IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received June 27, 2021, accepted July 20, 2021, date of publication July 26, 2021, date of current version August 2, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3099528

A Robust H_∞ Fault Tolerant Controller for Uncertain Systems Described by Linear Fractional Transformation Model

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ABSTRACT In this article, a robust H_∞ fault tolerant control law is addressed for a class of the uncertain dynamical systems represented via linear fractional transformation. To this objective, a state-feedback controller law is utilized for achieving the control objective. Thus, a linear matrix inequality based performance condition would be derived to guarantee that the disturbance suppression is accomplished in the uncertain system. Hence, the gains of the robust H_∞ controller would be suitably determined by checking the feasibility of such a linear matrix inequality problem. The proposed control technique is numerically simulated in two dynamical uncertain systems (i.e., a typical control system and a mechanical robotic arm). Considering the disturbance rejections and transient responses, the results illustrate the efficiency of the recommended robust technique compared with the existing control methods.

INDEX TERMS Robust H_∞ fault tolerant controller, uncertain control systems, linear fractional transformation, linear matrix inequality.

I. INTRODUCTION

Usually, in a typical control problem, the performance and stability of the closed-loop system would be destroyed due to the existence of unknown and uncertain expressions. However, many mathematical formulations and models have been developed to describe the system's uncertainty like additive type, multiplicative form, polytopic set, linear fractional transformation (LFT), and the others [1]. The key advantages of the LFT model over the other types are itemized as follows:

a) The uncertainties raised by the multiple sources can be easily handled as well.

b) The fractional and inverse expressions are formulated via a suitable selection of the LFT parameters.

c) Some types of uncertainties, as well as the additive and multiplicative, would be treated as a special form of the LFT model.

d) All system's uncertainties could be excluded into a single uncertain block.

Thus, in some control applications like the robotic manipulators [2], [3], the uncertainty can be emerged in the inverse form. Hence, such difficulty is effectively overcome using the LFT model. Firstly, the LFT mapping has been presented to formulate the uncertainties of the linear time-invariant systems (LTI) described by the transfer function. Then the stability and performance criterion is numerically checked via the μ-method [1].

The standard control problems can be studied in the uncertain systems like the regulation, tracking, stabilization, H_∞ control, and so on. Lately, the robust H_2 and H_∞ control (or fault-tolerant control) have attracted the attention of many researchers [4], [5]. An H_∞ robust predictive control is suggested to the linear systems under disturbances [6]. In these control methods, the initial conditions are ignored, and the effects of the external signals (like the disturbance rejection) would only be concerned on the closed-loop performance. Thus, the control parameters would be determined while

The associate editor coordinating the review of this manuscript and approving it for publication was Haibin Sun.

the H_2 or H_∞ norm of the closed-loop system is minimized. In the robust H_∞ fault tolerant control, the robust performance of the closed-loop system is guaranteed in the presence of some fault signals. Moreover, the control law is determined in the worst case of the fault signal. Hence, a numerical optimization problem has been solved to compute the gains of the control law.

In the uncertain systems, the linear matrix inequality (LMI) would be a systematic and powerful tool for the stability analysis and robust controller synthesis [7]. Some system's properties like H_2 and H_∞ norms can be directly determined by an LMI problem. Furthermore, the H_2 and H_∞ control design may also be formulated via the LMI solution [8], [9]. Similarly, the H_2 sampled-data control is designed for the uncertain linear system [10].

Nowadays, numerous robust H_∞ control laws have been suggested for some classes of uncertain systems. These briefly include stochastic systems [11], [12], switched systems [13]–[15], Markovian jump systems [16]–[18], implicit and fuzzy degenerate jump systems [19], [20], singular systems [21]–[23], uncertain linear systems with time-varying delays [24], nonlinear systems with time-delays [25], [26], networked control systems [27], [28], observer-based repetitive control systems [29], Takagi-Sugeno (TS) fuzzy systems [30], [31], synchronization of complex dynamical networks [32], stochastic systems [33], and so on. In these methods, the uncertainties are usually modeled as either additive or multiplicative or polytopic. As a result, the above-mentioned methods may destabilize the control system. Hence, they cannot be applied to the general types of uncertainty.

The controllers with disturbance rejection property are very interesting in practice. In the past years, the H_∞ controllers have been implemented in some industrial applications like the granulation process [34], islanded microgrid [35], energy internet [36], quadrotor UAV [37], active suspension system [38], double support balance system [39], horizontal wind turbine under various operating modes [40], multi-vehicle control system [41] and the others.

The LFT models would effectively cover the variations of the uncertain terms. So, they could be preferred in many control issues like regulation, tracking, and disturbance rejection. Quite recently, utilizing the LFT model, a robust predictive controller is designed based on the LMI [42], [43]. Based on the LFT framework, a tuning method is proposed for an impedance matching network [44]. Considering polytopic uncertainty, robust H_2 and H_∞ filters are designed for uncertain LFT systems [45]. The output-feedback controller of LFT system is studied in the presence of actuator saturations [46]. Then robust H_∞ output regulation of an uncertain LFT system is investigated in [47]. Lately, employing the additive uncertainties, an LMI method is addressed to the H_∞ control design [48]. But such a method would not be effective in the presence of other types of uncertainties. For example, an inverse uncertain term may emerge in the mass matrix of the robotics systems [2]. So, the features of system, as well as the stability property and performance index cannot be preserved or may not be preserved in presence of the uncertain terms. This point motivates the authors to use the LFT model by considering the other sources and forms of the uncertainties. Thus, a category of the uncertain control dynamics defined by LFT model is considered in this study. Therefore, an LMI condition would be derived to obtain the robust fault tolerant control law in the presence of the mentioned uncertainties. The main novelties of the paper in comparison to similar works can be listed as follows:

a) An LMI based approach is suggested to the robust fault tolerant H_∞ controller design in the uncertain nonlinear systems. The disturbance rejection could be theoretically proved by the proposed scheme.

b) The system uncertainties and unknown parameters are well-managed using the LFT model, as illustrated in the simulation.

The remainder of this paper is prepared as follows: in Section 2, some notations and preliminaries are briefly introduced. In Section 3, the robust H_∞ fault tolerant control synthesis is formulated in the uncertain systems. The main results of the research are addressed in Section 4. In Section 5, some control examples are simulated to show the applicability of the suggested technique. The concluding remarks are summarized in the last section.

II. NOTATIONS AND DEFINITIONS

Throughout the paper, I_n is $n \times n$ identity matrix, and the star symbol $(*)$ means symmetric property of a matrix. The two-norm of a vector $\nu \in \mathbb{R}^n$ is defined as $\|\nu\| \overset{\text{def}}{=} \sqrt{\nu^T \nu}$ and also the two-norm of a signal $w(t) \in \mathbb{R}^n$ is defined as $\|w(t)\| \overset{\text{def}}{=} \sqrt{\int_0^{+\infty} w^T(t)\, w(t)\, dt}$. The two-norm of a matrix $\Theta \in \mathbb{R}^{q\times p}$ is defined as $\|\Theta\| \overset{\text{def}}{=} \overline{\sigma}(\Theta)$ where $\overline{\sigma}(\Theta) = \sqrt{\lambda_{max}\left(\Theta^T \Theta\right)}$ and $\lambda_{max}(.)$ signifies the maximum eigenvalues of matrix.

Consider the following mathematical model:

$$\begin{cases} \begin{bmatrix} z \\ \phi \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix} \begin{bmatrix} d \\ \omega \end{bmatrix} \\ \omega = \Delta\phi \end{cases} \tag{1}$$

where $d \in \mathbb{R}^p$, $\omega \in \mathbb{R}^s$ are inputs, and $z \in \mathbb{R}^q$, $\phi \in \mathbb{R}^r$ are outputs of Eq. (1). The term $\Delta \in \mathbb{R}^{s\times r}$ would be an uncertain matrix. A block diagram of the Eq. (1) is depicted as Fig.1.

Define the lower LFT $\mathcal{F}_L(\mathcal{M}, \Delta)$ as follows:

$$\mathcal{F}_L(\mathcal{M}, \Delta) \overset{\text{def}}{=} \mathcal{M}_{11} + \mathcal{M}_{12}\Delta\left(I_r - \mathcal{M}_{22}\Delta\right)^{-1} \mathcal{M}_{21} \tag{2}$$

Here $\mathcal{M}_{11}$ is nominal value while $\mathcal{F}_L(\mathcal{M}, \Delta)$ is uncertain. It is clear that the matrices $\mathcal{M}_{12}$, $\mathcal{M}_{21}$, and $\mathcal{M}_{22}$ reveal the deviations of $\mathcal{F}_L(\mathcal{M}, \Delta)$ about $\mathcal{M}_{11}$. In a similar way, the upper LFT can also be defined as well [1]. Eq. (1) is written as:

$$z = \mathcal{F}_L(\mathcal{M}, \Delta)\, d \tag{3}$$

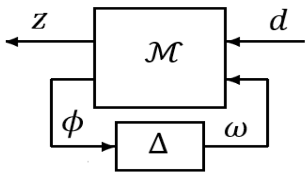

FIGURE 1. The representation of lower LFT.

where the matrix $\mathcal{M}$ is partitioned as follows:

$$\mathcal{M}=\begin{bmatrix}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \mathcal{M}_{21} & \mathcal{M}_{22}\end{bmatrix}$$

The LFT model (3) is well-posed when $\det\left(I_r - \mathcal{M}_{22}\Delta\right)\neq 0$. Otherwise, the LFT term tends to infinity.

The structured singular value (SSV) or simply the μ operator is defined for given matrices $\mathcal{N}\in\mathbb{R}^{r\times s}$ and $\Delta\in\mathbb{R}^{s\times r}$, as the following form:

$$\mu_\Delta(\mathcal{N})\overset{\text{def}}{=}\frac{1}{\min\{\overline{\sigma}(\Delta),\det(I_r-\mathcal{N}\Delta)=0\}} \tag{4}$$

and $\mu_\Delta(\mathcal{N})=0$ if $\det(I_r-\mathcal{N}\Delta)\neq 0$. It can be shown that $\mu_\Delta(\mathcal{N})\leq\overline{\sigma}(\mathcal{N})$ [1].

In the next section, in order to design a robust H_∞ controller, the control problem will be formulated for a category of the uncertain systems.

III. PROBLEM DESCRIPTION

Consider the uncertain nonlinear systems as follow:

$$\begin{cases}\begin{bmatrix}\dot{x}\\ \phi\end{bmatrix}=\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix}\begin{bmatrix}x\\ \omega\end{bmatrix}+\begin{bmatrix}\mathcal{B}_1\\ \mathcal{B}_2\end{bmatrix}u\\ \qquad +\begin{bmatrix}E_1\\ E_2\end{bmatrix}d+\begin{bmatrix}f_1(x)\\ f_2(x)\end{bmatrix}\\ \omega=\Delta\phi\\ z=Hx+H_u u\end{cases} \tag{5}$$

where $x(t)\in\mathbb{R}^n$ denotes the states vector, and $u(t)\in\mathbb{R}^m$ signifies the input vector of uncertain system (5). The vector $d(t)\in\mathbb{R}^p$ is an exogenous signal. It can be an external input, fault signals, disturbance, noise, or any norm bounded signals (i.e., $\|d(t)\|$ is finite). The term $z(t)\in\mathbb{R}^q$ would be a linear combination of the states and control inputs chosen as the system's output. An overall view of uncertain system (5) is illustrated as Fig. 2. So, a large class of uncertain dynamics are expressed by Eq. (5) as well.

The uncertain term $\Delta\in\mathbb{R}^{s\times r}$ would be extracted by a suitable selection of some internal signals $\phi(t)\in\mathbb{R}^r$ and $\omega(t)\in\mathbb{R}^s$. Compared to the existing studies [8], [48], [49], the differential equation (5) would be a more general form to represent the uncertainty. Thus many uncertain systems (for instants [8], [48]) can be described via Eq. (5).

Assumption 1: The nonlinear functions $f_1(.)$ and $f_2(.)$ are supposed to be unknown terms and satisfy the subsequent inequalities:

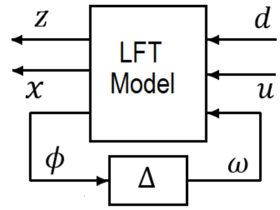

FIGURE 2. Schematic of uncertain system (5).

$$\begin{cases}\|f_1(x)\|\leq\|M_1x\|\\ \|f_2(x)\|\leq\|M_2x\|\end{cases} \tag{6}$$

Furthermore, the mappings $f_1(.)$ and $f_2(.)$ would be zero at zero (i.e., $f_1(0)=0, f_2(0)=0$).

Assumption 2: The uncertain nonlinear system (5) is supposed to be stabilized. Additionally, the state trajectories of uncertain system (5) are completely available for control objective.

The stabilizability is a necessary condition for the controller synthesis. So, it is supposed that a control law exists to stabilize the uncertain system (5).

Utilizing the LFT definition, the dynamical system (5) is rewritten as

$$\begin{cases}\dot{x}=\mathcal{A}(\Delta)x+\mathcal{B}(\Delta)u+E(\Delta)d+f_\Delta(x)\\ z=Hx+H_u u\end{cases} \tag{7}$$

where

$$\begin{cases}\mathcal{A}(\Delta)=\mathcal{A}_{11}+\mathcal{A}_{12}\Delta(I_r-\mathcal{A}_{22}\Delta)^{-1}\mathcal{A}_{21}\\ \mathcal{B}(\Delta)=\mathcal{B}_1+\mathcal{A}_{12}\Delta(I_r-\mathcal{A}_{22}\Delta)^{-1}\mathcal{B}_2\\ E(\Delta)=E_1+\mathcal{A}_{12}\Delta(I_r-\mathcal{A}_{22}\Delta)^{-1}E_2\\ f_\Delta(x)=f_1(x)+\mathcal{A}_{12}\Delta(I_r-\mathcal{A}_{22}\Delta)^{-1}f_2(x)\end{cases} \tag{8}$$

The above expressions are well-posed and bounded LFT terms. Then, through Assumption 1, it can be shown that there exists matrix $\overline{M}$ such that the following inequality is satisfied:

$$\|f_\Delta(x)\|\leq\left\|\overline{M}x\right\| \tag{9}$$

To design H_∞ fault tolerant controller, the control law is taken as the following feedback form:

$$u=Fx \tag{10}$$

The controller input $u(t)$ is designed when the matrix F is found. A simplified schematic of the uncertain system (7) with control law (10) is exposed in Fig. 3.

The uncertain matrices $\mathcal{A}(\Delta)$, $\mathcal{B}(\Delta)$, $E(\Delta)$, and $f_\Delta(x)$ would be well-posed if the following condition holds:

$$\det(I_r-\mathcal{A}_{22}\Delta)\neq 0 \tag{11}$$

The equation (11) is a necessary condition for the well-posedness [1]. It is strongly dependent on the structure of the

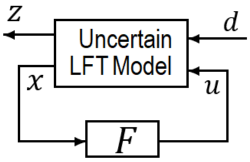

FIGURE 3. Simplified schematic of the control problem.

uncertain term Δ. Hence, a sufficient condition is found for the well-posedness via the small gain theorem as:

$$\|\mathcal{A}_{22}\| \,.\, \|\Delta\| < 1 \tag{12}$$

and a less-conservative sufficient condition can be found by the μ analysis as the following:

$$\mu_\Delta(\mathcal{A}_{22}) \,.\, \|\Delta\| < 1 \tag{13}$$

Consequently, the following inequality would hold when the LFT terms are well-defined:

$$\|\Delta\| < \frac{1}{\|\mathcal{A}_{22}\|} \leq \frac{1}{\mu_\Delta(\mathcal{A}_{22})} \tag{14}$$

Assumption 3: The uncertain matrix Δ is supposed to be norm bounded as follows:

$$\|\Delta\| \leq \delta \tag{15}$$

where the positive constant δ is satisfying the inequality: $\delta\,\|\mathcal{A}_{22}\| < 1$.

Fact 1: The overall gain (from the input $d(t)$ to the output $z(t)$) of the uncertain system (7) can be defined as $\gamma_0 = \|z\| / \|d\|$. The system's gain is less than or equal to γ if there exists a Lyapunov function $V(x)$ such that the following inequality holds for given $\gamma > 0$ [50]:

$$\dot{V}(x) + \|z\|^2 - \gamma^2\|d\|^2 \leq 0 \tag{16}$$

The constant γ would be an upper-bound of the system's gain γ_0. Hence, it is associated with the worst possible cases (i.e., $\gamma_0 \leq \gamma$).

Fact 2: Consider the matrices $\mathcal{X} \in \mathbb{R}^{n_1\times n_2}$, $\mathcal{H} \in \mathbb{R}^{n_2\times n_2}$ and $\mathcal{Y} \in \mathbb{R}^{n_2\times n_1}$. For any $\rho > 0$, the subsequent inequality holds:

$$\mathcal{X}\mathcal{H}\mathcal{Y} + \mathcal{Y}^T\mathcal{H}^T\mathcal{X}^T \leq \rho\mathcal{X}\mathcal{X}^T + \frac{1}{\rho}\mathcal{Y}^T\mathcal{H}^T\mathcal{H}\mathcal{Y} \tag{17}$$

Fact 3: For given matrices $\mathcal{X} \in \mathbb{R}^{n_1\times n_2}$, and $\mathcal{H} \in \mathbb{R}^{n_1\times n_1}$, the following inequality holds:

$$\mathcal{X}^T\mathcal{H}\mathcal{X} \leq \|\mathcal{H}\|\,\mathcal{X}^T\mathcal{X} \tag{18}$$

Next, the control law (10) will be designed to remove the impacts of the exogenous signals on the uncertain system (7) with the gain of less than or equal to γ.

IV. MAIN RESULTS

In the present part, an LMI condition is derived to guarantee that the system's gain meets the control goals. Then, a robust $\mathcal{H}_\infty$ fault tolerant controller is determined by the LMI feasibility checking.

Theorem 1: The uncertain nonlinear system (7) and Assumptions 1-3 are considered. For given $\gamma > 0$ and $\varepsilon > 0$, if there exists a positive-definite (PD) matrix $Y \in \mathbb{R}^{n\times n}$, a rectangular matrix $G \in \mathbb{R}^{m\times n}$ and the positive coefficients ρ, β such that following LMI is feasible:

$$\begin{bmatrix} \Omega_{11} & Y\overline{M}^T & \Omega_{13} & E_1 & \Omega_{15} & E_1E_2^T & \alpha\mathcal{A}_{12} & \Omega_{18} \\ * & -\beta I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\rho I_r & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma I_p & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma I_q & 0 & 0 & 0 \\ * & * & * & * & * & -\frac{1}{\varepsilon}\gamma I_r & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon\gamma I_s & 0 \\ * & * & * & * & * & * & * & -\gamma I_s \end{bmatrix} \leq 0 \tag{19}$$

where

$$\begin{aligned} \Omega_{11} &= \mathcal{A}_{11}Y + \mathcal{B}_1G + Y\mathcal{A}_{11}^T + G^T\mathcal{B}_1^T + \beta I_n + \rho\alpha^2\mathcal{A}_{12}\mathcal{A}_{12}^T, \\ \Omega_{13} &= Y\mathcal{A}_{21}^T + G^T\mathcal{B}_2^T, \\ \Omega_{15} &= YH^T + G^TH_u^T, \\ \Omega_{18} &= \alpha\,\|E_2\|\,\mathcal{A}_{12}, \\ \alpha &= \delta\left(1 - \|\mathcal{A}_{22}\|\,\delta\right)^{-1}, \end{aligned}$$

then an H_∞ controller would be found by the control law (10) with $F = GY^{-1}$. Additionally, the infinite norm of the uncertain system (7) would also be less than or equal to γ.

Proof: The quadratic Lyapunov functional $V(x) = x^TPx$ with $P = P^T > 0$ is considered. The inequality (16) is expressed as

$$\begin{aligned} &x^TP\left(\mathcal{A}(\Delta) + \mathcal{B}(\Delta)F\right)x + x^T\left(\mathcal{A}^T(\Delta) + F^T\mathcal{B}^T(\Delta)\right)Px \\ &\quad + x^TPE(\Delta)\,d + d^TE^T(\Delta)\,Px + x^TPf_\Delta(x) + f_\Delta^T(x)\,Px \\ &\quad + x^T(H + H_uF)^T(H + H_uF)\,x - \gamma^2d^Td \leq 0 \end{aligned} \tag{20}$$

Using Fact 2 and Assumption 1, the following inequality is implied:

$$\begin{aligned} x^TPf_\Delta(x) + f_\Delta^T(x)\,Px &\leq \beta_0x^TP^2x + \frac{1}{\beta_0}f_\Delta^T(x)\,f_\Delta(x) \\ &\leq x^T\left(\beta_0P^2 + \frac{1}{\beta_0}\overline{M}^T\overline{M}\right)x \end{aligned} \tag{21}$$

where β_0 is a positive constant. Thus, Eq. (20) would hold if the subsequent matrix inequality is fulfilled:

$$\begin{bmatrix} \overline{\Omega} & PE(\Delta) \\ * & -\gamma^2I_p \end{bmatrix} \leq 0 \tag{22}$$

where

$$\begin{aligned} \overline{\Omega} = &\,P\left(\mathcal{A}(\Delta) + \mathcal{B}(\Delta)F\right) + \mathcal{A}^T(\Delta)P + F^T\mathcal{B}^T(\Delta)P \\ &+ \beta_0P^2 + \frac{1}{\beta_0}\overline{M}^T\overline{M} + (H + H_uF)^T(H + H_uF). \end{aligned}$$

Let pre-post-multiply (22) by the symmetric matrix diag$\left\{\gamma^{\frac{1}{2}}P^{-1}, \gamma^{\frac{-1}{2}}I_p\right\}$.

$$\begin{bmatrix} \tilde{\Omega} & E(\Delta) \\ * & -\gamma I_p \end{bmatrix} \le 0 \tag{23}$$

where

$$\begin{aligned}\tilde{\Omega} &= \gamma\mathcal{A}(\Delta)P^{-1} + P^{-1}\gamma F^T\mathcal{B}^T(\Delta) + \gamma\mathcal{B}(\Delta)FP^{-1} \\ &\quad + \gamma P^{-1}\mathcal{A}^T(\Delta) + \beta_0\gamma I_n + \frac{\gamma}{\beta_0}P^{-1}\overline{M}^T\overline{M}P^{-1} \\ &\quad + \gamma P^{-1}(H+H_uF)^T(H+H_uF)P^{-1}.\end{aligned}$$

Now define $Y \overset{\text{def}}{=} \gamma P^{-1}$, $G \overset{\text{def}}{=} \gamma FP^{-1}$, and $\beta = \beta_0\gamma$. Then the inequality (23) is modified as:

$$\begin{bmatrix} \hat{\Omega} & E(\Delta) \\ * & -\gamma I_p \end{bmatrix} \le 0 \tag{24}$$

where

$$\begin{aligned}\hat{\Omega} &= \mathcal{A}(\Delta)Y + \mathcal{B}(\Delta)G + Y\mathcal{A}^T(\Delta) + G^T\mathcal{B}^T(\Delta) + \beta I_n \\ &\quad + \frac{1}{\beta}Y\overline{M}^T\overline{M}Y + \frac{1}{\gamma}(HY+H_uG)^T(HY+H_uG).\end{aligned}$$

From the definitions above, the controller gain is computed as $F = GY^{-1}$. The inequality (24) is rewritten as:

$$\begin{bmatrix} \tilde{\Pi} & E(\Delta) \\ * & -\gamma I_p \end{bmatrix} + \frac{1}{\beta}Y\overline{M}^T\overline{M}Y + \frac{1}{\gamma}\begin{bmatrix} YH^T + G^TH_u^T \\ 0 \end{bmatrix} \times \begin{bmatrix} HY + H_uG & 0 \end{bmatrix} \le 0 \tag{25}$$

where $\tilde{\Pi} = \mathcal{A}(\Delta)Y + \mathcal{B}(\Delta)G + Y\mathcal{A}^T(\Delta) + G^T\mathcal{B}^T(\Delta) + \beta I_n$.

Applying the Schur complement lemma [7], the inequality (25) is equivalent to the subsequent LMI:

$$\begin{bmatrix} \tilde{\Pi} & Y\overline{M}^T & E(\Delta) & YH^T + G^TH_u^T \\ * & -\beta I_n & 0 & 0 \\ * & * & -\gamma I_p & 0 \\ * & * & * & -\gamma I_q \end{bmatrix} \le 0 \tag{26}$$

It is trivial that, if there exist $Y \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{m\times n}$ and $\beta > 0$ so that LMI (26) is feasible for all admissible Δ, then the control law (10) with $F = GY^{-1}$ would guarantee that the system's gain would be less than or equal to γ. The inequality (26) depends on the uncertain term Δ. Thus (26) cannot be checked via the usual tools.

The inequality (26) is modified via the Schur's complement lemma [7] as:

$$\begin{aligned}&\mathcal{A}(\Delta)Y + \mathcal{B}(\Delta)G + Y\mathcal{A}^T(\Delta) + G^T\mathcal{B}^T(\Delta) + \beta I_n \\ &\quad + \frac{1}{\beta}Y\overline{M}^T\overline{M}Y + \frac{1}{\gamma}E(\Delta)E^T(\Delta) \\ &\quad + \frac{1}{\gamma}(HY+H_uG)^T(HY+H_uG) \le 0\end{aligned} \tag{27}$$

Then

$$\begin{aligned}&\mathcal{A}_{11}Y + \mathcal{B}_1G + Y\mathcal{A}_{11}^T + G^T\mathcal{B}_1^T + \beta I_n + \frac{1}{\beta}Y\overline{M}^T\overline{M}Y + \mathcal{A}_{12} \\ &\times \Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}(\mathcal{A}_{21}Y + \mathcal{B}_2G) + \left(Y\mathcal{A}_{21}^T + G^T\mathcal{B}_2^T\right)^T \\ &\times (I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T + \frac{1}{\gamma}E_1E_1^T + \frac{1}{\gamma}E_1E_2^T \\ &\times (I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T + \frac{1}{\gamma}\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}E_2E_1^T \\ &+ \frac{1}{\gamma}\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}E_2E_2^T(I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T \\ &+ \frac{1}{\gamma}(HY+H_uG)^T(HY+H_uG) \le 0\end{aligned} \tag{28}$$

The uncertain terms of inequality (28) may also be bounded utilizing Assumption 2 and Facts 2-3. By means of Fact 2, for any $\rho > 0$, the following condition holds:

$$\begin{aligned}&\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}(\mathcal{A}_{21}Y + \mathcal{B}_2G) + \left(Y\mathcal{A}_{21}^T + G^T\mathcal{B}_2^T\right)^T \\ &\quad \times (I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T \\ &\le \frac{1}{\rho}(\mathcal{A}_{21}Y + \mathcal{B}_2G)^T(\mathcal{A}_{21}Y + \mathcal{B}_2G) \\ &\quad + \rho\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}(I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T,\end{aligned} \tag{29}$$

Similarly, the following inequality holds for any $\varepsilon > 0$:

$$\begin{aligned}&E_1E_2^T(I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T + \mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}E_2E_1^T \\ &\le \varepsilon E_1E_2^TE_2E_1^T + \frac{1}{\varepsilon}\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}(I_r - \mathcal{A}_{22}\Delta)^{-T} \\ &\quad \times \Delta^T\mathcal{A}_{12}^T,\end{aligned} \tag{30}$$

Fact 3 implies that the following inequality holds:

$$\begin{aligned}&\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}E_2E_2^T(I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T \\ &\le \left\|E_2E_2^T\right\|\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}(I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T\end{aligned} \tag{31}$$

Thus, the inequality (28) is modified as:

$$\begin{aligned}&\mathcal{A}_{11}Y + \mathcal{B}_1G + Y\mathcal{A}_{11}^T + G^T\mathcal{B}_1^T + \beta I_n + \frac{1}{\beta}Y\overline{M}^T\overline{M}Y + \frac{1}{\rho} \\ &\quad \times (\mathcal{A}_{21}Y + \mathcal{B}_2G)^T(\mathcal{A}_{21}Y + \mathcal{B}_2G) + \frac{1}{\gamma}E_1E_1^T + \frac{1}{\gamma}\varepsilon E_1E_2^T \\ &\quad \times E_2E_1^T + \frac{1}{\gamma}(HY+H_uG)^T(HY+H_uG) \\ &\quad + \left(\rho + \frac{1}{\gamma}\frac{1}{\varepsilon} + \frac{1}{\gamma}\left\|E_2E_2^T\right\|\right)\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1} \\ &\quad \times (I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T \le 0\end{aligned} \tag{32}$$

The following condition is deduced by Assumption 2:

$$(I_r - \mathcal{A}_{22}\Delta) \ge (1 - \|\mathcal{A}_{22}\|\,\delta)I_r \tag{33}$$

Then

$$(I_r - \mathcal{A}_{22}\Delta)^{-1}(I_r - \mathcal{A}_{22}\Delta)^{-T} \le (1 - \|\mathcal{A}_{22}\|\,\delta)^{-2}I_r \tag{34}$$

Hence the uncertainty in (32) can be a bounded term as

$$\begin{aligned}&\mathcal{A}_{12}\Delta(I_r - \mathcal{A}_{22}\Delta)^{-1}(I_r - \mathcal{A}_{22}\Delta)^{-T}\Delta^T\mathcal{A}_{12}^T \\ &\le (1 - \|\mathcal{A}_{22}\|\,\delta)^{-2}\mathcal{A}_{12}\Delta\Delta^T\mathcal{A}_{12}^T \\ &\le (1 - \|\mathcal{A}_{22}\|\,\delta)^{-2}\delta^2\mathcal{A}_{12}\mathcal{A}_{12}^T\end{aligned} \tag{35}$$

Defining $\alpha \overset{\text{def}}{=} \delta\left(1-\|\mathcal{A}_{22}\|\,\delta\right)^{-1}$, then Eq. (32) is expressed as

$$\begin{aligned}&\mathcal{A}_{11}Y+\mathcal{B}_1G+Y\mathcal{A}_{11}^T+G^T\mathcal{B}_1^T+\beta I_n+\frac{1}{\beta}Y\overline{M}^T\overline{M}Y+\frac{1}{\rho}\\&\times\left(\mathcal{A}_{21}Y+\mathcal{B}_2G\right)^T\left(\mathcal{A}_{21}Y+\mathcal{B}_2G\right)+\frac{1}{\gamma}E_1E_1^T+\frac{1}{\gamma}\varepsilon E_1E_2^T\\&\times E_2E_1^T+\frac{1}{\gamma}\left(HY+H_uG\right)^T\left(HY+H_uG\right)\\&+\left(\rho+\frac{1}{\gamma}\frac{1}{\varepsilon}+\frac{1}{\gamma}\left\|E_2E_2^T\right\|\right)\alpha^2\mathcal{A}_{12}\mathcal{A}_{12}^T\le 0\end{aligned}\tag{36}$$

Applying the Schur complement lemma [7], Eq. (36) would be equivalent to the LMI (19). It completes the proof.

Corollary 1: The nominal system of the differential equation (5) would be found by substituting $\mathcal{A}_{12}=0$, $\mathcal{A}_{21}=0$, $\mathcal{A}_{22}=0$, $\mathcal{B}_2=0$, $E_2=0$, and $f_2(x)=0$ as follows:

$$\begin{cases}\dot{x}=\mathcal{A}_{11}x+\mathcal{B}_1u+E_1d+f_1(x)\\ z=Hx+H_uu\end{cases}\tag{37}$$

Thus, for given $\gamma>0$, if there exist a PD matrix $Y\in\mathbb{R}^{n\times n}$, the rectangular matrix $G\in\mathbb{R}^{m\times n}$ and $\beta>0$ such that following LMI is feasible:

$$\begin{bmatrix}\Pi & Y\overline{M}^T & E_1 & YH^T+G^TH_u^T\\ * & -\beta I_n & 0 & 0\\ * & * & -\gamma I_p & 0\\ * & * & * & -\gamma I_q\end{bmatrix}\le 0\tag{38}$$

where $\Pi=\mathcal{A}_{11}Y+\mathcal{B}_1G+Y\mathcal{A}_{11}^T+G^T\mathcal{B}_1^T+\beta I_n$, then an H_∞ fault tolerant controller would be found by the control law (10) with $F=GY^{-1}$. Additionally, the gain of the uncertain system (7) would be less than or equal to γ.

As seen, ignoring the nonlinear term of Eq. (37), the presented method reduced to the standard H_∞ control problem [8].

Corollary 2: The proposed approach is employed to design a stabilizing control signal in the following uncertain dynamics:

$$\dot{x}=\mathcal{A}(\Delta)x+\mathcal{B}(\Delta)u+f_\Delta(x)\tag{39}$$

Thus, if there exist a PD matrix $Y\in\mathbb{R}^{n\times n}$, a rectangular matrix $G\in\mathbb{R}^{m\times n}$ and positive coefficients ρ, β so that the subsequent LMI has a feasible solution:

$$\begin{bmatrix}\Pi & Y\overline{M}^T & Y\mathcal{A}_{21}^T+G^T\mathcal{B}_2^T\\ * & -\beta I_n & 0\\ * & * & -\rho I_r\end{bmatrix}\le 0\tag{40}$$

where $\Pi=\mathcal{A}_{11}Y+\mathcal{B}_1G+Y\mathcal{A}_{11}^T+G^T\mathcal{B}_1^T+\beta I_n+\rho\alpha^2\mathcal{A}_{12}\mathcal{A}_{12}^T$, then the system is asymptotically stable via the controller signal (10) with $F=GY^{-1}$.

Remark 1: The previous work [48] is a special case of the uncertain system (5) with $\mathcal{A}_{22}=0$, $\mathcal{B}_2=0$, $E_2=0$, and $f_2(x)=0$. Thus, it is considered as follows:

$$\begin{cases}\begin{bmatrix}\dot{x}\\ \phi\end{bmatrix}=\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & 0\end{bmatrix}\begin{bmatrix}x\\ \omega\end{bmatrix}+\begin{bmatrix}\mathcal{B}_1\\ 0\end{bmatrix}u\\ \qquad+\begin{bmatrix}E_1\\ 0\end{bmatrix}d+\begin{bmatrix}f_1\\ 0\end{bmatrix}\\ \omega=\Delta\phi\\ z=Hx+H_uu\end{cases}\tag{41}$$

It is modified to the following additive uncertain system:

$$\begin{cases}\dot{x}=\left(\mathcal{A}_{11}+\mathcal{A}_{12}\Delta\mathcal{A}_{21}\right)x+\mathcal{B}_1u+E_1d+f_1(x)\\ z=Hx+H_uu\end{cases}\tag{42}$$

For given $\gamma>0$ and $\rho>0$, it can be shown that if there exist a PD matrix $Y\in\mathbb{R}^{n\times n}$, rectangular matrix $G\in\mathbb{R}^{m\times n}$ and positive coefficients ρ, β such that following LMI is feasible:

$$\begin{bmatrix}\Pi & Y\overline{M}^T & \mathcal{A}_{12} & Y\mathcal{A}_{21}^T & YH^T+G^TH_u^T & E_1\\ * & -\beta I_n & 0 & 0 & 0 & 0\\ * & * & -\frac{1}{\rho}I_s & 0 & 0 & 0\\ * & * & * & -\rho I_r & 0 & 0\\ * & * & * & * & -I_q & 0\\ * & * & * & * & * & -\gamma^2I_p\end{bmatrix}\le 0\tag{43}$$

where $\Pi=\mathcal{A}_{11}Y+\mathcal{B}_1G+Y\mathcal{A}_{11}^T+G^T\mathcal{B}_1^T$, then an H_∞ fault tolerant controller would be determined by the control law (10) with $F=GY^{-1}$. Additionally, it is guaranteed that the infinite gain of the uncertain system (7) would be less than or equal to γ. Consequently, ignoring the nonlinear function, the results are reduced to the existing method [48].

V. NUMERICAL SIMULATION

In the present part, two instances are given to display the applicability of the proposed controller approach. Hence, a typical uncertain control system and a translational robotic manipulator are considered as control benchmarks to prove the efficacy of the suggested controller scheme.

Example 1: Consider an uncertain system (5) with the subsequent parameters:

$$\mathcal{A}_{11}=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 1 & -2 & 1 & 0 & -5\end{bmatrix},\quad \mathcal{A}_{12}=\begin{bmatrix}0 & 0.1 & 0\\ 0 & -0.1 & 0\\ 0 & 0.1 & 0\\ 0.3 & 0 & 0.1\\ 0.2 & 0 & 0.1\end{bmatrix},$$

$$\mathcal{A}_{21}=\begin{bmatrix}0 & 1 & 0 & 0 & -2\\ 0 & 0 & 1 & 0 & 0\\ 2 & 1 & 0 & -1 & 1\end{bmatrix},\quad \mathcal{A}_{22}=\begin{bmatrix}-0.1 & 0.2 & 0.1\\ -0.1 & 0.3 & 0.2\\ 0.1 & 0 & 0.1\end{bmatrix},$$

$$\mathcal{B}_1=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & -1\\ 0 & -1\\ 5 & 3\end{bmatrix},\quad \mathcal{B}_2=\begin{bmatrix}0 & 0\\ -1 & 0\\ 0 & 1\end{bmatrix},\ E_1=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 3 & 5\end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 3 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix},$$

$$H_u = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad f_1(x) = \frac{1}{10 + \|x\|^2} \begin{bmatrix} \sin(x_1) \\ 0 \\ \cos(2x_3) \\ 0 \\ x_5 \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The uncertain matrix is taken as follows:

$$\Delta = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & -0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

It is checked that $\|\Delta\| = 0.9239$ and $\|\mathcal{A}_{22}\| = 0.4460$. Hence $\delta = 0.95$ would be a reasonable upper-bound of the uncertain matrix Δ. The constant α can be calculated as $\alpha = \delta(1 - \|\mathcal{A}_{22}\|\,\delta)^{-1} = 1.6483$. The well-posedness condition is satisfied as follows:

$$\|\mathcal{A}_{22}\| \,.\, \|\Delta\| \le \|\mathcal{A}_{22}\|\,\delta = 0.4237 < 1 \tag{44}$$

The disturbance signals are taken as:

$$d_1(t) = \begin{cases} 1 - e^{-0.3t}, & 0 \le t < 50 \\ 0, & 50 \le t \le 100, \end{cases}$$

$$d_2(t) = \begin{cases} -1 + e^{-0.3t}, & 0 \le t < 50 \\ 0, & 50 \le t \le 100. \end{cases}$$

In numerical simulation, the initial conditions are set to zero, and the integration time is 1 millisecond. Applying the proposed control method with $\varepsilon = 1$ and $\gamma = 1$, the matrix variables Y and G are found as:

$$Y = \begin{bmatrix} 64.3825 & -41.1316 & 21.3301 & -28.1454 & -39.1270 \\ -41.1316 & 27.1019 & -17.1142 & 17.3906 & 22.7823 \\ 21.3301 & -17.1142 & 30.4299 & -4.7661 & 2.6337 \\ -28.1454 & 17.3906 & -4.7661 & 241.6484 & 19.2295 \\ -39.1270 & 22.7823 & 2.6337 & 19.2295 & 42.5997 \end{bmatrix}$$

$$G = \begin{bmatrix} -38.4926 & 22.3234 & -72.2243 & -94.0433 & -18.4328 \\ -58.2924 & 37.1284 & 101.4939 & 201.8884 & 58.5818 \end{bmatrix}$$

Then the controller gain matrix is calculated as:

$$F = \begin{bmatrix} -21.3819 & -36.6113 & -8.0590 & -0.4193 & 0.1957 \\ 27.6992 & 61.6033 & 19.3200 & 0.6142 & -7.6008 \end{bmatrix}$$

The gain of the closed-loop system is computed and compared in Table 1. The simulations are displayed in Figs. 4-12, and the outcomes are compared with the existing procedures. The state trajectories of uncertain system are seen in Figs. 4-8. The applied controller signals are depicted in Figs. 9-10. The outputs of uncertain system are seen in figs. 11-12.

The numerical and simulation outcomes verify that the proposed H_∞ control technique outperforms over similar control methods regarding the transient response, disturbance rejection, and the computed system's gain.

TABLE 1. Comparison of the system's gain.

Computed gain	Proposed method	Standard H_∞ control [8]	Shi's approach [48]
$\gamma_0 = \frac{\|z\|}{\|d\|}$	0.0784	0.1041	0.3018

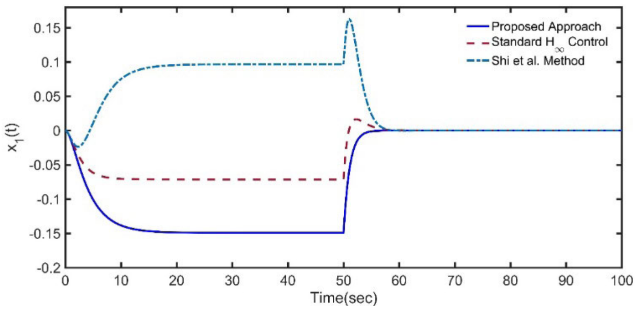

FIGURE 4. Time trajectory $x_1(t)$.

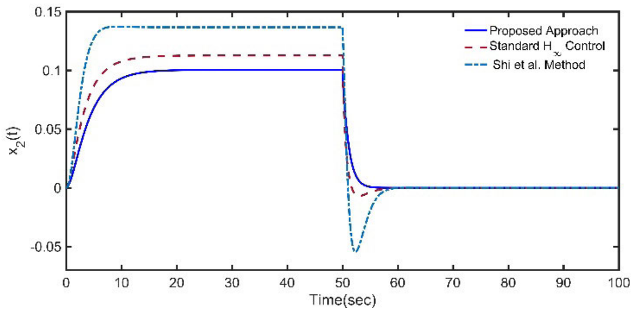

FIGURE 5. Time trajectory $x_2(t)$.

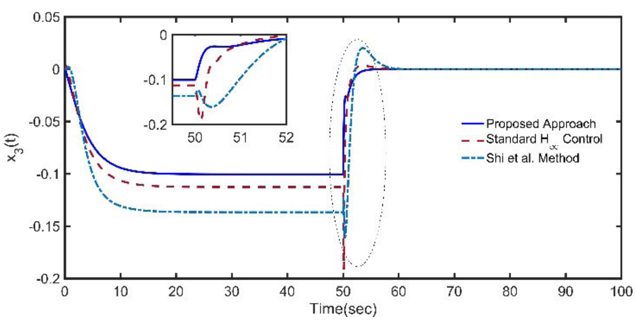

FIGURE 6. Time trajectory $x_3(t)$.

Example 2: Consider an uncertain translational robotic system described by the subsequent mass-spring-damper model [3]:

$$\begin{cases} M\ddot{\xi} + C\dot{\xi} + K\xi + \Gamma d = u \\ z = H_1\xi + H_2\dot{\xi} \end{cases} \tag{45}$$

where M, C, and K are some uncertain terms. The vectors $\xi(t)$ and $\dot{\xi}(t)$ are the positions and velocities of the robot in each robot axis. The vector $u(t)$ denotes the applied force

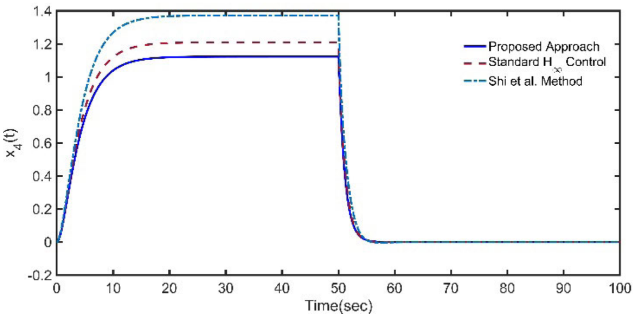

FIGURE 7. Time trajectory $x_4(t)$.

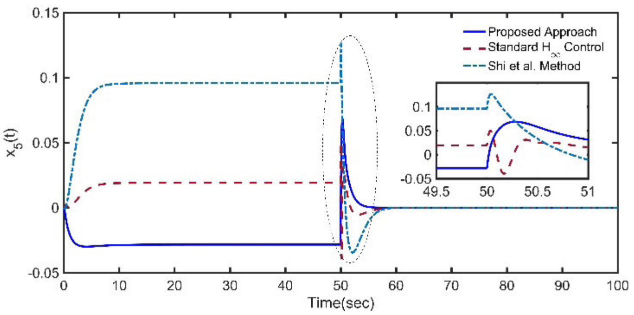

FIGURE 8. Time trajectory $x_5(t)$.

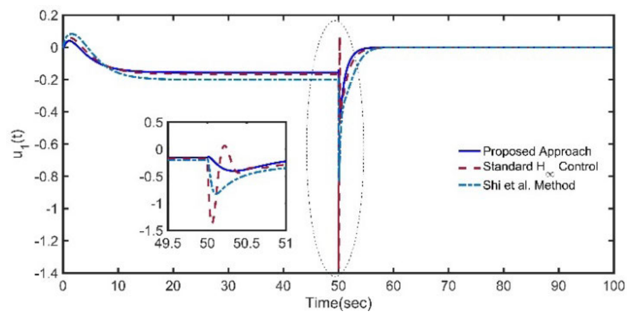

FIGURE 9. Time history of controller signal $u_1(t)$.

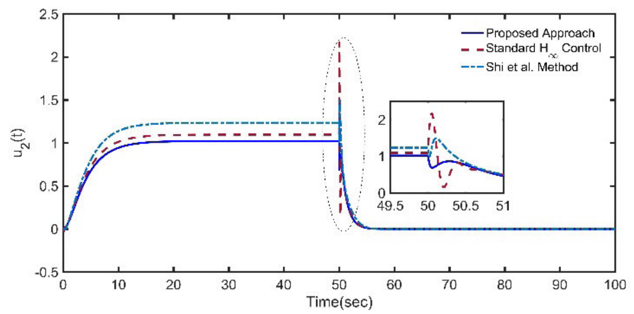

FIGURE 10. Time history of controller signal $u_2(t)$.

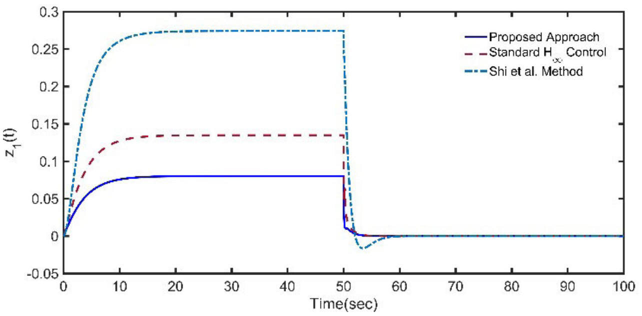

FIGURE 11. The system's response $z_1(t)$.

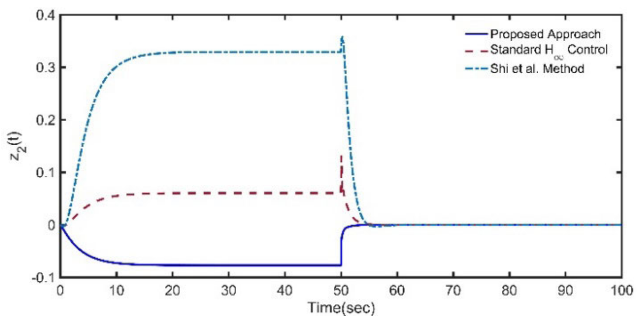

FIGURE 12. The system's response $z_2(t)$.

to each robot axis. The vector $z(t)$ is taken as the system's outputs. The external signal $d(t)$ is supposed to be norm bounded as $\|d(t)\| \leq 1$. The uncertain matrices is constructed as follows:

$$\begin{cases} M = M_0 + M_L \Delta_M M_R \\ C = C_0 + C_L \Delta_C C_R \\ K = K_0 + K_L \Delta_K K_R \end{cases} \tag{46}$$

where M_L, M_R, C_L, C_R, K_L, and K_R are some compatible matrices of weights. Furthermore, it is assumed that M_0 and M_R are non-singular matrices. Let define $x \overset{\text{def}}{=} \begin{bmatrix} x_1 \ x_2 \end{bmatrix}^T$, where $x_1 = \xi$ and $x_2 = \dot{\xi}$, then the uncertain system (45) is written as:

$$\dot{x} = \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}C \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u - \begin{bmatrix} 0 \\ M^{-1}\Gamma \end{bmatrix} d \tag{47}$$

The uncertain inverse term M^{-1} would had an adverse effect on the closed-loop performance. It can be represented as the LFT form as follows:

$$M^{-1} = \mathcal{F}_L(N, \Delta_M) \tag{48}$$

where

$$N = \begin{bmatrix} M_0^{-1} & -M_0^{-1}M_L \\ M_R M_0^{-1} & -M_R M_0^{-1} M_L \end{bmatrix}$$

Here, the uncertain matrix Δ, the vectors ω and ϕ take the following forms:

$$\Delta = \begin{bmatrix} \Delta_M & 0 & 0 \\ 0 & \Delta_C & 0 \\ 0 & 0 & \Delta_K \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_M \\ \omega_C \\ \omega_K \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_M \\ \phi_C \\ \phi_K \end{bmatrix},$$
$$\omega = \Delta\phi. \tag{49}$$

A simple block diagram of a mass-spring-damper system (45) is illustrated as Fig. 13.

Therefore, an LFT representation of the uncertain system (45) is found as the following:

$$\begin{bmatrix} \dot{x} \\ \phi \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} u + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} d \tag{50}$$

where

$$\mathcal{A}_{11} = \begin{bmatrix} 0 & I_n \\ -M_0^{-1}K_0 & -M_0^{-1}C_0 \end{bmatrix},$$

$$\mathcal{A}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ -M_0^{-1}M_L & -M_0^{-1}C_L & -M_0^{-1}K_L \end{bmatrix},$$

$$\mathcal{A}_{21} = \begin{bmatrix} -M_RM_0^{-1}K_0 & -M_RM_0^{-1}C_0 \\ 0 & C_R \\ K_R & 0 \end{bmatrix},$$

$$\mathcal{A}_{22} = \begin{bmatrix} -M_RM_0^{-1}M_L & -M_RM_0^{-1}C_L & -M_RM_0^{-1}K_L \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}_1 = \begin{bmatrix} 0 \\ M_0^{-1} \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} M_RM_0^{-1} \\ 0 \\ 0 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0 \\ -M_0^{-1}\Gamma \end{bmatrix},$$

$$E_2 = \begin{bmatrix} -M_RM_0^{-1}\Gamma \\ 0 \\ 0 \end{bmatrix}.$$

The well-posedness of the LFT terms would be checked when the following condition is satisfied:

$$\det(I_6 + \mathcal{A}_{22}\Delta) \neq 0 \tag{51}$$

Applying the well-known small-gain theorem [50], a sufficient well-posedness condition would be presented by

$$\overline{\sigma}(\Delta)\,\overline{\sigma}(\mathcal{A}_{22}) < 1 \tag{52}$$

where $\overline{\sigma}(\Delta) = \max\{\overline{\sigma}(\Delta_M), \overline{\sigma}(\Delta_C), \overline{\sigma}(\Delta_K)\}$. The condition (51) is reduced to the following form:

$$\det(I_2 + M_RM_0^{-1}M_L\Delta_M) \neq 0 \tag{53}$$

Similarly, a sufficient condition to the well-posedness of the LFT terms would be found by the small gain theorem as:

$$\overline{\sigma}(\Delta_M)\,\overline{\sigma}\left(M_RM_0^{-1}M_L\right) < 1 \tag{54}$$

However, a less-conservative well-posedness condition can be found by applying the μ concept as follows:

$$\overline{\sigma}(\Delta_M)\,\mu_{\Delta_M}\left(M_RM_0^{-1}M_L\right) < 1 \tag{55}$$

Therefore, the allowable variations of the uncertain matrix Δ is found as the following:

$$\overline{\sigma}(\Delta_M) < \frac{1}{\overline{\sigma}\left(M_RM_0^{-1}M_L\right)} \leq \frac{1}{\mu_{\Delta_M}\left(M_RM_0^{-1}M_L\right)} \tag{56}$$

It is assumed $\overline{\sigma}(\Delta_M) < \delta$ where $\overline{\sigma}\left(M_RM_0^{-1}M_L\right)$. $\delta < 1$. The parameters of the nominal system are taken as follows [3]:

$$M_0 = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \; K_0 = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$

TABLE 2. Comparison of the system's gain.

Computed gain	Proposed method	H_∞ control [8]	Shi's approach [48]
$\gamma_0 = \dfrac{\Vert z\Vert}{\Vert d\Vert}$	0.0084	0.0099	0.0453

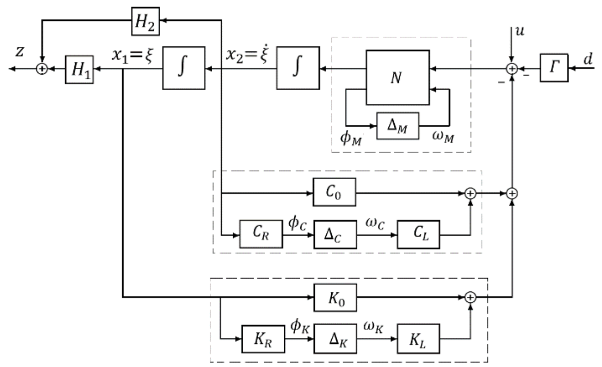

FIGURE 13. Block diagram of an uncertain mass-spring-damper system.

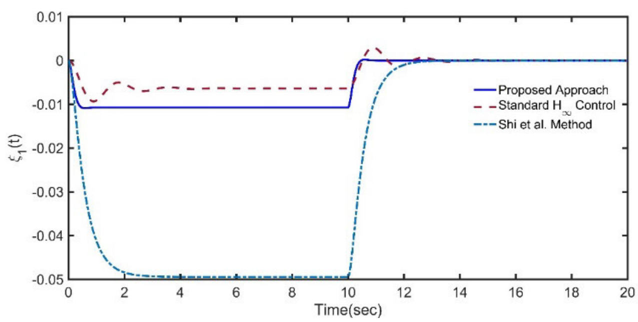

FIGURE 14. The position $\xi_1(t)$ in the robotic system.

$$\Gamma = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The weight matrices are selected as:

$$M_L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad K_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; K_R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The disturbance signals are applied with the following profiles:

$$d_1(t) = \begin{cases} 1, & 0 \leq t < 10 \\ 0, & 10 \leq t < 20, \end{cases}$$

$$d_2(t) = \begin{cases} -1, & 0 \leq t < 10 \\ 0, & 10 \leq t < 20. \end{cases}$$

The uncertain terms are considered as follows:

$$\Delta_M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Delta_C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; \Delta_K = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}.$$

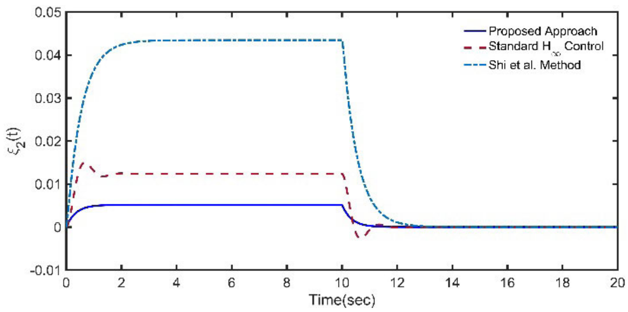

FIGURE 15. The position $\xi_2(t)$ in the robotic system.

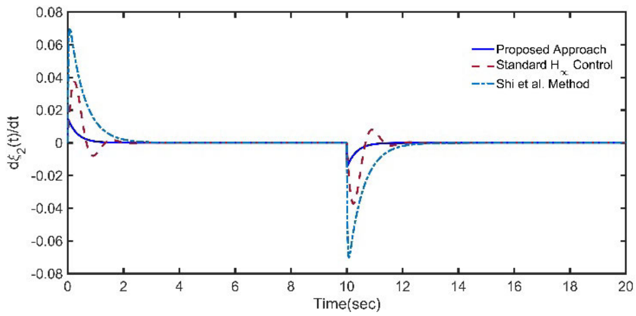

FIGURE 17. The velocity $\dot{\xi}_2(t)$ in the robotic system.

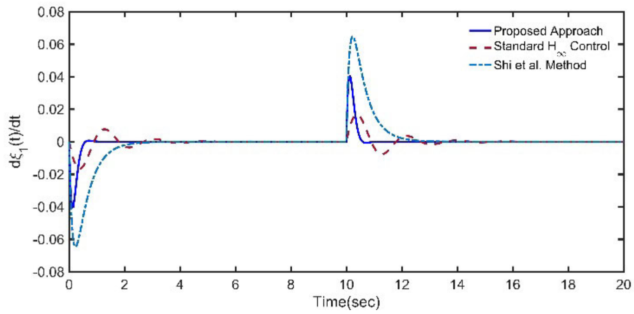

FIGURE 16. The velocity $\dot{\xi}_1(t)$ in the robotic system.

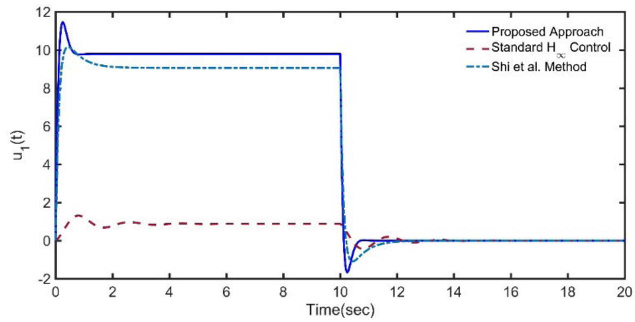

FIGURE 18. The applied input $u_1(t)$ in the robotic system.

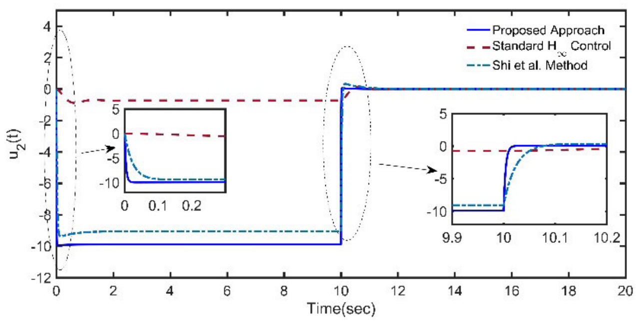

FIGURE 19. The applied input $u_2(t)$ in the robotic system.

Calculating $\|\mathcal{A}_{22}\| = \sqrt{0.24} = 0.4899$, $\mu_{\Delta_M}\left(M_R M_0^{-1} M_L\right) = 0.2$ and $\overline{\sigma}\left(M_R M_0^{-1} M_L\right) = 0.4$, the inequality (56) would be satisfied with the upper-bound $\delta = 1$ (i.e. $\overline{\sigma}(\Delta_M) \leq 1$). The initial conditions are set to zero. The integration time is 1 millisecond. The proposed control method is implemented with $\gamma = 1$ and $\varepsilon = 0.05$. The decision variables are calculated as:

$$Y = \begin{bmatrix} 6.3574 & 0 & -30.0780 & 0 \\ 0 & 3.3431 & 0 & -9.5279 \\ -30.0780 & 0 & 259.9260 & 0 \\ 0 & -9.5279 & 0 & 47.5456 \end{bmatrix},$$

$$G = 10^4 \begin{bmatrix} -0.0968 & 0 & -1.4329 & 0 \\ 0 & -0.0129 & 0 & -1.3073 \end{bmatrix}.$$

Then the controller gains are found as:

$$F = 10^3 \begin{bmatrix} -0.9127 & 0 & -0.1607 & 0 \\ 0 & -1.9172 & 0 & -0.6592 \end{bmatrix}.$$

The gain of the closed-loop system is computed and compared in Table 2.

The simulation results are plotted in Figs. 14-18. The robot positions are demonstrated in Figs. 14-15. The robot velocities are seen in figs. 16-17.

The applied control inputs are depicted in Figs. 18-19. The numerical and simulation results would verify that the suggested H_∞ control method outperforms similar methods regarding the disturbance rejection, transient response, and the computed system's gain.

VI. CONCLUSION AND FUTURE WORKS

A robust H_∞ fault tolerant controller (i.e., control law with disturbance rejection property) is obtained for a class of the nonlinear uncertain systems. For this purpose, the system's uncertainties are described via the LFT model. Hence, an LMI condition would be derived to guarantee the control objectives. Then, the robust H_∞ controller would be found by solving a convex LMI optimization problem. The suggested technique is successfully implemented in some uncertain control systems as well as a robotic manipulator. The simulation and quantitative outcomes validate the usefulness of the planned H_∞ control approaches compared to similar control methods in terms of the disturbance rejection, transient response, and the computed system's gain. As seen in Examples 1-2, the disturbance suppression would be the main control objective of this paper. Thus the other performances, as well as the extremum points of the system's output and actuator limitations, are not taken into account in the control problem. Some categories of the uncertain dynamic systems, as well as nonlinear and fuzzy, Markovian jump, time-delayed systems, can be described via the LFT tool. In future works, it is recommended that the control method is extended to such uncertain systems. The physical limitations, actuator nonlinearities, observer-based schemes, and multi-objective control policy may also be considered in the control problem. Consequently, effective and systematic control policy would be found to handle the system's uncertainties.

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