of the developed theory.

INDEX TERMS Resilient control, uncertain systems, robust stabilization, nonlinear systems, networked control system.

I. INTRODUCTION

Physical systems control is the subject of applying developed control theory these days. Control of such systems are based on obtaining suitable mathematical model using simplification or linearization which create uncertainty in their models. In addition, noisy input or output data are another source of uncertainty. For other sources of uncertainty, see for example [1]. It is clear that external disturbances affect the control performances in networked control systems (NCSs). Presence of unknown parameters in the mathematical model is one way to characterize physical systems. The analysis and synthesis of these systems are to determine possible behaviors of the system and to design robust control strategies for a family of admissible values of the uncertainty. In June 2009, a technical committee that pay attention and encourage research on uncertain systems was initiated. It aims to provide researchers working in this area with a network of resources, events, contacts, and others, [2]. The controller is also exposed to some of the uncertainty for many reasons, such as that of trying to reduce its order, imprecise implementation, actuator degradation, or due to the requirement of re-adjustment of its gains during the implementation stage. One way to model such uncertainties/perturbations in both the system or the controller is by additive uncertainty.

In the area of Networked Control System (NCS), researchers concentrate on two main problems, control of the network and control over the network. The first concerned with network problems such as networking protocol, congestion, routing, efficient communication, etc, while the later concentrates on the problems that are faced in real time while controlling industrial systems over the network. They study the effects of the network time delay, packet dropout or disorder of packet arrival on the controlled systems. The effect of these problems are required to be minimized by the designed controller. It is the goal of researchers to attain a high Quality of the Services (QoS) and a high Quality of Control (QoC). Our work in this research is devoted to the second type, QoC, i.e control of industrial systems over the network. These days, NCSs applications cover a wide range of industrial areas such as environmental monitoring, autonomous robots, industrial automation, smart grids, mobile communications, just to mention a few. These wide areas make control over a network successful due to its many advantages such as low cost, simple installation, reduced wiring and high reliability, see for example [3] and the references therein.

Researchers focus their work on studying the stability of Linear Networked Control Systems (LNCS), see for example [4]–[6] and the references therein. Nonlinear Networked

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Control Systems (NNCS) methods that study nonlinear systems are extensions of some techniques that were proposed to control LNCS under suitable conditions and assumptions.

The authors in [7], presented a *certain* observer-based H_∞ control for a special class of *certain* NNCS with packet losses. The packet losses are modeled as either occurred or not occurred using Bernoulli distribution. They presented their results in the form of Linear Matrix Inequality (LMI).

In [8], a decentralized resilient H_∞ observer based controller for a class of uncertain interconnected symmetric composite continuous-time systems with nonlinearities was presented. The uncertainty assumed in this was only in the system state matrix A.

Non-fragile robust control of a class of nonlinear networked control systems (NNCSs) with long time-varying delay was investigated with *certain* state feedback controller in [9]. The uncertainty was only considered in the nonlinearity term.

In [10], the authors discussed the same problem they proposed in [9], but with a short time delay and a static output feedback controller.

In [11], the authors discussed the non-fragile robust H_∞ control NNCSs with unknown actuator failures and time-varying delay whose upper and lower bounds are known. The uncertainties in the system are assumed in the A and B matrices, plus the controller type is *certain* state feedback.

In [12], resilient observer-based control for networked nonlinear T-S fuzzy *certain* systems with hybrid-triggered scheme was discussed where the nonlinear system is represented by a set of T-S fuzzy linear systems. The uncertainties are considered in the controller gains only.

In [13], only the uncertainty in the state and output parameters A and C of the system are considered. The authors designed an observer–based controller that control the error states, and they assumed the observer parameters are certain except the observer gain.

The paper in [14] presents new results to control uncertain nonlinear networked control systems with random packet loss. The controller designed is an observer-based H_∞ with certain parameters.

This paper discusses the resilient H_∞ control and stability of uncertain NNCS which suffer from packet losses in both directions. The resilient H_∞ controller is an observer-based. The work in [14] will be extended here such that the controller is resilient. The mathematical formulation of the problem resulted into solving Bilinear Matrix Inequality (BMI). Two procedures for converting Bilinear Matrix Inequality (BMI) to Linear Matrix Inequality (LMI) will be presented. To the knowledge of the author, the problem of designing a resilient H_∞ observer based dynamic controller for NNCS with uncertainties in all its parameters has not been discussed yet.

Finally, the effectiveness of the theory presented in this paper will be demonstrated by a numerical example.

The paper is organized as follows. Section 2, presents the formulation of the problem. Section 3, presents the main results. Subsection 3.1 covers the stability analysis, Subsection 3.2 studies the H_∞ performance and Subsection 3.3 presents new techniques to obtain the gains of the controller and the observer. To demonstrate the effectiveness of the proposed work, a numerical example is given in Section 4, while Section 5 concludes the work. The proofs of the two main theorems are listed in Appendices A and B.

Standard notations will be used in this paper. Let $\Pr\{\cdot\}$ denotes the occurrence of the probability of the event "$\cdot$". The expectation of the stochastic variable x will be denoted by $\mathbb{E}\{x\}$, while the sets of positive integers, the set of real numbers, the n-dimensional of Euclidean space and all $n \times m$ real matrices are denoted by $\mathbb{I}^+$, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$. The maximum and the minimum eigenvalues of matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ respectively. $\|\cdot\|$ denotes the Euclidean vector norm or the induced matrix $2-norm$. I is the identity matrix with appropriate dimension. Block-diagonal matrix is denoted by $diag(a_1, a_2, \ldots .a_n)$ and "$*$" is used in symmetric block matrices.

II. PROBLEM FORMULATION

The proposed layout of the nonlinear networked control system (NNCS) with packet dropout in the network is shown in Figure 1. The controlled plant is nonlinear and the uncertainty is assumed in all its parameters. The random packet losses, from the sensor to the controller or from the controller to the actuator, occur simultaneously over the communication network. The data is assumed to be transmitted in a single-packet manner with the same transmission length with a point-to-point network allowable data dropout rate.

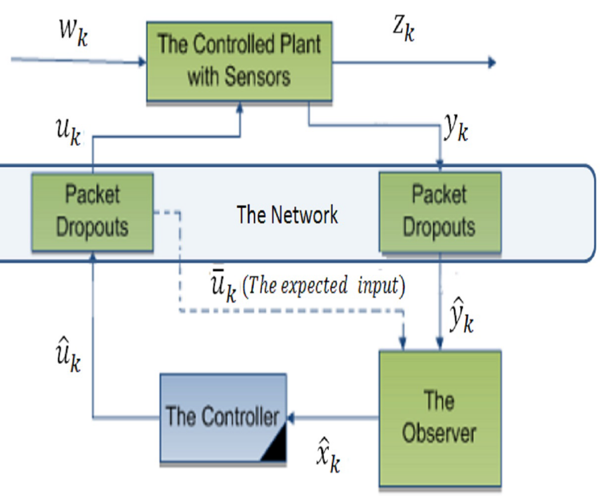

FIGURE 1. The layout of the system.

The considered uncertain nonlinear networked control systems is given as follows:

$$
\begin{aligned}
x_{k+1} &= (A + \Delta A)x_k + (B + \Delta B)\, u_k + f(k, x_k) + Dw_k \\
z_k &= (C_1 + \Delta C_1)x_k + D_1 w_k
\end{aligned} \tag{1}
$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input vector, $z_k \in \mathbb{R}^r$ is the controlled output vector, $w_k \in \mathbb{R}^q$ is the disturbance input belong to $l_2[0, \infty)$, and A, $\Delta A \in \mathbb{R}^{n\times n}$, B, $\Delta B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times q}$, C_1, $\Delta C_1 \in \mathbb{R}^{r\times n}$, $D_1 \in \mathbb{R}^{r\times q}$ where the system matrices (A, B, C_1, D, D_1) are known real constant, $(\Delta A, \Delta B, \Delta C_1)$ are the uncertainties in the matrices A, B, and C_1. $f(k, x_k)$ is a nonlinear function of the state variable that satisfies the global Lipschitz condition for a known real constant matrix G:

$$\|f(k, x)\| \leq \|Gx\| \tag{2}$$

$$\|f(k, x) - f(k, y)\| \leq \|G(x - y)\| \tag{3}$$

The following equation describes the measurement with random communication packet loss

$$\hat{y}_k = \alpha_k(C_2x_k) + D_2w_k \tag{4}$$

where $\widehat{y}_k \in \mathbb{R}^p$ is the measured output vector, $C_2 \in \mathbb{R}^{p\times n}$, $D_2 \in \mathbb{R}^{p\times q}$ are real constant matrices. The linear stochastic variable $\alpha_k \in \mathbb{R}$ is a Bernoulli distribution white sequence which simulates the packet dropout from the sensor to the controller with the following probability and variance

$$\Pr\{\alpha_k = 1\} = \mathbb{E}\{\alpha_k\} = \bar{\alpha} \tag{5a}$$

$$\Pr\{\alpha_k = 0\} = 1 - \mathbb{E}\{\alpha_k\} = 1 - \bar{\alpha} \tag{5b}$$

$$var\{\alpha_k\} = \mathbb{E}\{(\alpha_k - \bar{\alpha})^2\} = (1 - \bar{\alpha})\bar{\alpha} = \alpha_1^2 \tag{5c}$$

The stability analysis and controller synthesis for the NNCS system given in (1-5c) with random packet losses is very important in both theory and applications, and it is also a very challenging problem.

The proposed resilient dynamic controller is observer-based and is described by the following equations, [15].

$$\begin{aligned}\hat{x}_{k+1} &= (A + \Delta A)\hat{x}_k + (B + \Delta B)\bar{u}_k + f(k, \hat{x}_k)\\ &\quad + (L + \Delta L)\,(\hat{y}_k - \bar{\alpha}(C_2\hat{x}_k))\\ \bar{u}_k &= \bar{\beta}\hat{u}_k\end{aligned} \tag{6}$$

and

$$\begin{aligned}\hat{u}_k &= -\,(K + \Delta K)\,\hat{x}_k\\ u_k &= \beta_k\hat{u}_k\end{aligned} \tag{7}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the observer state, $\bar{u}_k \in \mathbb{R}^m$ is the control input to the observer, $\hat{u}_k \in \mathbb{R}^m$ is the output of the resilient state feedback controller, $u_k \in \mathbb{R}^m$ is the control input of the controlled system, $L \in \mathbb{R}^{n\times p}$ is the observer gain and $K \in \mathbb{R}^{m\times n}$ is the state feedback controller gain. The stochastic variable $\beta_k \in \mathbb{R}$ is a linear Bernoulli distribution white sequence which simulates the packet dropout from the controller to the actuator with the following properties.

$$\Pr\{\beta_k = 1\} = \mathbb{E}\{\beta_k\} = \bar{\beta} \tag{8a}$$

$$\Pr\{\beta_k = 0\} = 1 - \mathbb{E}\{\beta_k\} = 1 - \bar{\beta} \tag{8b}$$

$$\begin{aligned}var\{\beta_k\} &= \mathbb{E}\{(\beta_k - \bar{\beta})^2\} = (1 - \bar{\beta})\bar{\beta}\\ &= \beta_1^2\end{aligned} \tag{8c}$$

The linear stochastic variables α_k and β_k are assumed to be different.

The uncertainty matrices are defined as follows:

$$\begin{aligned}\begin{bmatrix}\Delta A & \Delta B\\ 0 & (\Delta K)^T\end{bmatrix} &= \begin{bmatrix}M_c & 0\\ 0 & M_k\end{bmatrix}\begin{bmatrix}\Delta_k & 0\\ 0 & \Delta_k\end{bmatrix}\\ &\quad * \begin{bmatrix}N_1 & N_2\\ 0 & N_2\end{bmatrix},\end{aligned} \tag{9}$$

$$\begin{aligned}\Delta L &= M_L\Delta_k N_L, \quad \Delta C_1 = M_3\Delta_k N_3\\ M_c &\in \mathbb{R}^{n\times l}, M_k \in \mathbb{R}^{n\times l}, \quad M_L \in \mathbb{R}^{n\times l}\\ &\quad \Delta_k \in \mathbb{R}^{l\times \upsilon}, N_1 \in \mathbb{R}^{\upsilon\times n},\\ N_2 &\in \mathbb{R}^{\upsilon\times m}, N_L \in \mathbb{R}^{\upsilon\times n},\\ &\quad M_3 \in \mathbb{R}^{r\times l}, N_3 \in \mathbb{R}^{\upsilon\times n}\end{aligned} \tag{10}$$

where M_c, M_3, M_k, M_L, N_1, N_2, N_3 and N_L are known real matrices with appropriate dimensions. Δ_k is an unknown time-varying matrix with the following constraint.

$$\begin{aligned}\Delta_k\Delta_k^T &< I_{l\times l} \text{ or } \Delta_k^T\Delta_k < I_{k\times k}\\ N_2N_2^T &< I_{v\times v}\end{aligned} \tag{11}$$

Now, the state estimation error is defined as follows:

$$e_k = x_k - \hat{x}_k \tag{12}$$

From (1), (6), (7) and (12), the closed-loop equations for the nonlinear network system are obtained with simple manipulations as

$$\begin{aligned}x_{k+1} &= ((A + \Delta A) - \bar{\beta}\,(B + \Delta B)\,(K + \Delta K))x_k\\ &\quad + (\beta_k - \bar{\beta})\,(B + \Delta B)\,(K + \Delta K)\,e_k\\ &\quad - (\beta_k - \bar{\beta})\,(B + \Delta B)\,(K + \Delta K)\,x_k\\ &\quad + \bar{\beta}\,(B + \Delta B)\,(K + \Delta K)\,e_k + f(k, x_k) + Dw_k\\ e_{k+1} &= -(\beta_k - \bar{\beta})\,(B + \Delta B)\,(K + \Delta K)\,x_k\\ &\quad - (\alpha_k - \bar{\alpha})\,(L + \Delta L)\,C_2x_k\\ &\quad + ((A + \Delta A) - \bar{\alpha}\,(L + \Delta L)\,C_2)e_k\\ &\quad + (\beta_k - \bar{\beta})\,(B + \Delta B)\,(K + \Delta K)\,e_k\\ &\quad + F_k + (D - (L + \Delta L)\,D_2)w_k\end{aligned} \tag{13}$$

Define η_k and F_k as

$$\eta_k = \begin{bmatrix}x_k\\ e_k\end{bmatrix}, \quad F_k = f(k, x_k) - f(k, \hat{x}_k) \tag{14}$$

The closed-loop uncertain and resilient nonlinear network control system is written in the following compact form

$$\eta_{k+1} = \bar{A}\eta_k + (\beta_k - \bar{\beta})\hat{A}_1\eta_k + (\alpha_k - \bar{\alpha})\hat{A}_2\eta_k + \bar{F}_k + \bar{B}w_k \tag{15}$$

where

$$\begin{aligned}\bar{A} &= \begin{bmatrix}A_\Delta & \bar{\beta}\,(B + \Delta B)\,(K + \Delta K)\\ 0 & (A + \Delta A) - \bar{\alpha}\,(L + \Delta L)\,C_2\end{bmatrix}\\ A_\Delta &= (A + \Delta A) + \bar{\beta}\,(B + \Delta B)\,(K + \Delta K)\\ \hat{A}_1 &= \begin{bmatrix}-\,(B + \Delta B)\,(K + \Delta K) & (B + \Delta B)\,(K + \Delta K)\\ -\,(B + \Delta B)\,(K + \Delta K) & (B + \Delta B)\,(K + \Delta K)\end{bmatrix}\end{aligned}$$

$$\hat{A}_2 = \begin{bmatrix} 0 & 0 \\ -(L+\Delta L)C_2 & 0 \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} D \\ D-(L+\Delta L)D_2 \end{bmatrix}$$
$$\bar{F}_k = \begin{bmatrix} f(k,x_k) \\ F_k \end{bmatrix}$$

This compact system contains the stochastic parameters α_k, β_k.

III. THE MAIN RESULTS

The objectives of this paper are to design a resilient observer-based controller (6-7) for the uncertain nonlinear networked control system (1), such that, in the presence of disturbance and random packet losses, the closed-loop nonlinear networked system (15) is exponentially mean square stable and the H_∞ performance constraint

$$\sum_{k=0}^{\infty} \mathbb{E}\,\|z_k\|^2 < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\,\|w_k\|^2 \tag{16}$$

is satisfied for a minimum prescribed scalar $\gamma > 0$.

The following useful definition and lemmas are required in this paper.

Definition 1: [16] The closed-loop uncertain nonlinear networked control system (15) is said to be exponentially mean-square stable, when $w_k = 0$ if there exist constant $\phi > 0$ and $\tau \in (0,1)$such that

$$\mathbb{E}\{\|\eta_k\|^2\} \leq \phi\tau^k \mathbb{E}\{\|\eta_0\|^2\}, \quad \forall \eta_0 \in \mathbb{R}^n, k \in I^+ \tag{17}$$

Lemma 1 ([17] (S-Procedure)): Let $T_i \in \mathbb{R}^{n\times n}(i = 0, 1, 2, \ldots.., p)$ be symmetric matrices. The conditions on $T_i\ (i = 0, 1, 2, \ldots.., p)$, $\varsigma^T T_o \varsigma > 0,\ \forall \varsigma \neq 0\ s.t.\ \varsigma^T T_i \varsigma \geq 0\ \ (i = 0, 1, 2, \ldots.., p)$ hold if there exist $\tau_i \geq 0\ (i = 0, 1, 2, \ldots.., p)$ such that $T_o - \sum_{i=1}^{p} \tau_i T_i > 0$.

Lemma 2 [18]: For real matrices $M = M^T$, H and E with $F(t)$ satisfying $F(t)F^T(t) < I$, then $M + HFE + E^T F^T H^T < 0$, if and only if there exist a positive scalar $\epsilon > 0$ such that

$$M + \epsilon HH^T + \frac{1}{\epsilon} E^T E < 0 \tag{18}$$

or equivalently

$$\begin{bmatrix} M & \epsilon H & E^T \\ \epsilon H^T & -\epsilon I & 0 \\ E & 0 & -\epsilon I \end{bmatrix} < 0 \tag{19}$$

A. STABILITY ANALYSIS

In this subsection, a stability theorem will be stated which gives a sufficient condition for the exponential mean square stability of the closed-loop uncertain NNCS (15) with a resilient observer based dynamic controller.

Theorem 1: The closed loop uncertain nonlinear networked control system (15), with $w_k = 0$ is exponentially mean-square stable under the given resilient controller in (6-7) if for the given communication channel parameters $0 \leq \bar{\alpha} \leq 1, 0 \leq \bar{\beta} \leq 1$ there exist $P > 0$, $Q > 0$ and $\tau_1 > 0, \tau_2 > 0, \epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$ satisfying the following matrix inequality

$$\begin{bmatrix} \Xi_0 & \Upsilon_1 & \epsilon_1\chi_1^T & \epsilon_2\Upsilon_2 & \chi_2^T & \Upsilon_3 & \epsilon_3\chi_3^T \\ \Upsilon_1^T & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ \epsilon_1\chi_1 & 0 & -\epsilon_1 I & 0 & 0 & 0 & 0 \\ \epsilon_2\Upsilon_2^T & 0 & 0 & -\epsilon_2 I & 0 & 0 & 0 \\ \chi_2 & 0 & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \Upsilon_3^T & 0 & 0 & 0 & 0 & -\epsilon_3 I & 0 \\ \epsilon_3\chi_3 & 0 & 0 & 0 & 0 & 0 & -\epsilon_3 I \end{bmatrix} < 0 \tag{20}$$

where

$$\Xi_0 = \begin{bmatrix} \Xi_{01} & * \\ \Xi_{02} & \Xi_{03} \end{bmatrix}$$
$$\Xi_{01} = \begin{bmatrix} -P+\tau_1 G^T G & * & * & * \\ 0 & -Q+\tau_2 G^T G & * & * \\ 0 & 0 & -\tau_1 I & * \\ 0 & 0 & 0 & -\tau_2 I \end{bmatrix}$$
$$\Xi_{02} = \begin{bmatrix} PA-\bar{\beta}PBK & \bar{\beta}PBK & P & 0 \\ \beta_1 PBK & -\beta_1 PBK & 0 & 0 \\ \beta_1 QBK & -\beta_1 QBK & 0 & 0 \\ \alpha_1 QLC_2 & 0 & 0 & 0 \\ 0 & Q(A-\bar{\alpha}LC_2) & 0 & Q \end{bmatrix}$$
$$\Xi_{03} = \begin{bmatrix} -P & 0 & 0 & 0 & 0 \\ 0 & -P & 0 & 0 & 0 \\ 0 & 0 & -Q & 0 & 0 \\ 0 & 0 & 0 & -Q & 0 \\ 0 & 0 & 0 & 0 & -Q \end{bmatrix}$$
$$\Upsilon_1 = \left[\begin{array}{c} 0_{4\times 5} \\ \hline \Theta_{1(5\times 5)} \end{array}\right] = \left[\begin{array}{ccccc} \multicolumn{5}{c}{0_{4\times 5}} \\ \hline PM_c & -\bar{\beta}PBM_k & 0 & 0 & 0 \\ 0 & \beta_1 PBM_k & 0 & 0 & 0 \\ 0 & \beta_1 QBM_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 QM_L \\ 0 & 0 & QM_c & -\bar{\alpha}QM_L & 0 \end{array}\right]$$
$$\chi_1 = \left[\begin{array}{c|c} \Phi_{1(5\times 4)} & 0_{5\times 5} \end{array}\right] = \left[\begin{array}{cccc|c} N_1 & 0 & 0 & 0 & \\ N_k & -N_k & 0 & 0 & \\ 0 & 0 & 0 & N_1 & 0_{5\times 5} \\ 0 & 0 & 0 & N_L C_2 & \\ N_L C_2 & 0 & 0 & 0 & \end{array}\right]$$
$$\Upsilon_2 = \left[\begin{array}{c} 0_{4\times 1} \\ \hline \Theta_{2(5\times 1)} \end{array}\right] = \left[\begin{array}{c} 0_{4\times 1}[8pt] \\ \hline -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \end{array}\right]$$

$$\begin{aligned}\chi_2 &= \left[\, \Phi_{2(1\times 4)} \,\middle|\, 0_{1\times 5} \,\right] \\ &= \left[\, N_2K \quad -N_2K \quad 0 \quad 0 \,\middle|\; 0_{1\times 5} \,\right]\end{aligned}$$

$$\Upsilon_3 = \begin{bmatrix} 0_{4\times 1} \\ \hline \Theta_{3(5\times 1)} \end{bmatrix} = \begin{bmatrix} 0_{4\times 1} \\ \hline -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\chi_3 &= \left[\, \Phi_{3(1\times 4)} \,\middle|\, 0_{1\times 5} \,\right] \\ &= \left[\, N_2K \quad -N_2K \quad 0 \quad 0 \,\middle|\, 0_{1\times 5} \,\right]\end{aligned}$$

$$\alpha_1 = \sqrt{(1-\bar{\alpha})\bar{\alpha}} \ \textit{and} \ \beta_1 = \sqrt{(1-\bar{\beta})\bar{\beta}}.$$

The proof is given in Appendix A.

B. H_∞– PERFORMANCE

Next, the sufficient condition proved in Theorem 1 will be extended to show that the closed-loop uncertain nonlinear networked control system (15) is exponentially mean square stable and satisfies the H_∞-performance constraint stated in (16) under nonzero disturbance w_k.

Theorem 2: Given the communication channel parameters $0 \le \bar{\alpha} \le 1, 0 \le \bar{\beta} \le 1$. The closed loop uncertain nonlinear networked control system (15) with $w_k \neq 0$ is exponentially mean-square stable under the resilient observer-based dynamic controller stated in (6-7) and the H_∞ performance constraint (16) is ensured if there exist $\tau_1 > 0, \tau_2 > 0, \epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$ and matrices $P > 0$ and $Q > 0$ satisfying the following matrix inequality

$$\begin{bmatrix} \tilde{\Xi}_0 & \tilde{\Upsilon}_1 & \epsilon_1\tilde{\chi}_1^T & \epsilon_2\tilde{\Upsilon}_2 & \tilde{\chi}_2^T & \tilde{\Upsilon}_3 & \epsilon_3\tilde{\chi}_3^T \\ \tilde{\Upsilon}_1^T & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ \epsilon_1\tilde{\chi}_1 & 0 & -\epsilon_1 I & 0 & 0 & 0 & 0 \\ \epsilon_2\tilde{\Upsilon}_2^T & 0 & 0 & -\epsilon_2 I & 0 & 0 & 0 \\ \tilde{\chi}_2 & 0 & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \tilde{\Upsilon}_3^T & 0 & 0 & 0 & 0 & -\epsilon_3 I & 0 \\ \epsilon_3\tilde{\chi}_3 & 0 & 0 & 0 & 0 & 0 & -\epsilon_3 I \end{bmatrix} < 0 \tag{21}$$

where

$$\tilde{\Xi}_0 = \begin{bmatrix} \tilde{\Xi}_{011} & * \\ \tilde{\Xi}_{012} & \tilde{\Xi}_{022} \end{bmatrix}$$

$$\tilde{\Xi}_{011} = \begin{bmatrix} -P+\tau_1 G^T G^T & * & * & * & 0 \\ 0 & -Q+\tau_2 G^T G^T & * & * & 0 \\ 0 & 0 & -\gamma^2 I & * & 0 \\ 0 & 0 & 0 & -\tau_1 I & 0 \\ 0 & 0 & 0 & 0 & -\tau_2 I \end{bmatrix}$$

$$\tilde{\Xi}_{012} = \begin{bmatrix} PA-\bar{\beta}PBK & \bar{\beta}PBK & PD & P & 0 \\ \beta_1 PBK & -\beta_1 PBK & 0 & 0 & 0 \\ \beta_1 QBK & -\beta_1 QBK & 0 & 0 & 0 \\ \alpha_1 QLC_2 & 0 & 0 & 0 & 0 \\ 0 & QA-\bar{\alpha}QLC_2 & Q(D-LD_2) & 0 & Q \\ C_1 & 0 & D_1 & 0 & 0 \end{bmatrix}$$

$$\tilde{\Xi}_{022} = \begin{bmatrix} -P & 0 & 0 & 0 & 0 & 0 \\ 0 & -P & 0 & 0 & 0 & 0 \\ 0 & 0 & -Q & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q & 0 & 0 \\ 0 & 0 & 0 & 0 & -Q & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}$$

$$\tilde{\Upsilon}_1 = \begin{bmatrix} 0_{5\times 6} \\ \tilde{\Theta}_{1(6\times 6)} \end{bmatrix}, \quad \tilde{\chi}_1^T = \begin{bmatrix} \tilde{\Phi}^T_{1(6\times 5)} \\ 0_{6\times 6} \end{bmatrix}$$

$$\tilde{\Theta}_{1(6\times 6)} = \begin{bmatrix} PM_c & -\bar{\beta}PBN_2^T & 0 & 0 & 0 & 0 \\ 0 & \beta_1 PBN_2^T & 0 & 0 & 0 & 0 \\ 0 & \beta_1 QBN_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1 QM_L \\ 0 & 0 & 0 & QM_c & xQM_L & 0 \\ 0 & 0 & M_3 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\Phi}^T_{1(6\times 5)} = \begin{bmatrix} N_1 & 0 & 0 & 0 & 0 \\ M_k^T & -M_k^T & 0 & 0 & 0 \\ N_3 & 0 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 & 0 \\ 0 & N_LC_2 & -\frac{1}{x}N_LD_2 & 0 & 0 \\ N_LC_2 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\tilde{\Upsilon}_2 = \begin{bmatrix} 0_{5\times 1} \\ \tilde{\Theta}_{2(6\times 1)} \end{bmatrix}, \quad \tilde{\chi}_2^T = \begin{bmatrix} \tilde{\Phi}^T_{2(6\times 1)} \\ 0_{5\times 1} \end{bmatrix}$$

$$\tilde{\Theta}_{2(6\times 1)} = \begin{bmatrix} -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{\Phi}^T_{2(6\times 1)} = \begin{bmatrix} N_2K & -N_2K & 0 & 0 & 0 & 0 \end{bmatrix}^T_{(6\times 1)}$$

$$\tilde{\Upsilon}_3 = \begin{bmatrix} 0_{4\times 1} \\ \tilde{\Theta}_{3(5\times 1)} \end{bmatrix}, \quad \tilde{\chi}_3^T = \begin{bmatrix} \tilde{\Phi}^T_{3(6\times 1)} & 0_{1\times 5} \end{bmatrix}$$

$$\widetilde{\Theta}_{3(6\times 1)} = \begin{bmatrix} -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\widetilde{\Phi}^T_{3(6\times 1)} = \begin{bmatrix} N_2K & -N_2K & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

Proof: The proof is given in Appendix B. ■

C. CONTROLLER DESIGN

The matrix inequality (21) in Theorem 2 is bilinear in K, L, P and Q. In the following subsection we propose two methods to convert the MI in (21) into LMI such that we can solve for controller gain matrices K and L. In this Subsection, we will refer to some equations which are derived in the proof of Theorem 2, (Appendix B).

$\widetilde{\Xi}_0$ in (21) is given as

$$\widetilde{\Xi}_0 = \begin{bmatrix} \widetilde{\Xi}_{011} & \widetilde{\Xi}^T_{012} \\ \widetilde{\Xi}_{012} & \widetilde{\Xi}_{022} \end{bmatrix}$$

Then from (68-70), Appendix B, (21) is expanded as

$$\left[\begin{array}{cccccccc} \widetilde{\Xi}_{011} & \widetilde{\Xi}^T_{012} & [0]_{5\times 6} & \epsilon_1\widetilde{\Phi}^T_1 & [0]_{5\times 1} & \widetilde{\Phi}^T_2 & [0]_{5\times 1} & \epsilon_3\widetilde{\Phi}^T_{3(5\times 1)} \\ \widetilde{\Xi}_{012} & \widetilde{\Xi}_{022} & \widetilde{\Theta}_{1(6\times 6)} & [0]_{6\times 6} & \epsilon_2\widetilde{\Theta}_{2(6\times 1)} & [0]_{6\times 1} & \widetilde{\Theta}_{3(6\times 1)} & [0]_{6\times 1} \\ [0]^T_{6\times 5} & \widetilde{\Theta}^T_{1(6\times 6)} & -\epsilon_1 [I]_{6\times 6} & 0 & 0 & 0 & 0 & 0 \\ \epsilon_1\widetilde{\Phi}_{1(6\times 5)} & [0]_{6\times 6} & 0 & -\epsilon_1 [I]_{5\times 5} & 0 & 0 & 0 & 0 \\ 0^T_{5\times 1} & \epsilon_2\widetilde{\Theta}^T_{2(6\times 1)} & 0 & 0 & -\epsilon_2 [I]_{1\times 1} & 0 & 0 & 0 \\ \widetilde{\Phi}_{2(1\times 5)} & 0_{1\times 6} & 0 & 0 & 0 & -\epsilon_2 [I]_{1\times 1} & 0 & 0 \\ [0]_{1\times 5} & \widetilde{\Theta}^T_{3(1\times 6)} & 0 & 0 & 0 & 0 & -\epsilon_3 [I]_{1\times 1} & 0 \\ \epsilon_3\widetilde{\Phi}_{3(1\times 5)} & [0]_{1\times 6} & 0 & & 0 & 0 & 0 & -\epsilon_3 [I]_{1\times 1} \end{array}\right] < 0 \tag{22}$$

Let

$$\mathbb{T} = diag\{I, \mathcal{T}, I, I, I, I\} \tag{23}$$

where

$$\mathcal{T} = \begin{bmatrix} P^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & P^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & Q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{24}$$

Multiply (22) from left and right by $\mathbb{T}$, we get

$$\left[\begin{array}{cccccccc} \widetilde{\Xi}_{011} & \widetilde{\Xi}^T_{012}\mathcal{T} & 0_{5\times 6} & \epsilon_1\widetilde{\Phi}^T_1 & 0_{5\times 1} & \widetilde{\Phi}^T_2 & 0_{5\times 1} & \epsilon_3\widetilde{\Phi}^T_{3(5\times 1)} \\ \mathcal{T}\widetilde{\Xi}_{012} & \mathcal{T}\widetilde{\Xi}_{022}\mathcal{T} & \mathcal{T}\widetilde{\Theta}_{6\times 6} & 0_{6\times 6} & \epsilon_2\mathcal{T}\widetilde{\Theta}_{2(6\times 1)} & 0_{6\times 1} & \widetilde{\Theta}_{3(6\times 1)} & [0]_{6\times 1} \\ 0^T_{6\times 5} & \widetilde{\Theta}^T_{6\times 6}\mathcal{T} & -\epsilon_1 I_{5\times 5} & 0 & 0 & 0 & 0 & 0 \\ \epsilon_1\widetilde{\Phi}_{1(6\times 5)} & 0_{6\times 6} & 0 & -\epsilon_1 I_{5\times 5} & 0 & 0 & 0 & 0 \\ 0^T_{5\times 1} & \epsilon_2\widetilde{\Theta}^T_{2(6\times 1)}\mathcal{T} & 0 & 0 & -\epsilon_2 I_{1\times 1} & 0 & 0 & 0 \\ \widetilde{\Phi}_{2(1\times 5)} & 0_{1\times 6} & 0 & 0 & 0 & -\epsilon_2 I_{1\times 1} & 0 & 0 \\ [0]_{1\times 5} & \widetilde{\Theta}^T_{3(1\times 6)} & 0 & 0 & 0 & -\epsilon_3 [I]_{1\times 1} & 0 & \\ \epsilon_3\widetilde{\Phi}_{3(1\times 5)} & [0]_{1\times 6} & 0 & 0 & 0 & 0 & 0 & -\epsilon_3 [I]_{1\times 1} \end{array}\right] < 0 \tag{25}$$

where

$$\mathcal{T}\widetilde{\Xi}_{012} = \begin{bmatrix} A-\bar{\beta}BK & \bar{\beta}BK & D & I & 0 \\ \beta_1 BK & -\beta_1 BK & 0 & 0 & 0 \\ \beta_1 BK & -\beta_1 BK & 0 & 0 & 0 \\ \alpha_1 HC_2 & 0 & 0 & 0 & 0 \\ 0 & QA-\bar{\alpha}HC_2 & QD-HD_2 & 0 & Q \\ C_1 & 0 & D_1 & 0 & 0 \end{bmatrix}, \quad H = QL$$

$$\mathcal{T}\widetilde{\Xi}_{022}\mathcal{T} = diag\{-P^{-1}, -P^{-1}, -Q^{-1}, -Q, -Q, -I\}$$

and

$$\mathcal{T}\widetilde{\Theta}_{1(6\times 6)} = \begin{bmatrix} M_c & -\bar{\beta}BN^T_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 BN^T_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 BN^T_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1 QM_L \\ 0 & 0 & 0 & QM_c & -xQM_L & 0 \\ 0 & 0 & M_3 & 0 & 0 & 0 \end{bmatrix}$$

$$\epsilon_1\widetilde{\Phi}^T_{1(6\times 5)} = \epsilon_1 \begin{bmatrix} N_1 & 0 & 0 & 0 & 0 \\ M^T_k & -M^T_k & 0 & 0 & 0 \\ N_3 & 0 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 & 0 \\ 0 & N_LC_2 & -\frac{1}{x}N_LD_2 & 0 & 0 \\ N_LC_2 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$x = \bar{\alpha}$$

$$\epsilon_2 \mathcal{T} \widetilde{\Theta}_{2(6\times 1)} = \epsilon_2 \begin{bmatrix} -\bar{\beta} M_c \\ \beta_1 M_c \\ \beta_1 M_c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\widetilde{\Phi}^T_{2(1\times 6)} = \begin{bmatrix} N_2 K & -N_2 K & 0 & 0 & 0 & 0 \end{bmatrix}^T_{(6\times 1)}$$

The MI in (25) still contains P, P^{-1}, Q and Q^{-1} in the term $\mathcal{T}\widetilde{\Xi}_{022}\mathcal{T}$. To convert (25) to LMI, two approaches are proposed in this paper which are summarized in the following two cases. A comparison between their efficiency is given in Table 1, next section.

Case A : The following fact

$$(I - Z^{-1})(I - Z^{-1}) > 0 \iff -Z^{-1} < -2I + Z \quad (26)$$

will be used in next lemma.

Lemma 3: The MI given in (25) will be an LMI in P and Q if the fact given in (26) is used to replace P^{-1} and Q^{-1} in (24). The observer gain is given by $L = Q^{-1}H$.

Case B : In the second method, the $\mathbb{S}$-procedure [17] is used as follows. If $\Omega_0 < 0$, $\exists\, \Omega_1 \leq 0$ such that $\Omega_0 - \Omega_1 < 0$. For simplicity in explaining the concept, let the term $\mathcal{T}\widetilde{\Xi}_{022}\mathcal{T}$ in (25) be denoted by Ω_0 and Ω_1 be defined as

$$\Omega_1 = \begin{bmatrix} N_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\Omega_0 - \Omega_1 = \begin{bmatrix} -\left(P^{-1} + N_{11}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\left(P^{-1} + N_{22}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\left(Q^{-1} + N_{33}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q & 0 & 0 \\ 0 & 0 & 0 & 0 & -Q & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}$$

$$= diag\left[-\tilde{N}_{11} - \tilde{N}_{22}\ -\tilde{N}_{33}\ -Q - Q - I \right] \quad (27)$$

where $\tilde{N}_{11} = P^{-1} + N_{11}$, $\tilde{N}_{22} = P^{-1} + N_{22}$ and $\tilde{N}_{33} = Q^{-1} + N_{33}$. So we replace $\mathcal{T}\widetilde{\Xi}_{022}\mathcal{T}$ in (25) by $\Omega_0 - \Omega_1$ given in (27).

The following lemma summarize the results of this approach.

Lemma 4: The MI given in (25) will be an LMI in P and Q if the term $\mathcal{T}\widetilde{\Xi}_{022}\mathcal{T}$ in (25) is replaced by $\Omega_0 - \Omega_1$ given in (27). The observer gain is given by $L = Q^{-1}H$.

The gains of the observer-based resilient controller in (6) and (7) for the uncertain NNCS (1) with minimum H_∞ performance constraint γ are obtained by solving the following optimization problems.

$$\min_{P>0, Q>0, \tau_1>0, K, H, \tau_2>0, \epsilon_1>0, \epsilon_2>0} \gamma \quad (28)$$

subject to inequality given in (25) for case (A) and the approximation used in (26), and

$$\min_{P>0, Q>0, K, H, \tilde{N}_{11}>0, \tilde{N}_{22}>0, \tilde{N}_{33}>0, \tau_1>0, \tau_2>0, \epsilon_1>0, \epsilon_2>0} \gamma \quad (29)$$

subject to inequality given in (25) for case (B) and the approximation used in (27).

IV. SIMULATION

In this section, a matlab code has been developed to solve LMI in (25). For the system described by (1) and (4), the following parameters are used in this simulation.

$$A = \begin{bmatrix} 0.8226 & -0.633 & 0 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.5 \\ 0 \\ 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 23.738 & 20.287 & 0 \end{bmatrix}, \quad D_1 = 0.1 \quad D_2 = 0.2$$

$$f(k, x_k) = \begin{bmatrix} 0.01 \sin x_k^1 \\ 0.01 \sin x_k^2 \\ 0.01 \sin x_k^3 \end{bmatrix}, x_k = \begin{bmatrix} x_{1k} \\ x_{2k} \\ x_{3k} \end{bmatrix}$$

$$G = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.2 \end{bmatrix}$$

$$N_3 = \begin{bmatrix} 0.3 & 0 & 0 \end{bmatrix}$$

$$M_k^T = M_L = M_c = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix}$$

with the following initial conditions

$$x_0 = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.1 \end{bmatrix} \quad \hat{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\omega_k = 1/k^2$ is taken as the input disturbance.

The objective of this example is to design the proposed resilient controller in (7) for the system described in (1), such that the H_∞ performance index γ is minimized. The simulation was performed for the two different approaches proposed in Subsection 3.4; i.e. Case A and Case B. The results have been summarized in Table 1.

Looking for the values of the H_∞ performance γ in Table 1, we can see that the linearization approach described in Case B gives better results than the one described in Case A. The performance factor γ remained balanced in both cases for

TABLE 1. Summary for Case A and B LMI approaches.

Simulation Results for Case B					
Figure#	$\bar{\alpha}$	$\bar{\beta}$	K	L	$\gamma_{\min}$
2	0.98	0.9	$\begin{bmatrix} 0.3452 & -0.2454 & 0.0020 \end{bmatrix}$	$\begin{bmatrix} 0.0075 & 0.0124 & 0.0207 \end{bmatrix}^T$	1.3202
3	0.5	0.4	$\begin{bmatrix} 0.2914 & -0.1862 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0071 & 0.0122 & 0.0209 \end{bmatrix}^T$	1.8322
4	0.02	0.1	$\begin{bmatrix} 0.1788 & -0.1376 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0120 & 0.0121 & 0.0228 \end{bmatrix}^T$	1.8429
Simulation Results for Case A					
Figure#	$\bar{\alpha}$	$\bar{\beta}$	K	L	$\gamma_{\min}$
5	0.98	0.9	$\begin{bmatrix} 0.3594 & -0.2766 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0070 & 0.0122 & 0.0208 \end{bmatrix}^T$	2.0936
6	0.5	0.4	$\begin{bmatrix} 0.2914 & -0.1862 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0071 & 0.0122 & 0.0209 \end{bmatrix}^T$	2.3829
7	0.02	0.1	$\begin{bmatrix} 0.1788 & -0.1376 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.012 & 0.0121 & 0.0228 \end{bmatrix}^T$	2.3904

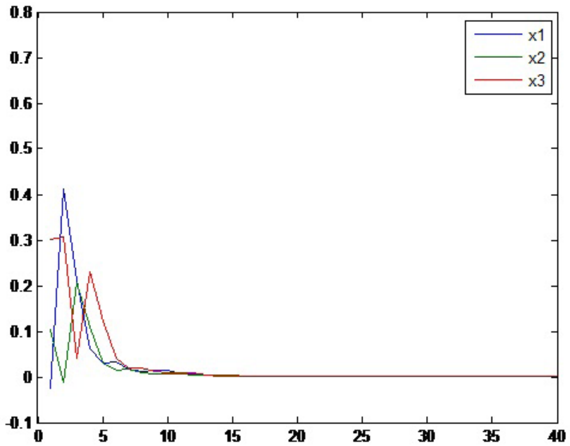

FIGURE 2. State trajectories using approach B with $\bar{\alpha} = 0.98$ and $\bar{\beta} = 0.9$.

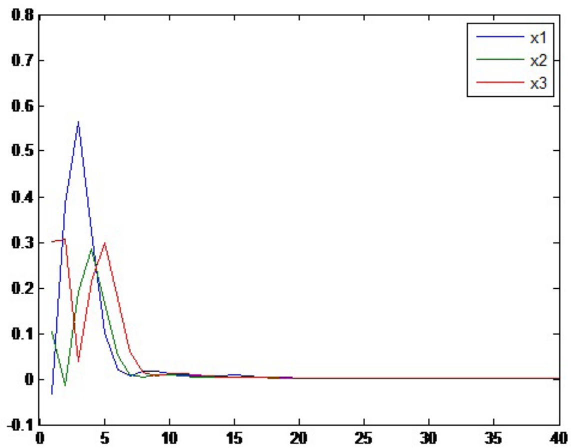

FIGURE 3. State trajectories using approach B with $\bar{\alpha} = 0.5$ and $\bar{\beta} = 0.4$.

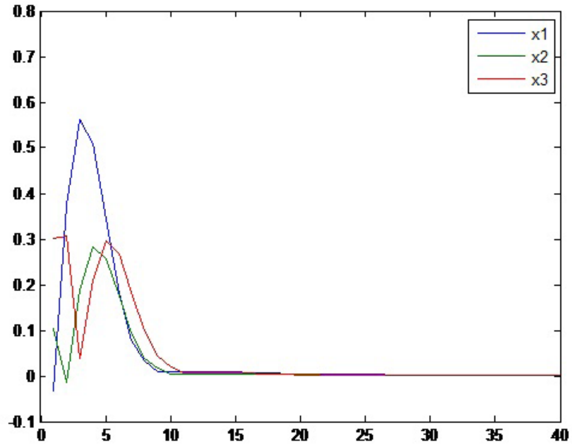

FIGURE 4. State trajectories using approach B with $\bar{\alpha} = 0.02$ and $\bar{\beta} = 0.1$.

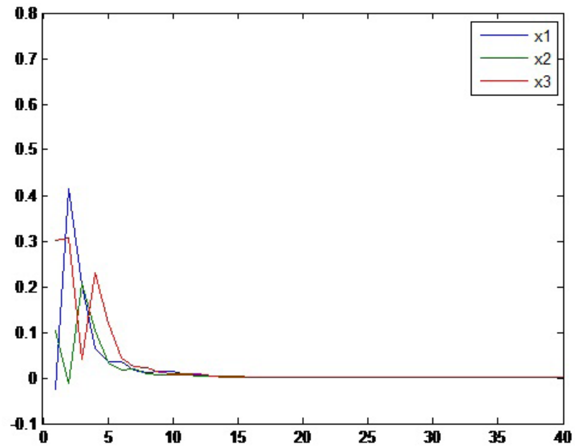

FIGURE 5. State trajectories using approach A with $\bar{\alpha} = 0.98$ and $\bar{\beta} = 0.9$.

different values of the packet losses probabilities α and β. The table shows that our proposed linearization given in Case B gives better results than case A. Figures 2 to 7 show that the state trajectories of the uncertain NNCS of the system by the resilient H_∞ observer-based controllers are satisfactory. There are only slight difference in the magnitude (overshot) and settling time (in seconds), but there is a big difference in the magnitude of the performance index γ. Comparing the results in Table 1 with those in [14] when certain controller was used to solve the same problem, we can see a very small increase in the value of the performance index &γ&.

Remark 1: : The formulation of this problem in its present form is not formulated nor solved in the literature. At this stage it is not feasible to compare our results. In a future work, we are working with different types of controllers to solve this problem. Then comparison will be reported.

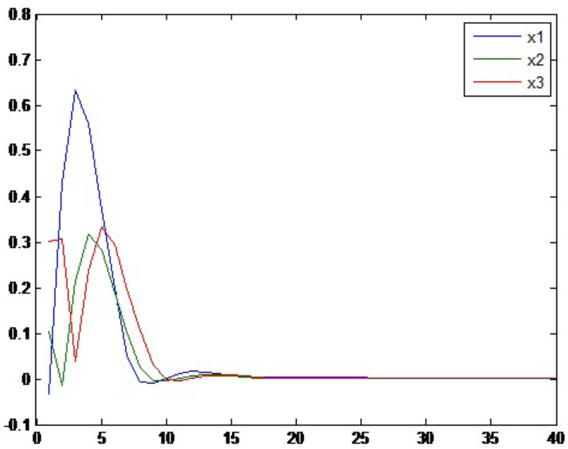

FIGURE 6. State trajectories using approach *A* with $\bar{\alpha} = 0.5$ and $\bar{\beta} = 0.4$.

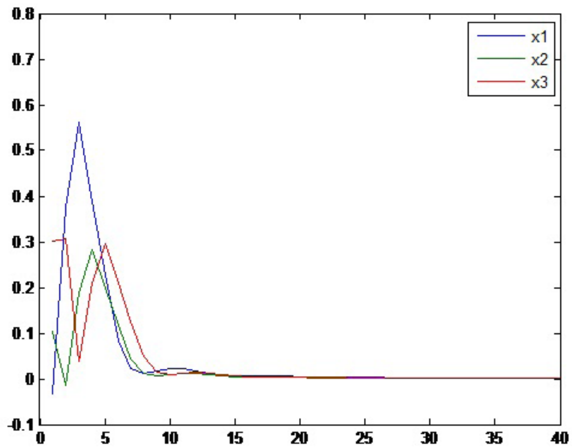

FIGURE 7. State trajectories using approach *A* with $\bar{\alpha} = 0.02$ and $\bar{\beta} = 0.1$.

V. CONCLUSION

This paper describes a new approach to solve the resilient observer-based H_∞ control problem of a class of uncertain nonlinear network control systems (NNCS) with packet dropouts. The packet dropouts over the network in two directions were modeled as two different Bernoulli distributions. New LMI's were derived to insure that the system is exponentially mean square stable and the H_∞ performance is minimized. A numerical example is presented to show the effectiveness of the derived LMIs. For the future work, two directions, theoretical and practical, are suggested. First, different types of control techniques such as Sliding mode control, Model Predictive Control, Backstopping control, etc., can be studied to solve the UNNCS problem. In the other direction, real time implementations of the results can be tested on small scale laboratory system. We are looking for a master student to help in implementing this task.

APPENDIX

A. APPENDIX:PROOF OF THEOREM 1

Proof: Assume the following Lyapunov function,

$$V_k = x_k^T P x_k + e_k^T Q e_k \tag{30}$$

where $P > 0$ and $Q > 0$. Then $\Delta V_k = V_{k+1} - V_k$ is given by

$$\begin{aligned}
\Delta V_k &= \mathbb{E}\{V_{k+1} | x_k, x_{k-1}, x_{k-2}, \ldots, x_0, e_k, e_{k-1}, \ldots, e_0\} - V_k \\
&= \mathbb{E}\{x_{k+1}^T P x_{k+1} + e_{k+1}^T Q e_{k+1}\} - x_k^T P x_k - e_k^T Q e_k \\
&= \mathbb{E}\left\{\left[\widetilde{V}_1\right]^T P \left[\widetilde{V}_1\right] + \left[\widetilde{V}_2\right]^T Q \left[\widetilde{V}_2\right]\right\} \\
&\quad - x_k^T P x_k - e_k^T Q e_k^T
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\widetilde{V}_1 &= ((A + \Delta A) - \bar{\beta}(B + \Delta B)(K + \Delta K))x_k \\
&\quad + (\beta_k - \bar{\beta})(B + \Delta B)(K + \Delta K)e_k \\
&\quad - (\beta_k - \bar{\beta})(B + \Delta B)(K + \Delta K)x_k \\
&\quad + \bar{\beta}(B + \Delta B)(K + \Delta K)e_k + f(k, x_k) \\
\widetilde{V}_2 &= -(\beta_k - \bar{\beta})(B + \Delta B)(K + \Delta K)x_k \qquad (32) \\
&\quad - (\alpha_k - \bar{\alpha})(L + \Delta L)C_2 x_k \\
&\quad + ((A + \Delta A) - \bar{\alpha}(L + \Delta L)C_2)e_k \\
&\quad + (\beta_k - \bar{\beta})(B + \Delta B)(K + \Delta K)e_k + F_k \qquad (33)
\end{aligned}$$

For simplicity in the derivations, introduce the following intermediate variables from (32-33)

$$\begin{aligned}
T_1 &= ((A + \Delta A) - \bar{\beta}(B + \Delta B)(K + \Delta K))x_k \\
&\quad + \bar{\beta}(B + \Delta B)(K + \Delta K)e_k + f(k, x_k) \\
T_2 &= (B + \Delta B)(K + \Delta K)e_k - (B + \Delta B)(K + \Delta K)x_k \\
T_3 &= ((A + \Delta A) - \bar{\alpha}(L + \Delta L)C_2)e_k + F_k \\
T_4 &= (L + \Delta L)C_2 x_k
\end{aligned} \tag{34}$$

Then ΔV_k can be rewritten as follows

$$\begin{aligned}
\Delta V_k &= \mathbb{E}\{\left[T_1 + (\beta_k - \bar{\beta})T_2\right]^T \times \left[PT_1 + (\beta_k - \bar{\beta})PT_2\right]\} \\
&\quad + \mathbb{E}\{\left[T_3 - (\alpha_k - \bar{\alpha})T_4 + (\beta_k - \bar{\beta})T_2\right]^T \\
&\quad \times \left[QT_3 - (\alpha_k - \bar{\alpha})QT_4 + (\beta_k - \bar{\beta})QT_2\right]\} \\
&\quad - x_k^T P x_k - e_k^T Q e_k^T
\end{aligned} \tag{35}$$

By expanding (35) with simplifications, we get

$$\begin{aligned}
\Delta V_k &= \mathbb{E}\left\{T_1^T P T_1 + (\beta_k - \bar{\beta})T_2^T P T_1\right\} \\
&\quad + \mathbb{E}\left\{(\beta_k - \bar{\beta})T_1^T P T_2 + (\beta_k - \bar{\beta})^2 T_2^T P T_2\right\} \\
&\quad + \mathbb{E}\left\{T_3^T Q T_3 - (\alpha_k - \bar{\alpha})T_4^T Q T_3 + (\beta_k - \bar{\beta})T_2^T Q T_3\right\} \\
&\quad - \mathbb{E}\left\{(\alpha_k - \bar{\alpha})T_3^T Q T_4 - (\alpha_k - \bar{\alpha})^2 T_4^T Q T_4\right. \\
&\quad \left. + (\alpha_k - \bar{\alpha})(\beta_k - \bar{\beta})T_2^T Q T_4\right\} \\
&\quad + \mathbb{E}\left\{(\beta_k - \bar{\beta})T_3^T Q T_2 - (\beta_k - \bar{\beta})(\alpha_k - \bar{\alpha})T_4^T Q T_2\right. \\
&\quad \left. + (\beta_k - \bar{\beta})^2 T_2^T Q T_2\right\} \\
&\quad - x_k^T P x_k - e_k^T Q e_k^T
\end{aligned} \tag{36}$$

Using the properties of the two independent stochastic Bernoulli distributions α_k and β_k, we get

$$\mathbb{E}\{(\beta_k-\bar\beta)\}=\mathbb{E}(\beta_k)-\bar\beta=\bar\beta-\bar\beta=0,\quad \textit{and}\ \ \mathbb{E}\{(\alpha_k-\bar\alpha)\}=0$$

Then using (5c) and (8c), (36) is reduced to

$$\Delta V_k=\left[T_1^TPT_1\right]+\left[\beta_1^2T_2^TPT_2\right]+\left[T_3^TQT_3\right]+\alpha_1^2T_4^TQT_4+\left[\beta_1^2T_2^TQT_2\right]-x_k^TPx_k-e_k^TQe_k^T \tag{37}$$

Let $\xi_k^T=\left[x_k^T\ e_k^T\ f^T(k,x_k)\ F_k^T\right]^T$. Then by substituting (34) in (37), and grouping similar terms, (37) can be written as

$$\Delta V_k=\begin{bmatrix}x_k\\e_k\\f(k,x_k)\\F_k\end{bmatrix}^T\Lambda\begin{bmatrix}x_k\\e_k\\f(k,x_k)\\F_k\end{bmatrix}\triangleq\xi_k^T\Lambda\xi_k \tag{38}$$

where

$$\Lambda=\begin{bmatrix}\Phi_{11}&*&*&*\\\Phi_{21}&\Phi_{22}&*&*\\\Phi_{31}&\bar\beta P(B+\Delta B)(K+\Delta K))&P&*\\0&Q((A+\Delta A)-\bar\alpha(L+\Delta L)C_2)&0&Q\end{bmatrix} \tag{39}$$

and

$$\begin{aligned}\Phi_{11}&=((A+\Delta A)-\bar\beta(B+\Delta B)(K+\Delta K))^TP((A+\Delta A)\\&\quad-\bar\beta(B+\Delta B)(K+\Delta K))\\&\quad+\beta_1^2(K+\Delta K)^T(B+\Delta B)^TP(B+\Delta B)(K+\Delta K)\\&\quad+\beta_1^2(K+\Delta K)^T(B+\Delta B)^TQ(B+\Delta B)(K+\Delta K)\\&\quad+\alpha_1^2C_2^T(L+\Delta L)^TQ(L+\Delta L)C_2-P\\\Phi_{22}&=\bar\beta^2(K+\Delta K)^T(B+\Delta B)^TP(B+\Delta B)(K+\Delta K)\\&\quad+\beta_1^2(K+\Delta K)^T(B+\Delta B)^TP(B+\Delta B)(K+\Delta K)\\&\quad+\beta_1^2(K+\Delta K)^T(B+\Delta B)^TQ(B+\Delta B)(K+\Delta K)\\&\quad+((A+\Delta A)-\bar\alpha(L+\Delta L)C_2)^TQ((A+\Delta A)\\&\quad-\bar\alpha(L+\Delta L)C_2)-Q\\\Phi_{21}&=\bar\beta(K+\Delta K)^T(B+\Delta B)^TP((A+\Delta A)\\&\quad-\bar\beta(B+\Delta B)(K+\Delta K))\\&\quad-\beta_1^2(K+\Delta K)^T(B+\Delta B)^TP(B+\Delta B)(K+\Delta K)\\&\quad-\beta_1^2(K+\Delta K)^T(B+\Delta B)^TQ(B+\Delta B)(K+\Delta K)\\\Phi_{31}&=P((A+\Delta A)-\bar\beta(B+\Delta B)(K+\Delta K))\end{aligned}$$

The constraints (2) and (3) could be rewritten as

$$\xi_k^T\begin{bmatrix}-G^TG&0&0&0\\0&0&0&0\\0&0&I&0\\0&0&0&0\end{bmatrix}\xi_k\triangleq\xi_k^T\Lambda_1\xi_k \tag{40}$$

$$\xi_k^T\begin{bmatrix}0&0&0&0\\0&-G^TG&0&0\\0&0&0&0\\0&0&0&I\end{bmatrix}\xi_k\triangleq\xi_k^T\Lambda_2\xi_k \tag{41}$$

By Lemma (1) and using the constraints (40) and (41), $\Delta V_k=\xi_k^T\Lambda\xi_k<0$ holds if there exist matrices $P>0$, $Q>0$ and scalars $\tau_1>0,\tau_2>0$ such that the following Matrix Inequality (MI) is satisfied

$$\Lambda-\tau_1\Lambda_1-\tau_2\Lambda_2<0 \tag{42}$$

which can be rewritten as $\Xi=\Lambda-\tau_1\Lambda_1-\tau_2\Lambda_2<0$ where

$$\Xi=\begin{bmatrix}-P+\tau_1G^TG&*&*&*\\0&-Q+\tau_2G^TG&*&*\\0&0&-\tau_1I&*\\0&0&0&-\tau_2I\end{bmatrix}+\begin{bmatrix}\breve\Phi_{11}&*&*&0\\\breve\Phi_{21}&\breve\Phi_{22}&*&*\\PX_1&\bar\beta PX_2&P&0\\0&QX_4&0&Q\end{bmatrix}<0 \tag{43}$$

and

$$\begin{aligned}\breve\Phi_{11}&=X_1^TPX_1+\beta_1^2X_2^TPX_2+\beta_1^2X_2^TQX_2+\alpha_1^2X_3^TQX_3\\\breve\Phi_{21}&=\bar\beta X_2^TPX_1-\beta_1^2X_2^TPX_2-\beta_1^2X_2^TQX_2\\\breve\Phi_{22}&=\bar\beta^2X_2^TPX_2+\beta_1^2X_2^TPX_2+\beta_1^2X_2^TQX_2+X_4^TQX_4\\X_1&=(A+\Delta A)-\bar\beta(B+\Delta B)(K+\Delta K))\\X_2&=(B+\Delta B)(K+\Delta K)\\X_3&=(L+\Delta L)C_2\\X_4&=(A+\Delta A)-\bar\alpha(L+\Delta L)C_2)\end{aligned}$$

Then, the second term in (43) is

$$\begin{bmatrix}X_1^TPX_1+\beta_1^2X_2^TPX_2+\beta_1^2X_2^TQX_2+\alpha_1^2X_3^TQX_3&\bar\beta X_1^TPX_2-\beta_1^2X_2PX_2^T-\beta_1^2X_2QX_2^T&*&0\\\bar\beta X_2^TPX_1-\beta_1^2X_2^TPX_2-\beta_1^2X_2^TQX_2&\bar\beta^2X_2^TPX_2+\beta_1^2X_2^TPX_2+\beta_1^2X_2^TQX_2+X_4^TQX_4&*&*\\PX_1&\bar\beta PX_2&P&0\\0&QX_4&0&Q\end{bmatrix}$$

which can be written as

$$\begin{bmatrix}X_1^TP&\beta_1X_2^TP&\beta_1X_2^TQ&\alpha_1X_3^TQ&0\\\bar\beta X_2^TP&-\beta_1X_2^TP&-\beta_1X_2^TQ&0&X_4^TQ\\P&0&0&0&0\\0&0&0&0&Q\end{bmatrix}*\begin{bmatrix}P^{-1}&0&0&0&0\\0&P^{-1}&0&0&0\\0&0&Q^{-1}&0&0\\0&0&0&Q^{-1}&0\\0&0&0&0&Q^{-1}\end{bmatrix}$$

$$
* \begin{bmatrix} PX_1 & \bar{\beta}PX_2 & P & 0 \\ \beta_1 PX_2 & -\beta_1 PX_2 & 0 & 0 \\ \beta_1 QX_2 & -\beta_1 QX_2 & 0 & 0 \\ \alpha_1 QX_3 & 0 & 0 & 0 \\ 0 & QX_4 & 0 & Q \end{bmatrix}
$$

By introducing $\Xi_{0\Delta}$ and using Schur Complement Ξ in (43) can be written as

$$
\Xi_{0\Delta} = \begin{bmatrix} \Xi_{01} & \Xi_{02\Delta}^T \\ \Xi_{02\Delta} & \Xi_{03} \end{bmatrix} \tag{44}
$$

where Ξ_{01} and Ξ_{03} are as defined in (20) and

$$
\Xi_{02\Delta} = \begin{bmatrix} PX_1 & \bar{\beta}PX_2 & P & 0 \\ \beta_1 PX_2 & -\beta_1 PX_2 & 0 & 0 \\ \beta_1 QX_2 & -\beta_1 QX_2 & 0 & 0 \\ \alpha_1 QX_3 & 0 & 0 & 0 \\ 0 & QX_4 & 0 & Q \end{bmatrix}
$$

Now we may start separating uncertainties from $\Xi_{02\Delta}$ as follows.

$$
\Xi_{02\Delta} = \Xi_{020} + \Xi_{02\Delta1} + \Xi_{02\Delta2} + \Xi_{02\Delta3}
$$

$$
\Xi_{020} = \begin{bmatrix} (PA - \bar{\beta}PBK) & \bar{\beta}PBK & P & 0 \\ \beta_1 PBK & -\beta_1 PBK & 0 & 0 \\ \beta_1 QBK & -\beta_1 QBK & 0 & 0 \\ \alpha_1 QLC_2 & 0 & 0 & 0 \\ 0 & Q(A - \bar{\alpha}LC_2) & 0 & Q \end{bmatrix}
$$

$$
\Xi_{02\Delta1} = \begin{bmatrix} (P\Delta A - \bar{\beta}PB\Delta K) & \bar{\beta}PB\Delta K & 0 & 0 \\ \beta_1 PB\Delta K & -\beta_1 PB\Delta K & 0 & 0 \\ \beta_1 QB\Delta K & -\beta_1 QB\Delta K & 0 & 0 \\ \alpha_1 Q\Delta LC_2 & 0 & 0 & 0 \\ 0 & Q\Delta A - \bar{\alpha}Q\Delta LC_2 & 0 & 0 \end{bmatrix}
$$

$$
\Xi_{02\Delta2} = \begin{bmatrix} -\bar{\beta}P\Delta BK & \bar{\beta}P\Delta BK & 0 & 0 \\ \beta_1 P\Delta BK & -\beta_1 P\Delta BK & 0 & 0 \\ \beta_1 Q\Delta BK & -\beta_1 Q\Delta BK & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\Xi_{02\Delta3} = \begin{bmatrix} -\bar{\beta}P\Delta B\Delta K & \bar{\beta}P\Delta B\Delta K & 0 & 0 \\ \beta_1 P\Delta B\Delta K & -\beta_1 P\Delta B\Delta K & 0 & 0 \\ \beta_1 Q\Delta B\Delta K & -\beta_1 Q\Delta B\Delta K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

Using the definition of the uncertainties ΔA, ΔB, ΔC_1, ΔK and ΔL defined in (9-11) and using simple manipulations we may write $\Xi_{02\Delta1}$, $\Xi_{02\Delta2}$ and $\Xi_{02\Delta3}$ as follows

$$
\Xi_{02\Delta1} = \begin{bmatrix} PM_c & -\bar{\beta}PBN_2^T & 0 & 0 & 0 \\ 0 & \beta_1 PBN_2^T & 0 & 0 & 0 \\ 0 & \beta_1 QBN_2^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 QM_L \\ 0 & 0 & QM_c & -\bar{\alpha}QM_L & 0 \end{bmatrix}
* \begin{bmatrix} \Delta_k & 0 & 0 & 0 & 0 \\ 0 & \Delta_k^T & 0 & 0 & 0 \\ 0 & 0 & \Delta_k & 0 & 0 \\ 0 & 0 & 0 & \Delta_k & 0 \\ 0 & 0 & 0 & 0 & \Delta_k \end{bmatrix}
* \begin{bmatrix} N_1 & 0 & 0 & 0 \\ M_k^T & -M_k^T & 0 & 0 \\ 0 & N_1 & 0 & 0 \\ 0 & N_L C_2 & 0 & 0 \\ N_L C_2 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= \Theta_1 \widehat{\Delta}_1 \Phi_1, \quad \Theta_1 \in \mathbb{R}^{5\times5}, \quad \widehat{\Delta}_1 \in \mathbb{R}^{5\times5}, \quad \Phi_1 \in \mathbb{R}^{5\times4}
$$

Similarly

$$
\Xi_{02\Delta2} = \begin{bmatrix} -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \end{bmatrix} \Delta_k \begin{bmatrix} N_2 K & -N_2 K & 0 & 0 \end{bmatrix} = \Theta_2 \widehat{\Delta}_2 \Phi_2
$$

and

$$
\Xi_{02\Delta3} = \begin{bmatrix} -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \end{bmatrix} \Delta_k N_2 N_2^T \Delta_k^T \begin{bmatrix} M_k^T & -M_k^T & 0 \end{bmatrix} = \Theta_3 \widehat{\Delta}_3 \Phi_3
$$

Note that since by assumption $N_2 N_2^T < I$ and $\Delta_k \Delta_k^T < I$, then $\Delta_k N_2 N_2^T \Delta_k^T < I$. For simplifications in completing the proof, let $\Xi_{0\Delta}$ in (44) be split as

$$
\Xi_{0\Delta} = \begin{bmatrix} \Xi_{01} & 0 \\ 0 & \Xi_{03} \end{bmatrix} + \begin{bmatrix} 0 & \Xi_{02\Delta}^T \\ \Xi_{02\Delta} & 0 \end{bmatrix} \tag{45}
$$

We may split he second term in (45) as

$$
\begin{bmatrix} 0 & \Xi_{02\Delta}^T \\ \Xi_{02\Delta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Xi_{020}^T \\ \Xi_{020} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Xi_{02\Delta1}^T \\ \Xi_{02\Delta1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Xi_{02\Delta2}^T \\ \Xi_{02\Delta2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Xi_{02\Delta3}^T \\ \Xi_{02\Delta3} & 0 \end{bmatrix}
$$

where

$$
\begin{bmatrix} 0 & \Xi_{02\Delta1}^T \\ \Xi_{02\Delta1} & 0 \end{bmatrix}
$$

$$
\begin{aligned}
&= \begin{bmatrix} 0_{4\times 4} & \Phi_1^T\widehat{\Delta}_1^T\Theta_1^T \\ \Theta_1\widehat{\Delta}_1\Phi_1 & 0_{5\times 5}\end{bmatrix} \\
&= \begin{bmatrix} 0_{4\times 5} \\ \Theta_{1(5\times 5)}\end{bmatrix}\widehat{\Delta}_{1(5\times 5)}\begin{bmatrix}\Phi_{1(5\times 4)} & 0_{5\times 5}\end{bmatrix} \\
&\quad + \begin{bmatrix}\Phi_1^T \\ 0\end{bmatrix}\widehat{\Delta}^T\begin{bmatrix}0 & \Theta_1^T\end{bmatrix} \qquad (46)\\
&= \Upsilon_1\widehat{\Delta}_1\chi_1 + \chi_1^T\widehat{\Delta}_1^T\Upsilon_1^T \\
&\le \epsilon_1^{-1}\Upsilon_1\Upsilon_1^T + \epsilon_1\chi_1^T\chi_1, \quad \text{for any } \epsilon_1 > 0
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\begin{bmatrix} 0 & \Xi_{02\Delta 2}^T \\ \Xi_{02\Delta 2} & 0\end{bmatrix} \\
&= \begin{bmatrix} 0_{4\times 4} & \Phi_2^T\widehat{\Delta}_2^T\Theta_2^T \\ \Theta_2\widehat{\Delta}_2\Phi_2 & 0_{5\times 5}\end{bmatrix} \\
&= \begin{bmatrix} 0_{4\times 1} \\ \Theta_{2(5\times 1)}\end{bmatrix}\widehat{\Delta}_{1\times 1}\begin{bmatrix}\Phi_{2(1\times 4)} & 0_{1\times 5}\end{bmatrix} \\
&\quad + \begin{bmatrix}\Phi_{2(4\times 1)}^T \\ 0\end{bmatrix}\widehat{\Delta}_{1\times 1}^T\begin{bmatrix}0 & \Theta_{2(1\times 5)}^T\end{bmatrix} \\
&= \Upsilon_2\widehat{\Delta}_2\chi_2 + \chi_2^T\widehat{\Delta}_2^T\Upsilon_2^T \\
&\le \epsilon_2\Upsilon_2\Upsilon_2^T + \epsilon_2^{-1}\chi_2^T\chi_2, \quad \text{for any } \epsilon_2 > 0 \qquad (47)
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{bmatrix} 0 & \Xi_{02\Delta 3}^T \\ \Xi_{02\Delta 23} & 0\end{bmatrix} \\
&= \begin{bmatrix} 0_{4\times 4} & \Phi_3^T\widehat{\Delta}_3^T\Theta_3^T \\ \Theta_3\widehat{\Delta}_3\Phi_3 & 0_{5\times 5}\end{bmatrix} \\
&= \begin{bmatrix} 0_{4\times 1} \\ \Theta_{3(5\times 1)}\end{bmatrix}\widehat{\Delta}_{1\times 1}\begin{bmatrix}\Phi_{3(1\times 4)} & 0_{1\times 5}\end{bmatrix} \\
&\quad + \begin{bmatrix}\Phi_{3(4\times 1)}^T \\ 0\end{bmatrix}\widehat{\Delta}_{1\times 1}^T\begin{bmatrix}0 & \Theta_{3(1\times 5)}^T\end{bmatrix} \\
&= \Upsilon_3\widehat{\Delta}_2\chi_3 + \chi_3^T\widehat{\Delta}_3^T\Upsilon_3^T \\
&\le \epsilon_3^{-1}\Upsilon_3\Upsilon_3^T + \epsilon_3\chi_3^T\chi_3, \quad \text{for any } \epsilon_3 > 0 \qquad (48)
\end{aligned}
$$

Finally, grouping the terms of $\Xi_{0\Delta}$ in (45), we get

$$
\begin{aligned}
\Xi_{0\Delta} &= \begin{bmatrix}\Xi_{01} & \Xi_{020}^T \\ \Xi_{020} & \Xi_{03}\end{bmatrix} + \begin{bmatrix} 0 & \Xi_{02\Delta 1}^T \\ \Xi_{02\Delta 1} & 0\end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \Xi_{02\Delta 2}^T \\ \Xi_{02\Delta 2} & 0\end{bmatrix} + \begin{bmatrix} 0 & +\Xi_{02\Delta 3}^T \\ \Xi_{02\Delta 3} & 0\end{bmatrix} \\
&\le \begin{bmatrix}\Xi_{01} & \Xi_{020}^T \\ \Xi_{020} & \Xi_{03}\end{bmatrix} + \epsilon_1^{-1}\Upsilon_1\Upsilon_1^T + \epsilon_1\chi_1^T\chi_1 \\
&\quad + \epsilon_2\Upsilon_2\Upsilon_2^T + \epsilon_2^{-1}\chi_2^T\chi_2 + \epsilon_3^{-1}\Upsilon_3\Upsilon_3^T + \epsilon_3\chi_3^T\chi_3 < 0 \\
&= \widetilde{\Xi}_0 + \epsilon_1^{-1}\Upsilon_1\Upsilon_1^T + \epsilon_1\chi_1^T\chi_1 + \epsilon_2\Upsilon_2\Upsilon_2^T + \epsilon_2^{-1}\chi_2^T\chi_2 \\
&\quad + \epsilon_3^{-1}\Upsilon_3\Upsilon_3^T + \epsilon_3\chi_3^T\chi_3 < 0 \qquad (49)
\end{aligned}
$$

Using Lemma (2) to combine all terms in (49), we get

$$
\begin{bmatrix}
\widetilde{\Xi}_0 & \Upsilon_1 & \epsilon_1\chi_1^T & \epsilon_2\Upsilon_2 & \chi_2^T & \Upsilon_3 & \epsilon_3\chi_3^T \\
\Upsilon_1^T & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\
\epsilon_1\chi_1 & 0 & -\epsilon_1 I & 0 & 0 & 0 & 0 \\
\epsilon_2\Upsilon_2^T & 0 & 0 & -\epsilon_2 I & 0 & 0 & 0 \\
\chi_2 & 0 & 0 & 0 & -\epsilon_2 I & 0 & 0 \\
\Upsilon_3^T & 0 & 0 & 0 & 0 & -\epsilon_3 I & 0 \\
\epsilon_3\chi_3 & 0 & 0 & 0 & 0 & 0 & -\epsilon_3 I
\end{bmatrix} < 0 \qquad (50)
$$

which is similar to (20) with $\widetilde{\Xi}_0 = \Xi_0$. Thus, it has been proved that

$$\Delta V_k = \xi_k^T\Lambda\xi_k < 0 \ if \ \ \Lambda < 0$$

i.e.

$$
\begin{aligned}
\Delta V_k &= \xi_k^T\Lambda\xi_k \le -\lambda_{\min}(-\Lambda)\xi_k^T\xi_k \\
\Delta V_k &\le -\lambda_{\min}(-\Lambda)(\eta_k^T\eta_k + f(k,x_k)^T f(k,x_k) + F_k^T F_k) \\
\Delta V_k &\le -\lambda_{\min}(-\Lambda)(\eta_k^T\eta_k + \|f(k,x_k)\|^2 + \|F_k\|^2) < -\alpha\eta_k^T\eta_k
\end{aligned}
$$

where

$$
\begin{aligned}
0 &< \alpha < \min\{\lambda_{\min}(-\Lambda), \sigma\} \\
0 &< \sigma < \min\{\lambda_{\min}(-\Lambda), \max\{\lambda_{\max}(P), \lambda_{\max}(Q)\}\}
\end{aligned}
$$

This proves that

$$\Delta V_k < -\alpha\eta_k^T\eta_k < -\frac{\alpha}{\sigma}V_k := -\psi V_k \qquad (51)$$

Therefore by Definition (1), the closed-loop uncertain nonlinear networked system (13) is exponentially mean square stable under the resilient observer based controller. ■

B. APPENDIX: PROOF OF THEOREM 2

Proof: Assume the following Lyapunov function,

$$V_k = x_k^T P x_k + e_k^T Q e_k \qquad (52)$$

where $P > 0$ and $Q > 0$. Then $\Delta V_k = V_{k+1} - V_k$ is given by

$$
\begin{aligned}
&\Delta V_k \\
&= \mathbb{E}\{V_{k+1}|x_k, x_{k-1}, x_{k-2}, \ldots, x_0, e_k, e_{k-1}, \ldots, e_0\} - V_k \\
&= \mathbb{E}\{x_{k+1}^T P x_{k+1} + e_{k+1}^T Q e_{k+1}\} - x_k^T P x_k - e_k^T Q e_k \\
&= \mathbb{E}\left\{\left[\widetilde{V}_1\right]^T P\left[\widetilde{V}_1\right] + \left[\widetilde{V}_2\right]^T Q\left[\widetilde{V}_2\right]\right\} \\
&\quad - x_k^T P x_k - e_k^T Q e_k^T \qquad (53)
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{V}_1 &= ((A+\Delta A) - \bar{\beta}(B+\Delta B)(K+\Delta K))x_k \\
&\quad + (\beta_k - \bar{\beta})(B+\Delta B)(K+\Delta K)e_k \\
&\quad - (\beta_k - \bar{\beta})(B+\Delta B)(K+\Delta K)x_k \\
&\quad + \bar{\beta}(B+\Delta B)(K+\Delta K)e_k + f(k,x_k) \qquad (54)\\
\widetilde{V}_2 &= -(\beta_k - \bar{\beta})(B+\Delta B)(K+\Delta K)x_k \\
&\quad - (\alpha_k - \bar{\alpha})(L+\Delta L)C_2 x_k \\
&\quad + ((A+\Delta A) - \bar{\alpha}(L+\Delta L)C_2)e_k \\
&\quad + (\beta_k - \bar{\beta})(B+\Delta B)(K+\Delta K)e_k + F_k \qquad (55)
\end{aligned}
$$

For simplicity in the derivations, introduce the following intermediate variables from (54-55)

$$\begin{aligned} T_1 &= ((A+\Delta A)-\bar{\beta}(B+\Delta B)(K+\Delta K))x_k \\ &\quad +\bar{\beta}(B+\Delta B)(K+\Delta K)e_k + f(k,x_k) \\ T_2 &= (B+\Delta B)(K+\Delta K)e_k-(B+\Delta B)(K+\Delta K)x_k \\ T_3 &= ((A+\Delta A)-\bar{\alpha}(L+\Delta L)C_2)e_k+F_k \\ T_4 &= (L+\Delta L)C_2x_k \end{aligned} \quad (56)$$

Consider also the following performance measure

$$J=\mathbb{E}\{z_k^Tz_k-\gamma^2w_k^Tw_k\} \quad (57)$$

For any nonzero w_k, and from (54-55) and (57) we have

$$\begin{aligned} &\mathbb{E}\{V_{k+1}\}-\mathbb{E}\{V_k\}+\mathbb{E}\{z_k^Tz_k\}-\gamma^2\mathbb{E}\{w_k^Tw_k\}<0 \\ &\quad \mathbb{E}\left\{[U_1]^TP[U_1]+[U_2]^TQ[U_2]\right\}-x_k^TPx_k-e_k^TQe_k^T \\ &\quad +[U_3]^T[U_3]-\gamma^2w_k^Tw_k^T<0 \end{aligned} \quad (58)$$

where

$$\begin{aligned} U_1 &= \widetilde{V}_1+Dw_k \\ U_2 &= \widetilde{V}_2+(D-(L+\Delta L)D_2)w_k \\ U_3 &= (C_1+\Delta C_1)x_k+D_1w_k \end{aligned}$$

For simplifying the derivation, introduce the following intermediate variables

$$\begin{aligned} T_{1w} &= T_1+Dw_k \\ T_{2w} &= T_2 \\ T_{3w} &= T_3+(D-(L+\Delta L)D_2)w_k \\ T_{4w} &= T_4 \end{aligned}$$

Then

$$\begin{aligned} &\mathbb{E}\{V_{k+1}\}-\mathbb{E}\{V_k\}+\mathbb{E}\{z_k^Tz_k\}-\gamma^2\mathbb{E}\{w_k^Tw_k\} \\ &= \mathbb{E}\left\{T_{1w}^TPT_{1w}+(\beta_k-\bar{\beta})T_{2w}^TPT_{1w}\right\} \\ &\quad +\mathbb{E}\left\{(\beta_k-\bar{\beta})T_{1w}^TPT_{2w}+(\beta_k-\bar{\beta})^2T_{2w}^TPT_{2w}\right\} \\ &\quad +\mathbb{E}\left\{T_{3w}^TQT_{3w}-(\alpha_k-\bar{\alpha})T_{4w}^TQT_{3w}+(\beta_k-\bar{\beta})T_{2w}^TQT_{3w}\right\} \\ &\quad -\mathbb{E}\left\{(\alpha_k-\bar{\alpha})T_{3w}^TQT_{4w}-(\alpha_k-\bar{\alpha})^2T_{4w}^TQT_{4w}\right\} \\ &\quad -\mathbb{E}\left\{(\alpha_k-\bar{\alpha})(\beta_k-\bar{\beta})T_{2w}^TQT_{4w}\right\} \\ &\quad +\mathbb{E}\left\{(\beta_k-\bar{\beta})T_{3w}^TQT_{2w}-(\beta_k-\bar{\beta})(\alpha_k-\bar{\alpha})T_{4w}^TQT_{2w}\right\} \\ &\quad +\mathbb{E}\left\{(\beta_k-\bar{\beta})^2T_{2w}^TQT_{2w}\right\}-x_k^TPx_k-e_k^TQe_k^T \\ &\quad +[(C_1+\Delta C_1)x_k+D_1w_k]^T[(C_1+\Delta C_1)x_k+D_1w_k] \\ &\quad -\gamma^2w_k^Tw_k^T<0 \end{aligned}$$

Using the properties of the two independent stochastic Bernoulli distributions α_k and β_k stated in (5a-5c) and (8a-8c) and due to the fact that

$$\mathbb{E}\{(\beta_k-\bar{\beta})\}=\mathbb{E}(\beta_k)-\bar{\beta}=\bar{\beta}-\bar{\beta}=0, \quad \textit{and}\ \mathbb{E}\{(\alpha_k-\bar{\alpha})\}=0$$

one gets

$$\begin{aligned} &\mathbb{E}\{V_{k+1}\}-\mathbb{E}\{V_k\}+\mathbb{E}\{z_k^Tz_k\}-\gamma^2\mathbb{E}\{w_k^Tw_k\} \\ &= \left[T_{1w}^TPT_{1w}\right]+\left[\beta_1^2T_{2w}^TPT_{2w}\right]+\left[T_{3w}^TQT_{3w}\right] \\ &\quad +\alpha_1^2T_{4w}^TQT_{4w}+\left[\beta_1^2T_{2w}^TQT_{2w}\right]-x_k^TPx_k-e_k^TQe_k^T \\ &\quad +[(C_1+\Delta C_1)x_k+D_1w_k]^T[(C_1+\Delta C_1)x_k+D_1w_k] \\ &\quad -\gamma^2w_k^Tw_k^T<0 \end{aligned}$$

Substitutions of T_{iw}, $i=1,2,3,4$, with simplifications and grouping similar terms, we have

$$\mathbb{E}\{V_{k+1}\}-\mathbb{E}\{V_k\}+\mathbb{E}\{z_k^Tz_k\}-\gamma^2\mathbb{E}\{w_k^Tw_k\}\triangleq\zeta_k^T\Omega\zeta_k \quad (59)$$

where

$$\Omega=\begin{bmatrix} \Omega_{11} & * & * & * & * & \\ \Omega_{21} & \Omega_{22} & * & * & * & \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * & * & \\ \Omega_{41} & \bar{\beta}P(B+\Delta B)(K+\Delta K) & PD & PD & P & * \\ 0 & \Omega_{52} & Q(D-(L+\Delta L)D_2) & 0 & Q & \end{bmatrix}<0$$

$$\begin{aligned} \zeta_k^T &= \begin{bmatrix} x_k^T & e_k^T & w_k^T & f^T(k,x_k) & F_k^T \end{bmatrix}^T \\ \Omega_{11} &= ((A+\Delta A)-\bar{\beta}((B+\Delta B)(K+\Delta K))^TP((A+\Delta A) \\ &\quad -\bar{\beta}(B+\Delta B)(K+\Delta K)) \\ &\quad +\beta_1^2((B+\Delta B)(K+\Delta K))^TP(B+\Delta B)(K+\Delta K) \\ &\quad +\beta_1^2((B+\Delta B)(K+\Delta K))^TQ(B+\Delta B)(K+\Delta K) \\ &\quad +\alpha_1^2C_2^T(L+\Delta L)^TQ(L+\Delta L)C_2 \\ &\quad +(C_1+\Delta C_1)^T(C_1+\Delta C_1)-P \\ \Omega_{22} &= \bar{\beta}^2((B+\Delta B)(K+\Delta K))^TP(B+\Delta B)(K+\Delta K) \\ &\quad +\beta_1^2((B+\Delta B)(K+\Delta K))^TP(B+\Delta B)(K+\Delta K) \\ &\quad +\beta_1^2((B+\Delta B)(K+\Delta K))^TQ(B+\Delta B)(K+\Delta K) \\ &\quad +((A+\Delta A)-\bar{\alpha}(L+\Delta L)C_2)^TQ((A+\Delta A) \\ &\quad -\bar{\alpha}(L+\Delta L)C_2)-Q \\ \Omega_{21} &= \bar{\beta}((B+\Delta B)(K+\Delta K))^TP((A+\Delta A) \\ &\quad -\bar{\beta}(B+\Delta B)(K+\Delta K)) \\ &\quad -\beta_1^2((B+\Delta B)(K+\Delta K))^TP(B+\Delta B)(K+\Delta K) \\ &\quad -\beta_1^2((B+\Delta B)(K+\Delta K))^TQ(B+\Delta B)(K+\Delta K) \\ \Omega_{31} &= D^TP((A+\Delta A)-\bar{\beta}(B+\Delta B)(K+\Delta K)) \\ &\quad +D_1^T(C_1+\Delta C_1) \\ \Omega_{32} &= \bar{\beta}D^TP(B+\Delta B)(K+\Delta K) \end{aligned}$$

$$
\begin{aligned}
&+(D-(L+\Delta L)D_2)^T Q((A+\Delta A)-\bar{\alpha}(L+\Delta L)C_2)\\
\Omega_{33} &= D^T PD+(D-(L+\Delta L)D_2)^T Q(D-(L+\Delta L)D_2)\\
&+D_1^T D_1-\gamma^2 I\\
\Omega_{41} &= P(A+\Delta A)-\bar{\beta}P(B+\Delta B)(K+\Delta K)\\
\Omega_{52} &= Q((A+\Delta A)-\bar{\alpha}(L+\Delta L)C_2)
\end{aligned}
$$

In the same manner as in (40) and (41), the constraints (2) are reshaped as

$$\zeta_k^T \begin{bmatrix} -G^TG & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \zeta_k \triangleq \zeta_k^T \Omega_1 \zeta_k \le 0 \qquad (60)$$

and

$$\zeta_k^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & -G^TG & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I \end{bmatrix} \zeta_k \triangleq \zeta_k^T \Omega_2 \zeta_k \le 0 \qquad (61)$$

Assume there exist real scalars $\tau_1 > 0$, $\tau_2 > 0$ and matrices $P > 0$ and $Q > 0$, then by the S-procedure, we have

$$\Omega - \tau_1\Omega_1 - \tau_2\Omega_2 < 0 \qquad (62)$$

That is

$$
\begin{aligned}
&\Omega - \tau_1\Omega_1 - \tau_2\Omega_2\\
&= \begin{bmatrix} -P+\tau_1 G^TG & * & * & * & 0\\ 0 & -Q+\tau_2 G^TG & * & * & 0\\ 0 & 0 & -\gamma^2 I & * & 0\\ 0 & 0 & 0 & -\tau_1 I & 0\\ 0 & 0 & 0 & 0 & -\tau_2 I \end{bmatrix}\\
&+ \begin{bmatrix} \tilde{\Omega}_{11} & * & * & * & *\\ \Omega_{21} & \tilde{\Omega}_{22} & * & * & *\\ \Omega_{31} & \Omega_{32} & \tilde{\Omega}_{33} & * & *\\ \Omega_{41} & \bar{\beta}P(B+\Delta B)(K+\Delta K) & PD & P & *\\ 0 & \Omega_{52} & Q(D-(L+\Delta L)D_2) & 0 & Q \end{bmatrix}\\
&< 0
\end{aligned}
$$

where

$$\tilde{\Omega}_{11} = \Omega_{11}+P, \quad \tilde{\Omega}_{22} = \Omega_{22}+Q, \quad \tilde{\Omega}_{33} = \Omega_{33}+\gamma^2 I$$

which can be rewritten in the following form.

$$\Pi = \Omega - \tau_1\Omega_1 - \tau_2\Omega_2 = \Gamma_{11} + \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \Gamma_{22}^{-1} \begin{bmatrix} \Phi_1^T \\ \Phi_2^T \end{bmatrix} < 0 \qquad (63)$$

where

$$\Gamma_{11} = \begin{bmatrix} -P+\tau_1 G^TG & * & * & * & 0\\ 0 & -Q+\tau_2 G^TG & * & * & 0\\ 0 & 0 & -\gamma^2 I & * & 0\\ 0 & 0 & 0 & -\tau_1 I & 0\\ 0 & 0 & 0 & 0 & -\tau_2 I \end{bmatrix}$$

$$\Gamma_{22}^{-1} = \begin{bmatrix} P^{-1} & 0 & 0 & 0 & 0 & 0\\ 0 & P^{-1} & 0 & 0 & 0 & 0\\ 0 & 0 & Q^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & Q^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & Q^{-1} & 0\\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

$$\Phi_1 = \begin{bmatrix} ((A+\Delta A)-\bar{\beta}((B+\Delta B)(K+\Delta K))^T P & \beta_1((B+\Delta B)(K+\Delta K))^T P\\ \bar{\beta}((B+\Delta B)(K+\Delta K))^T P & -\beta_1(B+\Delta B)(K+\Delta K))^T P\\ D^T P & 0\\ P & 0\\ 0 & 0 \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} \beta_1((B+\Delta B)(K+\Delta K))^T Q & \alpha_1 C_2^T (L+\Delta L)^T Q & 0 & C_1^T\\ -\beta_1((B+\Delta B)(K+\Delta K))^T Q & 0 & ((A+\Delta A)-\bar{\alpha}(L+\Delta L)C_2)^T Q & 0\\ 0 & 0 & (D-(L+\Delta L)D_2)^T Q & D_1^T\\ 0 & 0 & 0 & 0\\ 0 & 0 & Q & 0 \end{bmatrix}$$

Using the Schur Complement and separating the uncertainty terms with simplifications, Π can be rewritten as

$$\Pi = \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{12} & -\Gamma_{22} \end{bmatrix} + \begin{bmatrix} 0_{5\times 5} & \Gamma_{12\Delta}^T \\ \Gamma_{12\Delta} & 0_{6\times 6} \end{bmatrix} < 0$$

where

$$\Gamma_{12} = \begin{bmatrix} PA-\bar{\beta}PBK & \bar{\beta}PBK & \cdots\\ \beta_1 PBK & -\beta_1 PBK & \cdots\\ \beta_1 QBK & -\beta_1 QBK & \cdots\\ \alpha_1 QLC_2 & 0 & \cdots\\ 0 & QA-\bar{\alpha}QLC_2 & \cdots\\ C_1 & 0 & \cdots \end{bmatrix}$$

$$\begin{matrix} PD & P & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q(D-LD_2) & 0 & Q \\ D_1 & 0 & 0 \end{matrix}\Bigg]$$

$$\Gamma_{12\Delta} = \Gamma_{12\Delta 1} + \Gamma_{12\Delta 2} + \Gamma_{12\Delta 3} \in \mathbb{R}^{6\times 5}$$

$$\Gamma_{12\Delta 1} = \begin{bmatrix} P\Delta A - \bar{\beta}P(B)\Delta K & \bar{\beta}P(B)\Delta K & 0 & 0 & 0 \\ \beta_1 P(B)\Delta K & -\beta_1 P(B)\Delta K & 0 & 0 & 0 \\ \beta_1 Q(B)\Delta K & -\beta_1 Q(B)\Delta K & 0 & 0 & 0 \\ \alpha_1 Q\Delta LC_2 & 0 & 0 & 0 & 0 \\ 0 & Q\Delta A - \bar{\alpha}Q\Delta LC_2 & -Q\Delta LD_2 & 0 & 0 \\ \Delta C_1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_{12\Delta 2} = \begin{bmatrix} -\bar{\beta}P\Delta BK & \bar{\beta}P(\Delta B)K & 0 & 0 & 0 \\ \beta_1 P\Delta BK & -\beta_1 P(\Delta B)K & 0 & 0 & 0 \\ \beta_1 Q\Delta BK & -\beta_1 Q(\Delta B)K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_{12\Delta 3} = \begin{bmatrix} -\bar{\beta}P(\Delta B)\Delta K & \bar{\beta}P(\Delta B)\Delta K & 0 & 0 & 0 \\ \beta_1 P(\Delta B)\Delta K & -\beta_1 P(\Delta B)\Delta K & 0 & 0 & 0 \\ \beta_1 Q(\Delta B)\Delta K & -\beta_1 Q(\Delta B)\Delta K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now by substituting the uncertainties from (9) in $\Gamma_{12\Delta 1}$, $\Gamma_{12\Delta 2}$, $\Gamma_{12\Delta 3}$, each term can be written as

$$\Gamma_{12\Delta 1} = \begin{bmatrix} PM_c & -\bar{\beta}PBN_2^T & 0 & 0 & 0 & 0 \\ 0 & \beta_1 PBN_2^T & 0 & 0 & 0 & 0 \\ 0 & \beta_1 QBN_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1 QM_L \\ 0 & 0 & 0 & & -xQM_L & 0 \\ 0 & 0 & M_3 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} \Delta_k & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta_k^T & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_k & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_k & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_k \end{bmatrix} \tag{64}$$

$$* \begin{bmatrix} N_1 & 0 & 0 & 0 & 0 \\ M_k^T & -M_k^T & 0 & 0 & 0 \\ N_3 & 0 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 & 0 \\ 0 & N_LC_2 & -\frac{1}{x}N_LD_2 & 0 & 0 \\ N_LC_2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} x &= \bar{\alpha} \\ &= \widetilde{\Theta}_1\widetilde{\Delta}_{k1}\widetilde{\Phi}_1,\ \widetilde{\Theta}_1 \in \mathbb{R}^{6\times 6},\ \widetilde{\Delta}_{k1} \in \mathbb{R}^{6\times 6},\ \widetilde{\Phi}_1 \in \mathbb{R}^{6\times 5} \end{aligned} \tag{65}$$

Similarly

$$\begin{aligned} \Gamma_{12\Delta 2} &= \begin{bmatrix} -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta_k \begin{bmatrix} N_2K & -N_2K & 0 & 0 & 0 \end{bmatrix} \\ &= \widetilde{\Theta}_2\widetilde{\Delta}_{k2}\widetilde{\Phi}_2,\ \widetilde{\Theta}_2 \in \mathbb{R}^{6\times 1},\ \widetilde{\Delta}_{k2} \in \mathbb{R}^{1\times 1},\ \widetilde{\Phi}_2 \in \mathbb{R}^{1\times 5} \end{aligned} \tag{66}$$

and

$$\begin{aligned} &\Gamma_{12\Delta 3} \\ &= \begin{bmatrix} -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta_k N_2 N_2^T \Delta_k^T \begin{bmatrix} M_k^T & -M_k^T & 0 & 0 & 0 \end{bmatrix} \\ &= \widetilde{\Theta}_3\widetilde{\Delta}_{k3}\widetilde{\Phi}3,\quad \widetilde{\Theta}_3 \in \mathbb{R}^{6\times 1},\quad \widetilde{\Delta}_{k3} \in \mathbb{R}^{1\times 1},\quad \widetilde{\Phi}_3 \in \mathbb{R}^{1\times 5} \end{aligned} \tag{67}$$

Then

$$\begin{aligned} &\begin{bmatrix} 0 & \Gamma_{12\Delta}^T \\ \Gamma_{12\Delta} & 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} 0 & \Gamma_{12\Delta 1}^T \\ \Gamma_{12\Delta 1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Gamma_{12\Delta 2}^T \\ \Gamma_{12\Delta 2} & 0 \end{bmatrix} \\ &\qquad + \begin{bmatrix} 0 & \Gamma_{12\Delta 3}^T \\ \Gamma_{12\Delta 3} & 0 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \begin{bmatrix} 0 & \Gamma_{12\Delta 1}^T \\ \Gamma_{12\Delta 1} & 0 \end{bmatrix} &= \begin{bmatrix} 0_{5\times 5} & \widetilde{\Phi}_1^T\widetilde{\Delta}_{k1}^T\widetilde{\Theta}_1^T \\ \widetilde{\Theta}_1\widetilde{\Delta}_{k1}\widetilde{\Phi}_1 & 0_{6\times 6} \end{bmatrix} \\ &= \begin{bmatrix} 0_{5\times 6} \\ \widetilde{\Theta}_{16\times 6} \end{bmatrix} \widetilde{\Delta}_{k1(6\times 6)} \begin{bmatrix} \widetilde{\Phi}_{1(6\times 5)} & 0_{6\times 6} \end{bmatrix} \\ &\quad + \begin{bmatrix} \widetilde{\Phi}_1^T \\ 0_{6\times 6} \end{bmatrix} \widetilde{\Delta}_{k1}^T \begin{bmatrix} 0_{6\times 5} & \widetilde{\Theta}_1^T \end{bmatrix} \\ &= \widetilde{\Upsilon}_1\widetilde{\Delta}_{k1}\widetilde{\chi}_1 + \widetilde{\chi}_1^T\widetilde{\Delta}_{k1}^T\widetilde{\Upsilon}_1^T \\ &\le \epsilon_1^{-1}\widetilde{\Upsilon}_1\widetilde{\Upsilon}_1^T + \epsilon_1\widetilde{\chi}_1^T\widetilde{\chi}_1, \quad \text{for any } \epsilon_1 > 0 \end{aligned} \tag{68}$$

similarly

$$\begin{bmatrix} 0 & \Gamma_{12\Delta 2}^T \\ \Gamma_{12\Delta 2} & 0 \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 0_{4\times4} & \widetilde{\Phi}_2^T\widetilde{\Delta}_{k2}^T\widetilde{\Theta}_2^T \\ \widetilde{\Theta}_2\widetilde{\Delta}_{k2}\widetilde{\Phi}_2 & 0_{5\times5} \end{bmatrix} \\
&= \begin{bmatrix} 0_{5\times1} \\ \widetilde{\Theta}_{2(6\times1)} \end{bmatrix} \widetilde{\Delta}_{k2(1\times1)} \begin{bmatrix} \widetilde{\Phi}_{2(1\times6)} & 0_{1\times5} \end{bmatrix} \\
&\quad + \begin{bmatrix} \widetilde{\Phi}^T_{2(6\times1)} \\ 0_{5\times1} \end{bmatrix} \widetilde{\Delta}_{k2}^T \begin{bmatrix} 0_{1\times5} & \widetilde{\Theta}^T_{2(1\times6)} \end{bmatrix} \\
&= \widetilde{\Upsilon}_2\widetilde{\Delta}_{k2}\widetilde{\chi}_2 + \widetilde{\chi}_2^T\widetilde{\Delta}_{k2}^T\widetilde{\Upsilon}_2^T \qquad (69)\\
&\le \epsilon_2\widetilde{\Upsilon}_2\widetilde{\Upsilon}_2^T + \epsilon_2^{-1}\widetilde{\chi}_2^T\widetilde{\chi}_2, \quad \text{for any } \epsilon_2 > 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{bmatrix} 0 & \Gamma_{12\Delta3}^T \\ \Gamma_{12\Delta3} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0_{4\times4} & \widetilde{\Phi}_3^T\widetilde{\Delta}_{k3}^T\widetilde{\Theta}_3^T \\ \widetilde{\Theta}_3\widetilde{\Delta}_{k3}\widetilde{\Phi}_3 & 0_{5\times5} \end{bmatrix} \\
&= \begin{bmatrix} 0_{5\times1} \\ \widetilde{\Theta}_{3(6\times1)} \end{bmatrix} \widetilde{\Delta}_{k3(1\times1)} \begin{bmatrix} \widetilde{\Phi}_{3(1\times5)} & 0_{1\times6} \end{bmatrix} \\
&\quad + \begin{bmatrix} \widetilde{\Phi}^T_{3(5\times1)} \\ 0_{6\times1} \end{bmatrix} \widetilde{\Delta}_{k3(1\times1)}^T \begin{bmatrix} 0_{1\times5} & \widetilde{\Theta}^T_{3(1\times6)} \end{bmatrix} \\
&= \widetilde{\Upsilon}_3\widetilde{\Delta}_{k3}\widetilde{\chi}_3 + \widetilde{\chi}_3^T\widetilde{\Delta}_{k3}^T\widetilde{\Upsilon}_3^T \\
&\le \epsilon_3\widetilde{\Upsilon}_3\widetilde{\Upsilon}_3^T + \epsilon_3^{-1}\widetilde{\chi}_3^T\widetilde{\chi}_3, \quad \text{for any } \epsilon_2 > 0 \qquad (70)
\end{aligned}
$$

Then combining all terms of Π

$$
\begin{aligned}
\Pi &= \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{12} & \Gamma_{22} \end{bmatrix} + \begin{bmatrix} 0 & \Gamma_{12\Delta1}^T \\ \Gamma_{12\Delta1} & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \Gamma_{12\Delta2}^T \\ \Gamma_{12\Delta2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \Gamma_{12\Delta3}^T \\ \Gamma_{12\Delta3} & 0 \end{bmatrix} \\
&\le \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{12} & \Gamma_{22} \end{bmatrix} + \epsilon_1\widetilde{\Upsilon}_1\widetilde{\Upsilon}_1^T + \epsilon_1^{-1}\widetilde{\chi}_1^T\widetilde{\chi}_1 \\
&\quad + \epsilon_2\widetilde{\Upsilon}_2\widetilde{\Upsilon}_2^T + \epsilon_2^{-1}\widetilde{\chi}_2^T\widetilde{\chi}_2 + \epsilon_3\widetilde{\Upsilon}_3\widetilde{\Upsilon}_3^T + \epsilon_3^{-1}\widetilde{\chi}_3^T\widetilde{\chi}_3 < 0
\end{aligned} \qquad (71)
$$

where

$$
\widetilde{\Upsilon}_1 = \begin{bmatrix} 0_{5\times6} \\ \hline \widetilde{\Theta}_{1(6\times6)} \end{bmatrix}
= \left[\begin{array}{cccc|cc}
\multicolumn{4}{c|}{0_{5\times4}} & \multicolumn{2}{c}{0_{5\times2}} \\ \hline
PM_c & -\bar{\beta}PBN_2^T & 0 & 0 & 0 & 0 \\
0 & \beta_1 PBN_2^T & 0 & 0 & 0 & 0 \\
0 & \beta_1 QBN_2^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_1 QM_L \\
0 & 0 & 0 & QM_c & -xQM_L & 0 \\
0 & 0 & M_3 & 0 & 0 & 0
\end{array}\right]
$$

$$
\widetilde{\chi}_1^T = \begin{bmatrix} \widetilde{\Phi}_1^T \\ \hline 0_{6\times6} \end{bmatrix}
= \left[\begin{array}{ccccc}
N_1 & 0 & 0 & 0 & 0 \\
M_k^T & -M_k^T & 0 & 0 & 0 \\
N_3 & 0 & 0 & 0 & 0 \\
0 & N_1 & 0 & 0 & 0 \\
0 & N_L C_2 & -\frac{1}{x}N_L D_2 & 0 & 0 \\
N_L C_2 & 0 & 0 & 0 & 0 \\ \hline
\multicolumn{5}{c}{0_{6\times6}}
\end{array}\right]^T
$$

and

$$
\widetilde{\Upsilon}_2 = \begin{bmatrix} 0_{5\times1} \\ \hline \widetilde{\Theta}_{2(6\times1)} \end{bmatrix} = \begin{bmatrix} 0_{5\times1} \\ \hline -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{aligned}
\widetilde{\chi}_2^T &= \begin{bmatrix} \widetilde{\Phi}^T_{2(6\times1)} \\ \hline 0_{5\times1} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} N_2K & -N_2K & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \hline 0_{5\times1} \end{bmatrix}
\end{aligned}
$$

and

$$
\widetilde{\Upsilon}_3 = \begin{bmatrix} 0_{5\times1} \\ \hline \widetilde{\Theta}_{3(6\times1)} \end{bmatrix} = \begin{bmatrix} 0_{5\times1} \\ \hline -\bar{\beta}PM_c \\ \beta_1 PM_c \\ \beta_1 QM_c \\ 0 \\ 0 \\ 0 \end{bmatrix},
$$

$$
\widetilde{\chi}_3^T = \begin{bmatrix} \widetilde{\Phi}^T_{3(5\times1)} \\ \hline 0_{6\times1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} M_k^T & -M_k^T & 0 & 0 & 0 \end{bmatrix}^T \\ \hline 0_{6\times1} \end{bmatrix}
$$

Finally and according to Lemma (2), we have

$$
\begin{bmatrix}
\widetilde{\Xi}_0 & \widetilde{\Upsilon}_1 & \epsilon_1\widetilde{\chi}_1^T & \epsilon_2\widetilde{\Upsilon}_2 & \widetilde{\chi}_2^T & \widetilde{\Upsilon}_3 & \epsilon_3\widetilde{\chi}_3^T \\
\widetilde{\Upsilon}_1^T & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\
\epsilon_1\widetilde{\chi}_1 & 0 & -\epsilon_1 I & 0 & 0 & 0 & 0 \\
\epsilon_2\widetilde{\Upsilon}_2^T & 0 & 0 & -\epsilon_2 I & 0 & 0 & 0 \\
\widetilde{\chi}_2 & 0 & 0 & 0 & -\epsilon_2 I & 0 & 0 \\
\widetilde{\Upsilon}_3^T & 0 & 0 & 0 & 0 & -\epsilon_3 I & 0 \\
\epsilon_3\widetilde{\chi}_3 & 0 & 0 & 0 & 0 & 0 & -\epsilon_3 I
\end{bmatrix} < 0 \qquad (72)
$$

where

$$
\widetilde{\Xi}_0 = \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{12} & \Gamma_{22} \end{bmatrix} \qquad (73)
$$

Then (72) is similar to (21). At the end, it can be concluded from (59) that

$$
\mathbb{E}\{V_{k+1}\} - \mathbb{E}\{V_k\} + \mathbb{E}\{z_k^T z_k\} - \gamma^2\mathbb{E}\{w_k^T w_k\} < 0 \qquad (74)
$$

Taking the summation of (74) from 0 to ∞, yields

$$
\sum_{k=0}^{\infty}\mathbb{E}\{z_k^T z_k\} < \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{w_k^T w_k\} + \mathbb{E}\{V_0\} - \mathbb{E}\{V_\infty\} \qquad (75)
$$

where the CLNNC (closed-loop nonlinear networked control) system in (15) is exponentially mean square stable and for $\eta_0 = 0$, it can be concluded that

$$\sum_{k=0}^{\infty} \mathbb{E}\{z_k^T z_k\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{w_k^T w_k\} \tag{76}$$

Hence, the H_∞ performance constraints (16) are achieved. ■

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