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Many-to-Many Disjoint Paths in Augmented Cubes With Exponential Fault Edges

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ABSTRACT It is common knowledge that edge disjoint paths have close relationship with the edge connectivity. Motivated by the well-known Menger theorem, we find that the maximum cardinality of edge disjoint paths connecting any two disjoint connected subgraphs with g vertices in G can also define by the minimum modified edge-cut, called the g-extra edge-connectivity of G ($\lambda_g(G)$). It is the cardinality of the minimum set of edges in G, if such a set exists, whose deletion disconnects G and leaves every remaining component with at least g vertices. The n-dimensional augmented cube AQ_n is a variant of hypercube Q_n. In this paper, we observe that the g-extra edge-connectivity of the augmented cube for some exponentially large enough g exists a concentration behavior, for about 72.22 percent values of $g \le 2^{n-1}$, and that the g-extra edge-connectivity of AQ_n ($n \ge 3$) concentrates on $\lceil \frac{n}{2} \rceil - 1$ special values. Specifically, we prove that the exact value of g-extra edge-connectivity of augmented cube is a constant $2(\lceil \frac{n}{2} \rceil - r)2^{\lfloor \frac{n}{2} \rfloor + r}$ for each integer $2^{\lfloor \frac{n}{2} \rfloor + r} - l_r \le g \le 2^{\lfloor \frac{n}{2} \rfloor + r}$, where $n \ge 3$, $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$ and $l_r = \frac{2^{2r+1}-2}{3}$ if n is odd and $l_r = \frac{2^{2r+2}-4}{3}$ if n is even. The above upper and lower bounds of g are sharp. Moreover, we also obtain the exponential edge disjoint paths in AQ_n with edge faults.

INDEX TERMS Fault tolerance, many-to-many edge disjoint paths, interconnected networks, exponential fault edges.

I. INTRODUCTION

The edge disjoint path problems are applicable in many areas such as software testing, database design and code optimization. The edge-disjoint paths problem is a fundamental problem in networks, consisting of connecting as many demand pairs as possible in a graph via edge-disjoint paths. In disjoint paths problems, instead of considering the paired and unpaired many-to-many disjoint paths cover problem [9], [13], [20], [30], we focus on the problem of evaluation maximum number of many-to-many edge disjoint paths of a graph. Let g be a positive integer. We aim to find the maximum number of edge disjoint paths linking any two disjoint connected subgraphs with just g vertices in G. Very recently, the problem of edge-disjoint paths with faulty edges was investigated for the augmented cube [19], hypercubes and folded hypercubes [21]. Considering the problem of finding node or edge disjoint paths is largely concerned with the well-known Menger's theorem [21]. Menger's theorem is a description of the edge connectivity in finite graphs according to the maximum number of edge-disjoint paths that can be found between any two distinct pairs of vertices.

With the fast development trend of big data, the scale of parallel and distributed systems is increasing dramatically. The basic topology structure of a parallel and distributed system is usually modeled as an undirected simple graph $G = (V, E)$ with processors and physical links between the processors represented as the vertices and the edges of the G, respectively. We keep to the graph definition and notation of [1].

Processors and physical links faults may occur in designing such a system. Thus, it is significant to come up with essential fault measurements. The vertex connectivity and edge connectivity of interconnection networks of these systems are two traditional measurements. For a connected graph G, the connectivity $\kappa(G)$ or edge connectivity $\lambda(G)$, introduced by Menger [15], is defined as the minimum number of vertices or edges whose deletion disconnects the graph G.

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TABLE 1. A brief summary of previously known and current results for $\lambda_g(AQ_n)$.

g	$\lambda_g(AQ_n)$	Authors
2	$= 4n-4 (n \geq 2)$	Ma, Liu and Xu (2008) [18]
2	$= 4n-4 (n \geq 7)$	Gu, Hao and Cheng (2020) [6]
3	$= 6n-9 (n \geq 4)$	Ma and Yu (2021) [19]
3	$= 6n-9 (n \geq 16)$	Gu, Hao and Cheng (2020) [6]
4	$= 8n-16 (n \geq 29)$	Gu, Hao and Cheng (2020) [6]
$g \leq 2^{\lceil \frac{n}{2} \rceil} (n \geq 2)$	$= (2n-1)g - ex_g(AQ_n)$	Zhang, Xu and Yang (2020) [29]
$\frac{2^{n-1}+2^{2-f}}{3} \leq g \leq 2^{n-1} (n \geq 4)$ [1]	$= 2^n$	Zhang, Xu and Yang (2020) [29]
$2^{\lfloor \frac{n}{2} \rfloor + r} - l_r \leq g \leq 2^{\lfloor \frac{n}{2} \rfloor + r} (n \geq 3)$ [2]	$= 2(\lceil \frac{n}{2} \rceil - r)2^{\lfloor \frac{n}{2} \rfloor + r}$	Current authors

[1] If n is even, then $f=0$; if n is odd, then $f=1$.
[2] $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$, $l_r = \frac{2^{2r+1}-2}{3}$, n is odd; $l_r = \frac{2^{2r+2}-4}{3}$, n is even.

To make a comprehensive evaluation on the faulty interconnection network, in 1996, the concept of g-extra connectivity $\kappa_g(G)$ or g-extra edge-connectivity $\lambda_g(G)$ was first introduced by Fàbrega and Foil [4]. For a connected graph G, an edge subset $F \subseteq E(G)$ is called as a g-extra edge-cut of G, if $G-F$ is disconnected and every component of $G-F$ has at least g vertices. The g-extra edge-connectivity of G, written as $\lambda_g(G)$, is defined as the minimum cardinality among all the g-extra edge-cuts. In recent years, more works about determining the g-extra connectivity or g-extra edge-connectivity of famous networks are found in [5], [11], [12], [14], [26]–[28].

By the well-known Menger theorem, the maximum number of edge disjoint paths connecting any two disjoint connected subgraphs with g vertices in G can also define by the minimum modified edge-cut, called ($\lambda_g(G)$). The Menger's theorem is often required to find a maximum of edge-disjoint paths between two given distinct vertices of G. Motivated by this, we want to go even further, and consider cases on many-to-many edge disjoint paths of a connected graph G.

The augmented cube AQ_n, proposed by Choudum and Sunitha [3], is a variant of hypercube Q_n. Compared with Q_n, AQ_n retains some of the splendid properties. Some basic properties of the augmented cube, such as panconnectivity [17], [22], [24], pancyclicity [23], hamiltonicity [7], [16], [25], various diagnosability [2], [10], structure fault-tolerance [8], several familiar connectivities [18], [29]. The known and current results on the g-extra edge-connectivity of n-dimensional augmented cube AQ_n, $\lambda_g(AQ_n)$ have been showed in Table 1. In this paper, we pay our attention on $\lambda_g(AQ_n)$ for exponential $g = 2^c$, $c = \lfloor \frac{n}{2} \rfloor + r$, where $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$, $n \geq 3$ which is better to reflect the fault tolerance ability of the interconnection network than the traditional connectivity. Our result improves some previously known results on $\lambda_g(AQ_n)$ in a sense.

In this study, we study the g-extra edge-connectivity of the augmented cube AQ_n. Specifically, we obtain the edge disjoint paths with edge faults. Interestingly, we notice that for $2^{\lfloor \frac{n}{2} \rfloor + r} - l_r \leq g \leq 2^{\lfloor \frac{n}{2} \rfloor + r}$, where $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$ and $l_r = \frac{2^{2r+1}-2}{3}$ if n is odd and $l_r = \frac{2^{2r+2}-4}{3}$ if n is even, the g-extra edge-connectivity of the augmented cube AQ_n ($n \geq 3$) exists a concentration behavior: as n tends to infinity, for about 72.22% values of $g \leq 2^{n-1}$, the g-extra edge-connectivity of AQ_n ($n \geq 3$) concentrates on several special values $(2n-2c)2^c$, $c = \lfloor \frac{n}{2} \rfloor + r$.

II. PRELIMINARIES

In a connected graph G, the degree of a vertex v, denoted by $deg(v)$, is the number of neighborhoods of v in G. A graph G is k-regular if $deg(v) = k$ for all $v \in V(G)$. A graph S is called a subgraph of G if every vertex and edge in S is also in G. A maximal connected subgraph of G is called a component of G. For any edge set $F \subseteq E(G)$, the notation $G-F$ denotes the subgraph obtained after removing the edges in F from G. In particular, F is called an edge cut of G if $G-F$ is disconnected. Given a vertex set $X \subseteq V(G)$, we denote $G[X]$ the subgraph of G induced by X, and $\overline{X} = V(G) \setminus X$ the complement of X. For two vertex sets X and $\overline{X}$, we denote $[X, \overline{X}]$ the set of edges of G with one end in X and the other end in $\overline{X}$. If G_1 and G_2 have the same nodes and edges, and $\lambda_i(G_1) = \lambda_i(G_2)$, $i = 1, 2, \ldots, g-1$, $\lambda_g(G_1) > \lambda_g(G_2)$, then G_1 is more reliable than G_2. Let $\xi_m(G)$ =min$\{|[X, \overline{X}]| : |X| = m \leq \lfloor |V(G)|/2 \rfloor$, and $G[X]$ is connected$\}$. A connected graph is regarded as λ_g-optimal if $\lambda_g(G) = \xi_g(G)$ holds.

The definition of the n-dimensional augmented cube is given in the following.

Definition 1 [3]: Given $n \geq 1$ is an integer. The n-dimensional augmented cube AQ_n has 2^n vertices, each labeled by an n-bit binary string $a_1a_2\ldots a_n$. AQ_1 is a complete graph K_2 with the vertex set $\{0, 1\}$. For $n \geq 2$, AQ_n is obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and adding $2 \times 2^{n-1}$ edges between the two as follows:

Let $V(AQ^0_{n-1}) = \{0a_2a_3\ldots a_n : a_i = 0 or 1\}$ and $V(AQ^1_{n-1}) = \{1b_2b_3\ldots b_n : b_i = 0 or 1\}$. Two vertices $A = 0a_2a_3\ldots a_n$ of AQ^0_{n-1} and $B = 1b_2b_3\ldots b_n$ of AQ^1_{n-1} are adjacent if only and if either

1) $a_i = b_i$ for every i, $2 \leq i \leq n$; or
2) $a_i = \overline{b_i}$ for every i, $2 \leq i \leq n$.

Case (1) edges are called hypercube edges while case (2) edges are called complement edges. Clearly, we can see that every vertex of AQ^0_{n-1} has two neighbors in AQ^1_{n-1}. In fact, AQ_n can be viewed as $AQ^0_{n-1} \bigoplus AQ^1_{n-1}$ briefly. It has been

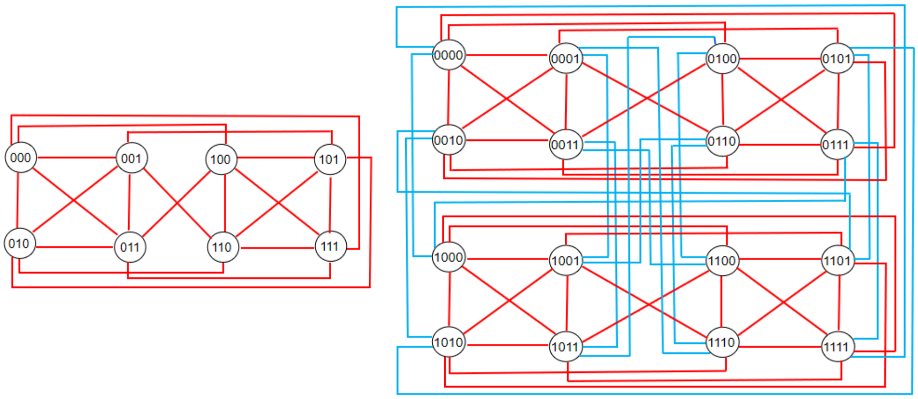

FIGURE 1. Two augmented cubes AQ_3 and AQ_4.

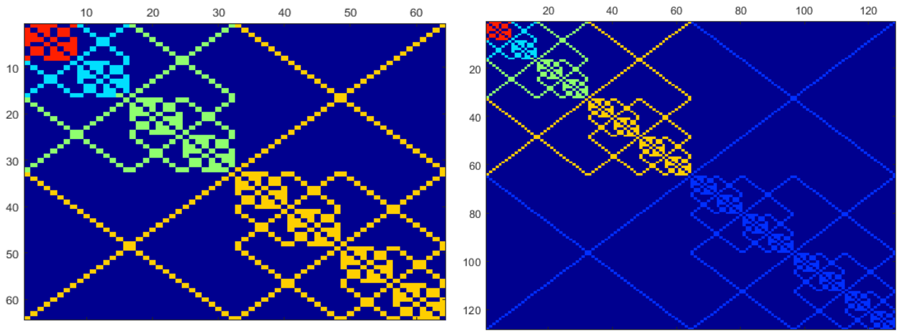

FIGURE 2. The images of adjacency matrices of AQ_6 and AQ_7.

shown that AQ_n is $(2n-1)-$regular. Hence, $|E(AQ_n)| = (2n-1)2^{n-1}$. Choudum and Sunitha [3] have defined and studied the augmented cube and its properties. For example, augmented cubes AQ_3 and AQ_4 are shown in Fig. 1. As the integer n exponentially grows, the scale of the AQ_n is more and more big, the topological structure of AQ_n is more and more complicated. Therefore, we use the adjacency matrix of AQ_n to represent the adjacent relationship between vertices of AQ_n. The images of adjacency matrix of AQ_6 and AQ_7 are exhibited in Fig. 2 (in two images, the dark blue pixel at position (i, j) corresponds to no edges between nodes A and B).

The $ex_m(G)$ is twice of the maximum number edges of the subgraph induced by m vertices in G, also is the maximum possible sum of the degrees of the vertices in the subgraph induced by m vertices in G. The exact values for the $ex_m(AQ_n)$ have been given [29], which plays a prominent role in studying the $\lambda_g(AQ_n)$.

Zhang *et al.* [29] have obtained the following major results:

Lemma 1 [29]: For a positive integer m, $1 \leq m \leq 2^n$, it can be written that $m = \sum_{i=0}^{s} 2^{t_i}$, where $t_0 > t_1 > \ldots > t_s \geq 0, 0 \leq i \leq s$. Then

$$\begin{aligned} &ex_m(AQ_n) \\ &\quad = 2|E(AQ_n[L_m^n])| \\ &\quad = \sum\nolimits_{i=0}^{s}(2t_i - 1)2^{t_i} + \sum\nolimits_{i=0}^{s} 4i2^{t_i} + \delta, \end{aligned} \tag{1}$$

where $t_0 = \lfloor \log_2 m \rfloor$, $t_i = \lfloor \log_2(m - \sum_{k=0}^{i-1} 2^{t_k}) \rfloor$ for $i \geq 1$, and $\delta = 0$ when m is even, $\delta = 1$ when m is odd.

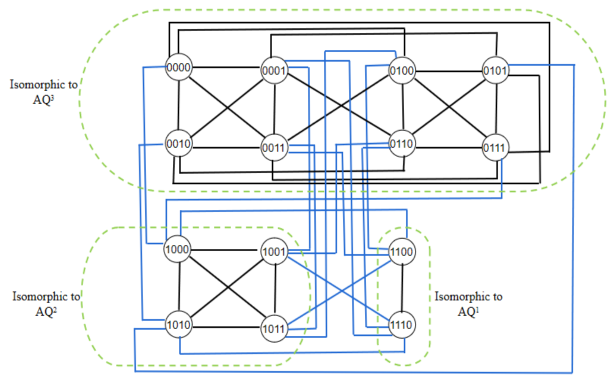

FIGURE 3. The induces subgraph by L_{14} in AQ_4.

If $x = x_1x_2\ldots x_n$ is a vertex of AQ_n, we can also be represented by decimal number $\sum_{i=1}^{n} x_i 2^{n-i}$, $x_i \in \{0, 1\}$. Let S_m be the set $\{0, 1, 2, \ldots, m-1\}$ (under decimal representation) and L_m^n the corresponding set represented by n-binary strings. Let $AQ_n[L_m^n]$ be the subgraph induced by L_m^n in AQ_n, $L_m^n \subset V(AQ_n)$. $AQ_n[L_m^n]$ and $AQ_n[\overline{L_m^n}]$ are connected in AQ_n. For example, for $n = 4$, if $S_{14} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, then $L_{14}^4 = \{0000,$

0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1110}. By equation (1), we will obtain $ex_{14}(AQ_4) = 2|E(AQ_4[L_{14}^4])| = (2\times3-1)2^3+(2\times2-1)2^2+(2\times1-1)2^1+4\times0\times2^3+4\times1\times2^2+4\times2\times2^1+0 = 86$. The subgraph of AQ_4 induced by L_{14}^4 is shown in Fig. 3. The edges between subcubes isomorphic to AQ^1, AQ^2 and AQ^3 are marked in blue.

In [29], they have introduced a method to construct $s+1$ t_i-dimensional disjoint augmented subcubes in AQ_n for $0 \le i \le s$ as follows:

$$AQ^0 : 0\ldots0\underbrace{Z_{t_0}\ldots Z_1}_{t_0}$$

$$AQ^1 : 0\ldots01\underbrace{0\ldots0\underbrace{Z_{t_1}\ldots Z_1}_{t_1}}_{t_0}$$

$$AQ^2 : 0\ldots010\ldots01\underbrace{0\ldots01\underbrace{0\ldots0\underbrace{Z_{t_2}\ldots Z_1}_{t_2}}_{t_1}}_{t_0}$$

$$\ldots$$

AQ^0 is given and note that AQ^i is taken from a t_{i-1}-dimensional augmented subcube which is obtained from AQ^{i-1} by changing the 0 of $(t_{i-1}+1)th$-coordinate of AQ^{i-1} to 1 for $i = 1, \ldots, s$. Hence, $V(AQ^i) \cap V(AQ^j) \doteq \emptyset$ for $i \ne j, i, j \in \{0, \ldots, s\}$ and $|V(AQ^0) \cup \ldots \cup V(AQ^s)| = \sum_{i=0}^{s} 2^{t_i} = m$. Define $G_1 = AQ_n[V(AQ^0) \cup \ldots \cup V(AQ^s)]$. It can be calculated that the number of edges of G_1 by considering the edges within AQ^i's $(\sum_{i=0}^{s}(2t_i-1)2^{t_i-1})$ and the edges between AQ^i's $(\sum_{i=0}^{s} 2i \cdot 2^{t_i})$ when $t_s > 0$; similarly, $|E(G_1)| = \sum_{i=0}^{s-1}(2t_i-1)2^{t_i-1} + \sum_{i=0}^{s} 2i \cdot 2^{t_i}$ when $t_s = 0$. Hence, $ex_m(G_1) = 2|E(G_1)| = \sum_{i=0}^{s}(2t_i-1)2^{t_i} + \sum_{i=0}^{s} 4i2^{t_i} + \delta$, where $\delta = 0$ when m is even, and $\delta = 1$ when m is odd. In fact, $G_1 = AQ_n[L_m^n]$.

Lemma 2 [29]: Note that each $V(AQ^i)$ is connected, for $0 \le i \le s$. Then $AQ_n[L_m^n]$ is connected.

Let $F = [L_m^n, \overline{L_m^n}]$ the set of edges of AQ_n with exactly one end vertex in L_m^n such that $AQ_n - F$ is disconnected and it has two components.

Lemma 3 [29]: The subgraph $AQ_n[\overline{L_m^n}]$ of AQ_n is connected.

III. SOME PROPERTIES OF THE FUNCTION $\xi_m(AQ_n)$

The $\lambda_g(AQ_n)$ is closely related to the monotonic intervals and fractal structure of the function $\xi_m(AQ_n)$. We obtain the image of the relationship between functions $\xi_g(AQ_{10})$ and $\lambda_g(AQ_{10})$, for $1 \le g \le 2^9$ in Fig. 4, image-magnification on internal [108, 128] is illustrated below it.

For positive integers $g \le m = \sum_{i=0}^{s} 2^{t_i} \le 2^{n-1}$,

$$\lambda_g(AQ_n) = min\{\xi_m(AQ_n) : g \le m \le 2^{n-1}\}. \tag{2}$$

In view of Handshaking lemma and regularity of an n-dimensional augmented cube, combining with connectedness of $AQ_n[L_m^n]$, it follows that

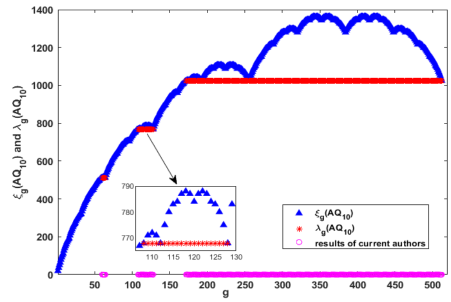

FIGURE 4. The plots of functions $\xi_g(AQ_{10})$ and $\lambda_g(AQ_{10})$.

$$\begin{aligned}\xi_m(AQ_n) &= (2n-1)m - ex_m(AQ_n) \\ &= (2n-1)m - [\sum_{i=0}^{s}(2t_i-1)2^{t_i} \\ &\quad + \sum_{i=0}^{s} 4i2^{t_i} + \delta],\end{aligned} \tag{3}$$

where $\delta = 0$ if m is even, and $\delta = 1$ if m is odd.

For the augmented cube, the g-extra edge-connectivity of AQ_n is well-defined for each $g \le 2^{n-1}$. In [29], they have applied the value of $ex_m(AQ_n)$ to obtain the g-extra edge-connectivity of AQ_n for $1 \le g \le 2^{\lfloor \frac{n}{2} \rfloor}(n \ge 2)$ and $\frac{2^{n-1}+2^{2-f}}{3} \le g \le 2^{n-1}(n \ge 4)$ where $f = 0$ if n is even, and $f = 1$ if n is odd. In this paper, we further consider the g-extra edge-connectivities of AQ_n for other values of g on the basis of their minds. The following lemmas of some properties of the function $\xi_m(AQ_n)$ are useful, which are need for giving the conclusions of $\lambda_g(AQ_n)$.

Lemma 4: For every integer $h = \sum_{i=0}^{s} 2^{a_i}, a_0 > a_1 > \ldots > a_t > a_{t+1} > \ldots > a_s \ge 0, t < s, h \le 2^{n-1}$, let $h = h_1 + h_2, h_1 = \sum_{i=0}^{t} 2^{a_i}$, we can get $ex_h(AQ_n) = ex_{h_1}(AQ_n) + ex_{h_2}(AQ_n) + 4(t+1)h_2$.

Proof: As a matter of convenience, we write $h_2 = h - h_1 = 2^{a_{t+1}} + 2^{a_{t+2}} + \ldots + 2^{a_s} = \sum_{i=0}^{s-t-1} 2^{a_{t+1+i}}$, it is easy to observe that the h_1 must be even and the h_2 is either odd or even. According to equation (1), we have

$$\begin{aligned} ex_{h_1}(AQ_n) &= \sum_{i=0}^{t}(2a_i-1)2^{a_i} + \sum_{i=0}^{t} 4i2^{a_i}, \\ ex_{h_2}(AQ_n) &= \sum_{i=0}^{s-t-1}(2a_{t+1+i}-1)2^{a_{t+1+i}} \\ &\quad + \sum_{i=0}^{s-t-1} 4i2^{a_{t+1+i}} \\ &\quad + \delta, \\ ex_{h_1+h_2}(AQ_n) &= \sum_{i=0}^{s}(2a_i-1)2^{a_i} + \sum_{i=0}^{s} 4i2^{a_i} + \delta \\ &= \sum_{i=0}^{t}(2a_i-1)2^{a_i} \\ &\quad + \sum_{i=0}^{s-t-1}(2a_{t+1+i}-1)2^{a_{t+1+i}} \\ &\quad + \sum_{i=0}^{t} 4i2^{a_i} \\ &\quad + \sum_{i=0}^{s-t-1} 4(t+1+i)2^{a_{t+1+i}} + \delta \\ &= ex_{h_1}(AQ_n) + ex_{h_2}(AQ_n) \\ &\quad + \sum_{i=0}^{s-t-1} 4(t+1)2^{a_{t+1+i}} \\ &= ex_{h_1}(AQ_n) + ex_{h_2}(AQ_n) + 4(t+1)h_2. \end{aligned}$$

So the lemma holds. □

For $0 \leq m < 2^n$, by the symmetry of cut, the equation $\xi_m(AQ_n) = \xi_{2^n-m}(AQ_n)$ holds. For any $n > n'$, $0 \leq m < 2^{n'}$, because the value of $ex_m(AQ_n)$ is uniquely determined by the decomposition of m, so $ex_m(AQ_n) = ex_m(AQ_{n'})$.

Lemma 5: Let $2^c < m \leq 2^{n-1}$ for $0 \leq c \leq n-2$. Then

$$\xi_m(AQ_n) \geq \xi_{2^c}(AQ_n).$$

Proof: It is sufficient to show that for $2^k < m \leq 2^{k+1}$, $k = c, c+1, \ldots, n-2$, $\xi_m(AQ_n) \geq \xi_{2^k}(AQ_n)$. Since

$$\begin{aligned}
&\xi_{2^{k+1}}(AQ_n) - \xi_{2^k}(AQ_n) \\
&\quad = (2n-1)2^{k+1} - ex_{2^{k+1}}(AQ_n) \\
&\qquad -[(2n-1)2^k - ex_{2^k}(AQ_n)] \\
&\quad = (2n-1)2^{k+1} - [2(k+1)-1]2^{k+1} - \delta \\
&\qquad -[(2n-1)2^k - (2k-1)2^k - \delta] \\
&\quad = (2n-2k-2)2^{k+1} - (2n-2k)2^k \\
&\quad = (n-k-2)2^{k+1} \\
&\quad \geq 0.
\end{aligned}$$

Equality holds if and only if $k = n-2$. Therefore, we may assume $2^k < m < 2^{k+1}$. So $0 < m - 2^k < 2^k$. Let $m = \sum_{i=0}^{s} 2^{t_i}$ and $m' = m - 2^k$. Clearly, $t_0 = k$ and $t_1 < k \leq n-2$. Then $m' = \sum_{i=1}^{s} 2^{t_i} = \sum_{i=0}^{s-1} 2^{t_{i+1}} < 2^k \leq 2^{n-2}$ and

$$\begin{aligned}
&\xi_m(AQ_n) - \xi_{2^k}(AQ_n) \\
&\quad = (2n-1)m - ex_m(AQ_n) - [(2n-1)2^k - ex_{2^k}(AQ_n)] \\
&\quad = (2n-1)\sum_{i=0}^{s} 2^{t_i} - \sum_{i=0}^{s}(2t_i-1)2^{t_i} \\
&\qquad - \sum_{i=0}^{s} 4i2^{t_i} - \delta - [(2n-1)2^k - (2k-1)2^k - \delta] \\
&\quad = (2n-1)\sum_{i=0}^{s-1} 2^{t_{i+1}} - \sum_{i=0}^{s-1}(2t_{i+1}-1)2^{t_{i+1}} \\
&\qquad - \sum_{i=0}^{s-1} 4(i+1)2^{t_{i+1}} \\
&\quad = (2n-5)m' - \sum_{i=0}^{s-1}(2t_{i+1}-1)2^{t_{i+1}} - \sum_{i=0}^{s-1} 4i2^{t_{i+1}} \\
&\quad = [2(n-2)-1]m' - \sum_{i=0}^{s-1}(2t_{i+1}-1)2^{t_{i+1}} \\
&\qquad - \sum_{i=0}^{s-1} 4(i+1)2^{t_{i+1}}.
\end{aligned}$$

For $0 < m' < 2^k$, we can get $ex_{m'}(AQ_n) = ex_{m'}(AQ_k) < (2k-1)m'$ and $\xi_m(AQ_n) - \xi_{2^k}(AQ_n) = [2(n-2)-1]m' - \sum_{i=0}^{s-1}(2t_{i+1}-1)2^{t_{i+1}} - \sum_{i=0}^{s-1} 4(i+1)2^{t_{i+1}} > 2(n-k-2)m' > 0$. □

Let $f = 0$ if n is odd, and $f = 1$ if n is even. For $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$, $n \geq 3$ and $0 \leq j \leq r$, we define $s_{r,j}$ as follows:

$$s_{r,j} = \begin{cases} 2^{\lfloor \frac{n}{2} \rfloor + r} & \textit{if } j = 0; \\ 2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f} & \textit{if } 1 \leq j \leq r. \end{cases}$$

One can check that $s_{r,j} - s_{r,j+1} = 2^{2r-2j-1+f}$, for any $0 \leq j \leq r-1$, and $s_{r,r} = 2^{\lfloor \frac{n}{2} \rfloor + r} - \frac{2^{2r+1+f} - 2^{1+f}}{3}$. In fact, if n is odd, then $\frac{2^{2r+1+f} - 2^{1+f}}{3} = \frac{2^{2r+1} - 2}{3}$; if n is even, then $\frac{2^{2r+1+f} - 2^{1+f}}{3} = \frac{2^{2r+2} - 4}{3}$. For the sake of simplicity, let $l_r = \frac{2^{2r+1} - 2}{3}$ if n is odd and $l_r = \frac{2^{2r+2} - 4}{3}$ if n is even. After simple calculation for $s_{r,j}$, we can get: $2^{\lfloor \frac{n}{2} \rfloor + r} - l_r = s_{r,r} < s_{r,r-1} < \ldots < s_{r,1} < s_{r,0} = 2^{\lfloor \frac{n}{2} \rfloor + r}$.

We give an example to illustrate the variability of r, j, and $s_{r,j}$ for $n = 9, 10$ (see the Table 2). If $n = 2$, we can get $r = 0$ and $j = 0$, so we only consider the situation of $n \geq 3$.

TABLE 2. The variability of r, j, and $s_{r,j}$ for $n = 9, 10$.

		$n = 9$	$n = 10$
r	j	$s_{r,j}$	$s_{r,j}$
4	0	$256 = 2^8$	$512 = 2^9$
4	1	$128 = 2^7$	$256 = 2^8$
4	2	$96 = 2^6 + 2^5$	$192 = 2^7 + 2^6$
4	3	$88 = 2^6 + 2^4 + 2^3$	$176 = 2^7 + 2^5 + 2^4$
4	4	$86 = 2^6 + 2^4 + 2^2 + 2^1$	$172 = 2^7 + 2^5 + 2^3 + 2^2$
3	0	$128 = 2^7$	$256 = 2^8$
3	1	$96 = 2^6 + 2^5$	$192 = 2^7 + 2^6$
3	2	$88 = 2^6 + 2^4 + 2^3$	$176 = 2^7 + 2^5 + 2^4$
3	3	$86 = 2^6 + 2^4 + 2^2 + 2^1$	$172 = 2^7 + 2^5 + 2^3 + 2^2$
2	0	$64 = 2^6$	$128 = 2^7$
2	1	$56 = 2^5 + 2^4 + 2^3$	$112 = 2^6 + 2^5 + 2^4$
2	2	$54 = 2^5 + 2^4 + 2^2 + 2^1$	$108 = 2^6 + 2^5 + 2^3 + 2^2$
1	0	$32 = 2^5$	$64 = 2^6$
1	1	$30 = 2^4 + 2^3 + 2^2 + 2^1$	$60 = 2^5 + 2^4 + 2^3 + 2^2$

Lemma 6: Given n, r and j are three integers, $r = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$ and $0 \leq j \leq r$, where $f = 0$ if n is odd, and $f = 1$ if n is even. Then

$$\xi_{s_{r,j}}(AQ_n) = 2(\lceil \frac{n}{2} \rceil - r)2^{\lfloor \frac{n}{2} \rfloor + r}. \tag{4}$$

Proof: It is easy to check on that $\xi_{s_{r,0}}(AQ_n) = \xi_{2^{\lfloor \frac{n}{2} \rfloor + r}}(AQ_n) = (2n-1)2^{\lfloor \frac{n}{2} \rfloor + r} - [2(\lfloor \frac{n}{2} \rfloor + r) - 1]2^{\lfloor \frac{n}{2} \rfloor + r} = 2(\lceil \frac{n}{2} \rceil - r)2^{\lfloor \frac{n}{2} \rfloor + r}$.

To deal with f, we note that $f = \lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{2} \rceil + 1$. When $1 \leq j \leq r$, $s_{r,j} = 2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}$, we obtain

$$\begin{aligned}
&\xi_{s_{r,j}}(AQ_n) \\
&\quad = (2n-1)s_{r,j} - ex_{s_{r,j}}(AQ_n) \\
&\quad = (2n-1)(2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}) \\
&\qquad - ex_{2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}}(AQ_n) \\
&\quad = (2n-1)(2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}) \\
&\qquad - ex_{2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}}(AQ_{2^{\lfloor \frac{n}{2} \rfloor + r}}) \\
&\quad = (2\lceil \frac{n}{2} \rceil - 2r)(2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}) \\
&\qquad + [2(\lfloor \frac{n}{2} \rfloor + r) - 1](2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}) \\
&\qquad - ex_{2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}}(AQ_{2^{\lfloor \frac{n}{2} \rfloor + r}}) \\
&\quad = (2\lceil \frac{n}{2} \rceil - 2r)(2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f}) \\
&\qquad + [2(\lfloor \frac{n}{2} \rfloor + r) - 1]\sum_{i=0}^{j-1} 2^{2r-2i-1+f} \\
&\qquad - ex_{\sum_{i=0}^{j-1} 2^{2r-2i-1+f}}(AQ_{2^{\lfloor \frac{n}{2} \rfloor + r}}) \\
&\quad = (2\lceil \frac{n}{2} \rceil - 2r)(2^{\lfloor \frac{n}{2} \rfloor + r} - \sum_{i=0}^{j-1} 2^{2r-2i-1+f})
\end{aligned}$$

$$+[2(\lfloor\frac{n}{2}\rfloor+r)-1]\sum_{i=0}^{j-1}2^{2r-2i-1+f}$$
$$-\sum_{i=0}^{j-1}[2(2r-2i-1+f)-1]2^{2r-2i-1+f}$$
$$-\sum_{i=0}^{j-1}4i2^{2r-2i-1+f}$$
$$=2(\lceil\frac{n}{2}\rceil-r)2^{\lfloor\frac{n}{2}\rfloor+r}+[2(\lfloor\frac{n}{2}\rfloor-\lceil\frac{n}{2}\rceil)+2-2f]$$
$$\times\sum_{i=0}^{j-1}2^{2r-2i-1+f}$$
$$=2(\lceil\frac{n}{2}\rceil-r)2^{\lfloor\frac{n}{2}\rfloor+r}.$$

Hence, $\xi_{s_{r,j}}(AQ_n)=2(\lceil\frac{n}{2}\rceil-r)2^{\lfloor\frac{n}{2}\rfloor+r}$. The proof is completed. □

Lemma 7: For any $2^{\lfloor\frac{n}{2}\rfloor+r}-l_r\le m\le 2^{\lfloor\frac{n}{2}\rfloor+r}$, where $r=1,2,\ldots,\lceil\frac{n}{2}\rceil-1$ and $l_r=\frac{2^{2r+1}-2}{3}$ if n is odd and $l_r=\frac{2^{2r+2}-4}{3}$ if n is even. Then

$$\xi_m(AQ_n)\ge\xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n). \quad (5)$$

Proof: For $2^{\lfloor\frac{n}{2}\rfloor+r}-l_r\le m\le 2^{\lfloor\frac{n}{2}\rfloor+r}$, there exists a positive integer m', where $0\le m'\le 2^{2r-2j-1+f}$ and $m=s_{r,j}-m'$, from equation (3), we can obtain that

$$\xi_m(AQ_n)$$
$$=(2n-1)m-ex_m(AQ_n)$$
$$=(2\lceil\frac{n}{2}\rceil-2r)m+[2(\lfloor\frac{n}{2}\rfloor+r)-1]m$$
$$-ex_m(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$=(2\lceil\frac{n}{2}\rceil-2r)m+[2(\lfloor\frac{n}{2}\rfloor+r)-1](2^{\lfloor\frac{n}{2}\rfloor+r}-m)$$
$$-ex_{2^{\lfloor\frac{n}{2}\rfloor+r}-m}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$=2(\lceil\frac{n}{2}\rceil-r)2^{\lfloor\frac{n}{2}\rfloor+r}-(2\lceil\frac{n}{2}\rceil-2r)(2^{\lfloor\frac{n}{2}\rfloor+r}-m)$$
$$+[2(\lfloor\frac{n}{2}\rfloor+r)-1](2^{\lfloor\frac{n}{2}\rfloor+r}-m)$$
$$-ex_{2^{\lfloor\frac{n}{2}\rfloor+r}-m}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$=\xi_{s_{r,j}}(AQ_n)+(4r-3+2f)(2^{\lfloor\frac{n}{2}\rfloor+r}-m)$$
$$-ex_{2^{\lfloor\frac{n}{2}\rfloor+r}-m}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}}).$$

In view of Lemma 4,

$$ex_{2^{\lfloor\frac{n}{2}\rfloor+r}-m}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$=ex_{2^{\lfloor\frac{n}{2}\rfloor+r}-s_{r,j}}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})+ex_{m'}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})+4jm'$$
$$=\sum_{i=0}^{j-1}[2(2r-2i-1+f)-1]2^{2r-2i-1+f}$$
$$+\sum_{i=0}^{j-1}4i2^{2r-2i-1+f}+ex_{m'}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})+4jm'$$
$$=(4r-3+2f)\sum_{i=0}^{j-1}2^{2r-2i-1+f}+ex_{m'}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$+4jm'$$
$$=(4r-3+2f)(2^{\lfloor\frac{n}{2}\rfloor+r}-s_{r,j})+ex_{m'}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$+4jm'.$$

Therefore,

$$\xi_m(AQ_n)$$
$$=(2n-1)m-ex_m(AQ_n)$$
$$=\xi_{s_{r,j}}(AQ_n)+(4r-3+2f)(s_{r,j}-m)$$
$$-ex_{m'}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})-4jm'$$
$$=\xi_{s_{r,j}}(AQ_n)+(4r-3+2f)m'-ex_{m'}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$-4jm'$$
$$=\xi_{s_{r,j}}(AQ_n)+(4r-3+2f-4j)m'$$
$$-ex_{m'}(AQ_{2^{\lfloor\frac{n}{2}\rfloor+r}})$$
$$=\xi_{s_{r,j}}(AQ_n)+[2(2r+\lfloor\frac{n}{2}\rfloor-\lceil\frac{n}{2}\rceil-2j)-1]m'$$
$$-ex_{m'}(AQ_{2r+\lfloor\frac{n}{2}\rfloor-\lceil\frac{n}{2}\rceil-2j})$$
$$=\xi_{s_{r,j}}(AQ_n)+\xi_{m'}(AQ_{2r+\lfloor\frac{n}{2}\rfloor-\lceil\frac{n}{2}\rceil-2j})$$
$$\ge\xi_{s_{r,j}}(AQ_n),$$

where the equality holds if and only if $m'=0$ or $m'=2^{2r-2j-1+f}$. Hence, the lemma follows. □

Remark 1: The upper and lower bounds of m are sharp for $2^{\lfloor\frac{n}{2}\rfloor+r}-l_r\le m\le 2^{\lfloor\frac{n}{2}\rfloor+r}$, where $r=1,2,\ldots,\lceil\frac{n}{2}\rceil-3$ and $l_r=\frac{2^{2r+1}-2}{3}$ and $f=0$, if n is odd; $l_r=\frac{2^{2r+2}-4}{3}$ and $f=1$, if n is even.

Let $m_1=2^{\lfloor\frac{n}{2}\rfloor+r}-l_r$, since $2^{\lfloor\frac{n}{2}\rfloor+r}-l_r=2^{\lfloor\frac{n}{2}\rfloor+r}-\sum_{i=0}^{r-1}2^{2r-2i-1+f}=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2-f}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j+f}+2^{f+1}$, so if n is even, $m_1-1=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-3}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j+1}+2^1+2^0$; if n is odd, $m_1-1=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j}+2^0$. If n is odd, by Lemma 4,

$$ex_{m_1}(AQ_n)$$
$$=ex_{\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j}+2^1}(AQ_n)$$
$$=ex_{\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j+f}}(AQ_n)$$
$$+ex_{2^1}(AQ_n)+4(n-3)$$
$$=ex_{\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j+f}}(AQ_n)$$
$$+4n-10,$$

and

$$ex_{m_1-1}(AQ_n)$$
$$=ex_{\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j+f}+2^0}(AQ_n)$$
$$=ex_{\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j+f}}(AQ_n)$$
$$+ex_{2^0}(AQ_n)+2(n-3)$$
$$=ex_{\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-r-2}2^{\lfloor\frac{n}{2}\rfloor+r-i-1}+\sum_{j=0}^{r-1}2^{2r-2j+f}}(AQ_n)$$
$$+2n-6,$$

by equation (3), $\xi_{m_1}(AQ_n)-\xi_{m_1-1}(AQ_n)=3$. If n is even, by Lemma 4, we can get $ex_{m_1}(AQ_n)=ex_{m_1-1}(AQ_n)+ex_{2^0}(AQ_n)+4[(\lfloor\frac{n}{2}\rfloor-r-2)+r+2]=ex_{m_1-1}(AQ_n)+ex_{2^0}(AQ_n)+2(n-1)$, then $\xi_{m_1}(AQ_n)-\xi_{m_1-1}(AQ_n)=(2n-1)m_1-ex_{m_1}(AQ_n)-[(2n-1)(m_1-1)-ex_{m_1-1}(AQ_n)]=1$. Therefore, the lower bound is sharp.

For any $r \le \lceil\frac{n}{2}\rceil - 3$, let $m = 2^{\lfloor\frac{n}{2}\rfloor+r} + m_0$, $m_0 < 2^{n-3}$. Then

$$\begin{aligned}
&\xi_m(AQ_n)\\
&\quad= (2n-1)m - ex_m(AQ_n)\\
&\quad= (2n-1)(2^{\lfloor\frac{n}{2}\rfloor+r} + m_0) - [ex_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n)\\
&\qquad+ex_{m_0}(AQ_n) + 4m_0]\\
&\quad= (2n-1)2^{\lfloor\frac{n}{2}\rfloor+r} - ex_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n)\\
&\qquad+(2n-1)m_0 - 4m_0 - ex_{m_0}(AQ_n)\\
&\quad= \xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n) + [2(n-2)-1]m_0\\
&\qquad-ex_{m_0}(AQ_{n-2})\\
&\quad= \xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n) + \xi_{m_0}(AQ_{n-2}),
\end{aligned}$$

by equation (3), the inequality $\xi_m(AQ_n) \ge \xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n)$ holds, so the upper bound is also sharp.

Remark 2: For the interval $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le m \le 2^{\lfloor\frac{n}{2}\rfloor+r}$, the lower bound of m is sharp, but the upper bound of m is not sharp for $r = \lceil\frac{n}{2}\rceil - 2, \lceil\frac{n}{2}\rceil - 1$.

For $r = \lceil\frac{n}{2}\rceil - 2, \lceil\frac{n}{2}\rceil - 1$, $2^{\lfloor\frac{n}{2}\rfloor+r} = 2^{n-2}$, $2^{\lfloor\frac{n}{2}\rfloor+r} = 2^{n-1}$ respectively. By Lemma 6, $\xi_{2^{n-2}}(AQ_n) = \xi_{2^{n-1}}(AQ_n) = 2^n$. In fact, if n is even, for $r = \lceil\frac{n}{2}\rceil - 2$ and $r = \lceil\frac{n}{2}\rceil - 1$, the lower bounds $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r = 2^{n-2} - \frac{2^{2(\frac{n}{2}-2)+2}-4}{3} = 2^{n-2} - \frac{2^{n-2}-4}{3} = \frac{2^{n-1}+4}{3} = \lceil\frac{2^{n-1}+2}{3}\rceil$ and $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r = 2^{n-1} - \frac{2^{2(\frac{n}{2}-1)+2}-4}{3} = \frac{2^{n-1}+4}{3} = \lceil\frac{2^{n-1}+2}{3}\rceil$ respectively. If n is odd, for $r = \lceil\frac{n}{2}\rceil - 2$ and $r = \lceil\frac{n}{2}\rceil - 1$, their corresponding intervals have the same lower bounds $\lceil\frac{2^{n-1}+2}{3}\rceil$. So $\lceil\frac{2^{n-1}+2}{3}\rceil \le m \le 2^{n-1}$ and $\lceil\frac{2^{n-1}+2}{3}\rceil \le m \le 2^{n-2}$ have overlaps. By Lemma 7, if $\lceil\frac{2^{n-1}+2}{3}\rceil \le m \le 2^{n-2}$, we can show $\xi_m(AQ_n) \ge \xi_{2^{n-2}}(AQ_n) = 2^n$. For any integer m, $\lceil\frac{2^{n-1}+2}{3}\rceil \le m \le 2^{n-1}$, we have $\xi_m(AQ_n) \ge \xi_{2^{n-2}}(AQ_n) = \xi_{2^{n-1}}(AQ_n) = 2^n$, thus the upper and lower bounds are sharp.

IV. THE g-EXTRA EDGE-CONNECTIVITY OF AUGMENTED CUBE

In this section, we get a simple method for calculating the g-extra edge-connectivity of augmented cube for $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor\frac{n}{2}\rfloor+r}$.

Theorem 1: Suppose $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor\frac{n}{2}\rfloor+r}$, where $r = 1, 2, \ldots, \lceil\frac{n}{2}\rceil - 1$ and $l_r = \frac{2^{2r+1}-2}{3}$ if n is odd and $l_r = \frac{2^{2r+2}-4}{3}$ if n is even, then

$$\lambda_g(AQ_n) = 2(\lceil\frac{n}{2}\rceil - r)2^{\lfloor\frac{n}{2}\rfloor+r}. \qquad (6)$$

Proof: On the one hand, recall that the definition of $AQ_n[L_m^n]$ and $F = [L_m^n, \overline{L_m^n}]$ the set of edges of AQ_n with exactly one end vertex in L_m^n and the other end in $\overline{L_m^n}$. Combining with Lemmas 2 and 3, $AQ_n[L_m^n]$ and $AQ_n[\overline{L_m^n}]$ are connected. If $|V(AQ_n[L_m^n])| = 2^{\lfloor\frac{n}{2}\rfloor+r}$, $|E(AQ_n[L_m^n])| = ex_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n)$, then F is a g-extra edge-cut of AQ_n and $|F| = \xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n) = (2n-1)2^{\lfloor\frac{n}{2}\rfloor+r} - ex_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n)$. On the other hand, for any $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor\frac{n}{2}\rfloor+r}$, where $r = 1, 2, \ldots, \lceil\frac{n}{2}\rceil - 1$ and $l_r = \frac{2^{2r+1}-2}{3}$ if n is odd, $l_r = \frac{2^{2r+2}-4}{3}$ if n is even, in view of Lemmas 5 and 7, and Remarks 1 and 2, the minimum value of $\{\xi_m(AQ_n) : 2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le m \le 2^{\lfloor\frac{n}{2}\rfloor+r}\}$ is $\xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n)$. Hence,

$$\begin{aligned}
\lambda_g(AQ_n) &= min\{\xi_m(AQ_n) : g \le m \le 2^{n-1}\}\\
&= min\{\xi_m(AQ_n) : g \le m \le 2^{\lfloor\frac{n}{2}\rfloor+r}\}\\
&= \xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n)\\
&= 2(\lceil\frac{n}{2}\rceil - r)2^{\lfloor\frac{n}{2}\rfloor+r}.
\end{aligned}$$

The proof is completed. □

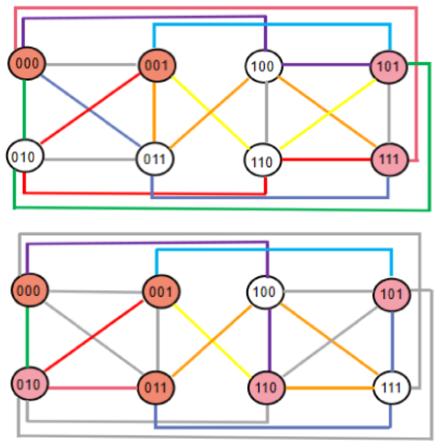

FIGURE 5. Many-to-many edge disjoint paths in two disjoint subgraphs of AQ_3.

Based on this result, for $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor\frac{n}{2}\rfloor+r}$, where $r = 1, 2, \ldots, \lceil\frac{n}{2}\rceil - 1$ and $l_r = \frac{2^{2r+1}-2}{3}$ if n is odd and $l_r = \frac{2^{2r+2}-4}{3}$ if n is even, we can find the maximum $2(\lceil\frac{n}{2}\rceil - r)2^{\lfloor\frac{n}{2}\rfloor+r}$ edge disjoint paths linking any two disjoint connected subgraphs with just g vertices in AQ_n. In order to understand this issue, we take an example, for $n = 3$, $g = 2$ and $g = 3$, there are 8 edge disjoint paths marked by different colours except gray in two disjoint subgraphs of AQ_3 shown in Fig. 5.

Unexpectedly, we find that the g-extra edge-connectivity of AQ_n exists a concentration behavior for some exponentially large enough g on the interval length of l_r, $r = 1, 2, \ldots, \lceil\frac{n}{2}\rceil - 1$ where $l_r = \frac{2^{2r+1}-2}{3}$ if n is odd and $l_r = \frac{2^{2r+2}-4}{3}$ if n is even. For convenience, we define a set $N_r = \{g : \xi_g(AQ_n) = \xi_{2^{\lfloor\frac{n}{2}\rfloor+r}}(AQ_n) = 2(\lceil\frac{n}{2}\rceil - r)2^{\lfloor\frac{n}{2}\rfloor+r}, n \ge 3, r = 1, 2, \ldots, \lceil\frac{n}{2}\rceil - 1\}$ on this interval $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor\frac{n}{2}\rfloor+r}$. Taken $n = 11$ as an example, the Fig. 6 shows the set N_r on the interval length of l_r. From this bar, it can be seen that as the integer r varies, the integer intervals exponentially increase for fixed n.

This paper focuses on the g-extra edge-connectivity of the n-dimensional augmented cube ($n \ge 3$) for $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor\frac{n}{2}\rfloor+r}$. Because intervals $l_{\lceil\frac{n}{2}\rceil-2}$ and $l_{\lceil\frac{n}{2}\rceil-1}$ have the same interval $l_{\lceil\frac{n}{2}\rceil-2}$, so denote that $s(n) = \sum_{r=1}^{\lceil\frac{n}{2}\rceil-1} l_r - l_{\lceil\frac{n}{2}\rceil-2}$ is sum of the number g for integer interval $2^{\lfloor\frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor\frac{n}{2}\rfloor+r}$, $r = 1, 2, \ldots, \lceil\frac{n}{2}\rceil - 1$ where $l_r = \frac{2^{2r+1}-2}{3}$ if n is

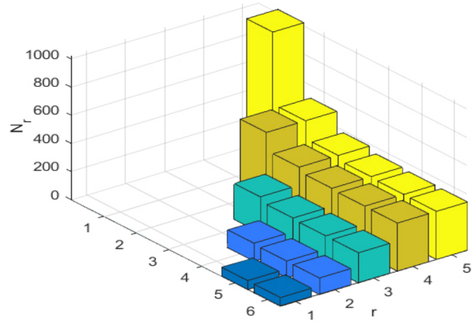

FIGURE 6. The set N_r on the interval length of l_r for $n = 11$.

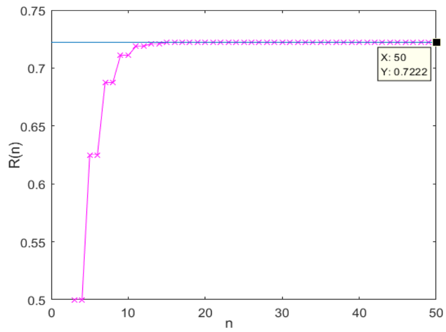

FIGURE 7. The function $R(n)$ for $3 \le n \le 50$.

TABLE 3. The values $s(n)$ and $R(n)$ for $3 \le n \le 20$.

n	3	4	5	6	7	8
$s(n)$	2	4	10	20	44	88
2^{n-1}	4	8	16	32	64	128
$R(n)$	0.500	0.500	0.6250	0.6250	0.6875	0.6875
n	9	10	11	12	13	14
$s(n)$	182	364	736	1472	2954	5908
2^{n-1}	256	512	1024	2048	4096	8192
$R(n)$	0.7109	0.7109	0.7188	0.7188	0.7212	0.7212
n	15	16	17	18	19	20
$s(n)$	11828	23656	47326	94652	189320	378640
2^{n-1}	16384	32768	65536	131072	262144	524288
$R(n)$	0.7219	0.7219	0.7221	0.7221	0.7222	0.7222

odd and $l_r = \frac{2^{2r+2}-4}{3}$ if n is even. Let $R(n) = s(n)/2^{n-1}$. Then $\lim_{n\to\infty} R(n) \approx 0.7222$. The function $R(n)$ for $3 \le n \le 50$ is shown in Fig. 7. We list the values of n, $s(n)$ and $R(n)$ for $3 \le n \le 20$ in Table 3.

V. CONCLUDING REMARKS

The g-extra edge-connectivity is an important subject for interconnection network's ability to fault edges. The g-extra edge-connectivity of G is defined as the maximum number of edge disjoint paths connecting any two disjoint connected subgraphs with g vertices in G. The problem of the g-extra edge-connectivity of AQ_n is well-defined for each $g \le 2^{n-1}$. In this paper, we can obtain that for about 72.22 percent values of $g \le 2^{n-1}$, the g-extra edge-connectivity of AQ_n ($n \ge 3$) concentrate on several special $\xi_{2^{\lfloor \frac{n}{2}\rfloor+r}}(AQ_n)$, for $r = 1, 2, \ldots, \lceil \frac{n}{2}\rceil - 1$. More specially, we prove that the values of the g-extra edge-connectivity of AQ_n are the constants $\xi_{2^{\lfloor \frac{n}{2}\rfloor+r}}(AQ_n) = 2(\lceil \frac{n}{2}\rceil - r)2^{\lfloor \frac{n}{2}\rfloor+r}$ for each integer $2^{\lfloor \frac{n}{2}\rfloor+r} - l_r \le g \le 2^{\lfloor \frac{n}{2}\rfloor+r}$, where $r = 1, 2, \ldots, \lceil \frac{n}{2}\rceil - 1$ and $l_r = \frac{2^{2r+1}-2}{3}$ if n is odd and $l_r = \frac{2^{2r+2}-4}{3}$ if n is even. Our results provide a more accurate measure for evaluating a large scale AQ_n network reliability and availability. Further research will focus on considering g-extra edge-connectivity of more generalized networks that also exist a concentration behavior.

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