Justified Representation for Perpetual Voting

LAURENT BULTEAU[1], NOAM HAZON [ID]2, RUTVIK PAGE [ID]3,
ARIEL ROSENFELD[4], AND NIMROD TALMON [ID]5

[1]Laboratoire d'Informatique Gaspard Monge, 77454 Marne-la-Vallée, France
[2]Department of Computer Science, Ariel University, Ariel 40700, Israel
[3]Department of Computer Science and Engineering, Indian Institute of Information Technology, Nagpur, Nagpur 440006, India
[4]Department of Information Science, Bar-Ilan University, Ramat Gan 5290002, Israel
[5]Ben Gurion University of the Negev, Be'er Sheva 8410501, Israel

Corresponding author: Noam Hazon (noam.hazon@g.ariel.ac.il)

ABSTRACT We study the social choice setting of perpetual voting where, based on voter preferences, decisions have to be taken over a finite horizon of consecutive points in time (e.g, days). We consider two complementary settings: a *Static* setting, in which voter preferences remain static over time, and a *Dynamic* setting, in which voter preferences may change over time. We adapt the well-established Justified Representation and Proportional Justified Representation axioms, commonly used in the social choice literature, to perpetual voting, resulting in two axiomatic variants for the static setting and four variants for the dynamic setting. We show that all of the axioms are always satisfiable, and that simple preference aggregation methods can be used to satisfy the axioms in both cases. We then conduct a large human study ($N = 190$) aimed at identifying what potential voters (i.e., ordinary people) deem as desirable outcomes in simple perpetual voting settings. Our results show that approximately half of our participants consider different interpretations of fairness that correspond to our axiomatic framework. Taken jointly, our results can be used to help the research community identify appropriate aggregation methods to use in practice.

I. INTRODUCTION

In a standard social choice setting there is a single collective decision to be made in a single point in time, such as selecting a president for a nation or a committee to act on behalf of a group of people. There are, however, many situations in which a set of collective decisions are to be taken sequentially, over a number of time periods.

As an illustrating example, consider a group of colleagues that want to decide upon a restaurant to dine in for, say, each day of a given week. One way by which the group might proceed is to do the following: Independently, for each day of the week, ask all colleagues for their preferences and employ a standard aggregation method to select a restaurant for that day. To be more concrete, assume that there are 4 candidate restaurants, denoted r_i, $i \in [4]$, and that each colleague, for each day, simply reports a subset of restaurants she is willing to dine in (i.e., restaurants she approves of for that day). Assume that we have 3 colleagues, v_i, $i \in [3]$, with

the following preferences for the first day of the week (i.e., Monday): $v_1 = \{r_1\}$, $v_2 = \{r_1\}$, $v_3 = \{r_3, r_4\}$. One natural way to decide upon the restaurant to go would be to employ *Approval voting*, namely, choose the restaurant (i.e., candidate) approved by the largest number of voters. In our case, this would mean selecting r_1. Unfortunately, using the above procedure for each day of the week separately may result in some colleagues being repeatedly unhappy. For example, now assume that the voter preferences remain static during the week – i.e., they do not change from day to day. Then, employing Approval voting on a daily basis would mean that the group of colleagues would go to r_1 each day of the week, leaving v_3 constantly unhappy; there is not even a single day in which her preferences are taken into account. Following Lackner [1], we refer to this setting as *Perpetual Voting* (PV).

A particularly important motivation for the study of perpetual voting stems from the temporal notions that it entails. For example, if the head of an institution has to be elected by voting (assuming that there are periodic elections), it is natural to think that for each election, only a certain faction of voters can be satisfied by the outcome, owing to the inherent presence

---

The associate editor coordinating the review of this manuscript and approving it for publication was Zhan Bu [ID].

of contradictory opinions amongst the voter factions. If left unabated, such collective decision making mechanisms tend to cater to only a limited subset of voter factions and tend to overlook others, which actually deserve representation in the form of election of a head of their choice. Proportionality in Perpetual Voting abates the dis-satisfaction of certain voter groups by putting in place certain mechanisms such that over a period of time, (i.e. conduction of multiple elections over time) all of the sufficiently large voter groups having similarly aligned opinions are either fully or approximately (i.e. equally) satisfied.

To the best of our knowledge, the existing literature that studies sequential collective decision making does not deliberate upon proportional outcome allocation over the set of voters. Consequently, there exists a scope to study this problem from the point of view of an axiomatic, algorithmic and an empirical standpoint. To bring about more desirable outcomes in PV settings, we first identify two complementary variants of PV: *Static PV*, in which voter preferences do not change over time (remain static); and *Dynamic PV*, in which voter preferences may change over time. We adapt the well-established Justified Representation (JR) and Proportional Justified Representation (PJR) axioms [2], which are commonly used in the social choice literature (see, e.g., [3] and the references within) to the two variants resulting in two axiomatic variants for Static PV and four variants for Dynamic PV. Our adaptations formulate the notion that any sufficiently-large and sufficiently-cohesive subgroup of voters cannot be underrepresented.

We first formally define the two variants of PV discussed above and derive JR and PJR axioms pertaining to these variants. We prove that the axioms are always satisfiable by providing *simple* preference aggregation methods that can be used to satisfy them under both PV variants. These results suggest that our axiomatic framework and preference aggregation methods are analytically sound and practically implementable. Our axiomatic framework is then taken to the field and examined empirically through a large scale human study ($N = 190$). Our study is aimed at identifying what potential voters (i.e., ordinary people of different backgrounds) deem as desirable outcomes in simple PV settings. While most potential voters have demonstrated a strong tendency to adopt a utilitarian approach (i.e., maximize the number of days each voter is satisfied), a large portion of whom have also expressed different notions (which we list and describe) of fairness which roughly correspond to our axiomatic framework.

Our paper argues that in PV settings, voters are proportionally represented in static variant of PV, even so in polynomial time and provides suitable preference aggregation methods for finding respective solutions. Further, it exhibits the existence of solutions for four axioms that we propose in alignment with dynamic variant of PV, two of which align with Justified Representation while the remaining aligning with Proportional Justified Representation. In addition to the provision of preference aggregation rules for producing

solutions that are in adherence with the proposed axioms thereby confirming the existence of solutions, our paper also proves one of these to be polynomial-time computable and the remaining three to be fixed-parameter tractable with respect to the number of voters.

II. RELATED WORK

There are only a handful of social choice studies on settings concerning repeated elections. One notable study is the work by Frege [4] who proposed a Plurality-based election approach, in which the voter set may change over time. Contrary to this, we consider Approval ballots in addition to Plurality ballots and while our voters might change their preferences over time, our set of voters is static. In a more recent paper, Lackner [1] analyzes a similar PV setting developed independently. Lackner suggests several voting rules and analyze them via three axiomatic properties, as well as a quantitative evaluation by computer simulations. The crucial difference between the approach taken by Lackner and the one proposed in our study is that the former concentrates on each *individual* voter and analyzes her (dis)satisfaction along the decision sequence; in contrast, our approach concentrates on the representation that *subgroups* of voters achieve along the decision sequence.

In particular, our approach is to adapt the well-studied axioms of Justified Representation (JR) and Proportional Justified Representation (PJR) to the PV setting. In the context of multiwinner elections [3], for which JR and PJR were originally proposed, these axioms require that sufficiently-large and cohesive groups of voters shall not be disregarded by the multiwinner voting rule, but shall get adequate representation in the winning committees. Since their introduction [2], JR and PJR have been studied quite extensively in the context of multiwinner elections [5], [6].

In our PV settings, we are interested in selecting several candidates as winners, one for each point in time. As such, our settings have some connections with multiwinner elections [3] and apportionment rules [7]. However, while in multiwinner elections the outcome is an unordered set of candidates, in a PV setting the outcome is a vector of candidates in which certain candidates might appear more than once.

Additional works related to ours include dynamic social choice [8], in which a stochastic model is adapted, however proportionality is not a concern. A setting of repeated choice is also considered in fair division [9], [10] which is generalized in the study of dynamic mechanisms [11]. Notably, Freeman *et al.* [12] consider an online setting in which a decision for time period i has to be made based only on the preferences up to period i. Some notions of proportionality, similar to the ones proposed in this paper, were achieved for the investigated settings.

A particularly relevant resource allocation problem to our settings is studied by Conitzer *et al.* [13]. Their study involves the maximization of long term Nash Welfare due to a simultaneous decision on a given number of issues by the aggregation of preferences using a utility function reportage. While our

settings also set out with an objective of long term fair representation, our model differs with them in the sense that it studies sequential decision making rather than simultaneous decision making along with approval ballots rather than the reporting of utility functions.

Our human study is inspired by a variety of user studies performed in the larger context of computational social choice research, which have helped to better situate and understand the theoretical and computation advances made in various settings. For example, human voting behavior has been the focus of various experimental studies such as that by Scheuerman *et al.* [14], [15], who has identified different manipulation aspects in human approval voting; Tal *et al.* [16], who have demonstrated different voting behaviors in various online settings under the Plurality rule; Zou *et al.* [17], who have studied voter behavior in Doodle pools; and Grandi *et al.* [18], who have studied iterative human voting in combinatorial domains. Common to these and similar works is the understating that any intelligent automated decision necessitates the understanding of real-world users (in our case, voters) and their judgment (in our case, what people deem appropriate) [19].

III. FORMAL MODEL

In PV settings, we have a set $A = \{a_1, \ldots, a_m\}$ of *alternatives*, a horizon of *k points in time*, and a set $V = \{v_1, \ldots, v_n\}$ of *voters*. The output is a sequence $W = [w^1, \ldots, w^k]$, where $w^j \in A$ for each $j \in [k]$; w^j is referred to as the *winner* at time *j*. We consider the Approval model of elections, adapted to our scenario. Formally, each voter v_i reports $\vec{v}_i = [V_i^1, \ldots, V_i^k]$, where $V_i^j \subseteq A$ for each $j \in [k]$; V_i^j is a set containing those alternatives that voter v_i approves of at time *j*. Therefore, the aggregate choice of a subgroup of voters $V' \subseteq V$ is denoted by $\cap_{v \in V'} \vec{v}$. With a slight abuse of notation, we represent the number of points in time that have a non-empty intersection between the voters' approval ballots and the output sequence W by $|\cap_{v \in V'} \vec{v} \cap W|$.

We identify two complementary PV settings:

1) A *Static* setting of PV, in which voter preferences do not change over time. Namely, $V_i^j = V_i^{j'}$ for every voter (*i*) and two point in time (*j*, *j'*). As such, in the Static PV setting, $\vec{v}_i$ will simply be a set of approved alternatives throughout the time horizon.

2) A *Dynamic* setting of PV, in which voter preferences may change over time. Note that we consider the case in which voter preferences are given completely as an input to the aggregation algorithm; thus, in particular, voter preferences for time *t* cannot depend on the aggregation choice of the selected winners at any other time steps. (Put differently, we consider an *offline* model and not an *online* model.)

We argue that the intuition behind JR and PJR [2] are natural for the PV setting. In particular, as illustrated by our example in Section I, a desired solution would require some form of proportionality and representation from any adequate aggregation method. To that end, in the next two sections, we adapt each of the desired axioms (JR and PJR) to both the Static and Dynamic PV settings.

A. STATIC PV
We begin the analysis of PV settings by considering the static variant.

1) STATIC JR
In multiwinner elections, a multiwinner voting rule that elects *k* winners satisfies JR if for each group of at least n/k voters with non-empty intersecting Approval sets there is non-empty intersection between the union of the Approval sets of the voters in the group and the winning committee. In other words, sufficiently-large and sufficiently-cohesive groups of voter are not completely ignored. The JR axiom translates to the following in the Static PV setting;

Definition 1 (Static JR): A rule $\mathcal{R}$ in the Static PV setting satisfies Static Justified Representation (Static JR) *if for each group of voters V' that (1) has a non-empty intersection (i.e., for which $\cap_{v \in V'} \vec{v} \neq \emptyset$), and that (2) contains at least n/k voters (i.e., for which $|V'| \geq n/k$), it holds that $|\cup_{v \in V'} \vec{v} \cap W| \neq 0$.*

It turns out, that Static JR is a rather weak axiom; in particular, the following simple rule satisfies it.

Definition 2 (GreedyCC): The GreedyCC *rule proceeds in iterations, where in each iteration we compute the Approval scores of all alternatives, pick the alternative with the most Approvals, put it as the next winner (i.e., in iteration i, we set this alternative to be w^i), and remove from further consideration all voters that approve it. If, after strictly less than k iterations we are left with no voters, then we choose the remaining winners arbitrarily.*

Proposition 1: GreedyCC satisfies Static JR.

Proof: The claim follows by observing that: (1) there cannot be more than *k* distinct groups of at least n/k voters with non-empty intersection of approval ballots; (2) GreedyCC ensures that all of these groups are represented at least once. These groups are exactly the groups that the Static JR axiom requires a rule to represent. ∎

Thus, Static JR always exists and can be found in polynomial time.

2) STATIC PJR
With the Static JR being a weak axiom, we continue by adapting the PJR to the Static PV. Recall that in multiwinner elections, a multiwinner voting rule that elects *k* winners satisfies the PJR axiom if for each group V' of voters with non-empty intersecting Approval sets there are at least $\lfloor \frac{|V'|}{|V|} k \rfloor$ members of the winning committee from the union of the Approval sets of the voters in *V*. In other words, sufficiently-cohesive groups of voters are *adequately* represented based on their sizes.

Definition 3 (Static PJR): A rule $\mathcal{R}$ in the static setting satisfies Static Proportional Justified Representation

(Static PJR) *if for each group of voters V' that has a non-empty intersection (i.e., for which $\cap_{v \in V'} \vec{v} \neq \emptyset$), it holds that $|\cup_{v \in V'} \vec{v} \cap W| \geq \lfloor \frac{|V'|}{|V|} k \rfloor$.*

It turns out that even this stronger axiom is satisfied by a slight modification of GreedyCC (Definition 2).

Observation 1: There is a polynomial-time algorithm that always finds a solution satisfying Static PJR.

Proof: We propose a slight modification of GreedyCC: In each iteration, look for a candidate c approved by the most number of voters and denote the corresponding voter set by V'. Set $\lfloor k|V'|/|V| \rfloor$-many winners arbitrarily to be c, remove V' from further consideration and reiterate.

In every iteration, the algorithm sets $\lfloor k|V'|/|V| \rfloor$ copies of candidate with the highest approval score to be the winner at those many arbitrarily chosen days, each satisfying disjoint sets of $|V|/k$ voters and since there are at most k disjoint sets of voters of size $|V|/k$, all voters would be satisfied in at most k iterations of this modified algorithm. ∎

Taken jointly, we can conclude that:

Corollary 1: A solution satisfying Static PJR (and thereby, Static JR) always exist and can be found in polynomial time.

B. DYNAMIC PV

In Section 3 we considered the Static PV setting in which voter preferences do not change over time. In this section, we extend our analysis to the Dynamic PV setting, in which voter preferences may change over time. Unlike the Static setting, here the number of periods in which there is an intersection between voters in each group matters. We propose two possible temporal interpretations for the JR and PJR axioms in the Dynamic PV setting: (1) we adapt the JR and PJR axioms while taking care only of groups V' that intersect on *all periods*; (2) we consider stronger adaptations of the JR and PJR axioms, in which we take into account the number of periods in which there is a non-empty intersection for the voters in V'.

1) DYNAMIC ALL-PERIODS-INTERSECTION
We start by adapting the JR axiom to the Dynamic PV under the two temporal interpretations described above. We start with the first, and weaker, interpretation which requires that intersections occur over the entire time horizon.

a: DYNAMIC ALL-PERIODS-INTERSECTION JR
Adapting the first temporal interpretation, the JR axiom would require that all sufficiently-large and sufficiently-cohesive *over all the entire time horizon* voter groups should not be completely ignored.

Definition 4 (Dynamic All-periods-intersection JR): A rule $\mathcal{R}$ in the dynamic setting satisfies Dynamic All-periods-intersection Justified Representation (Dynamic All-periods-intersection JR) *if for each group of voters V' that (1) has a non-empty intersection in all periods (i.e., for which it holds that, for each $j \in [k]$, $\cap_{V'} V^j \neq \emptyset$), and that (2) contains at least $|V|/k$ voters (i.e., for which $|V'| \geq |V|/k$), it holds that there exists at least one $j \in [k]$ such that $w_j \in \cup_{v \in V'} V^j$.*

It turns out that Dynamic All-periods-intersection JR is also satisfied by another simple modification of GreedyCC (Definition 2). Note that the algorithm GreedyCC does not work since the candidate selected in each of its iteration is not necessarily a one which lies in the intersection of the voter group's approval ballots on all days. Hence, that group of voters which does not have the same candidate in the intersection of its approval sets on all days is effectively overlooked by the algorithm.

Proposition 2: There is a polynomial-time algorithm that always finds a solution satisfying Dynamic All-periods-intersection JR.

Proof: Consider the following algorithm: Set w^1 to be a candidate c with the most number of Approvals in the first time period; then remove the voters which approve c in the first period from further consideration, and continue to the next time period. If at some iteration i, the approval score of any candidate is $\geq n/k$, fill the winners in the remaining positions arbitrarily and terminate the algorithm.

For correctness, first consider one iteration in which we remove voters approving some c in some period; denote these voters by V'. Note that any group of voters containing at least one voter $v \in V'$ is already satisfied (including, in particular, V', which we take care of explicitly as we set a winner in that period to be c). Now, note that if at a certain point the number of Approvals for the candidate c is strictly less than n/k, then it means in particular that there are no more groups that still need to be taken care of, as we should only care of groups of size at least n/k. Then, as long as there are groups of size at least n/k, in each iteration of the algorithm we "pay" one winner to remove at least n/k voters. As we have n voters and k days, it follows that the algorithm will halt at a point in which there are strictly less than n/k voters not explicitly taken care of. ∎

Thus, also Dynamic All-periods-intersection JR always exists and can be found in polynomial time.

b: DYNAMIC ALL-PERIODS-INTERSECTION PJR
Using the same temporal interpretation, the PJR axiom would require that all sufficiently-large and sufficiently-cohesive *over the entire time horizon* voter groups would be *adequately* represented based on their sizes.

Definition 5 (Dynamic All-Periods-Intersection PJR): A rule $\mathcal{R}$ in the dynamic setting satisfies Dynamic All-periods-intersection Proportional Justified Representation (Dynamic All-periods-intersection PJR) *if for each group of voters V' that has a non-empty intersection in all periods (i.e., for which it holds that, for each $j \in [k]$, $\cap_{v_i \in V'} V_i^j \neq \emptyset$), it holds that $|\{j \in [k] : w^j \in \cup_{v_i \in V'} V_i^j\}| \geq \lfloor \frac{|V'|}{|V|} k \rfloor$.*

Unfortunately, it seems that GreedyCC defined in Definition 2 (and therefore, its stronger variant which guarantees a solution satisfying proportional representation mentioned in Observation 1) cannot be adapted to satisfy Dynamic All-periods-intersection PJR, as described by the following example:

Example 1: Consider this input:

$$v_1 : \{1\}\{3\}\{3\}\{3\}$$
$$v_2 : \{1\}\{3\}\{3\}\{3\}$$
$$v_3 : \{1, 2\}\{2\}\{2\}\{2\}$$
$$v_4 : \{2\}\{2\}\{2\}\{2\}$$

GreedyCC would select 1 *as the winner of the first day. Then, however, it would not necessarily take care of* $V' = \{v_3, v_4\}$ *which deserves two winners.*

We conjecture that satisfying this axiom is NP-hard, however we do not yet have NP-hardness proof for it. The basis of our conjecture is that it seems that we should consider larger cohesive voter groups first, however, finding a large group that has a non-empty intersection on all periods is intractable; a sketch of a reduction from the NP-hard Clique problem [20] for this is as follows: given a graph, consider $A = \{0, 1\}$; let each voter each time step be vertices, and set V_i^j to be $\{1\}$ if $i = j$, $\{0, 1\}$ if i and j are adjacent, and $\{0\}$ otherwise; then, groups of voters with all-periods-intersections are exactly cliques of the graph.

Conjecture 1: No polynomial-time algorithm solves Dynamic All-periods-intersection PJR unless P = NP.

While we conjecture general intractability for Dynamic All-periods-intersection PJR, we still show that it is possible to satisfy this axiom, albeit with a super polynomial algorithm. In fact, the algorithm is fixed-parameter tractable wrt. the number n of voters.

Proposition 3: Dynamic All-periods-intersection PJR is fixed-parameter tractable wrt. the number n of voters.

Proof: For a parameterization by the number n of voters, we can consider all groups of voters separately (as there are 2^n groups). For each such group V', we check whether V' has non-empty intersection in all periods. Then, we consider only those groups V' with non-empty intersection in all periods and order them by their size (i.e., by $|V'|$). Now, we consider the largest such V', arbitrarily assign to it the number of winners it deserves (i.e., $\lfloor \frac{|V'|}{|V|} k \rfloor$ many winners), remove those voters from further consideration, and re-iterate. We halt when no groups with non-empty intersection in all periods are left or when the size of the largest such group is less than n/k. We choose the remaining winners arbitrarily.

For correctness, note first that the algorithm takes care of disjoint groups as after it takes care of a group of sufficiently large size, it removes all voters belonging to it. Thus, in particular, the algorithm always has sufficient "budget" – a free period to select its winner in each iteration. Now, any voter v the algorithm removes explicitly takes care of all groups V' that contain v; note that this holds, in particular, as we consider bigger groups, that deserve more winners, first. Thus, in particular, the algorithm takes care of any group of voters containing at least one of those voters that the algorithm explicitly removed during its operation. At the end of the algorithm there are strictly less than n/k voters left, thus there are in particular no more groups that needs to be taken care of. ∎

Indeed, the above fixed-parameter algorithm implies in particular that there is always a solution satisfying Dynamic All-periods-intersection PJR, albeit with only a superpolynomial-time algorithm.

2) DYNAMIC SOME-PERIODS-INTERSECTION
Next, we adapt the second, and stronger, temporal interpretation requiring that voter preference intersections occur over sufficiently-many periods of time.

a: DYNAMIC SOME-PERIODS-INTERSECTION JR
Using the second temporal interpretation, our adaptation of JR requires that all groups that are sufficiently-large and also sufficiently-cohesive *over sufficiently-many periods of time* will not be ignored.

Definition 6 (Dynamic Some-Periods-Intersection JR): A rule $\mathcal{R}$ in the dynamic setting satisfies Dynamic Some-periods-intersection Justified Representation (Dynamic Some-periods-intersection JR) *if for each group of voters V' that has (1) a non-empty intersection for k' periods (i.e., there are at least k' j's for which $\cap_{v_i \in V'} V_i^j \neq \emptyset$); and for which (2) $\lfloor \frac{|V'|}{|V|} k' \rfloor > 0$; it holds that $\{j \in [k] : w^j \in \cup_{v_i \in V'} V_i^j\} \neq \emptyset$.*

Remark 1: Note that Dynamic Some-periods-intersection JR is stronger than Dynamic All-periods-intersection JR. This is so as, while Dynamic All-periods-intersection JR requires representation only for groups of voter with non-empty intersection in all periods, Dynamic Some-periods-intersection JR requires representation for all groups (of sufficient size and intersection): In particular, it requires to reserve at least one winner for groups of voters that has non-empty intersection in at least some k' periods, provided that their size is at least $|V|/k'$. Indeed, this means, also, that any algorithm satisfying Some-periods-intersection JR satisfies also Dynamic All-periods-intersection JR. However, Dynamic Some-periods-intersection JR is incomparable to All-periods-intersection PJR.

Theorem 1: There is a polynomial-time algorithm that always finds a solution satisfying Dynamic Some-periods-intersection JR.

Proof: Consider a PV setting with alternatives A, k time steps, and voters V. Let $T = [k]$ be the set of time steps. Given an alternative a, a set of voters $V' \subseteq V$, and a time $t \in T$, say that the *worth* of a at time t for V', denoted by $\text{worth}_{V'}(a, t)$, is $|\{v_i \in V' \mid a \in V_i^t\}|$; i.e. the number of voters among set V' that approve a at time t.

Consider the following algorithm: start with $j = 0$, $V_0 = V$, and $T_0 = T$. Pick $(a_j, t_j) = \arg\max_{a \in A, t \in T_j} \text{worth}_{V_j}(a, t)$. Set $w_{t_j} = a_j$, $T_{j+1} = T_j \setminus \{t_j\}$, $V_j' = \{v_i \mid a_j \in V_i^{t_j}\}$, $V_{j+1} = V_j \setminus V_j'$, and $j = j + 1$. Start again unless one of V_j or T_j is empty. For each remaining $t \in T_j$ (if any), set $w_t = a$ for any alternative a. Then, w_t is defined for each $t \in T$; return the corresponding solution W.

Note first that this algorithm terminates in polynomial time.

We now show that the solution W satisfies Dynamic Some-periods-intersection JR. We say that a group V' of voters is k'-*important* over $V'' \subseteq V$ and $T' \subseteq T$ if it satisfies the conditions of Dynamic Some-periods-intersection JR with integer k' over the reduced instance containing only voters V'' and time points T'. Formally, V' has a non-empty intersection at k' different time points among T' and $\lfloor \frac{|V'|}{|V''|} k' \rfloor > 0$; i.e. $|V'| \geq \frac{|V''|}{k'}$.

We can now show by induction over $|T_j|$ that the algorithm gives at least one winner to each k'-important groups of voters over V_j and T_j with $1 \leq k' \leq k$. First note that, if either T_j or V_j is empty, then no such group V' exist, and the claim is proven for the base case. In general, consider such a set V'. If some voter v_i is both in V' and in the set V'_j picked by the algorithm, then $a_j \in V_i^{t_j}$, and setting $w_{t_j} = a_j$ gives a winner to $v_i \in V'$ at time t_j. If no such voter exists, i.e. $V' \cap V'_j = \emptyset$, then we claim that V' is $(k'-1)$-important in the sub-instance with voters V_{j+1} and time points T_{j+1}. Note also that $k' > 1$ (since $k' = 1$ yields $|V'| = |V_j|$ and $V' = V_j$, which necessarily intersects V'_j), so $k' - 1 \geq 1$ and, by induction, a winner is given to V' in a subsequent step of the algorithm.

In remains to prove the claim above. First note that there exists at least one alternative with worth $|V'|$ or more (since V' self-intersects at least once). Since the size of V'_j is the maximum worth of alternatives over V_j and T_j, we have $|V'_j| \geq |V'| \geq \frac{|V_j|}{k'}$, so $|V_{j+1}| \leq \frac{(k'-1)|V_j|}{k'}$. Thus, using $|V'| \geq \frac{|V_j|}{k'}$, we have $|V'| \geq \frac{|V_{j+1}|}{k'-1}$. Furthermore, V' self intersects on k' time points among T_j, so it intersects on at least $k' - 1$ time points among $T_j \setminus \{t_j\}$. Overall V' is $(k'-1)$-important over V_{j+1} and T_{j+1}. ∎

b: DYNAMIC SOME-PERIODS-INTERSECTION PJR
We complete our axiomatic framework with the adaptation of the PJR axiom to the Dynamic setting and the second temporal interpretation. The Dynamic Some-periods-intersection PJR axiom requires that all groups that are sufficiently-large and also sufficiently-cohesive *over sufficiently-many points in time* would be adequately represented, based on their sizes and the points in time in which they intersect. Note that the periods in which a group V' "gets a winner" need not necessarily be the same periods in which there is a non-empty intersection.

Definition 7 (Dynamic Some-Periods-Intersection PJR): A rule $\mathcal{R}$ in the dynamic setting satisfies Dynamic Some-periods-intersection Proportional Justified Representation (Dynamic Some-periods-intersection PJR) *if for each group of voters V' that has a non-empty intersection for k' periods (i.e., there are at least k' j's for which $\cap_{v_i \in V'} V_i^j \neq \emptyset$), it holds that* $|\{j \in [k] : w^j \in \cup_{v_i \in V'} V_i^j\}| \geq \lfloor \frac{|V'|}{|V|} k' \rfloor$.

Following Conjecture 1, here we also conjecture hardness.

Conjecture 2: No polynomial-time algorithm solves Dynamic Some-periods-intersection PJR unless $P = NP$.

As before, we show that Dynamic Some-periods-intersection PJR is always satisfiable, and can be computed efficiently if there are not too many voters.

Theorem 2: Dynamic Some-periods-intersection PJR is fixed-parameter tractable wrt. the number n of voters.

Proof: The proof follows by the following algorithm. The algorithm works in two phases. In the first phase, we do an iterative greedy algorithm, as follows: We consider the set of all groups of voters that deserve some representation (i.e., groups V' for which $|V'|/k'$ is at least n); denote this set by $\mathcal{V}$. Now, in each iteration we consider a group $V' \in \mathcal{V}$, where we take the group V' that has the highest value of $|V'|/k'$ (since $|V|$ is a constant and the higher is the value of $|V'|/k'$, more 'demanding' is the voter group in that sense). While iterating over these $V' \in \mathcal{V}$, we maintain a set of constraints $\mathcal{C}$ (initially empty) where each constraint $c \in \mathcal{C}$ is of the form "burn i places out of the places $x \in [k]$". For this group V', we add the constraint: "burn $\lfloor \frac{|V'|}{|V|} k' \rfloor$ places out of the places $x \in [k]$", where x is the index of each day on which the group V' has nonempty intersection. Then we remove all voters of V' from further consideration (in particular, prune $\mathcal{V}$ to not include any group containing at least one voter from the current V') and reiterate. We end this first phase whenever we have strictly less than n/k voters left (and then, in particular $\mathcal{V}$ will be empty).

In the second phase of the algorithm, we first take the constraints $\mathcal{C}$, and split each constraint of the form "burn i places out of x" into i constraints, each of the form "burn 1 place out of x" This splitting keeps correctness.

Now, we go over the constraints of $\mathcal{C}$ in increasing order of their corresponding k' (observe that each constraint $c \in \mathcal{C}$ corresponds to some group of voters V' with some k'). Now, we greedily burn places as needed.

Next we show that indeed we can always "burn" places. Furthermore it is important to show that, for each V' we consider, there exist enough days to satisfy voters of V' – out of those days in which V' has nonempty intersection – that is not already fixed (by groups of voters that we explicitly took care of in earlier iterations). To this end, assume, towards a contradiction that, during a run of the algorithm we encounter now some V^* that intersects in some k^* periods, however, the winners of all these k^* periods are already fixed. As all these k^* periods are already taken, it means in particular that the algorithm already took care explicitly of at least k^* voter groups, each with nonempty intersection in at least $k' \leq k^*$ periods (as we consider the groups in $\mathcal{V}$ in increasing values of their k'). Now, as for each V' with its k' it holds that $k' \leq k^*$, it also holds that each V' is such that $|V'| \geq |V^*|$ (as groups that deserve winners are those with $|V'|/k' \geq n$, so if k' gets smaller it means that V' shall become larger). Thus, this means that the algorithm already explicitly took care of at least k^* groups V', each with size at least equal to $|V^*|$; crucially, this holds as all these groups V' are disjoint in voters (as the algorithm prunes out $\mathcal{V}$, the group of satisfied voters). Furthermore, observe that $|V^*| \geq n/k^*$ (as V^* is a group that deserves a winner). Thus, this means that the number of voters

the algorithm already took care of is at least $k^* \cdot (n/k^*) = n$; however, this contradicts the assumption that the algorithm now encounters some V^*, a group of unsatisfied voters.

In effect, the algorithm iterates over the 2^n possible subsets of voters and prunes the satisfied groups of voters before commencing each iteration while selecting disjoint subsets of voters. Thus, the number of iterations that the algorithm performs is bounded by 2^n. ∎

IV. EMPIRICAL DIFFICULTY

Following Conjectures 1, 2 and 3, we set to investigate the empirical difficulty of satisfying the associated axioms using simple algorithms on synthetic instances. Specifically, we evaluate the extent to which the Dynamic All-periods-intersection PJR axiom (DynamicAllPeriodPJR for short; Definition 5), Some-periods-intersection JR axiom (DynamicSomePeriodJR for short; Definition 6) and the Dynamic Some-periods-intersection PJR axiom (Dynamic-SomePeriodPJR for short; Definition 7) can be satisfied by the simple GreedyCC algorithm (Definition 2), a naïve modification of GreedyCC (which we describe next, denoted PGreedyCC) and a baseline random winner determination algorithm (denoted Rand) which uniformly selects a winner for each time period. The PGreedyCC works as follows:

PGreedyCC. In each iteration, look at the first time period for which the winner is still undetermined (j). Look for a candidate c approved by the most number of voters in that period, denoted V'. Count the number of periods in which voters V' has a non-empty intersection (k'). Starting at period j, for $\lfloor \frac{|V'|}{|V|} k' \rfloor$ consecutive periods, choose an arbitrary winner from the intersection of V' preferences on each period (if non-empty, otherwise choose from the union), and remove V' from further consideration.

To generate a sensible dataset we use the following procedure: For $m \in \{5, 10\}$ alternatives, $k \in \{10, 30, 50\}$ periods, and $n \in \{10, 15\}$ voters, we first generate a random non-empty subset of approved alternatives (V_i^j) for each of the voters. Then, for the Dynamic settings, for each period $k' > 1$ we (slightly) modify each voter's approved alternatives based on the voter's approved alternatives in period $k' - 1$: Each alternative approved in period $k' - 1$ continues to be approved in period k' with probability p and each non-approved alternative in period $k - 1$ joins the set of approved alternatives in period k' with probability q. For the sake of conciseness, we only present the results for $p = 0.5, q = 0.25$ (which are very similar to other reasonable p, q values we tested). The process is repeated $1,000$ times, resulting in $12,000$ instances.

For each instance, each of the three algorithms (Rand, GreedyCC and PGreedyCC) are used to determine a winner for each period. Then, we iterate over all the exponentially-many possible subgroups of voters to examine which of the axioms are satisfied by each of the algorithms' outputs.

To generate a sensible dataset we use the following procedure: For $m \in \{5, 10\}$ alternatives, $k \in \{10, 30, 50\}$ periods, and $n \in \{10, 15\}$ voters, we first generate a random non-empty subset of approved alternatives (V_i^j) for each of the voters. Then, for the Dynamic settings, for each period $k' > 1$ we (slightly) modify each voter's approved alternatives based on the voter's approved alternatives in period $k' - 1$: Each alternative approved in period $k' - 1$ continues to be approved in period k' with probability p and each non-approved alternative in period $k - 1$ joins the set of approved alternatives in period k' with probability q. For the sake of conciseness, we only present the results for $p = 0.5, q = 0.25$ (which are very similar to other reasonable p, q values we tested). The process is repeated $1,000$ times, resulting in $12,000$ instances.

For each instance, each of the three algorithms (Rand, GreedyCC and PGreedyCC) are used to determine a winner for each period. Then, we iterate over all the exponentially-many possible subgroups of voters to examine which of the axioms are satisfied by each of the algorithms' outputs.

A. RESULTS

Starting with the DynamicAllPeriodPJR axiom, we find that all evaluated algorithms, including Rand, were able to satisfy 100% of the instances. Specifically, despite Conjecture 1, it seems that the requirement is very easily satisfiable from a practical perspective, even by randomly selecting winners.

Continuing with the more demanding DynamicSomePeriodJR axiom, we see that the Rand algorithm was able to comply with the axiom for 73% of the dataset. The GreedyCC complied with the axiom slightly more times, satisfying 79% of the dataset and its adaptation PGreedyCC satisfied 84% of the times. To compare between the three conditions we use the repeated measures analysis of variance (ANOVA) test [21] followed by the Tukey's HSD post-hoc test [22]. All of the measures for both conditions were found to be distributed normally according to the Shapiro–Francia normality test [23]. The ANOVA test determined that there were significant differences between the algorithms ($p < 0.05$). The post-hoc test revealed that the difference is attributed to PGreedyCC which was found to outperform both other algorithms, which in turn, do not differ significantly ($p < 0.05$). See Figure 1 for an illustration. Analysing the cases in which each algorithm failed to comply with axiom reveals that the PGreedyCC algorithm was not only able to comply more often, but the average number of unrepresented groups of voters *in the unsatisfied instances* was significantly lower compared to both other algorithms. Specifically, PGreedyCC averaged 1.6 unrepresented groups (in the cases which the axiom was not satisfied) while the Rand and GreedyCC averaged 2 unrepresented groups. The difference is found to conform with the above statistical analysis, with PGreedyCC outperforming the other algorithms which, in turn, do not differ significantly. No statistically significant difference was found between the average and maximal sizes of unrepresented groups among the examined algorithms in cases when the axiom is violated.
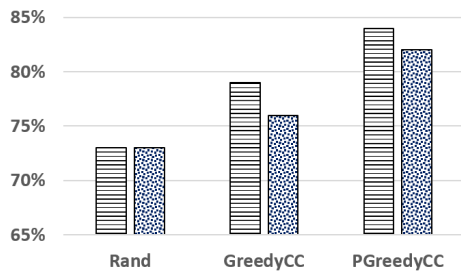
FIGURE 1. Compliance rates according to our empirical evaluation. Bars with horizontal lines indicate the compliance with the DynamicAllPeriodJR axiom, doted bars indicate the compliance with the DynamicAllPeriodPJR axiom.

Turning to the stronger DynamicSomePeriodPJR axiom, we see very similar results. Once more, the Rand was able to satisfy the axiom for 73% of the dataset. The GreedyCC complied with the axiom slightly more times, satisfying 76% of the dataset and its adaptation PGreedyCC satisfied 82% of the times. Using the same statistical analysis as before, we find the PGreedyCC algorithm outperforms both other algorithms which, in turn, do not differ significantly ($p < 0.05$). See Figure 1 for an illustration. In addition, the PGreedyCC algorithm again outperforms both other algorithms in terms of the number of under-represented groups of voters (in cases where the axiom is violated) averaging 1.7 compared to 2.1 averaged by both Rand and GreedyCC algorithms ($p < 0.05$). Finally, no statistically significant difference was found between the average and maximal sizes of unrepresented groups among the examined algorithms in cases when the axiom is violated.

V. HUMAN STUDY

In the following, we examine what ordinary people consider to be desirable outcomes in PV settings. To this end, we devise a simple yet natural PV scenario where, given voter preferences, ordinary people are tasked with choosing the most appropriate outcome. Specifically, we use the following motivational scenario in our experiment:

Motivating Scenario: Five students are about to study towards a test for the next five days. In order to save time, in each of the study days, the students wish to decide on a single restaurant to order food from. There are five potential restaurants (Restaurant A, B, . . ., E; e.g., Pizza, Hamburger, etc.). To this end, each student has indicated, for each day, which restaurant(s) she wants to order from. Your task is to help the students decide where to order from in each of the five days.[1]

The above scenario was devised following this rationale: by using generic restaurants and student identities we seek to avoid participants casting their own preferences into the decision setting. In addition, since no explicit utility is defined, we leave it up to the participants to decide what they

---

[1] The number of days and restaurants in our scenario were chosen empirically following a short informal trial-and-error investigation with students in our labs.

constitute as appropriate in this generic setting without introducing potentially biasing terms such as "fairness".

Following the motivating scenario's text, participants were presented with an example instance, as shown in Figure 2.

A. DESIGN

Following the motivational scenario outlined above, we first randomly generated two sets of PV instances (one for each of the examined PV variants – Static and Dynamic), each including 5 voters with preferences over 5 options, for 5 days. To examine different underlining aggregation axioms, each generated instance was first solved using five different aggregation methods, as detailed next. Then, six instances for which none of the five outputs coincide were selected. Overall, 2 Static PV instances and 4 Dynamic PV instances (2 for each of interpretations discussed in Section III-B) were selected.

Starting with the Static PV variant, each instance was solved using the following five aggregation methods:

- MAX: For each day, the candidate with the maximal number of approvals is chosen. This aggregation method maximizes the social utility if we assume that the utility of each voter is defined to be the number of days a restaurant from her ballot is selected (i.e., the number of days that voter is satisfied).
- JR: We use the GreedyCC algorithm (Definition 2), which satisfies JR.
- PJR: We use the slightly modified GreedyCC algorithm (Observation 1), which satisfies PJR.
- *P.Quota*. We use an algorithm recently proposed by [4], which satisfies JR but not PJR. The algorithm iterates over days, successively decreasing the weights of already satisfied voters on previous days. Perhaps this algorithm can be characterized as a *weighted*-MAX rule, where a voter's marginal utility diminishes as she gets more satisfied with the solution. While there are many temporally axiomatic characteristics allied with this rule as mentioned in [4], here we concentrate on the ability of the algorithm to elect justifiably representative solutions albeit in temporal settings.
- RAND: We randomly choose a solution such that the social utility, as measured by the number of days each voter is satisfied, falls between 50% and 70% of that achieved by MAX. The outcome is thus "reasonable" in terms of social utility.

Two instances for which the outcomes did not coincide were randomly selected – i.e., we looked for instances for which each of the rules selects a different outcome, and chose 2 such instances.

Similarly, each of the Dynamic PV instances was solved using the same five methods discussed; yet, this time, it was solved *twice* - once for each of the interpretations discussed in Section III-B. Namely, each instance was solved using the following 5 aggregation methods where JR and PJR are once used following their all-periods-intersection

	Monday	Tuesday	Wednesday	Thursday	Friday
Student 1	C	C	C	C	A,C
Student 2	A	A	A	A	B
Student 3	A,B	A	A,C	A	A
Student 4	A	A	B	C	A
Student 5	B,C	B	B	B,C	A

FIGURE 2. An example instance presented to our participants.

interpretation and once, separately, under the same-period-intersection interpretation.

- MAX.
- JR: We use either the algorithm provided in Proposition 2 (for the All-Period-Intersection interpretation) or the algorithm provided in Theorem 1 (for the Some-Period-Intersection interpretation), both satisfying JR under the corresponding interpretation.
- PJR: We use the algorithm described in Proposition 3 (for the All-Period-Intersection interpretation) or the algorithm described in Theorem 2, both satisfying PJR under the corresponding interpretation.
- *P.Quota* (see [4]).
- RAND.

Two instances for which the outcomes under the All-Period-Intersection interpretation did not coincide and additional two instances for which the outcomes under the Some-Period-Intersection interpretation did not coincide were randomly selected.

Prior to the presentation of the six instances (the instances appear in the Appendix), participants were asked to provide their age, gender, and field of studies (as discussed next, all participants were university students). Then, the selected six instances were presented to the participants, in a random order, along with the motivational scenario discussed above. Participants were asked to choose the most appropriate solution, i.e., decide where the students should order food from in each day. To simplify the process, each instance was accompanied by the five outputs calculated using the aggregation methods above, in a random order as well, from which each participant had to choose the most appropriate one (as she sees fit).

Furthermore, following the six instances, participants were asked to describe the reasoning behind their selection in free text (this question was not mandatory). So, our survey has two parts: one, in which participants see an instance and select the solution they see as most appropriate for the instance; and another, in which participants verbally explain their choices.

The survey was administered during the months of October and November 2020 to three participant groups:

1) Information Science Master's students from Bar-Ilan University (20 students, 18 male, average age of 35); 2) Industrial Engineering and Management Bachelor's students from Ben-Gurion University (22 students, 8 male, average age of 26); and 3) Computer Science students from Ariel and Bar-Ilan Universities (108 students, 78 male, average age of 26, approximately half from each university). The use of three groups of students allows us to examine potential differences between the groups (e.g., the former two groups consist of more female students than male, the first group is older on average, etc.). All groups were recruited by posting ads on the courses' webpages given by the authors, offering them a chance of winning one of three gift-cards, each of 100NIS (~$30), in a raffle.

B. RESULTS
1) PREPROCESSING

We start our analysis by omitting all answers of participants who completed the survey in an unreasonable time (in particular, in less than 3 minutes) as to avoid possibly under-quality responses. Fortunately, only very few participants were omitted in this phase (3 from all groups combined).

Recall that, in addition to their selections, participants were asked to describe the reasoning behind their selection in free text. In order annotate the provided text, the authors have randomly selected a subset of about two dozens of the provided explanations and identified 5 underlining rationales: *Maximizing social welfare* (e.g., "satisfy as many students as possible every day"); this roughly corresponds to the MAX and RAND aggregation methods, *Egalitarian considerations* (e.g., "Every student should get food from one of her requested restaurants at least once"; this roughly corresponds to the JR, PJR and PQUOTA aggregation methods), *Temporal considerations* (e.g., "A student should not be unhappy two days in a row (if possible)".), *Ballot size considerations* (e.g., "'Flexible' students who approve many restaurants should be prioritized" and *Diversity considerations* (e.g., "The same restaurant should not be selected too often"). The authors AR and NT have separately annotated each of the provided texts to the appropriate subset of rationals expressed in the text. Out

TABLE 1. The table shows the distribution of the aggregation method selected by the participants (rows) divided by their groups (columns). Columns do not necessarily sum up to 1 due to rounding.

Method	Information Sci.	Computer Sci.	Industrial Eng.	Overall
MAX	38.9%	43.5%	48.4%	43.4%
JR	5.6%	7%	4%	6.6%
PJR	17.6%	18.1%	18.3%	18%
P.QUOTA	30.6%	25.7%	27%	26.5%
RAND	7.4%	5.6%	2.4%	5.5%

of 172 provided texts, the 2 annotations perfectly matched on 152 (88%). The author NH settled the remaining 20 instances.

2) CHOSEN OUTCOMES

Effectively, each participant has selected only one "most appropriate" outcome in each of the six presented instances. Through that selection, the participant has implicitly indicated which aggregation method had brought about the most appropriate outcome (according to her taste). We summed the number of times each aggregation method was chosen by the participants of each participant group. The results are presented in Table 1.

Starting with the Information Sci. group, using the Friedman test followed by post hoc Wilcoxon signed-rank test[2] with Bonferroni correction, reveals that the MAX, PJR and PQouta methods were chosen significantly more often than the JR and RAND methods, $p < 0.05$. None of these three methods was found to be chosen significantly more often than the other, mainly due to the low number of participants in this group. Similarly, no significant difference was found between JR and RAND. Using the same statistical analysis, more significant differences are found in the Computer Sci. and Industrial Eng. groups. Specifically, for both groups separately, we find that the MAX method was chosen significantly more often than any other method, $p < 0.05$. In turn, the PJR and PQouta methods were chosen significantly more often than the JR and RAND methods, which in turn, do not significantly differ. No significant difference is found between the PJR and PQouta methods.

As can be seen in Table 1 and in the above statistical analysis, there is very little difference between the examined groups. The results suggest that MAX is generally preferred to any other examined method by our study participants, regardless of the examined group. At the same time, *more than half* of the participants' choices correspond to one of the "fairness" interpretations that are directly captured in our proposed theoretical framework (i.e., JR, PJR, and PQuota – recall that the latter satisfies JR). (Note that, while PQuota also satisfies JR, the outcomes are not necessarily coinciding as there might be several solutions satisfying JR.) Combining these results with the fact that no significant difference were found when comparing PJR and PQuota leads us to believe that our participants display two central tendencies: first, outcomes which maximize social welfare are generally deemed

TABLE 2. Participants' individual consistency in their selections. Rows denote the maximal number of participant's selections, which coincide on the same aggregation method divided by participant group (columns). For example, in the Industrial Eng. group, 25% of the participants chose exactly four outcomes (out of 6) that coincide on the same aggregation method. Columns do not necessarily sum up to 0 due to rounding.

Method	Information Sci.	Computer Sci.	Industrial Eng.	Overall
6	0%	4.3%	5%	3.8%
5	11.8%	11.3%	20%	12.6%
4	35.3%	29.1%	25%	29%
3	29.4%	39.7%	40%	39.4%
2	23.5%	15.6%	10%	15.3%

appropriate by our participants; second, outcomes that satisfy some notion of fair representation are generally preferred to those that do not satisfy them.

3) INDIVIDUAL CONSISTENCY

We further examine the participants' *individual consistency*: that is, we look at how consistent participants were in their six selections, in terms of the preferred aggregation methods. In particular, we say that a participant is:

- *inconsistent* if no more than 2 of her selections correspond to the same aggregation method;
- *reasonably consistent* if 3 or 4 of her selections coincide on the same method;
- and *consistent* if at least 5 of her selections coincide on the same method.

For all groups, approximately 65% of the participants were reasonably consistent; see Table 2.

Overall, despite the inherent complexity of the task, most participants were reasonably consistent in their choices. This result strengthen our confidence in our analysis.

For all groups, no significant differences were found between men and women. Similar results are observed when all participants are grouped together.

4) EXPLANATIONS

We turn to analyze the participants' provided explanations as to what guided their selections. Recall that the provided free-form texts were annotated by the authors. Specifically, each text was labeled using at least one of the following underlining rationales: *Maximizing social welfare*, *Egalitarian considerations*, *Temporal considerations*, *Ballot size considerations*, and *Diversity considerations*.

Out of the 172 provided texts (recall, this question was not mandatory), 141 (82%) were labeled as *Maximizing social welfare*, 73 (42%) were labeled as *Egalitarian*

---

[2]The use of Friedman and Wilcoxon tests is since normality cannot be adequately assumed for the collected data.

יום ה'	יום ד'	יום ג'	יום ב'	יום א'	
A,B	A,B	A,B	A,B	A,B	סטודנט 1
A,C	A,C	A,C	A,C	A,C	סטודנט 2
B,C	B,C	B,C	B,C	B,C	סטודנט 3
A	A	A	A	A	סטודנט 4
B	B	B	B	B	סטודנט 5
A	C	A	C	C	אפשרות 1
A	A	B	B	B	אפשרות 2
A	C	B	A	B	אפשרות 3
B	B	B	B	B	אפשרות 4
B	B	B	A	B	אפשרות 5

FIGURE 3. Static instance: 1 out of 2.

considerations, 8 (5%) were labeled as *Temporal considerations*, 12 (7%) were labeled as *Ballot size considerations* and 14 (8%) were labeled as *Diversity considerations*. Interestingly, 75% of those who mentioned Egalitarian considerations have also indicated that they are maximizing social welfare. In addition, all participants who have mentioned one of the last three consideration types (i.e., Temporal, Ballot size, or Diversity) have indicated only one of which and have also mentioned maximizing social welfare in their answer.

The above results are very much aligned with the participants' selections analyzed above. Specifically, the vast majority of participants have indicated that they are interested in maximizing some form of social welfare, a statement which can be easily expressed in many natural ways. However, at the same time, nearly half of the participants have also indicated Egalitarian considerations. It is important to note that one cannot expect ordinary people to articulate different notions of fairness (in particular, JR or PJR), especially without prior knowledge in the field and as part of a short Internet-based survey. The fact that very few participants have expressed any other type of consideration (i.e., Temporal, Ballot size, or Diversity) further supports our previous results, indicating that our participants consider both aspects desirable.

No significant differences were found between the examined groups or between men and women.

VI. CONCLUSION
We proposed and studied several adaptations of the well-known multiwinner axioms of JR and PJR to two perpetual voting settings: Static and Dynamic. For the Static case, where voter preferences do not change over time, we showed that our natural adaptations of JR and PJR can always be satisfied in polynomial time. For the more complex and perhaps more realistic Dynamic setting, where voter preference do change over time, we showed that some of our adaptations can always be satisfied in polynomial time while the rest can be always satisfied in general. Using very simple heuristics, we showed that, despite having no polynomial time algorithm for our more demanding adaptations, all our adaptations can nevertheless be empirically satisfied in a large majority of the cases we considered.

The human study provides strong evidence to support the conclusion that fairness, potentially in the form of JR and PJR, is desirable.

A. FURTHER WORK AND EXTENSIONS
We believe that PV, as studied here and through additional interpretations and possible extensions, is an important yet currently understudied social choice setting with various application to the study of multi-agent decision-making, human-agent interaction, and mechanism design, to name a few. Next, we briefly discuss some directions for future research.

1) FIELD EXPERIMENTATION
Following [24], we plan to conduct a field experiment in a Mars analogue environment. The analogue mimics real space habitation by applying artificially delayed

יום ה'	יום ד'	יום ג'	יום ב'	יום א'	
B	B	B	B	B	סטודנט 1
A,C	A,C	A,C	A,C	A,C	סטודנט 2
B,C	B,C	B,C	B,C	B,C	סטודנט 3
A	A	A	A	A	סטודנט 4
A,C	A,C	A,C	A,C	A,C	סטודנט 5
A	A	A	B	A	אפשרות 1
B	B	A	A	A	אפשרות 2
A	A	A	A	A	אפשרות 3
B	C	A	B	A	אפשרות 4
B	C	B	C	C	אפשרות 5

FIGURE 4. Static instance: 2 out of 2.

יום ה'	יום ד'	יום ג'	יום ב'	יום א'	
C	C	C	C	A,C	סטודנט 1
A	A	A	A	B	סטודנט 2
A,B	A	A,C	A	A	סטודנט 3
A	A	B	C	A	סטודנט 4
B,C	B	B	B,C	A	סטודנט 5
A	A	A	A	A	אפשרות 1
A	C	A	A	A	אפשרות 2
A	A	A	C	A	אפשרות 3
C	A	A	C	A	אפשרות 4
A	B	B	A	B	אפשרות 5

FIGURE 5. Dynamic instance: 1 out of 4.

communication with ground control, enforcing strict operation protocols (e.g., the use of spacesuits and relevant equipment), and other means aimed at replicating space conditions. We propose to integrate a decision-support system, relying on our findings, as part of a future mission through which we could examine individual satisfaction and social fairness in this real-world perpetual voting setting.

יום ה'	יום ד'	יום ג'	יום ב'	יום א'	
A,B	A	A	A,C	A,B	סטודנט 1
A,B	A,B	C	C	B,C	סטודנט 2
A,C	A,C	A	A	C	סטודנט 3
B,C	C	B,C	B,C	B,C	סטודנט 4
C	C	B	B	B	סטודנט 5
A	A	A	C	B	אפשרות 1
A	B	B	B	B	אפשרות 2
C	C	B	B	C	אפשרות 3
A	C	C	A	B	אפשרות 4
A	A	A	A	B	אפשרות 5

FIGURE 6. Dynamic instance: 2 out of 4.

יום ה'	יום ד'	יום ג'	יום ב'	יום א'	
A,C	A,C	A,B,C	B,C	B,C	סטודנט 1
B	B	C	A	A,B,C	סטודנט 2
A	A,C	A,B	A	A,C	סטודנט 3
C	A	A,B	A,B	A	סטודנט 4
B	C	C	C	A	סטודנט 5
B	A	C	B	A	אפשרות 1
A	A	A	B	A	אפשרות 2
A	C	B	B	A	אפשרות 3
A	B	B	B	C	אפשרות 4
A	A	A	A	A	אפשרות 5

FIGURE 7. Dynamic instance: 3 out of 4.

2) DESIGNING EFFICIENT ALGORITHMS

As some of our adaptations seem computationally intractable, it is natural to study further parameterizations, as well as the possibility of approximation algorithms.

3) DOMAIN RESTRICTIONS

In fact, our Static PV setting can be considered as a domain restriction. Generally speaking, studying further domain restrictions for PV might help in better understanding PV in

יום ה'	יום ד'	יום ג'	יום ב'	יום א'	
A,C,E	A,C,E	A	A,B,C,D	B,D	סטודנט 1
C,D	B,C,D	B,C,D	B,C,D	B,D	סטודנט 2
A,B,C	A,B,E	A,E	A,D,E	A,B,C,D	סטודנט 3
D	C	C	C,D,E	A,B	סטודנט 4
A,B	A,B,D	A	A,D	A,D,E	סטודנט 5
B	D	A	A	E	אפשרות 1
D	B	A	D	A	אפשרות 2
A	A	A	D	B	אפשרות 3
A	E	A	D	B	אפשרות 4
A	A	A	A	B	אפשרות 5

FIGURE 8. Dynamic instance: 4 out of 4.

general. E.g., it is natural to consider single-peaked preferences in the context of PV.

4) BALANCING SOCIAL WELFARE AND FAIRNESS

A natural future study would be to investigate algorithms that aim at balancing social welfare and proportionality; this is especially meaningful given our human study results.

5) ORDINAL PERPETUAL VOTING

PV settings in which voters provide ordinal preferences, as opposed to the Approval model considered here, is another natural model for PV. In particular, adapting the proportionality axioms of this paper to the ordinal preferences model may reveal new insights for PV in general.

6) ONLINE PERPETUAL VOTING

Here we concentrated on an *offline* model, in which voter preferences are given completely as an input to the aggregation method. A natural variant of our model is an *online* model, in which the algorithm has to make its decision for each day based only on the preferences of the voters up to that day, and voter preferences for a specific day may depend on the selected winners so far.

APPENDIX
HUMAN STUDY INSTANCES

The six instances used in our human study (see Section V) appear from Fig. 3 - Fig. 8. The instances are presented in Hebrew (and thus should be read from right to left) yet they are easily understandable. For each instance, the first five rows indicate the voter's preferences over the five days (columns) whereas the next five rows indicate the five outputs of the examined aggregation methods.

REFERENCES

[1] M. Lackner, "Perpetual voting: Fairness in long-term decision making," in *Proc. AAAI*, 2020, pp. 2103–2110.

[2] H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh, "Justified representation in approval-based committee voting," *Social Choice Welfare*, vol. 48, no. 2, pp. 461–485, Feb. 2017.

[3] P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon, "Multiwinner voting: A new challenge for social choice theory," in *Trends in Computational Social Choice*, U. Endriss, Ed. Segundo, CA, USA: AI Access Foundation, 2017.

[4] P. Harrenstein, M.-L. Lackner, and M. Lackner, "A mathematical analysis of an election system proposed by Gottlob Frege," 2019, *arXiv:1907.03643*. [Online]. Available: http://arxiv.org/abs/1907.03643

[5] M. Brill, R. Freeman, S. Janson, and M. Lackner, "Phragmén's voting methods and justified representation," in *Proc. AAAI*, 2017, pp. 406–413.

[6] L. Sánchez-Fernández, E. Elkind, M. Lackner, N. Fernández, J. A. Fisteus, P. B. Val, and P. Skowron, "Proportional justified representation," in *Proc. AAAI*, 2017, pp. 670–676.

[7] M. Brill, J.-F. Laslier, and P. Skowron, "Multiwinner approval rules as apportionment methods," *J. Theor. Politics*, vol. 30, no. 3, pp. 358–382, Jul. 2018.

[8] D. C. Parkes and A. D. Procaccia, "Dynamic social choice with evolving preferences," in *Proc. AAAI*, 2013, pp. 767–773.

[9] I. Kash, A. D. Procaccia, and N. Shah, "No agent left behind: Dynamic fair division of multiple resources," *J. Artif. Intell. Res.*, vol. 51, pp. 579–603, Nov. 2014.

[10] M. Guo, V. Conitzer, and D. M. Reeves, "Competitive repeated allocation without payments," in *Proc. WINE*, 2009, pp. 244–255.

[11] D. C. Parkes, R. Cavallo, F. Constantin, and S. Singh, "Dynamic incentive mechanisms," *AI Mag.*, vol. 31, no. 4, pp. 79–94, 2010.

[12] R. Freeman, S. M. Zahedi, and V. Conitzer, "Fair social choice in dynamic settings," in *Proc. IJCAI*, 2017, pp. 4580–4587.

[13] V. Conitzer, R. Freeman, and N. Shah, "Fair public decision making," in *Proc. ACM Conf. Econ. Comput.*, Jun. 2017, pp. 629–646.

[14] J. Scheuerman, J. L. Harman, N. Mattei, and K. B. Venable, "Heuristic strategies in uncertain approval voting environments," 2019, *arXiv:1912.00011*. [Online]. Available: http://arxiv.org/abs/1912.00011

[15] J. Scheuerman, J. Harman, N. Mattei, and K. B. Venable, "Modeling voters in multi-winner approval voting," 2020, *arXiv:2012.02811*. [Online]. Available: http://arxiv.org/abs/2012.02811

[16] M. Tal, R. Meir, and Y. K. Gal, "A study of human behavior in online voting," in *Proc. Int. Conf. Auto. Agents Multiagent Syst.*, 2015, pp. 665–673.

[17] J. Zou, R. Meir, and D. Parkes, "Strategic voting behavior in Doodle polls," in *Proc. 18th ACM Conf. Comput. Supported Cooperat. Work Social Comput.*, 2015, pp. 464–472.

[18] U. Grandi, J. Lang, A. Ozkes, and S. Airiau, "Voting behavior in one-shot and iterative multiple referenda," *SSRN Electron. J.*, to be published.

[19] A. Rosenfeld and S. Kraus, "Predicting human decision-making: From prediction to action," *Synth. Lectures Artif. Intell. Mach. Learn.*, vol. 12, no. 1, pp. 1–150, Jan. 2018.

[20] M. R. Garey and D. S. Johnson, *Computers and Intractability*, vol. 29. 2002.

[21] E. R. Girden, *ANOVA: Repeated measures*. no. 84. Newbury Park, CA, USA: Sage, 1992.

[22] H. Abdi and L. J. Williams, "Tukey's honestly significant difference (HSD) test," in *Encyclopedia of Research Design*. Thousand Oaks, CA, USA: Sage, 2010, pp. 1–5.

[23] S. S. Shapiro and R. S. Francia, "An approximate analysis of variance test for normality," *J. Amer. Stat. Assoc.*, vol. 67, no. 337, pp. 215–216, Mar. 1972.

[24] A. Rosenfeld, "Human-agent interaction for human space exploration," in *Proc. Adjunct Publication 27th Conf. User Modeling, Adaptation Personalization UMAP Adjunct*, 2019, pp. 9–12.

RUTVIK PAGE is currently pursuing the degree with the Department of Computer Science and Engineering, Indian Institute of Information Technology, Nagpur, India. He serves as a Research Assistant with the Dr. Talmon's Social Choice Laboratory, Ben Gurion University of the Negev, Be'er Sheva, Israel.

ARIEL ROSENFELD received the B.Sc. degree *(magna cum laude)* in computer science and economics from Tel-Aviv University, Israel, and the Ph.D. degree in computer science from Bar-Ilan University, Israel. Before joining Bar-Ilan, he was a Koshland Postdoctoral Fellow with the Computer Science and Applied Mathematics Department, Weizmann Institute of Science, Israel. He is currently an Assistant Professor with the Information Science Department, Bar-Ilan University. His research interest includes human-agent interaction and he has coauthored over 40 articles on the topic. He chaired the 18th European Conference on Multi-Agent Systems and served as a senior program committee member and a program committee member in other leading AI conferences. He has a rich lecturing background which spans over a decade and has recently been awarded the Best Lecturer Award for Outstanding Teaching by Bar-Ilan University. He was a recipient of the IFAAMAS Distinguished Dissertation Award for 2017.

LAURENT BULTEAU received the Ph.D. degree in computer science from Nantes University. He has been a CNRS Research Fellow with Université Gustave Eiffel, France, since 2015. Previously, he was a Postdoctoral Fellow with TU Berlin and Lyon University. His research interests include exact algorithms and complexity theory, with a special interest on questions based on sequence analysis.

NOAM HAZON received the B.Sc., M.Sc., and Ph.D. degrees in computer science from Bar Ilan University, Israel, in 2003, 2005, and 2010, respectively. He was a Postdoctoral Fellow with CMU, Pittsburgh, PA. During his M.Sc. degree, he worked as a Software Engineer with Applied Materials, Israel. He is currently a Senior Lecturer with Ariel University, Israel. He has won an ISF grant, in 2014, a planning grant from Volkswagen Foundation, in 2018, and a grant from the Ministry of Science, Technology and Space, Israel, in 2019. He has coauthored over 40 articles. His research interests include computational social choice, human–computer interactions, coalitional game theory, stochastic search, influence maximization, and multi-robot coverage. He serves as a senior program committee member and a program committee member for the leading AI conferences.

NIMROD TALMON received the Ph.D. degree in computer science from TU Berlin. He is currently an Assistant Professor with the Industrial Engineering and Management Department, Ben-Gurion University, Israel. Before joining Ben-Gurion University, he was a Postdoctoral Fellow with the Computer Science and Applied Mathematics Department, Weizmann Institute of Science, Israel. His research interests include computational social choice, social networks, and combinatorial optimization. He has Erdos number three, Sabbath number seven, and Bacon number six.

• • •