IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received June 3, 2021, accepted June 18, 2021, date of publication June 28, 2021, date of current version July 13, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3092745

Interval Solution to Fuzzy Relation Inequality With Application in P2P Educational Information Resource Sharing Systems

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This work was supported in part by the National Natural Science Foundation of China under Grant 61877014; in part by the Department of Education of Guangdong Province under Grant 2017KTSCX124, Grant 2020GXJK385, Grant ZD201802, and Grant 2019KZDXM013; and in part by Hanshan Normal University under Grant XS202008.

ABSTRACT Max-min fuzzy relation inequalities have recently been introduced to describe the peer-to-peer (P2P) educational information resource sharing systems. It is well known that the complete solution set of the max-min fuzzy relation system is fully determined by its minimal solutions. However, solving all the minimal solutions has been proven to be equivalent to the set-covering problem, which is NP-hard. Without solving the complete solution set, some specific solutions can be obtained through the corresponding fuzzy relation optimization problems. However, these solutions are usually unstable and fragile. Any minor fluctuations to the components of these specific solutions will cause them to be no longer feasible. In this work, we define and study the widest interval solution of a max-min fuzzy relation inequality system for the first time. The interval solution allows the solution to fluctuate within some range. The fluctuation range is determined by the width of the interval solution. We propose a novel resolution method for searching for the widest interval solution. The resolution method is carried out by detailed procedures and illustrated by a numerical example.

INDEX TERMS Fuzzy relation inequality, fuzzy relation equation, max-min composition, interval solution, widest interval solution.

I. INTRODUCTION

In classical algebra, the composition operation is typical addition-multiplication. However, in various application fields, it has been found that the addition-multiplication composition is defective and unsuitable for modeling the quantitative relation. The max-min composition became an effective substitute for the addition-multiplication relation. Correspondingly, the relevant max-min algebra has attracted some scholars' attention [14]–[17], [35]. The max-min composition and mechanism have been widely applied in engineering management and control [8]–[13].

The max-min composition was introduced to the linear equation system by Sanchez [19], [20] for the first time. It was named max-min fuzzy relation equations. The structure of the solution set for a max-min fuzzy relation equation system was much different from that for a classical linear equation system. It has been formally proven that the complete solution set of a consistent max-min system is a nonconvex set in most cases (when its minimal solutions are not unique). The solution set is fully determined by its unique maximum solution and a finite number of minimal solutions.

When first investigated by E. Sanchez, fuzzy relation equations were applied to medical diagnoses [20]. Fuzzy relation systems, including both equation systems and inequality systems, have been successfully applied for dealing with various kinds of practical problems, such as fuzzy inference systems [21], image compression and reconstruction [22], [23], medical diagnosis [24], [25], knowledge engineering [26], three-tier media streaming systems using HTTP protocols [27], peer-to-peer (P2P) network systems [28], BitTorrent-like peer-to-peer (BTP2P) file-sharing systems [29], [36]–[41], foodstuff supply [30]–[32], and wireless communication systems [33], [34].

Recently, the max-min fuzzy relation system was applied to educational information resource allocation [18], [46]

The associate editor coordinating the review of this manuscript and approving it for publication was Yiming Tang.

(see Fig. 1.). Assume the educational information resources stored in terminals, denoted by $A_1, A_2, \ldots, A_n$, are unbalanced. Each terminal is connected to any other terminal and free to download its required educational information resources. The bandwidth between the terminals A_i and A_j is assumed to be a_{ij}. That is, when the ith terminal A_i downloads its required resources from the jth terminal A_j, the actual quality level is

$$a_{ij} \wedge x_j,$$

where x_j (measure: Mbps) represents the quality level on which A_j shares (sends out) its local resources. $a_{ij} \wedge x_j$ denotes the receiving quality level at A_i from A_j. In general, A_i will select the terminal with the highest receiving quality level to download the resources. Additionally, we assume that the download traffic requirement is no less than b_i and no more than d_i. Then, the A_i requirement can be written as

$$b_i \leq (a_{i1} \wedge x_1) \vee (a_{i2} \wedge x_2) \vee \cdots \vee (a_{in} \wedge x_n) \leq d_i.$$

Without loss of generality, we also assume that some of the terminals, i.e., $\{A_1, A_2, \ldots, A_m\}$, request their required resources for downloading. As a consequence, their requirements can be represented by the following max-min fuzzy relation inequalities after normalization:

$$\begin{cases} b_1 \leq (a_{11} \wedge x_1) \vee (a_{12} \wedge x_2) \vee \cdots \vee (a_{1n} \wedge x_n) \leq d_1, \\ b_2 \leq (a_{21} \wedge x_1) \vee (a_{22} \wedge x_2) \vee \cdots \vee (a_{2n} \wedge x_n) \leq d_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_m \leq (a_{m1} \wedge x_1) \vee (a_{m2} \wedge x_2) \vee \cdots \vee (a_{mn} \wedge x_n) \leq d_m, \end{cases} \quad (1)$$

where $a_{ij}, x_j \in [0, 1]$, $0 < b_i \leq d_i \leq 1$, $i \in I, j \in J$, and

$$I = \{1, 2, \ldots, m\}, \quad J = \{1, 2, \ldots, n\}.$$

In [18], the authors investigated the approximate solution(s) of the max-min fuzzy relation equations system in cases where the system is inconsistent. In some existing works, the total deviations, i.e., the sum of all deviations with respect to each equation, were applied to characterize the approximate solution. Instead of the total deviations, the authors defined the approximate solution based on the largest deviation [18]. They also proposed a linear searching algorithm for obtaining the approximate solution of the inconsistent system.

Different from the inconsistent system studied in [18], the above system (1) considered in this paper is assumed to be consistent. We investigate the interval solution of system (1) in this work.

The novelty and contribution of this work can be summarized as follows.

(i) In this paper, the fluctuation of a given solution to the max-min fuzzy relation inequality system, i.e., system (1), is considered for the first time.

(ii) We define the interval solution concept for system (1). The width of an interval solution is designed to characterize the corresponding fluctuation range.

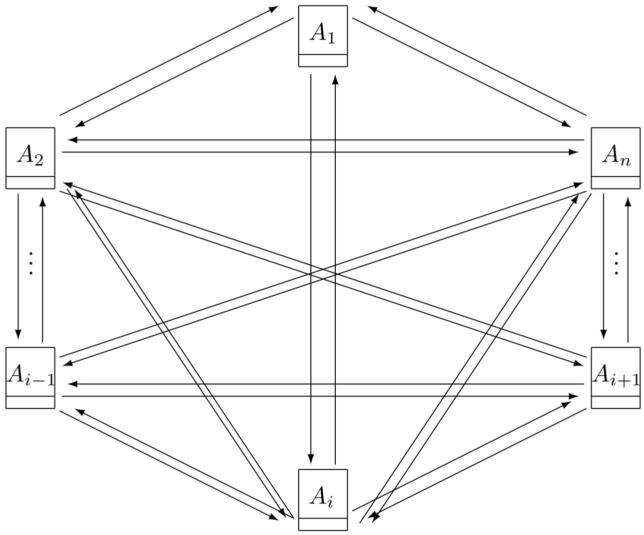

FIGURE 1. P2P educational information resource sharing system.

(iii) To maximize the solution fluctuation range, we define and investigate the widest interval solution of system (1). An effective and detailed resolution algorithm is proposed for searching the widest interval solution of system (1).

The remainder of this manuscript is organized as follows. Section 2 presents some existing concepts and results on max-min fuzzy relation inequalities. In Section 3, we define the widest interval solution and provide a sufficient and necessary condition for its existence. In Section 4, we propose a novel resolution method for obtaining the widest interval solution of system (1). We also give detailed resolution procedures and a numerical illustrative example. Section 5 concludes.

II. PRELIMINARY

System (1) can be written in its matrix form as

$$b^T \leq A \circ x^T \leq d^T,$$

where $A = (a_{ij})_{m\times n}$, $x = (x_1, x_2, \ldots, x_n)$, $b = (b_1, b_2, \ldots, b_m)$, $d = (d_1, d_2, \ldots, d_m)$ and "$\circ$" represents the max-min composition. Here, the order relation $\leq$ is defined as follows. For any $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in [0, 1]^n$, $x \leq y$ means that $x_j \leq y_j$ for all $j \in J$. In what follows, the set of all solutions of system (1) is denoted by $X(A, b, d)$, i.e.,

$$X(A, b, d) = \{x \in [0, 1]^n | b^T \leq A \circ x^T \leq d^T\}. \quad (2)$$

Definition 1 ([7]Consistency): System (1) is said to be consistent, if $X(A, b, d) \neq \emptyset$. Otherwise, it is said to be inconsistent.

Definition 2 ([7]Maximum/Minimal Solution): For system (1), a solution $\hat{x} \in X(A, b, d)$ is said to be the maximum solution if $x \leq \hat{x}$ for any $x \in X(A, b, d)$; a solution $\check{x} \in X(A, b, d)$ is said to be a minimal solution if for any $x \in X(A, b, d)$, $x \leq \check{x}$ implies that $x = \check{x}$.

To check the consistency of system (1), we introduce the vector $\hat{x}$ as follows. For arbitrary $j \in J$, let

$$I_j = \{i \in I | a_{ij} > d_i\}. \tag{3}$$

Furthermore, denote $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$, where

$$\hat{x}_j = \begin{cases} 1, & I_j = \emptyset, \\ \bigwedge_{i \in I_j} d_i, & I_j \neq \emptyset. \end{cases} \tag{4}$$

Then, we obtain the following Theorem 1.

Theorem 1 ([5]–[7]): System (1) is consistent if and only if the above-defined vector $\hat{x}$ is a solution of system (1). Moreover, if system (1) is consistent, $\hat{x}$ is exactly its unique maximum solution.

In fact, when system (1) is consistent, it always has a unique maximum solution and a finite number of minimal solutions. Moreover, the complete solution set of system (1) can be represented as indicated in the following Theorem 2

Theorem 2 ([5]–[7]): If system (1) is consistent, then its solution set can be represented by

$$X(A, b, d) = \bigcup_{\check{x} \in \check{X}(A,b,d)} [\check{x}, \hat{x}]. \tag{5}$$

Here, $\hat{x}$ is its unique maximum solution, and $\check{X}(A, b, d)$ denotes the set of all its minimal solutions.

As shown in Theorem 2, solving system (1) is equivalent to searching all its minimal solutions. However, it is difficult to obtain all the minimal solutions since they have been proven to be highly related to the set-covering problem [1]–[4]. Moreover, the number of minimal solutions to a consistent max-min fuzzy relation inequality (or equation) system is exponentially associated with the size of the system. It is hard and unnecessary to compute and represent the complete solution set of system (1).

III. WIDEST INTERVAL SOLUTION DEFINITION

As noted in the previous section, obtaining all the (minimal) solutions is difficult and unnecessary. Some researchers have paid attention to optimization problems subject to the fuzzy relation system [5], [6], [27], [35], [42]–[45]. By solving the corresponding fuzzy relation optimization problems, some specific solutions to the fuzzy relation system can be found. However, these specific solutions are usually unstable and fragile. When some minor fluctuation or perturbation occurs in any component of the specific solution, it might no longer be a max-min fuzzy relation system solution. To study the fluctuation in the solution of system (1), we define and investigate the widest interval solution in this work as follows.

Definition 3 (Width of an Interval): Let $[x', x''] \subseteq [0, 1]$ be an interval with $x' \leq x''$. We say $\min_{j \in J}\{x''_j - x'_j\}$ the width of the interval $[x', x'']$. Moreover, we denote the width of $[x', x'']$ by $w[x', x'']$.

Definition 4 (Interval Solution): Let $[x', x''] \subseteq [0, 1]$ be an interval with $x' \leq x''$. We say $[x', x'']$ an interval solution of system (1), if $[x', x''] \subseteq X(A, b, d)$.

Definition 5 (Widest Interval Solution): Let $[x^*, x^{**}] \subseteq X(A, b, d)$ be an interval solution of system (1). $[x^*, x^{**}]$ is the widest interval solution if $w[x^*, x^{**}] \geq w[x', x'']$ for any interval solution $[x', x''] \subseteq X(A, b, d)$.

Proposition 1: Let $[x', x''], [y', y''] \subseteq X(A, b, d)$ be two interval solutions of system (1). If $[x', x''] \subseteq [y', y'']$, i.e. $x' \geq y'$ and $x'' \leq y''$, then we have $w[x', x''] \leq w[y', y'']$.

Proof: The proof is straightforward following Definition 3. □

Theorem 3: Assume system (1) is consistent with the maximum solution $\hat{x}$. Then, there exists a minimal solution $\check{x}^* \in X(A, b, d)$, such that $[\check{x}^*, \hat{x}]$ is the widest interval solution of system (1).

Proof: Let $[x', x''] \subseteq X(A, b, d)$ be an arbitrary interval solution of system (1) with $x' \leq x''$. It is clear that $x', x'' \in X(A, b, d)$. Note that $\hat{x}$ is the maximum solution, which is

$$x'' \leq \hat{x}. \tag{6}$$

According to Theorem 2, $X(A, b, d) = \bigcup_{\check{x} \in \check{X}(A,b,d)} [\check{x}, \hat{x}]$, where $\check{X}(A, b, d)$ represents the minimal solution set. Since $x' \in X(A, b, d)$, there exists a minimal solution $\check{x}' \in \check{X}(A, b, d)$ such that $x' \in [\check{x}', \hat{x}]$, i.e.,

$$\check{x}' \leq x'. \tag{7}$$

(6) and (7) indicates that $[x', x''] \subseteq [\check{x}', \hat{x}]$. It follows from 1 that

$$w[x', x''] \leq w[\check{x}', \hat{x}]. \tag{8}$$

Note that $\check{X}(A, b, d)$ is a finite set. There exists $\check{x}^* \in \check{X}(A, b, d)$ such that

$$w[\check{x}^*, \hat{x}] = \max_{\check{x} \in \check{X}(A,b,d)} w[\check{x}, \hat{x}]. \tag{9}$$

Since $\check{x}' \in \check{X}(A, b, d)$, we have

$$\max_{\check{x} \in \check{X}(A,b,d)} w[\check{x}, \hat{x}] \geq w[\check{x}', \hat{x}]. \tag{10}$$

(8), (9) and (10) contribute to $w[\check{x}^*, \hat{x}] \geq w[x', x'']$. Due to the arbitrariness of $[x', x'']$, $[\check{x}^*, \hat{x}]$ is the widest interval solution of system (1). □

It is indicated in Theorem 3 that when system (1) is consistent, the widest interval solution can be obtained by selecting it from the minimal solution set by pairwise comparison. However, as noted in the last section, obtaining the minimal solution set is hard to achieve. To overcome such hardness, we propose a novel resolution method to find the widest interval solution of system (1) in the following.

IV. RESOLUTION METHOD BASED ON THE MAXIMUM SOLUTION AND INDEX SETS

In this section, to obtain the widest interval solution of system (1), we propose a resolution method based on the maximum solution and some index sets.

A. THEORETICAL ANALYSIS AND PROOF

Based on the maximum solution $\hat{x}$, we define the following index sets:

$$J_i = \{j \in J | a_{ij} \wedge \hat{x}_j \geq b_i\}, \tag{11}$$

for $i = 1, 2, \ldots, m$. Moreover, denote

$$P = J_1 \times J_2 \times \cdots \times J_m. \tag{12}$$

Proposition 2: If system (1) is consistent, then it holds that $J_i \neq \emptyset$ for any $i \in I$, i.e., $P \neq \emptyset$.

Proof: If system (1) is consistent, then it follows from Theorem 1 that $\hat{x} \in X(A, b, d)$. Observing system (1), it holds that

$$b_i \leq a_{i1} \wedge \hat{x}_1 \vee \cdots \vee a_{in} \wedge \hat{x}_n \leq d_i, \quad \forall i \in I. \tag{13}$$

Hence, for any $i \in I$, there exists some $j_i \in J$ such that $a_{ij_i} \wedge \hat{x}_{j_i} \geq b_i$. By (11), we have

$$J_i \neq \emptyset, \quad \forall i \in I, \tag{14}$$

i.e., $P \neq \emptyset$. □

Theorem 4: Let $X = [x', x''] \subseteq X(A, b, d)$ be an arbitrary interval solution of system (1). Then, there exists $p = (p_1, p_2, \ldots, p_m) \in P$ such that $[x', x''] \subseteq [x^p, \hat{x}]$, where $x^p = (x_1^p, x_2^p, \ldots, x_n^p)$ and

$$x_j^p = \begin{cases} 0, & \text{if } I_j^p \triangleq \{i \in I | p_i = j\} = \emptyset, \\ \bigvee_{i \in I_j^p} b_i, & \text{if } I_j^p \triangleq \{i \in I | p_i = j\} \neq \emptyset. \end{cases} \tag{15}$$

Proof: Since $[x', x'']$ is an interval solution and $\hat{x}$ is the maximum solution of system (1), it is obvious that

$$x' \leq x'' \leq \hat{x}, \tag{16}$$

and

$$b_i \leq (a_{i1} \wedge x_1') \wedge (a_{i2} \wedge x_2') \wedge \cdots \wedge (a_{in} \wedge x_n') \leq d_i, \quad \forall i \in I. \tag{17}$$

Hence, for any $i \in I$, it holds that

$$a_{ij} \wedge x_j' \leq d_i, \quad \forall j \in J, \tag{18}$$

and there exists $p_i \in J$ such that

$$a_{ip_i} \wedge x_{p_i}' \geq b_i. \tag{19}$$

Since $x' \leq \hat{x}$, we have

$$a_{ip_i} \wedge \hat{x}_{p_i} \geq a_{ip_i} \wedge x_{p_i}' \geq b_i, \quad \forall i \in I. \tag{20}$$

It follows from (11) that $p_i \in J_i$, $\forall i \in I$. Thus, $p = (p_1, p_2, \ldots, p_m) \in P$.

Next, we check that $x^p \leq x'$. Take arbitrary $j \in J$. Case 1. If $I_j^p = \emptyset$, then it is clear that $x_j^p = 0 \leq x_j'$. Case 2. If $I_j^p \neq \emptyset$, then we have $x^p = \bigvee_{i \in I_j^p} b_i$. For any $i \in I_j^p \triangleq \{i \in I | p_i = j\}$, it is clear that $p_i = j$. It follows from (19) that

$$x_j' = x_{p_i}' \geq a_{ip_i} \wedge x_{p_i}' \geq b_i, \quad \forall i \in I_j^p. \tag{21}$$

This implies that $x_j' \geq \bigvee_{i \in I_j^p} b_i = x_j^p$. Following Cases 1 and 2, it holds that $x_j^p \leq x_j'$, $\forall j \in J$. So we obtain

$$x^p \leq x'. \tag{22}$$

Inequalities (16) and (22) contribute to $x^p \leq x' \leq x'' \leq \hat{x}$. Hence, $[x', x''] \subseteq [x^p, \hat{x}]$. □

Define $p^* = (p_1^*, p_2^*, \ldots, p_m^*)$, where

$$p_i^* = \arg\max_{j \in J_i} \{\hat{x}_j\}, \tag{23}$$

$i = 1, 2, \ldots, m$. It is clear that $p^* \in P = J_1 \times J_2 \times \cdots \times J_m$ and

$$\hat{x}_{p_i^*} = \max_{j \in J_i} \{\hat{x}_j\}, \quad \forall i \in I. \tag{24}$$

Furthermore, we define the following vector x^{p^*} based on p^*. Denote $x^{p^*} = (x_1^{p^*}, x_2^{p^*}, \ldots, x_n^{p^*})$ and

$$x_j^{p^*} = \begin{cases} 0, & \text{if } I_j^{p^*} \triangleq \{i \in I | p_i^* = j\} = \emptyset, \\ \bigvee_{i \in I_j^{p^*}} b_i, & \text{if } I_j^{p^*} \triangleq \{i \in I | p_i^* = j\} \neq \emptyset. \end{cases} \tag{25}$$

Next, we verify that the above-defined vector x^{p^*} is exactly a solution of system (1). As a consequence, we obtain an interval solution of system (1) as $[x^{p^*}, \hat{x}]$.

Lemma 1: For arbitrary $i \in I$ and $j \in J$, it holds that $a_{ij} \wedge \hat{x}_j \leq d_i$.

Proof: Take arbitrary $j \in J$.

Case 1. If $I_j = \emptyset$, then by (3) and (4), $\hat{x}_j = 1$ and $a_{ij} \leq d_i$ for all $i \in I$. So we have

$$a_{ij} \wedge \hat{x}_j = a_{ij} \wedge 1 = a_{ij} \leq d_i, \quad \forall i \in I. \tag{26}$$

Case 2. If $I_j \neq \emptyset$, then by (4), $\hat{x}_j = \bigwedge_{k \in I_j} d_k$. When $i \notin I_j$, it follows from (3) that $a_{ij} \leq d_i$. Hence, $a_{ij} \wedge \hat{x}_j \leq a_{ij} \leq d_i$. When $i \in I_j$, we have $a_{ij} \wedge \hat{x}_j = a_{ij} \wedge (\bigwedge_{k \in I_j} d_k) \leq \bigwedge_{k \in I_j} d_k \leq d_i$.

Combining cases 1 and 2, we have $a_{ij} \wedge \hat{x}_j \leq d_i, \forall i \in I$. □

Theorem 5: Let x^{p^*} be the vector defined by (25) based on $p^* = (p_1^*, p_2^*, \ldots, p_m^*)$. Then, $[x^{p^*}, \hat{x}]$ is an interval solution of system (1).

Proof: In fact, we only have to verify that x^{p^*} is a solution of system (1). Take arbitrary $k \in I$.

Denote $j_k = p_k^* \in J_k$. By (25), it is clear that $k \in I_{j_k}^{p^*} \neq \emptyset$. So we have

$$x_{j_k}^{p^*} = \bigvee_{i \in I_{j_k}^{p^*}} b_i \geq b_k. \tag{27}$$

It follows from $j_k \in J_k$ and (11) that $a_{kj_k} \wedge \hat{x}_{j_k} \geq b_k$. This indicates that

$$a_{kj_k} \geq a_{kj_k} \wedge \hat{x}_{j_k} \geq b_k. \tag{28}$$

Inequalities (27) and (28) contribute to

$$(a_{k1} \wedge x_1^{p^*}) \vee (a_{k2} \wedge x_2^{p^*}) \vee \cdots \vee (a_{kn} \wedge x_n^{p^*}) \geq a_{kj_k} \wedge x_{j_k}^{p^*} \geq b_k. \tag{29}$$

For any $j \in J$, we further check the inequality that $a_{kj} \wedge x_j^{p^*} \le d_k$ in two cases. Case 1. If $I_j^{p^*} = \emptyset$, then $a_{kj} \wedge x_j^{p^*} = a_{kj} \wedge 0 = 0 \le d_k$. Case 2. If $I_j^{p^*} \neq \emptyset$, then $x_j^{p^*} = \bigvee_{i \in I_j^{p^*}} b_i$. For any $i \in I_j^{p^*}$, it follows from (25) that

$$j = p_i^* \in J_i. \tag{30}$$

Furthermore, according to (11), we have

$$\hat{x}_j \ge a_{ij} \wedge \hat{x}_j \ge b_i, \quad \forall i \in I_j^{p^*}. \tag{31}$$

This indicates that

$$x_j^{p^*} = \bigvee_{i \in I_j^{p^*}} b_i \le \bigvee_{i \in I_j^{p^*}} \hat{x}_j = \hat{x}_j. \tag{32}$$

Following Lemma 1,

$$a_{kj} \wedge x_j^{p^*} \le a_{kj} \wedge \hat{x}_j \le d_k. \tag{33}$$

Due to the arbitrariness of j, we obtain

$$(a_{k1} \wedge x_1^{p^*}) \vee (a_{k2} \wedge x_2^{p^*}) \vee \cdots \vee (a_{kn} \wedge x_n^{p^*}) \le d_k. \tag{34}$$

Considering inequalities (29) and (34), it is obvious that x^{p^*} is a solution of system (1). Hence, $[x^{p^*}, \hat{x}]$ is an interval solution. □

Theorem 6: The width of the interval solution $[x^{p^*}, \hat{x}]$ is

$$w[x^{p^*}, \hat{x}] = \min_{i \in I}\{\hat{x}_{p_i^*} - b_i\} \wedge \min_{j \in J - J^{p^*}}\{\hat{x}_j\},$$

where $J^{p^*} = \{p_1^*, p_2^*, \ldots, p_m^*\} \subseteq J$.

Proof: According to Definition 3, the width of the interval solution $[x^{p^*}, \hat{x}]$ is

$$\begin{aligned} w[x^{p^*}, \hat{x}] &= \min_{j \in J}\{\hat{x}_j - x_j^{p^*}\} \\ &= \min_{j \in J^{p^*}}\{\hat{x}_j - x_j^{p^*}\} \wedge \min_{j \in J - J^{p^*}}\{\hat{x}_j - x_j^{p^*}\} \end{aligned} \tag{35}$$

If $j \in J - J^{p^*}$, then $j \notin J^{p^*}$. This indicates that there does not exist any $i \in I$ such that $p_i^* = j$. Note that $I_j^{p^*} = \{i \in I | p_i^* = j\}$. We have $I_j^{p^*} = \emptyset$. It follows from (25) that $x_j^{p^*} = 0$. Hence

$$\min_{j \in J - J^{p^*}}\{\hat{x}_j - x_j^{p^*}\} = \min_{j \in J - J^{p^*}}\{\hat{x}_j\}. \tag{36}$$

If $j \in J^{p^*}$, then there exists $i \in I$ such that $p_i^* = j$. Thus, $I_j^{p^*} \neq \emptyset$. Note that $J^{p^*} = \{p_1^*, p_2^*, \ldots, p_m^*\} \subseteq J$ and $I_j^{p^*} = \{i \in I | p_i^* = j\}$. It is clear that

$$\bigcup_{j \in J^{p^*}} I_j^{p^*} = I. \tag{37}$$

For any $j \in J^{p^*}$, since $I_j^{p^*} \neq \emptyset$, it follows from (25) that $x_j^{p^*} = \bigvee_{i \in I_j^{p^*}} b_i$. Hence

$$\begin{aligned} \min_{j \in J^{p^*}}\{\hat{x}_j - x_j^{p^*}\} &= \min_{j \in J^{p^*}}\{\hat{x}_j - \bigvee_{i \in I_j^{p^*}} b_i\} \\ &= \min_{j \in J^{p^*}}\{\bigwedge_{i \in I_j^{p^*}}\{\hat{x}_j - b_i\}\} \\ &= \min_{j \in J^{p^*}} \min_{i \in I_j^{p^*}}\{\hat{x}_j - b_i\}. \end{aligned} \tag{38}$$

According to $I_j^{p^*} = \{i \in I | p_i^* = j\}$, it holds for arbitrary $j \in J^{p^*}$ that

$$p_i^* = j, \quad \forall i \in I_j^{p^*}. \tag{39}$$

Hence, for any $j \in J^{p^*}$, we have

$$\min_{j \in J^{p^*}} \min_{i \in I_j^{p^*}}\{\hat{x}_j - b_i\} = \min_{j \in J^{p^*}} \min_{i \in I_j^{p^*}}\{\hat{x}_{p_i^*} - b_i\}. \tag{40}$$

Considering (37), (38) and (40), we further obtain

$$\begin{aligned} \min_{j \in J^{p^*}}\{\hat{x}_j - x_j^{p^*}\} &= \min_{j \in J^{p^*}} \min_{i \in I_j^{p^*}}\{\hat{x}_j - b_i\} \\ &= \min_{j \in J^{p^*}} \min_{i \in I_j^{p^*}}\{\hat{x}_{p_i^*} - b_i\} \\ &= \min_{i \in I}\{\hat{x}_{p_i^*} - b_i\}. \end{aligned} \tag{41}$$

Equalities (35), (36) and (41) contribute to

$$w[x^{p^*}, \hat{x}] = \min_{i \in I}\{\hat{x}_{p_i^*} - b_i\} \wedge \min_{j \in J - J^{p^*}}\{\hat{x}_j\}.$$

The proof is complete. □

Theorem 7: For any $p = (p_1, p_2, \ldots, p_m) \in P$, define $x^p = (x_1^p, x_2^p, \ldots, x_n^p)$, where

$$x_j^p = \begin{cases} 0, & \text{if } I_j^p \triangleq \{i \in I | p_i = j\} = \emptyset, \\ \bigvee_{i \in I_j^p} b_i, & \text{if } I_j^p \triangleq \{i \in I | p_i = j\} \neq \emptyset. \end{cases} \tag{42}$$

Then, it holds that $w[x^p, \hat{x}] \le w[x^{p^*}, \hat{x}]$.

Proof: Let $J^{p^*} = \{p_1^*, p_2^*, \ldots, p_m^*\} \subseteq J$. Then

$$w[x^{p^*}, \hat{x}] = \min_{k \in J^{p^*}}\{\hat{x}_k - x_k^{p^*}\} \wedge \min_{k \in J - J^{p^*}}\{\hat{x}_k - x_k^{p^*}\}. \tag{43}$$

Take arbitrary $k \in J$.

Case 1. If $k \notin J^{p^*}$, i.e., $k \in J - J^{p^*}$, then according to the proof of Theorem 6, we have $I_k^{p^*} = \emptyset$ and $x_k^{p^*} = 0$. Hence

$$\hat{x}_k - x_k^{p^*} = \hat{x}_k \ge \hat{x}_k - x_k^p \ge \min_{j \in J}\{\hat{x}_j - x_j^p\} = w[x^p, \hat{x}]. \tag{44}$$

Case 2. If $k \in J^{p^*}$, then according to the proof of Theorem 6, we have $I_k^{p^*} \neq \emptyset$ and $x_k^{p^*} = \bigvee_{i \in I_k^{p^*}} b_i$. Obviously, there exists $i' \in I_k^{p^*}$ such that $b_{i'} = \bigvee_{i \in I_k^{p^*}} b_i$. Notice that $i' \in I_k^{p^*}$ indicates

$$p_{i'}^* = k. \tag{45}$$

Hence

$$\hat{x}_k - x_k^{p^*} = \hat{x}_{p_{i'}^*} - b_{i'}. \tag{46}$$

Denote

$$p_{i'} = k'. \tag{47}$$

It is clear that $k, k' \in J_{i'}$. It follows from (24) that

$$\hat{x}_{p^*_{i'}} \geq \hat{x}_{k'} = \hat{x}_{p_{i'}}. \tag{48}$$

$p_{i'} = k'$ also indicates $i' \in I^p_{k'} \neq \emptyset$. Thus

$$x^p_{k'} = \bigvee_{i \in I^p_{k'}} b_i \geq b_{i'}. \tag{49}$$

Considering (46)-(49), we have

$$\begin{aligned} \hat{x}_k - x^{p^*}_k &= \hat{x}_{p^*_{i'}} - b_{i'} \\ &\geq \hat{x}_{p_{i'}} - b_{i'} \\ &\geq \hat{x}_{p_{i'}} - \bigvee_{i \in I^p_{k'}} b_i \\ &= \hat{x}_{k'} - x^p_{k'} \\ &\geq \min_{j \in J}\{\hat{x}_j - x^p_j\} \\ &= w[x^p, \hat{x}]. \end{aligned} \tag{50}$$

Combining Cases 1 and 2, we have

$$\hat{x}_k - x^{p^*}_k \geq w[x^p, \hat{x}], \quad \forall k \in J. \tag{51}$$

Hence, $w[x^{p^*}, \hat{x}] = \min_{k \in J}\{\hat{x}_k - x^{p^*}_k\} \geq w[x^p, \hat{x}]$. □

Theorem 8: Let x^{p^*} be defined by (23) and (25), and $\hat{x}$ be the maximum solution of system (1). Then, $[x^{p^*}, \hat{x}]$ is the widest interval solution of (1).

Proof: According to Theorem 5, $[x^{p^*}, \hat{x}]$ is an interval solution of system (1). Moreover, its width is

$$w[x^{p^*}, \hat{x}] = \min_{i \in I}\{\hat{x}_{p^*_i} - b_i\} \wedge \min_{j \in J - J^{p^*}}\{\hat{x}_j\},$$

by Theorem 6. Let $[x', x''] \in X(A, b, d)$ be an arbitrary interval solution of (1). Then, by Theorem 4, there exists $p \in P$ such that $[x', x''] \subseteq [x^p, \hat{x}]$. It follows from Proposition 1 and Theorem 7 that

$$w[x', x''] \leq w[x^p, \hat{x}] \leq w[x^{p^*}, \hat{x}]. \tag{52}$$

Due to the arbitrariness of $[x', x'']$, it follows from Definition 5 that $[x^{p^*}, \hat{x}]$ is the widest interval solution of system (1). □

B. RESOLUTION PROCEDURES

Based on the theoretical results presented in the previous subsection, we summarize the resolution procedures for the widest interval solution of system (1) as follows.

Step 1. Compute the index sets $I_1, I_2, \ldots, I_n$ by (3).

Step 2. Compute the potential maximum solution $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ by (4).

Step 3. Following Theorem 1, check the consistency of system (1) by the above-obtained vector $\hat{x}$. If $\hat{x} \in X(A, b, d)$, then system (1) is consistent and goes to the next step. Otherwise, system (1) is inconsistent and has no widest interval solution.

Step 4. Compute the index sets $J_1, J_2, \ldots, J_m$ by (11).

Step 5. Compute the optimal indexes $p^*_1, p^*_2, \ldots, p^*_m$ by (23) and denote $p^* = (p^*_1, p^*_2, \ldots, p^*_m)$.

Step 6. Based on the above-obtained p^*, compute the index sets $I^{p^*}_1, I^{p^*}_2, \ldots, I^{p^*}_n$, where $I^{p^*}_j = \{i \in I | p^*_i = j\}, \forall j \in J$.

Step 7. Based on the above-obtained index sets $I^{p^*}_1, I^{p^*}_2, \ldots, I^{p^*}_n$, compute the vector $x^{p^*} = (x^{p^*}_1, x^{p^*}_2, \ldots, x^{p^*}_n)$ by (25).

Step 8. Combining x^{p^*} and $\hat{x}$, we find the widest interval solution of system (1) as $[x^{p^*}, \hat{x}]$ according to Theorem 8.

Our proposed resolution procedures are represented in Fig. 2.

C. NUMERICAL EXAMPLE

Example 1: Assume a P2P educational information resource sharing system with six terminals is reduced into the following max-min fuzzy relation inequalities: system (53).

$$\begin{cases} 0.55 \leq (0.5 \wedge x_1) \vee (0.7 \wedge x_2) \vee (0.5 \wedge x_3) \vee (0.4 \wedge x_4) \\ \vee (0.3 \wedge x_5) \vee (0.8 \wedge x_6) \leq 0.8, \\ 0.6 \leq (0.7 \wedge x_1) \vee (0.6 \wedge x_2) \vee (0.5 \wedge x_3) \vee (0.6 \wedge x_4) \\ \vee (0.8 \wedge x_5) \vee (0.4 \wedge x_6) \leq 0.7, \\ 0.7 \leq (0.6 \wedge x_1) \vee (0.9 \wedge x_2) \vee (0.8 \wedge x_3) \vee (0.3 \wedge x_4) \\ \vee (0.5 \wedge x_5) \vee (0.7 \wedge x_6) \leq 0.8, \\ 0.75 \leq (0.8 \wedge x_1) \vee (0.7 \wedge x_2) \vee (0.6 \wedge x_3) \vee (0.95 \wedge x_4) \\ \vee (0.8 \wedge x_5) \vee (0.5 \wedge x_6) \leq 0.9, \\ 0.7 \leq (0.5 \wedge x_1) \vee (0.85 \wedge x_2) \vee (0.3 \wedge x_3) \vee (0.7 \wedge x_4) \\ \vee (0.9 \wedge x_5) \vee (0.8 \wedge x_6) \leq 0.8, \\ 0.65 \leq (0.9 \wedge x_1) \vee (0.4 \wedge x_2) \vee (0.6 \wedge x_3) \vee (0.5 \wedge x_4) \\ \vee (0.8 \wedge x_5) \vee (0.6 \wedge x_6) \leq 0.75, \end{cases} \tag{53}$$

We aim to find the widest interval solution of the system (53).

Solution:

The matrix form of the above system (53) is

$$b^T \leq A \circ x^T \leq d^T,$$

in which

$$A = \begin{bmatrix} 0.5 & 0.7 & 0.5 & 0.4 & 0.3 & 0.8 \\ 0.7 & 0.6 & 0.5 & 0.6 & 0.8 & 0.4 \\ 0.6 & 0.9 & 0.8 & 0.3 & 0.5 & 0.7 \\ 0.8 & 0.7 & 0.6 & 0.95 & 0.8 & 0.5 \\ 0.5 & 0.85 & 0.3 & 0.7 & 0.9 & 0.8 \\ 0.9 & 0.4 & 0.6 & 0.5 & 0.8 & 0.6 \end{bmatrix},$$

and $x = (x_1, x_2, \ldots, x_6)$, $b = (0.55, 0.6, 0.7, 0.75, 0.7, 0.65)$, $d = (0.8, 0.7, 0.8, 0.9, 0.8, 0.75)$.

Next, we attempt to find the widest interval solution of system (53) following our proposed resolution procedures.

Step 1. By (3), it is easy to compute the index sets as $I_1 = \{6\}$, $I_2 = \{3, 5\}$, $I_3 = \emptyset$, $I_4 = \{4\}$, $I_5 = \{2, 5, 6\}$, and $I_6 = \emptyset$.

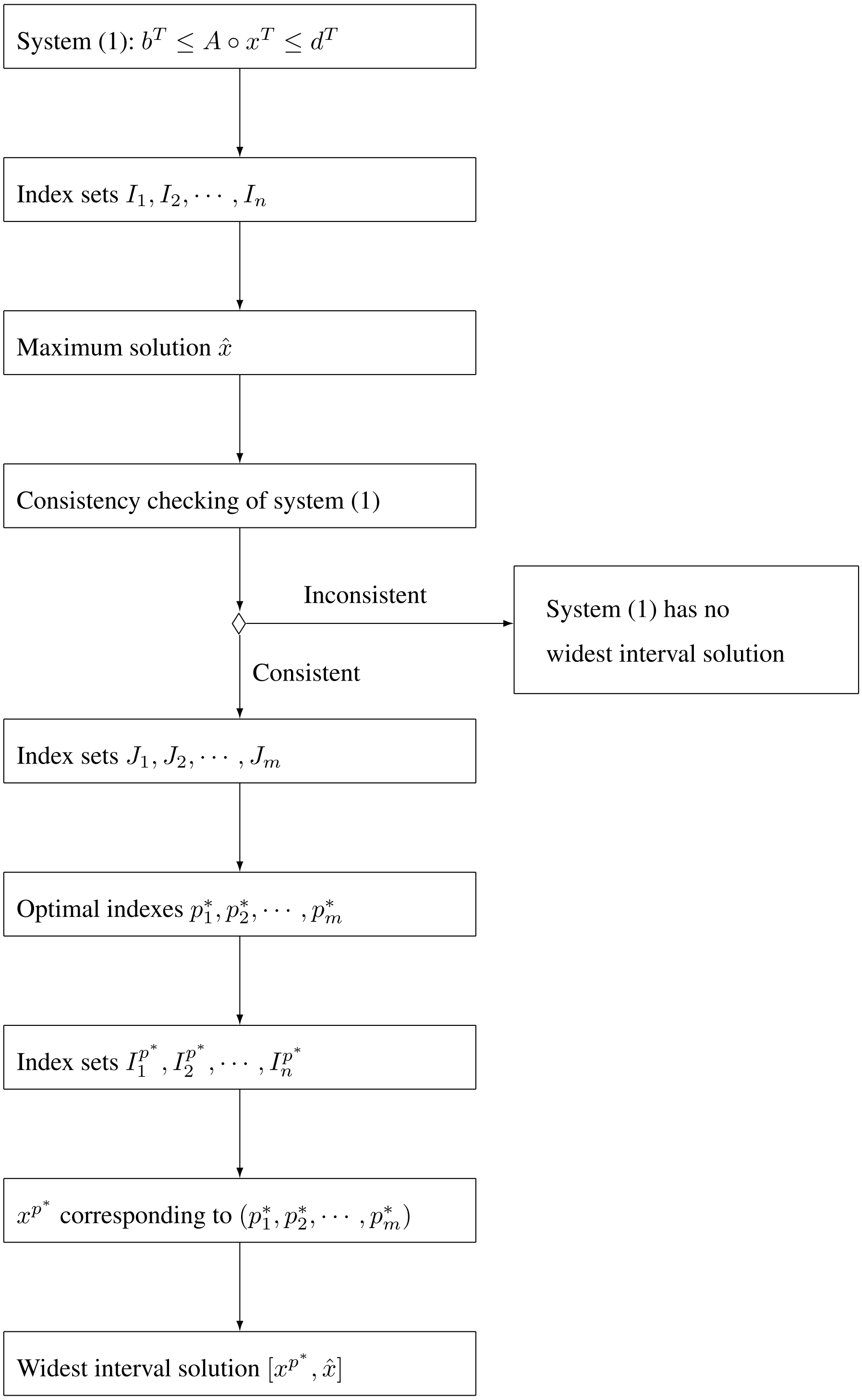

FIGURE 2. Flowchart of resolution procedures.

Step 2. Based on the index sets obtained in Step 1, we can compute the potential maximum solution $\hat{x}$ by (4) as follows. Since $I_3 = I_6 = \emptyset$, it is clear that $\hat{x}_3 = \hat{x}_6 = 1$. In addition,

$$\begin{aligned} \hat{x}_1 &= \bigwedge_{i \in I_1} d_i = d_6 = 0.75, \\ \hat{x}_2 &= \bigwedge_{i \in I_2} d_i = d_3 \wedge d_5 = 0.8 \wedge 0.8 = 0.8, \\ \hat{x}_4 &= \bigwedge_{i \in I_4} d_i = d_4 = 0.9, \\ \hat{x}_5 &= \bigwedge_{i \in I_5} d_i = d_2 \wedge d_5 \wedge d_6 = 0.7 \wedge 0.8 \wedge 0.75 = 0.7. \end{aligned} \tag{54}$$

Hence, the potential maximum solution is $\hat{x} = (0.75, 0.8, 1, 0.9, 0.7, 1)$.

Step 3. It is easy to determine that $\hat{x}$ satisfies all inequalities in (53). Thus, $\hat{x} \in X(A, b, d)$ is a solution of system (53). It follows from Theorem 1 that system (53) is consistent, and we continue to the next step.

Step 4. By (11), we compute the index sets $J_1, J_2, \ldots, J_6$ as $J_1 = \{2, 6\}$, $J_2 = \{1, 2, 4, 5\}$, $J_3 = \{2, 3, 6\}$, $J_4 = \{1, 4\}$, $J_5 = \{2, 4, 5, 6\}$, $J_6 = \{1, 5\}$.

Step 5. According to (23), we have

$$\begin{aligned} p_1^* &= \arg\max_{j\in J_1}\{\hat{x}_j\} = \arg\max\{\hat{x}_2, \hat{x}_6\} \\ &= \arg\max\{0.8, 1\} = 6, \\ p_2^* &= \arg\max_{j\in J_2}\{\hat{x}_j\} = \arg\max\{\hat{x}_1, \hat{x}_2, \hat{x}_4, \hat{x}_5\} \\ &= \arg\max\{0.75, 0.8, 0.9, 0.7\} = 4, \\ p_3^* &= \arg\max_{j\in J_3}\{\hat{x}_j\} = \arg\max\{\hat{x}_2, \hat{x}_3, \hat{x}_6\} \\ &= \arg\max\{0.8, 1, 1\} = 3, \\ p_4^* &= \arg\max_{j\in J_4}\{\hat{x}_j\} = \arg\max\{\hat{x}_1, \hat{x}_4\} \\ &= \arg\max\{0.75, 0.9\} = 4, \\ p_5^* &= \arg\max_{j\in J_5}\{\hat{x}_j\} = \arg\max\{\hat{x}_2, \hat{x}_4, \hat{x}_5, \hat{x}_6\} \\ &= \arg\max\{0.8, 0.9, 0.7, 1\} = 6, \\ p_6^* &= \arg\max_{j\in J_6}\{\hat{x}_j\} = \arg\max\{\hat{x}_1, \hat{x}_5\} \\ &= \arg\max\{0.75, 0.7\} = 1. \end{aligned} \tag{55}$$

Hence, $p^* = (6, 4, 3, 4, 6, 1)$.

Step 6. Compute the index set $I_j^{p^*}$ by $I_j^{p^*} = \{i \in I | p_i^* = j\}$, for $j = 1, 2, \ldots, 6$. Since

$$p_6^* = 1, \quad p_3^* = 3, \quad p_2^* = p_4^* = 4, \quad p_1^* = p_5^* = 6,$$

we have $I_1^{p^*} = \{6\}$, $I_2^{p^*} = \emptyset$, $I_3^{p^*} = \{3\}$, $I_4^{p^*} = \{2, 4\}$, $I_5^{p^*} = \emptyset$, $I_6^{p^*} = \{1, 5\}$.

Step 7. Compute the vector $x^{p^*} = (x_1^{p^*}, x_2^{p^*}, \ldots, x_6^{p^*})$ by (25), we have $x_2^{p^*} = x_5^{p^*} = 0$ since $I_2^{p^*} = I_5^{p^*} = \emptyset$. Additionally,

$$\begin{aligned} x_1^{p^*} &= \bigvee_{i\in I_1^{p^*}} b_i = b_6 = 0.65, \\ x_3^{p^*} &= \bigvee_{i\in I_3^{p^*}} b_i = b_3 = 0.7, \\ x_4^{p^*} &= \bigvee_{i\in I_4^{p^*}} b_i = b_2 \vee b_4 = 0.6 \vee 0.75 = 0.75, \\ x_6^{p^*} &= \bigvee_{i\in I_6^{p^*}} b_i = b_1 \vee b_5 = 0.55 \vee 0.7 = 0.7. \end{aligned} \tag{56}$$

Hence, $x^{p^*} = (0.65, 0, 0.7, 0.75, 0, 0.7)$. Moreover, it follows from Theorem 6 that the width of $[x^{p^*}, \hat{x}]$ is $w[x^{p^*}, \hat{x}] = 0.1$.

Step 8. Following Theorem 8,

$$\begin{aligned} &[x^{p^*}, \hat{x}] \\ &= ([0.65, 0.75], [0, 0.8], [0.7, 1], [0.75, 0.9], [0, 0.7], [0.7, 1]) \end{aligned}$$

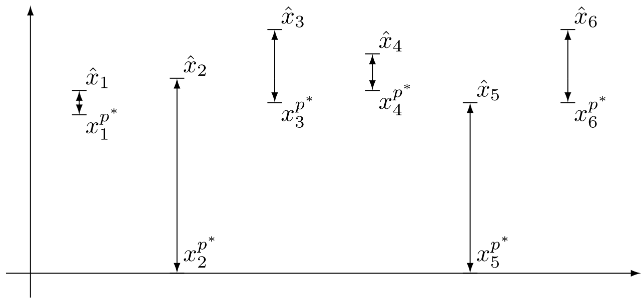

FIGURE 3. The widest interval solution of system (53).

is the widest interval solution (see Fig. 3.) of system (53) with width 0.1.

V. COMPARISON WITH THE EXISTING WORKS

In this section, we compare our studied problems with those in the relevant existing works.

(i) In the existing works [18], [46], [47], the system of max-min fuzzy relation inequalities or equations was assumed to be inconsistent. Under this assumption, there was no solution for such a system. As a consequence, the so-called approximate solution was defined and studied for the inconsistent fuzzy relation system with max-min composition [18], [47].

(ii) Solving the complete solution set of max-min fuzzy relation inequalities or equations is always an important research topic [1], [19], [26], [35]. The solution set of a consistent max-min system can be represented by a union of a finite number of closed intervals. However, it has been verified that solving the complete solution set is hard in most cases [1], [4]. Instead of solving the complete solution set, we focus only on the widest interval solution of the max-min fuzzy relation system in this paper.

(iii) Some of the existing works focused on the minimal solution(s) of a fuzzy relation system [48]–[51]. It is well known that the set of all minimal solutions is exactly a finite set. However, the number of minimal solutions might increase exponentially associated with the increase in the problem size. Hence, it is not easy and it is unnecessary to obtain all the minimal solutions. In this paper, our resolution method successfully avoids obtaining all the minimal solutions.

(iv) Other works have attempted to search for the optimal solution of the optimization problem subject to fuzzy relation inequalities or equations [5]–[7], [27], [28], [35], [45]. In fact, these optimal solutions can be viewed as specific solutions to the fuzzy relation constraints. However, these optimal solutions are usually unstable and fragile. Any minor fluctuations in the components of these optimal solutions will make them no longer feasible. Any minor fluctuation is not permitted for the optimal solution. To overcome the frangibility of the optimal solution, this paper further studies the so-called interval solution, which allows the solution to fluctuate in a certain range.

VI. CONCLUSION

In the existing works [18], [46], [47], the max-min fuzzy relation inequalities, i.e., system (1), were introduced to model a P2P sharing system. The authors only considered the inconsistent case and studied the approximate solution of the max-min fuzzy relation system. In theory, the complete solution of system (1) can be obtained; however, this is difficult, since it is equivalent to the set-covering problem, a famous NP-hard problem. Instead of solving system (1) completely, optimal solutions to some kinds of optimization problems with fuzzy relation constraints were investigated [5], [6], [27], [35], [42]–[45]. However, no perturbation was permitted to these optimal solutions. Minor fluctuations in their components will make them no longer optimal or even no longer feasible in the max-min fuzzy relation system. In this paper, we study the interval solution, which allows the solution to fluctuate to some degree. The fluctuation range is determined by the width of the interval solution. To maximize the allowable fluctuation range, we define and investigate the widest interval solution. A detailed resolution method is proposed to find the widest interval solution of system (1). The resolution procedures are illustrated by a numerical example. In the future, we will further extend the concept of the widest interval solution to other types of fuzzy relation systems.

ACKNOWLEDGMENT

The authors are grateful to the Associate Editor and the anonymous referees for their valuable comments and suggestions, which helped improve their manuscript.

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